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LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS

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NASA TM-76585 |  |  | LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS | October 1981 |  | V.I. Korolev |  | Leo Kanner Associates Redwood City, California 94063 |  | NASw-3541 | Translation |  |  | Translation of "Sloistyye anizotropnyye plastinki i obolochki iz armirovannykh plastmass," Mashinostroyeniye Press, Moscow, 1965, pp. 1-272 | Basic technical theories and engineering calculation equations for anisotropic plates and shells made of rigid reinforced plastics, mainly laminated fiberglass, are presented and discussed. Solutions are given for many problems of design of structural plates and shells, including curved sections and tanks, as well as two chapters on selection of the optimum materials, are given. Accounting for interlayer shearing and transverse separation, which are new engineering properties, are discussed. Application of the results obtained to thin three-ply plates and shells with a light elastic filler is presented and discussed. |  | Unlimited-Unclassified | Unclassified | Unclassified |
The basic technical theories of anisotropic plates and shells made of rigid reinforced plastics are reported in the book. Solutions of numerous technical problems most often encountered in engineering practice are obtained, with recommendations on efficient design of elastic reinforced plastic parts. Some sections are entirely devoted to questions of selection of the optimum structure of the material.

The results obtained are valid for thin three ply plates and shells, if appropriate substitution is made of the rigidity parameters, which are among the most efficient stress schemes for reinforced plastic structures.

The book is intended for engineering and technical workers who are engaged in the development of thin walled reinforced plastic structures.
Shells of laminated plastics made on a base of continuous fillers and various synthetic binders are multiply anisotropic, heterogeneous elastic systems. Some simplifying hypotheses permit study of deformed, stressed states of a shell to be reduced to study of bending and deformation of a surface of reduction. In this case, the system of stresses which develop in normal sections of the shell are replaced by a statically equivalent system of elastic forces and moments.

It follows from such a reduction that, to within the assumptions made, a structurally nonuniform laminated shell can be considered a uniform shell, i.e., laminated plastics can be considered uniform materials, which have some reduced properties which are determined by the properties of the initial components and the mutual location and orientation of the reinforcing filler.

Skillful use of the abovementioned properties permits the development of extremely efficient laminated plastic structures which, in many cases, are not inferior in efficiency and technical characteristics to excellent structures made of traditional materials. The defects inherent in welded metal structures, which significantly reduce the critical loads or result in premature destruction, are absent in thin walled laminated structures.

With the appearance of new structural materials, fiberglass plastics, the field of use of laminated plastics in engineering broadened substantially, and the technical and economic advantages of their use increased sharply. The development of methods of calculation of thin walled laminated plastic structures becomes of great practical importance in this connection. The attention of investigators was first drawn to these questions by the founder of the Soviet School of Anisotropic Plates S.G. Lekhnitskiy, the results of many years of study of which are reported in his monographs [16, 17].

The laminated plastics used in engineering have symmetrical elastic properties in the majority of cases, i.e., they are orthotropic materials. However, their principal directions of anisotropy may not coincide with the directions of the coordinate axes and, consequently, it becomes necessary to consider the elasticity relationships which correspond to the general case of anisotropy. For orthotropic materials, there are reliable methods of determination of the necessary mechanical characteristics in two principal directions of anisotropy. Moreover, fundamentally new characteristics of the laminated orthotropic material must be known, which usually do not have to be dealt with in isotropic uniform shells, namely: the shearing strength by layer and the transverse separation strength. These new characteristics of laminated plastics are associated with their structural inhomogeneity and the significant difference of the elastic and strength properties under various types of loads.

This book reports an approximate method of accounting for the effect of interlayer shear on the stressed, deformed state of laminated anisotropic plates and shells. In selection of simplifying hypotheses
for study of thin laminated shells, it was considered that the elastic properties of existing cements and binders are appreciably less than the corresponding elastic characteristics of the reinforcing fillers and, consequently, the interlayer shears which develop in the bending of laminated shells can significantly distort the pattern of the deformed state described by the hypotheses of nondeformable normals extensively used in the theory of shells, especially when the shell operates under heating conditions.

The results of thorough studies of thin laminated anisotropic shells, with interlayer shear and transverse deformation taken into account, are reported in the monograph of S.A. Ambartsumyan [1]. Since the corresponding rigidity parameters of a laminated sheet differ significantly, allowance for transverse deformation gives an extremely insignificant correction, and we will disregard its effect.

The proposed approximate method of calculation of laminated shells was used in study of three ply plates and shells, and it showed satisfactory correspondence with experimental results. Besides the usual elastic characteristics of a laminated shell, two new ones appear $K_1$ and $K_2$, which define the connection of the cross forces to interlayer shears of the mean surface and characterize the resistance of the laminated shell to such shears in two mutually orthogonal directions. Laminated shell rigidity parameters $K_1$ and $K_2$ are determined experimentally in transverse bending tests of laminated strips and, consequently, they somewhat compensate the errors which are tolerated by the initial hypotheses adopted.

The results obtained in the work are valid for three ply plates and shells with a light elastic filler, if appropriate rigidity parameters are used. This question is presented in detail in the last chapter, where some stability problems characteristic only of three ply plates and shells with elastic fillers also are discussed.

In distinction from the traditional courses on the theory of shells, the author attempted to discuss problems connected with efficient design of plates and shells made of reinforced plastics, subjected to the effects of the loads most frequently encountered in engineering practice. Chapters 8 and 13 are completely devoted to questions of selection of the optimum structure of the material of cylindrical shells operating under axisymmetric loads.

The present work does not cover many questions raised by modern engineering practice and the needs of machine building. Problems connected with large displacements of the mean surface of a shell, including problems of stability are not touched on. Nonlinear elastic and inelastic deformations of laminated shells are not discussed, and questions of nonlinear oscillations are not covered. There is no doubt that they will be treated in the works of other investigators in the near future.
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LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS

V.I. Korolev

CHAPTER 1. BASIC EQUATIONS OF TECHNICAL THEORY OF ANISOTROPIC PLATES AND SHELLS

1. Initial Hypotheses and Basic Relationships

We will consider laminated shells produced on a base of continuous fillers and synthetic binders as uniform anisotropic systems, the elastic constants of which are determined by conventional methods.

We use the rectilinear element hypothesis [14] for description of the deformed state of the shell. This is satisfactorily consistent with experimental results for three ply plates with a light elastic filler.

Based on this hypothesis and the assumption that the normal stresses are independent of interlayer shear for the stresses which arise in normal sections of a shell, we have the following expressions

\[
\begin{align*}
\sigma_1 &= c_{11}(\varepsilon_1 + z\kappa_1^e) + c_{12}(\varepsilon_2 + z\kappa_2^e) + c_{13}(\omega + 2z\kappa_3^e); \\
\sigma_2 &= c_{22}(\varepsilon_1 + z\kappa_1^e) + c_{23}(\varepsilon_2 + z\kappa_2^e) + c_{33}(\omega + 2z\kappa_3^e); \\
\tau &= c_{13}(\varepsilon_1 + z\kappa_1^e) + c_{23}(\varepsilon_2 + z\kappa_2^e) + c_{33}(\omega + 2z\kappa_3^e),
\end{align*}
\]

where \(c_{ij}\) (\(i,j=1, 2, 3\)) are the elastic constants of the material; \(\varepsilon_1, \varepsilon_2, \omega\) are the relative elongation and shear of the mean surface of the shell; \(\kappa_1^e, \kappa_2^e, \kappa_3^e\) are the effective changes in curvature and torsion of the mean surface of the shell.

As in [14], for shearing stresses which act between the layers of the shell, the following expressions can be obtained

\[
\begin{align*}
\tau_1 &= \frac{3}{2} \frac{(\delta^3 - \delta)^2}{\delta^3} Q_1; \\
\tau_2 &= \frac{3}{2} \frac{(\delta^3 - \delta)^2}{\delta^3} Q_2.
\end{align*}
\]

In distinction from thin three ply shells with an elastic filler, shearing through the layers in a laminate shell change by a parabolic relationship and disappear at the bounding surfaces \(z=\pm\delta/2\).

We will account for the effect of surface load \(X, Y\) on the shearing stresses in the membrane solution, on the assumption, as in [1], that the shearing stresses change linearly through the thickness of the shell.

Numbers in the margin indicate pagination in the foreign text.
shell.

We will characterize the shearing stresses between the layers which correspond to shearing stresses (2) by the maximum shears which arise in the mean surface of the shell. Like the stresses, shears through the thickness of the shell change parabolically.

By reducing stress system (1)-(2) to the statically equivalent system of elastic forces \( T_1, T_2, S \) and moments \( G_1, G_2, H \), the following elasticity relationships can be obtained for a laminated anisotropic shell

\[
T_1 = B_{11} \varepsilon_1 + B_{12} \varepsilon_2 + B_{13} \omega;
\]
\[
T_2 = B_{21} \varepsilon_1 + B_{22} \varepsilon_2 + B_{23} \omega;
\]
\[
S = B_{31} \varepsilon_3 + B_{32} \varepsilon_2 + B_{33} \omega;
\]
\[
G_1 = -D_{11} \chi_1^1 - D_{12} \chi_2^1 - 2D_{13} \chi_3^1;
\]
\[
G_2 = -D_{12} \chi_1^2 - D_{22} \chi_2^1 - 2D_{23} \chi_3^1;
\]
\[
H = -D_{13} \chi_1^3 - D_{23} \chi_2^3 - 2D_{33} \chi_3^3;
\]
\[
Q_1 = -K_1 Y_1; \quad Q_2 = -K_2 Y_2.
\]

where shell rigidity parameters \( D_{ij}, B_{ij}, K_1, K_2 \) (1, \( j=1, 2, 3 \)), and shell thickness \( \delta \) by the following expressions

\[
B_{ij} = c_{ij}; \quad D_{ij} = \frac{1}{12} c_{ij} \delta^4 (i, j = 1, 2, 3);
\]
\[
K_1 = \frac{5}{6} G_{13} \delta; \quad K_2 = \frac{5}{6} G_{23} \delta.
\]

where \( G_{13}, G_{23} \) are the moduli of elasticity in interlayer shearing.

The rigidity parameters of a laminated shell also can be determined from the simplest experiments, which are described schematically in Section 8.

By solving expressions (3) for the components of deformation, the known relationships, which will be needed subsequently, can be obtained

\[
\varepsilon_1 = \frac{1}{6} (a_{11} T_1 + a_{12} T_2 + a_{13} S);
\]
\[
\varepsilon_2 = \frac{1}{6} (a_{21} T_1 + a_{22} T_2 + a_{23} S);
\]
\[
\varepsilon_3 = \frac{1}{6} (a_{31} T_1 + a_{32} T_2 + a_{33} S).
\]
In the general case of anisotropy, elastic constants \( a_{ij}(i, j=1, 2, 3) \) are independent, and they are expressed through elastic constants \( c_{ij} \) by the expressions

\[
\begin{align*}
a_{11} &= \frac{c_{11} - c_{12}}{\Delta} ; \\
a_{12} &= \frac{c_{11} - c_{13}}{\Delta} ; \\
a_{22} &= \frac{c_{11} - c_{22}}{\Delta} ; \\
a_{13} &= \frac{c_{11} - c_{13}}{\Delta} ; \\
\Delta &= c_{11}c_{22}c_{33} + 2c_{11}c_{12}c_{13} - c_{11}c_{33} - \\
&- c_{12}c_{22} - c_{23}c_{13}.
\end{align*}
\] (9)

The deformed state of a laminated shell is defined by five random functions: three components of displacement \( u, v, w \) in the directions of coordinate axes \( a_1, a_2 \), which coincide with the lines of curvature of the mean surface of the shell, and by the external normals to the mean surface and by two functions \( \phi, \psi \) which characterize the bending of the shell without allowance for the effect of interlayer shearing.

\[
\begin{align*}
\epsilon_1 &= \frac{1}{A_1} \frac{\partial u}{\partial a_1} + \frac{\nu}{A_1A_2} \frac{\partial A_1}{\partial a_2} + \frac{\omega}{R_t} ; \\
\epsilon_2 &= \frac{1}{A_2} \frac{\partial v}{\partial a_2} + \frac{\nu}{A_1A_2} \frac{\partial A_2}{\partial a_1} + \frac{\omega}{R_t} ; \\
\omega &= \frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left( \frac{v}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial a_2} \left( \frac{u}{A_2} \right) ; \\
\chi^e_1 &= \frac{1}{A_1} \frac{\partial \phi}{\partial a_1} + \frac{\psi}{A_1A_2} \frac{\partial A_1}{\partial a_2} ; \\
\chi^e_2 &= \frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\phi}{A_1A_2} \frac{\partial A_2}{\partial a_1} ; \\
2\chi^e_3 &= \frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left( \frac{\psi}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial a_2} \left( \frac{\phi}{A_2} \right) ; \\
\gamma_1 &= \phi + \frac{1}{A_1} \frac{\partial \omega}{\partial a_1} - \frac{u}{R_t} ; \\
\gamma_2 &= \psi + \frac{1}{A_2} \frac{\partial \omega}{\partial a_2} - \frac{v}{R_t} .
\end{align*}
\] (10) (11) (12)

As curvilinear coordinates \( a_1, a_2 \) on the mean surface of the shell, it is advisable to consider only coordinates which form a rectilinear regular grid of coordinate lines. Geometrically, this condition is reduced to the requirement that vectors \( \overrightarrow{a_1}, \overrightarrow{a_2} \), which are tangent to coordinate lines \( a_1, a_2 \), not the collinear. For this, it is necessary analytically that, of the three Jacobians...
(where x, y, z are coordinates in the vector parametric representation of the surface) at least one be different from zero.

If the coordinate lines coincide with the lines of curvature of the mean surface, the curvilinear coordinate is orthogonal, and we subsequently will use primarily such a coordinate system.

In such a coordinate system, Lame's constants $A_1$, $A_2$ are determined by the expressions

$$
A_1 = \sqrt{\left(\frac{\partial z}{\partial u_1}\right)^2 + \left(\frac{\partial z}{\partial u_2}\right)^2 + \left(\frac{\partial z}{\partial u_3}\right)^2};
$$

$$
A_2 = \sqrt{\left(\frac{\partial z}{\partial v_1}\right)^2 + \left(\frac{\partial z}{\partial v_2}\right)^2 + \left(\frac{\partial z}{\partial v_3}\right)^2}. \tag{14}
$$

The equations of continuity of deformation for laminated anisotropic shells are written as in the case of isotropic shells, and they have the form

$$
\begin{align*}
\frac{\partial A_1}{\partial u_1} x_1 - \frac{1}{A_1} \frac{\partial A_2}{\partial u_1} x_2 - \frac{\partial A_3}{\partial u_1} x_3 - \\
- \frac{1}{H_1} \left( \frac{\partial A_1}{\partial u_1} \varepsilon_1 - \frac{1}{A_1} \frac{\partial A_2}{\partial u_1} \frac{\omega}{2} - \frac{\partial A_3}{\partial u_1} \varepsilon_3 \right) + \\
+ \frac{\partial}{\partial u_1} \left( A_2 \frac{\omega}{2} \right) + \frac{1}{H_2} \frac{\partial A_1}{\partial u_2} \omega = 0;
\end{align*}
$$

$$
\begin{align*}
\frac{\partial A_1}{\partial u_2} x_2 - \frac{1}{A_2} \frac{\partial A_2}{\partial u_2} x_2 - \frac{\partial A_3}{\partial u_2} x_3 - \\
- \frac{1}{H_2} \left( \frac{\partial A_1}{\partial u_2} \varepsilon_2 - \frac{1}{A_2} \frac{\partial A_2}{\partial u_2} \frac{\omega}{2} - \frac{\partial A_3}{\partial u_2} \varepsilon_3 \right) + \\
+ \frac{\partial}{\partial u_2} \left( A_1 \frac{\omega}{2} \right) + \frac{1}{H_1} \frac{\partial A_1}{\partial u_1} \omega = 0;
\end{align*}
$$

$$
\begin{align*}
\frac{\partial A_1}{\partial u_3} x_3 - \frac{1}{A_3} \frac{\partial A_2}{\partial u_3} x_2 - \frac{\partial A_3}{\partial u_3} x_2 - \\
- \frac{1}{H_3} \left( \frac{\partial A_1}{\partial u_3} \varepsilon_3 - \frac{1}{A_3} \frac{\partial A_2}{\partial u_3} \frac{\omega}{2} - \frac{\partial A_3}{\partial u_3} \varepsilon_3 \right) + \\
+ \frac{\partial}{\partial u_3} \left( A_2 \frac{\omega}{2} \right) = 0;
\end{align*}
$$

here $\varepsilon_1$, $\varepsilon_2$, $\omega$ are the components of deformation and shear of the mean surface of the shell determined by expressions (10), and $\kappa_1$, $\kappa_2$, $\kappa_3$ are
the total changes of curvature and torsion of the mean surface, which can be determined by the following expressions

\[ \begin{align*}
x_1 &= x_1^* - \frac{1}{A_1} \frac{\partial y_1}{\partial a_3} - \frac{y_1}{A_1 A_3} \frac{\partial A_3}{\partial a_3}; \\
x_2 &= x_2^* - \frac{1}{A_2} \frac{\partial y_2}{\partial a_3} - \frac{y_2}{A_2 A_3} \frac{\partial A_3}{\partial a_3}; \\
2\kappa_2 &= 2\kappa_2^* - \left[ \frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left( \frac{y_1}{A_1} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial a_2} \left( \frac{y_1}{A_1} \right) \right].
\end{align*} \] (16)

2. Equations of Equilibrium of Laminated Shell and Boundary Conditions

If the positive directions of the elastic forces and moments are selected as indicated in Fig. 1, the equations of equilibrium in the system of curvilinear orthogonal coordinates selected can be written in conventional form

\[ \begin{align*}
\frac{\partial (A_2 T_1)}{\partial a_1} + \frac{\partial (A_1 S)}{\partial a_2} - T_2 \frac{\partial A_2}{\partial a_3} + S \frac{\partial A_1}{\partial a_3} - A_1 A_3 \frac{Q_1}{R_1} + & \\
& + A_1 A_3 X = 0;
\end{align*} \]

\[ \begin{align*}
\frac{\partial (A_2 T_2)}{\partial a_1} + \frac{\partial (A_1 S)}{\partial a_2} - T_2 \frac{\partial A_2}{\partial a_3} + S \frac{\partial A_1}{\partial a_3} - A_1 A_3 \frac{Q_2}{R_2} + A_1 A_3 Y = & 0;
\end{align*} \]

\[ \begin{align*}
\frac{\partial (A_2 G_1)}{\partial a_1} + \frac{\partial (A_1 G_1)}{\partial a_2} + A_1 A_3 \left( \frac{T_1}{R_1} + \frac{T_2}{R_2} \right) - A_1 A_3 Z = & 0;
\end{align*} \]

\[ \begin{align*}
\frac{\partial (A_2 G_2)}{\partial a_1} + \frac{\partial (A_1 G_2)}{\partial a_2} - G_2 \frac{\partial A_2}{\partial a_3} + H \frac{\partial A_1}{\partial a_3} - A_1 A_3 Q_1 = & 0;
\end{align*} \]

\[ \begin{align*}
\frac{\partial (A_2 G_3)}{\partial a_1} + \frac{\partial (A_1 G_3)}{\partial a_2} - G_3 \frac{\partial A_2}{\partial a_3} + H \frac{\partial A_1}{\partial a_3} - A_1 A_3 Q_2 = & 0;
\end{align*} \] (17)

in addition, for determination of the normal transverse stress \( \sigma_z \), the following expression can be obtained, which characterizes equilibrium of the internal and external forces in the normal direction

\[ \begin{align*}
\sigma_z = \frac{1}{28} \left[ Z^+ (6^3 + 36z - 4z^3) - Z^- (6^3 - 36z + 4z^3) \right] + & \\
& + \frac{(6-4z^2)}{28} \left[ 3 \left( \frac{C}{R_1} + \frac{G_1}{R_2} \right) - z \left( \frac{T_1}{R_1} + \frac{T_2}{R_2} \right) \right],
\end{align*} \] (18)

where \( Z^+, Z^- \) are components of the normal surface load applied to the upper \( z=\delta/2 \) and lower \( z=-\delta/2 \) bounding surfaces of the shell, respectively.

The boundary conditions for an anisotropic laminated shell, the deformed state of which is described by the rectilinear element hypotheses, differ significantly from the boundary conditions for shells which are deformed by the direct normals principle. We subsequently limit ourselves to cases when the edge of the shell coincides with the coordinate lines. In the case of closed shells or shells closed in one direction,
the boundary conditions are replaced by the corresponding periodicity conditions, which should ensure well defined movement at any point of the closed coordinate line.

Fig. 1. Adopted positive directions of elastic forces and moments.

Variation of the potential energy of a laminated shell is determined by the following expression

$$
\delta U = \iint (T_1 \delta z_1 + T_2 \delta z_2 + S \delta \omega - G_1 \delta \kappa^1 - G_2 \delta \kappa^2 - 2H \delta \kappa^3 - Q_1 \delta \gamma_1 - Q_2 \delta \gamma_2 + X \delta u + Y \delta v + Z \delta w) A_1 A_2 da_1 da_2.
$$

(19)

By substituting relationships (10)-(12) in Eq. (19), we obtain

$$
\delta U = \iint \left[ T_1 \delta \left( \frac{1}{A_4} \frac{\partial u}{\partial a_4} + \frac{\nu}{A_1 A_4} \frac{\partial A_1}{\partial a_4} + \frac{\nu}{R_1} \right) + 
T_2 \delta \left( \frac{1}{A_2} \frac{\partial v}{\partial a_2} + \frac{u}{A_1 A_3} \frac{\partial A_3}{\partial a_3} + \frac{u}{R_2} \right) + 
+ \frac{C}{A_2} \delta \left[ \frac{A_2}{A_1} \frac{\partial \varphi}{\partial a_1} \left( \frac{\nu}{A_1} \right) + \frac{A_1}{A_3} \frac{\partial \varphi}{\partial a_3} \left( \frac{u}{A_1} \right) \right] - 
- G_1 \delta \left( \frac{1}{A_2} \frac{\partial \varphi}{\partial a_2} + \frac{\psi}{A_1 A_2} \frac{\partial A_2}{\partial a_2} \right) - G_2 \delta \left( \frac{1}{A_3} \frac{\partial \varphi}{\partial a_3} + \frac{\psi}{A_1 A_3} \frac{\partial A_3}{\partial a_3} \right) - 
- H \delta \left[ \frac{A_2}{A_1} \frac{\partial \varphi}{\partial a_1} \left( \frac{\psi}{A_2} \right) + \frac{A_1}{A_4} \frac{\partial \varphi}{\partial a_4} \left( \frac{\nu}{A_1} \right) \right] - 
- Q_1 \delta \left( \varphi + \frac{1}{A_4} \frac{\partial v}{\partial a_4} - \frac{u}{R_1} \right) - Q_2 \delta \left( \psi + \frac{1}{A_3} \frac{\partial w}{\partial a_3} - \frac{u}{R_2} \right) + X \delta u + 
+ Y \delta v - p \delta w \right] A_1 A_2 da_1 da_2.
$$

(20)

By partial integration, we obtain
\[ \delta U = \int \left[ T_{1} A_{1} \delta u + S A_{1} \delta v - G_{1} A_{1} \delta \varphi - H A_{1} \delta \psi - Q_{1} A_{1} \delta w \right] da_{1} + \]
\[ + \int \left[ S A_{1} \delta u - G_{2} A_{1} \delta \psi + H A_{1} \delta \varphi - Q_{2} A_{1} \delta w \right] da_{1} + \]
\[ + \int \left[ \left[ \frac{\partial A_{2} T_{1}}{\partial a_{1}} + \frac{\partial A_{2} S}{\partial a_{1}} - T_{1} \frac{\partial A_{1}}{\partial a_{1}} + S \frac{\partial A_{1}}{\partial a_{1}} - A_{1} A_{2} \frac{Q_{1}}{R_{1}} + A_{1} A_{2} x \right] \delta u + \right. \]
\[ + \left. \left[ \frac{\partial A_{2} T_{1}}{\partial a_{2}} + \frac{\partial A_{2} S}{\partial a_{2}} - T_{2} \frac{\partial A_{1}}{\partial a_{2}} + S \frac{\partial A_{1}}{\partial a_{2}} - A_{1} A_{2} \frac{Q_{2}}{R_{2}} + A_{1} A_{2} y \right] \delta v - \right. \]
\[ - \left[ \frac{\partial A_{2} Q_{1}}{\partial a_{1}} + \frac{\partial A_{2} Q_{2}}{\partial a_{2}} + A_{1} A_{2} \left( \frac{T_{1}}{R_{1}} + \frac{T_{2}}{R_{2}} \right) - A_{1} A_{2} p \right] \delta w - \]
\[ - \left[ \frac{\partial A_{2} G_{1}}{\partial a_{1}} + \frac{\partial A_{2} H}{\partial a_{1}} - G_{2} \frac{\partial A_{1}}{\partial a_{1}} + H \frac{\partial A_{1}}{\partial a_{1}} - A_{1} A_{2} Q_{1} \right] \delta \varphi - \]
\[ - \left[ \frac{\partial A_{2} G_{2}}{\partial a_{2}} + \frac{\partial A_{2} H}{\partial a_{2}} - G_{2} \frac{\partial A_{1}}{\partial a_{2}} + H \frac{\partial A_{1}}{\partial a_{2}} - A_{1} A_{2} Q_{2} \right] \delta \psi \right] da_{1} da_{2}. \]

On the strength of the independence of variations of \( \delta u, \delta v, \delta w, \delta \varphi \) and \( \delta \psi \), the boundary conditions and equilibrium Eq. (17) follow as a result of this.

We present the boundary conditions for an edge which is bounded by the coordinate line \( a_1 = \text{const} \). The boundary conditions are decomposed into boundary conditions for tangential forces and movements and boundary conditions for normal forces, moments, deflection and displacement functions \( \varphi, \psi \).

The following uniform boundary conditions can occur for the tangential forces and movements
\[
\begin{align*}
T_{1} &= 0; \quad S = 0; \quad u = 0; \quad S = 0; \\
T_{2} &= 0; \quad v = 0; \quad u = 0; \quad v = 0.
\end{align*}
\]
\[
(22)
\]

Correspondingly, for the normal forces, moments, deflection and displacement functions \( \varphi, \psi \), the following uniform boundary conditions (canonical form) can occur
\[
\begin{align*}
G_{1} &= H = Q_{1} = 0; \quad \varphi = H = Q_{1} = 0; \\
G_{2} &= H = w = 0; \quad \varphi = H = w = 0; \\
G_{3} &= \psi = Q_{2} = 0; \quad \psi = \psi = Q_{2} = 0; \\
G_{4} &= \psi = w = 0; \quad \psi = w = 0.
\end{align*}
\]
\[
(23)
\]

In the case of flat plates, conditions (22) and (23) determine the boundary conditions for the two dimensional problem and for bending, respectively. For shells, boundary conditions (22) and (23) are set, i.e., when the deformed state of a laminated shell is described by the rectilinear element hypotheses, an extremely great diversity of boundary conditions can occur, namely, 3! different cases of canonical edge supporting anchors.
Thus, for example:

a. unsupported edge

\[ T_1 = S = Q_1 = H = Q_1 = 0; \]

b. rigidly fastened edge

\[ u = v = w = \phi = \psi = 0; \]

c. edge unsupported in the tangential direction and rigidly fastened in the transverse direction

\[ T_1 = S = w = \phi = \psi = 0; \]

d. an edge rigidly fastened in the tangential direction and unsupported in the transverse direction

\[ u = v = Q_1 = H = Q_1 = 0 \]

etc.

There is a diversity of boundary conditions because, compared with the undeformable normal hypotheses, the rectilinear element hypotheses take into account two additional degrees of freedom of movement, which are characterized by the magnitudes of the interlayer shears in two directions.

3. Laminated Shells of Varied Orthotropic Structure

As has been noted, laminated shells produced on a continuous filler base can have diverse structures, depending on the mutual location and orientation of the filler.

We will assume each unit layer to be uniform and orthotropic, with elastic constants \( E_1, E_2, G, \nu_1, \nu_2 \); the well known relationship \( E_1 \nu_2 = E_2 \nu_1 \) occurs here.

1. If the principal axes of anisotropy coincide with the coordinate axis in production of the shell, the material of the laminated shell will be orthotropic, its elastic constants will coincide with the elastic constants of the unit layer and basic relationships (3)-(5), which connect the deformed and stressed states of the shell, are simplified and take the form

\[
\begin{align*}
T_1 &= B_1 (\varepsilon_1 + \nu_1 \varepsilon_2); \\
T_2 &= B_2 (\varepsilon_2 + \nu_2 \varepsilon_1); \\
S &= B_3 \omega; \\
G_1 &= -D_1 (\kappa_1 + \nu_1 \kappa_2); \\
G_2 &= -D_1 (\kappa_2 + \nu_2 \kappa_1); \\
H &= -2D_3 \kappa_3; \\
\end{align*}
\]

(24)
where the rigidity parameters of the shell are determined by the expressions

\[
\begin{align*}
B_1 &= \frac{E_1 \delta}{1 - \nu_1 \nu_2}; \\
B_2 &= \frac{E_2 \delta}{1 - \nu_1 \nu_2}; \\
B_3 &= G \delta; \\
D_1 &= \frac{E_1 \delta^3}{12(1 - \nu_1 \nu_2)}; \\
D_2 &= \frac{E_2 \delta^3}{12(1 - \nu_1 \nu_2)}; \\
D_3 &= \frac{G \delta^3}{12}; \\
K_1 &= \frac{5}{6} G_{11} \delta; \\
K_2 &= \frac{5}{6} G_{22} \delta.
\end{align*}
\]

By solving Eq. (24) for \( \epsilon_1, \epsilon_2, \omega \), the following relationships can be obtained for an orthotropic shell

\[
\begin{align*}
\epsilon_1 &= \frac{T_1 - \nu_1 T_2}{B_1(1 - \nu_1 \nu_2)}; \\
\epsilon_2 &= \frac{T_2 - \nu_2 T_1}{B_2(1 - \nu_1 \nu_2)}; \\
\omega &= \frac{S}{B_2}.
\end{align*}
\]

2. If the shell is produced in such a way that the principal direction of anisotropy with modulus of elasticity \( E_1 \) forms angle \( \phi \) with coordinate axis \( \alpha_1 \) (Fig. 2), the basic relationships which connect the stress and deformed states of the shell are defined by general expressions (3)-(5). However, since the shell material is orthotropic, elastic constants \( c_{ij} \) will not be independent, but they will be determined through four independent, for example, \( E_1, E_2, G, \nu_1 \) by the known expressions

\[
\begin{align*}
c_{11} &= E_1 \cos^4 \phi + (E_2 - E_1) \sin^2 \phi \cos^2 \phi + 2E_2 \sin^4 \phi; \\
c_{12} &= E_1 \nu_1 + (E_1 + E_2 - 2E_3) \sin^2 \phi \cos^2 \phi; \\
c_{22} &= E_1 \sin^4 \phi + (E_2 - E_1) \sin^2 \phi \cos^2 \phi + E_2 \cos^4 \phi; \\
c_{33} &= G + (E_1 + E_2 - 2E_3) \sin^2 \phi \cos^2 \phi; \\
c_{13} &= \frac{1}{2} (E_2 \sin^2 \phi - E_1 \cos^2 \phi + E_3 \cos 2\phi) \sin 2\phi; \\
c_{23} &= \frac{1}{2} (E_2 \cos^2 \phi - E_1 \sin^2 \phi - E_3 \cos 2\phi) \sin 2\phi; \\
E_1 &= \frac{E_1}{1 - \nu_1 \nu_2}; \\
E_2 &= \frac{E_2}{1 - \nu_1 \nu_2}; \\
E_3 &= 2G + E_3 \nu_1.
\end{align*}
\]
Elastic constants $a_{ij}$ ($i, j=1, 2, 3$) in Eq. (8) also are not independent, and they are determined by the formulas

$$
\begin{align*}
a_{11} &= \frac{\cos^2 \varphi}{E_1} + \left(\frac{1}{G} - \frac{2v_1}{E_1}\right) \sin^2 \varphi \cos \varphi + \frac{\sin^2 \varphi}{E_1}; \\
a_{12} &= \frac{\sin^2 \varphi}{E_1} + \left(\frac{1}{G} - \frac{2v_1}{E_1}\right) \sin^2 \varphi \cos \varphi + \frac{\cos^2 \varphi}{E_1}; \\
a_{13} &= \frac{\cos^2 \varphi}{E_1} + \left(\frac{1 + v_1}{E_1} + \frac{1 + v_2}{E_2}\right) \sin \varphi; \\
a_{22} &= -\left[ \frac{\sin^2 \varphi}{E_2} - \frac{\cos^2 \varphi}{E_2} + \frac{1}{2} \left(\frac{1}{G} - \frac{2v_1}{E_1}\right) \cos 2\varphi \right] \sin 2\varphi; \\
a_{33} &= \frac{\cos^2 \varphi}{E_3} - \frac{\sin^2 \varphi}{E_3} - \frac{1}{2} \left(\frac{1}{G} - \frac{2v_1}{E_1}\right) \cos 2\varphi \sin 2\varphi.
\end{align*}
$$

3. Still another case of practical importance can be presented, when the structure of a laminated plastic ensures that it has orthotropic elastic properties [14]. With a large number of unit layers which are cross laid at angles $\pm \theta$ (Fig. 3), the laminated plastic can be considered orthotropic.

In this case, the basic relationships which connect the stressed and deformed states of the shell are represented by expressions (24)-(26). The rigidity parameters of the shell are determined by Eq. (27), where $E_1$, $E_2$, $G$, $v_1$ are the elastic constants of the laminated material.

If the elastic constants of a unit orthotropic layer of filler $E_1^0$, $E_2^0$, $G^0$, $v_1^0$ are known, the following should be assumed in Eq. (27)

$$
\begin{align*}
E_1 &= c_{11}; & E_2 &= c_{22}; & G &= c_{33}; \\
v_2 &= \frac{c_{12}}{c_{11}}; & v_1 &= \frac{c_{12}}{c_{22}},
\end{align*}
$$

where $c_{11}, c_{22}, c_{33}, c_{12}$ are determined by relationships (29), i.e., they depend on the cross laying angle of the filler.
CHAPTER 2. CYLINDRICAL BENDING OF RECTANGULAR PLATES

4. General Expressions for Calculation of Bending of Laminated Beams

Some simplest cases of determination of the deformed and stressed states in cylindrical bending of rectangular plates with bending rigidity $D$ and rigidity $K$ with respect to interlayer shear are discussed in this chapter. For extremely narrow plates, the Poisson coefficients in the expressions for $D$ should be considered to equal zero. All the results obtained below are presented for a strip of unit width.

The coordinate system and symbols adopted in this chapter are indicated in Fig. 4. It is assumed that one of the principal directions of elasticity coincides with the $x$ axis. The basic elasticity relationships in accordance with Eq. (24)-(26) have the form

$$G = -D \varphi'; \quad Q = -K (\varphi + \omega').$$

From Eq. (33) and the equilibrium equations

$$Q' = -p; \quad G' = Q,$$

general expressions can be obtained for determination of the stressed and deformed states of rectangular strips in cylindrical bending. Since system of Eq. (33), (34) is equivalent to one fourth order differential equation four random integration constants appear in the general solution, which are determined from the boundary conditions

$$Q = -px + C_i; \quad G = -\frac{px^3}{2D} + C_i x + C_j;$$

$$\varphi = \frac{px^3}{6D} - \frac{C_i x^2}{2D} - \frac{C_j x}{D} + C_k;$$

$$w = -\frac{px^4}{24D} + \frac{C_i x^2}{6D} + \left(\frac{p}{K} + \frac{C_j}{D}\right) \frac{x^2}{2} - \left(\frac{C_k}{K} + C_l\right) x + C_m.$$

General Eq. (35) are valid along the entire length of the strip, and only constants $C_i(i=1, 2, 3, 4)$ differ in sections which differ by the nature of the loading or by rigidities $D$ or $K$. To determine the new random constants with each such section, the conditions of smoothness and continuity of conjugation

$$w_k = w_{k+1}; \quad \varphi_k = \varphi_{k+1}; \quad G_k = G_{k+1}; \quad Q_k = Q_{k+1}.$$

are added to the boundary conditions.
5. Bending of Cantilever Strip with Concentrated and Uniformly Distributed Loads

a. Bending of cantilever by concentrated force applied on unsupported end (Fig. 5). After determination of constants $C_j (j=1, 2, 3, 4)$ from the boundary conditions

$$w=0; \quad \phi=0 \text{ at } x=0$$

$$G=0; \quad Q=p \text{ at } x=l;$$

from Eq. (35), with $p=0$, we obtain

$$w = \frac{P_x}{6D} (x^3 - 3lx - 6 \frac{D}{K})$$

$$\varphi = \frac{P_x}{2D} (2l - x).$$

The maximum deflection of the strip at the unsupported end

$$|w|_{max} = \frac{p l^3}{3D} (1 + 3\gamma)$$

Fig. 5. Diagram of bending of cantilever by force applied to unsupported end.

Here and subsequently, the symbol for the relative give of the strip by interlayer shearing is introduced

$$\gamma = \frac{D}{Kh}. \quad (39)$$

Correspondingly, the greatest values of the normal and shearing stresses are

$$\sigma_{max} = \frac{6G}{8l^2} = \frac{6Pl}{8l^2} \quad \text{and} \quad \tau_{max} = \frac{3Q}{26} = \frac{3P}{26}. \quad (40)$$

b. Bending of cantilever by uniformly distributed load (Fig. 6). By determining the random constants in a manner similar to the preceding from the boundary conditions

$$w=0, \quad \phi=0 \text{ at } x=0;$$

$$G=0, \quad Q=0 \text{ at } x=l,$$

we obtain
\[ Q = p(l - z); \quad G = -\frac{p}{2} (z - l)^2; \]
\[ \varphi = \frac{p}{6d} (x^3 - 3lx + 3l^2); \]
\[ w = -\frac{p}{12d} [a^3 - 4a^2 + 6a^2 - 12\gamma (z - 2l)]l^3. \]  \tag{41}

The maximum deflection of the unsupported end of the beam is
\[ |w|_{\text{max}} = \frac{pl}{8d} (1 + 4\gamma). \]  \tag{42}

6. Bending of Hinge Supported Strip

We consider several cases of loading of a strip which are most often encountered in various engineering applications.

a. Pure bending of strip (Fig. 7). Integration constants \( C_1 \) (\( i = 1, 2, 3, 4 \)) can be determined, for example, from these boundary conditions

\[
\begin{align*}
    w &= 0, \quad Q = 0, \quad G = M_0 \quad \text{at } z = 0; \\
    \varphi &= 0 \quad \text{at } z = \frac{l}{2}; \\
    Q &= 0, \quad G = M_0; \\
    \varphi &= \frac{M_0}{2l} (l - 2z); \\
    w &= \frac{M_0}{2B} (x - l). \\
\end{align*}
\]  \tag{43}

The maximum deflection of the center section of the beam
\[ |w|_{\text{max}} = \frac{M_0 l}{8B}. \]  \tag{44}

In pure bending, interlayer shear is absent, and the greatest normal stresses are constant over the length of the strip.

b. Bending of strip by concentrated force applied in middle section (Fig. 8). Because of symmetry, the left half of the strip alone can be considered \( 0 \leq x \leq l/2 \).

Boundary conditions:
\[
\begin{align*}
    \text{at } x = 0 & \quad w = 0, \quad G = 0, \quad Q = \frac{p}{2}; \\
    \text{at } x = \frac{l}{2} & \quad \varphi = 0.
\end{align*}
\]
After determination of random constants $C_j (j=1, 2, 3, 4)$, we obtain

\[
\begin{align*}
Q &= \frac{P}{2}; \quad G = \frac{Px}{2}; \\
\varphi &= \frac{P}{16D} (l^2 - 4x^2); \\
\omega &= \frac{Pz}{48D} (4x^2 - 3l^2 - 24\gamma l^4).
\end{align*}
\]  

(45)

The maximum deflection in the center of the strip

\[|w|_{max} = \frac{P\beta}{48D} (1 + 12\gamma).\]  

(46)

The greatest shearing stresses are constant in each half of the beam, and the absolute value is

\[\tau_{max} = \frac{3P}{4\beta}.\]  

(47)

The normal stress in the critical section of the beam

\[\sigma_{max} = \frac{3P\beta}{28D}.\]  

(48)

c. Bending of strip under uniformly distributed load (Fig. 9). After determination of the random constants from the boundary conditions

at $x = 0 \quad w = 0, \quad G = 0$;

at $x = \frac{l}{2} \quad \varphi = 0, \quad Q = 0$,

we obtain

\[
\begin{align*}
Q &= \frac{P}{2} (l - 2x); \quad G = \frac{Px}{2} (l - x); \\
\varphi &= \frac{P}{24D} (4x^2 - 6lx^2 + l^3); \\
\omega &= \frac{Pz}{24D} (l^3 - 2lx^2 + 12\gamma (l - x)^2). \\
\end{align*}
\]

(49)

The maximum deflection of the strip in the center

\[|w|_{max} = \frac{5P\beta}{384D} (1 + 48\gamma).\]  

(50)

The greatest shearing stress between the layers occurs at the ends of the strip

\[\tau_{max} = \frac{3P\beta}{48}.\]  

(51)
The normal stresses in the critical section
\[ \sigma_{\text{max}} = \frac{3}{4} \frac{p}{b}. \]  

(52)

7. Bending of Rigidly Fastened Strip

In the case of rigid fastening of the ends in the simplest cases of loading, it is especially easy to obtain expressions for the elastic forces and deflection.

a. Bending of strip by concentrated force applied in center (Fig. 10).

The boundary conditions

at \( x = 0 \) \( w = 0 \), \( \phi = 0 \), \( Q = \frac{P}{2} \);

at \( x = \frac{l}{2} \) \( \phi = 0 \).

After determination of the random constants, we have

\[ Q = \frac{P}{2}; \quad G = \frac{P}{b} (4x - l); \]
\[ \phi = \frac{P}{8D} (2x - l); \]
\[ w = \frac{P}{48D} (4x^2 - 3lx - 24y). \]

(53)

The maximum deflection in the center of the strip

\[ |w|_{\text{max}} = \frac{P}{192D} (1 + 48y). \]  

(54)

Fig. 10. Diagram of bending of strip with rigidly fastened ends by concentrated force.

Fig. 11. Diagram of bending of strip with rigidly fastened ends by uniform loading.

The shearing stresses are constant over the length

\[ \tau_{\text{max}} = \frac{3}{4} \frac{P}{b}. \]  

(55)

The normal stresses in the critical section

\[ \sigma_{\text{max}} = \frac{3Pb}{48y}. \]  

(56)

b. Bending of strip by uniformly distributed load (Fig. 11).

The boundary conditions
Accordingly, for the elastic forces and deformations of a sheet, the following expressions can be obtained

\[
\begin{align*}
Q &= \frac{P}{2} (l - 2x); \\
G &= -\frac{P}{12} (6x^2 - 6lx + l^2); \\
\varphi &= \frac{P}{12l} (2x^2 - 3lx + l^2); \\
w &= -\frac{P}{24l} [x^2 (x - l)^2 + 12y x (l - x) l^2].
\end{align*}
\]  

(57)

The maximum deflection in the center

\[|w|_{\text{max}} = \frac{P l^4}{384 E I} (1 + 48y).\]  

(58)

The greatest shearing stresses arise at the ends of the strip

\[\tau_{\text{max}} = \frac{3pl}{6l}.\]  

(59)

The normal stresses in the critical center section of the strip

\[\sigma_{\text{max}} = \frac{P l^2}{8d^3}.\]  

(60)

8. Experimental Determination of Elastic and Rigidity Parameters of Orthotropic Laminated Shells

For determination of the complete set of rigidity characteristics of a laminated orthotropic material, tensile, torsion, clean and transverse bending tests of rectangular strips cut in the principal directions of anisotropy are required.

With standard specimens under tension, tensile rigidities \(B_1, B_2,\) moduli of elasticity \(E_1, E_2\) and Poisson coefficients \(\nu_1, \nu_2\) are obtained, which should satisfy the condition \(E_1\nu_2 = E_2\nu_1.\)

Flexural rigidities of the laminated shell \(D_1, D_2\) are determined by Eq. (27), if the reduced flexural and tensile and flexural moduli of elasticity are the same. Otherwise, flexural rigidities must be determined in clean bending tests of rectangular specimens according to the symmetrical two cantilever beam system (Fig. 12).

It evidently is advisable to provide for clean bending tests in the principal directions of anisotropy in all cases, if only as control tests, the more so that they are the simplest.
Fig. 12. Diagram of clean bending test.

If the deflection of a strip in the center section measured under load is \( w_0 \) and the width of the strip is \( b \), the flexural rigidity is determined by the formula

\[
D = \frac{E I}{w_0}. \tag{61}
\]

The torsional rigidity of a plate \( D_3 \), shear modulus \( G \) and, consequently, shear rigidity \( B_3 \) are determined by torsion tests or by transverse bending tests of rectangular or square plates loaded with four equal balanced concentrated forces applied to the corners of the plate (Fig. 13).

Fig. 13. Diagram of torsion tests.

Torsion tests of specimens cut in the other principal direction of anisotropy are control tests.

As the results obtained above show, the effect of interlayer shearing of laminated plates in the deformed and stressed states depends on the relative thickness of the plate, the boundary conditions and the nature of the load. It is evident that, for determination of the rigidity parameters of a plate, it is more advisable to use transverse bending tests by systems of hinge supported or rigidly fastened beams.

If, for example, the deflection of a hinge supported strip of width \( b \), measured in transverse bending tests with loading force \( P \) applied in the center of the span is \( w_0 \), the rigidity of the laminated plate with respect to shearing between the layers is determined by the formula

\[
K = \frac{12DP}{4EDbw_0 - Ph} \cdot \tag{62}
\]

By conducting such tests of strips cut in both principal directions of anisotropy, we obtain rigidities \( K_1, K_2 \).

In this manner, the simplest mechanical tests of rectangular strips cut from a laminated plate completely solve the problems associated with determination of the elastic and rigidity characteristics of laminated shells.

If a laminated shell is bent in one or both directions, the fabrication of flat control samples should be provided for, which are cut under the same technological conditions and go through the same heat treatment as the shell itself.
To obtain control samples, it is advisable to provide for technological margins in the fabrication of an actual structure, which are then cut into samples for mechanical testing.
CHAPTER 3. AXISYMMETRIC BENDING OF CIRCULAR PLATES WITH CYLINDRICAL ANISOTROPY

9. Equations of Bending of Circular Plate and General Solution under Axisymmetric Loading

We consider a circular plate of constant thickness made of a cylindrically orthotropic material (Fig. 14).

It is assumed that the axis of anisotropy passes through the center of the plate perpendicular to the mean surface, and that the principal axes of elasticity coincide with the axis of anisotropy, with radial and circular directions. A load distributed symmetrically about the z axis acts on the plate normal to the mean surface.

In conformance with Eq. (10)-(12), (24), (26), in axisymmetric deformation of the plate, the basic relationships which connect the stressed and deformed states of the plate have the form

\[
\begin{align*}
T_1 &= B_1 \left( u' + v_s \frac{u}{r} \right); \\
T_2 &= B_1 \left( \frac{u}{r} + v_t u' \right); \\
G_1 &= -D_1 \left( \varphi' + v_s \frac{\varphi}{r} \right); \\
G_2 &= -D_1 \left( \frac{\varphi}{r} + v_t \varphi' \right). \\
Q_1 &= -K_1 (w' + \varphi). 
\end{align*}
\]

For determination of radial displacement \( u \), plate deflection \( w \) and deformation function \( \phi \), we have three equations of equilibrium

\[
\begin{align*}
(rT_1)' - T_s &= 0; \\
(rG_1)' - G_2 &= 0; \\
(rQ_1)' + pr &= 0. 
\end{align*}
\]

System of differential equations (66) is decomposed into two systems relative to \( u \) and \( w \), and \( \phi \)

\[
\begin{align*}
\left( \frac{u''}{r} - \frac{u'}{r^2} u \right) &= 0; \\
\frac{\varphi''}{r} - \frac{\varphi'}{r^2} \varphi &= \frac{pr}{2D_1} - \frac{C_1}{D_1} \frac{1}{r}; \\
\left( w' + \varphi \right) &= \frac{pr}{2K_1} - \frac{C_1}{K_1} \frac{1}{r}. 
\end{align*}
\]
where $C_1$ is the random integration constant and

$$\lambda^* = \frac{E_1}{E_1}.$$  \hspace{1cm} (69)

General solutions of Eq. (68) have the form

$$Q_1 = -\frac{p_1}{2} + \frac{C_1}{r};$$

$$\varphi = \frac{p_1r^2}{2(\alpha - \lambda^*) D_1} + \frac{C_1}{(\lambda^* - 1) D_1} + C_2 r^{\lambda^*} + C_3 r^{-\lambda^*};$$

$$w = \frac{p_2}{4K_1} - \frac{p_3}{8(\alpha - \lambda^*) D_1} - \frac{C_2}{(\lambda^* - 1) D_1} - \frac{C_3}{\lambda + 1} r^{\lambda^* - 1} + \frac{C_4}{\lambda - 1} r^{1-\lambda} + C_4;$$

$$G_1 = -D_1 \left[ \frac{p_2(3 + \nu_2) r^2}{2(\alpha - \lambda^*) D_1} + \frac{(1 + \nu_2) C_1}{(\lambda^* - 1) D_1} + C_2 \lambda^* - 1 - C_3 (1 - \lambda^*) r^{-(\lambda^* - 1)} \right];$$

$$G_2 = -D_2 \left[ \frac{(1 + \nu_1) p_2}{2(\alpha - \lambda^*) D_1} + \frac{(1 + \nu_2) C_1}{(\lambda^* - 1) D_1} + C_2 \lambda^* - 1 + C_3 (1 - \lambda^*) r^{-(\lambda^* - 1)} \right];$$

Integration constants $C_1, C_2, C_3, C_4$ are determined from the boundary conditions at the edges of the plate. In the case of a solid plate, the boundary condition on the inner profile is replaced by the condition in the center of the plate with $r=0$, which is reduced to the requirement of limitation of deflection of the plate or the finite nature of the bending moments, or the cutting force as a function of the type of load.

General solutions (70) permit various cases of symmetrical loading of the plate to be considered with diverse boundary conditions.

10. Bending of Solid Circular Plate by Uniform Load

Let a solid circular plate be bent by normal pressure uniformly distributed over the upper boundary of the plate (see Fig. 14).

In this case, by virtue of the finite nature of the deflection in the center of the plate and the absence of rotation of the normal, $C_1 = C_3 = 0$ and, consequently,

$$G_1 = -D_1 \left[ \frac{(3 + \nu_2) p_2}{2(\alpha - \lambda^*) D_1} + C_2 (\lambda + \nu_2) r^{\lambda^* - 1} \right];$$

$$G_2 = -D_2 \left[ \frac{(1 + \nu_1 + \nu_2) p_2}{2(\alpha - \lambda^*) D_1} + C_2 (1 + \nu_1) r^{\lambda^* - 1} \right];$$

$$w = \frac{p_2}{4K_1} - \frac{p_3}{8(\alpha - \lambda^*) D_1} - \frac{C_2}{\lambda + 1} r^{\lambda^* - 1} + \frac{C_4}{\lambda - 1} r^{1-\lambda} + C_4;$$

$$\varphi = \frac{p_3}{2(\alpha - \lambda^*) D_1} + C_2 r^{\lambda^*}. $$

(71)
a. Edge of Plate Hinge Supported

By determining constants $C_2$, $C_4$ from the boundary conditions with $r=a$, $w=0$, $G_1=0$, we obtain

$$w = \frac{pa^4}{8(\lambda - \lambda_1)} \left[ \frac{(\lambda - 3)(\lambda + 4 + v_1)}{(\lambda + v_3)(\lambda + 1)} - \left(\frac{r}{a}\right)^4 + \frac{4(3 + v_1)}{(\lambda + 1)(\lambda + v_3)} \left(\frac{r}{a}\right)^{\lambda+1} + \frac{2(9 - \lambda^2)D_1}{\lambda_1 a^2} \left(\frac{r}{a}\right)^4 - 1 \right];$$

$$\varphi = \frac{pa^2}{2(9 - \lambda^2)D_1} \left[ \left(\frac{r}{a}\right)^3 - \frac{3 + v_3}{\lambda + v_3} \left(\frac{r}{a}\right)^{\lambda+1} \right];$$

$$G_1 = -\frac{(3 + v_1)a^4}{2(9 - \lambda^2)} \left[ \left(\frac{r}{a}\right)^3 - \left(\frac{r}{a}\right)^{\lambda+1} \right];$$

$$G_2 = -\frac{a^4(1 + 3v_3)}{2(9 - \lambda^2)} \left[ \left(\frac{r}{a}\right)^3 - \frac{1 + \lambda v_1}{1 + 3v_1} \left(\frac{r}{a}\right)^{\lambda+1} \right].$$

The maximum deflection in the center of the plate is determined by the expression

$$|w|_{\text{max}} = \frac{pa^4(\lambda + 4 + v_1)}{8(\lambda + 3)(\lambda + v_3)(\lambda + 1)D_1} + \frac{pa^2}{4K_1};$$

and, at the edge of the plate ($r=a$)

$$G_1 = 0; \quad G_2 = \frac{pa^4(1-v_1^2v_3)}{2(\lambda + v_3)(\lambda + 3)}.$$

In the center of the plate, bending moments $G_1=G_2=0$ if $\lambda>1$, or they increase indefinitely if $\lambda<1$, i.e., the stressed state depends essentially on the nature of the anisotropy.

b. Edge of Plate Rigidly Fastened

In this case, we obtain

$$w = \frac{pa^4}{8(\lambda - \lambda_1)(\lambda + 1)D_1} \left[ \lambda - 3 + 4 \left(\frac{r}{a}\right)^{\lambda+1} - (\lambda + 1) \left(\frac{r}{a}\right)^4 + \frac{p(r^2-a^2)}{4K_1}; \right.$$

$$\left. \varphi = \frac{pa^2}{2(9 - \lambda^2)D_1} \left[ \left(\frac{r}{a}\right)^3 - \left(\frac{r}{a}\right)^{\lambda+1} \right]; \right.$$

$$G_1 = -\frac{(3 + v_1)a^4}{2(9 - \lambda^2)} \left[ \left(\frac{r}{a}\right)^3 - \frac{3 + v_3}{\lambda + v_3} \left(\frac{r}{a}\right)^{\lambda+1} \right];$$

$$G_2 = -\frac{a^4(1 + 3v_3)}{2(9 - \lambda^2)} \left[ \left(\frac{r}{a}\right)^3 - \frac{1 + \lambda v_1}{1 + 3v_1} \left(\frac{r}{a}\right)^{\lambda+1} \right].$$

The greatest deflection in the center of the plate

$$|w|_{\text{max}} = \frac{pa^4}{8(\lambda + 3)(\lambda + 1)D_1} + \frac{pa^2}{4K_1};$$

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The bending moments at the edge of the plate

\[ G_1 = -\frac{P a^2}{2(3+\lambda)}; \quad G_4 = v_G G_1. \] (77)

In this case, the stressed state of the plate depends essentially on the nature of the anisotropy. If \( \lambda^2 > 1 \), the maximum stresses develop at the edge of the plate. If \( \lambda^2 < 1 \), the stresses in the center increase indefinitely.

Thus, if a laminated, cylindrically orthotropic, circular plate operates under uniform normal pressure, the modulus of elasticity of the material in the circular direction should be greater than the modulus of elasticity of the material in the radial direction. It is advisable to use laminated plates which do not satisfy this condition for the manufacture of circular parts. If, for some reason, a plate should be made of a material, for which \( \lambda^2 = E_2/E_1 < 1 \), the center of the plate must be reinforced with an absolutely rigid disk, i.e., a disk the flexural rigidity of which is considerably greater than the flexural rigidity of the plate.

11. Bending of Solid Circular Plate by Concentrated Force

Let a solid circular plate made of a cylindrically orthotropic laminated plastic be bent by normal concentrated force \( P \) applied to the center of the plate (Fig. 15). In this case, \( p = 0 \), \( C_1 = -P/2\pi \), \( C_3 = 0 \) must be set in Eq. (70). We then obtain

\[
G_1 = D_1 \left[ \frac{P (1 + v_4)}{2\pi (\lambda^3 - 1) D_1} - C_1 (\lambda + v_4) \rho^{\lambda - 1} \right];
\]

\[
G_3 = D_1 \left[ \frac{P (1 + v_1)}{2\pi (\lambda^3 - 1) D_1} - C_3 (1 + \lambda v_1) \rho^{\lambda - 1} \right];
\]

\[
w = \frac{P r^2}{4\pi (\lambda^3 - 1) D_1} + \frac{P \ln r}{2\pi K_1} - \frac{C_3}{\lambda + 1} \rho^{\lambda + 1} + C_4.
\] (78)

a. Edge of Plate \( r = a \) Hinge Supported

The stressed, deformed state of the plate is determined by the expressions

\[
G_1 = \frac{P (1 + v_4)}{2\pi (\lambda^3 - 1)} \left[ 1 - \left( \frac{r}{a} \right)^{\lambda - 1} \right];
\]

\[
G_3 = \frac{P (1 + v_1) \lambda^3}{2\pi (\lambda^3 - 1)} \left[ 1 - \frac{(1 + v_4) (1 + \lambda v_1)}{(\lambda + v_4) (1 + v_1)} \left( \frac{r}{a} \right)^{\lambda - 1} \right];
\]

\[
\psi = -\frac{P a}{2\pi (\lambda^3 - 1) D_1} \left[ \left( \frac{r}{a} \right)^{\lambda - 1} - \frac{1 + v_4}{\lambda + v_4} \left( \frac{r}{a} \right)^{\lambda} \right];
\]

\[
w = \frac{P a^3}{4\pi (\lambda^3 - 1) D_1} \left[ \left( \frac{r}{a} \right)^{\lambda + 1} - \frac{2 (1 + v_4)}{(\lambda + v_4) (1 + v_1)} \left( \frac{r}{a} \right)^{\lambda + 1} \right]
\]

\[ - \frac{(\lambda - 1) (\lambda + 2 + v_4)}{(\lambda + v_4) (1 + v_1)} \] + \frac{P \ln \frac{r}{a}}{2\pi K_1}.
\] (79)
The deflection in the center of the plate reverts to infinity. Consequently, in transmission of the force, distribution of the load over a definite area must be ensured, or a rigid disk must be incorporated in the center of the plate.

b. Edge of Plate Rigidly Fastened

In this case, the load also should be transmitted through a rigid disk in the center or it should be distributed over a certain area in the center of the plate.

Relative to stress distribution, the same conclusions are valid as those made in the preceding section. For the fabrication of circular plates operating under locally distributed load applied in the center of the plate, it is advisable to use plates the modulus of elasticity in the radial direction of which is greater than the modulus of elasticity in the annular direction of which is greater than the modulus of elasticity, i.e., $\lambda<1$.

It is of interest to note that precisely such anisotropy of elastic properties develops in circular disks strengthened with radial stiffening ribs. However, radial ribs which converge in the center of the disk form a rigid hub. For more favorable stress distribution in the reinforced disks, annular strengthening ribs should be provided.

12. Bending of Circular Plate with Rigid Disk in Center by Uniform Pressure

We now consider a circular plate made of cylindrically orthotropic laminated plastic, subjected to uniformly distributed normal pressure. The inner profile of the plate is rigidly fastened to a massive disk located in the center (Fig. 16).

In this case, we have $C_1=0$, $Q=-pr/2$ and, consequently,
\[ w = \frac{p_a^3 \left( \frac{r_e}{a_e} - 1 \right)}{4k_1} \left( \frac{r_e}{a_e} \right)^{\lambda+1} + \frac{p_b^3 \left( \frac{r_b}{a_b} - 1 \right)}{8(9-\lambda^2)D_i} \left( \frac{r_b}{a_b} \right)^{\lambda+1} \]

\[ C_a \left( \left( \frac{r_e}{a_e} \right)^{\lambda+1} - 1 \right) \]

\[ C_b \left( \left( \frac{r_b}{a_b} \right)^{\lambda+1} - 1 \right) \]

\[ \varphi = \frac{p_a^3}{2(9-\lambda^2)D_i} + C_a^3 + C_b^3 \]

\[ G_1 = -D_i \left[ \frac{3 + 3v_3}{2(9-\lambda^2)} \right] + C_b(\lambda + v_3) r^{\lambda-1} -

\[ - C_b(\lambda - v_3) r^{-(\lambda+1)} \]

\[ G_2 = -D_i \left[ \frac{1 + 3v_3}{2(9-\lambda^2)} \right] + C_b(\lambda + v_3) r^{\lambda-1} +

\[ + C_b(\lambda - v_3) r^{-(\lambda+1)} \]

\[ a. \text{ Edge of plate } r=a \text{ hinge supported (see Fig. 16). By satisfying boundary conditions } w=0, G_1=0, \text{ we obtain} \]

\[ C_1 = -\frac{p_a^3 - \lambda}{2(9-\lambda^2)} \frac{(\lambda - v_3) q^{\lambda+3} + 3 + v_3}{(\lambda - v_3) q^{2\lambda + \lambda + v_3}} \]

\[ C_2 = \frac{p_b^{\lambda+3}}{2(9-\lambda^2)} \frac{3 + 3v_3)(\lambda + v_3) q^{\lambda-g} (\lambda + v_3)}{(\lambda - v_3) q^{2\lambda + \lambda + v_3}} \]

where \( q = \frac{b}{a} \).

At edge \( r=a \) of the plate, the bending moments

\[ G_1 = 0, G_2 = -\frac{p_a^3(1 - v_3 v_3) q\lambda^3}{2(9-\lambda^2)} \frac{(\lambda + v_3) q^{2\lambda - 2}\lambda^{\lambda+3} + \lambda - 3}{(\lambda - v_3) q^{2\lambda + \lambda + v_3}} \]

At the edge of the plate around the inner disk

\[ G_1 = -\frac{p_b^3}{2(9-\lambda^2)} \frac{(\lambda - v_3)(3 - \lambda) q^{2\lambda - 2}\lambda(3 + v_3) q^{\lambda-g} + (\lambda + 3)(\lambda + v_3)}{(\lambda - v_3) q^{2\lambda + \lambda + v_3}} \]

\[ G_2 = v_3 G_1 \]

b. \text{ Edge of plate } r=0 \text{ rigidly fastened (Fig. 17).} \]

\[ C_a = -\frac{p_a^3 - \lambda}{2(9-\lambda^2)} \frac{1 - q^{\lambda+3}}{1 - q^{2\lambda}} \]

\[ C_b = -\frac{p_b^{\lambda+3}}{2(9-\lambda^2)} \frac{1 - q^{\lambda-3}}{1 - q^{2\lambda}} \]

where \( p = b/a \).
At the edge of plate \( r = a \), bending moments

\[
G_1 = \frac{p a^2}{2 (\alpha - \lambda)} \left( \frac{\lambda - 3 - 2 \lambda q^{3} + \lambda + 3}{1 - q^{2}} \right); \quad G_2 = \nu_G G_1. \tag{86}
\]

Correspondingly at the edge \( r = b \), we obtain

\[
G_1 = -\frac{p b^2}{2 (\alpha - \lambda)} \left( \frac{\lambda + 3 - 2 \lambda q^{3} - \lambda + 3}{1 - q^{2}} \right); \quad G_2 = \nu_G G_1. \tag{87}
\]

In both cases considered, the greatest tangential stresses arise on the outer profile of the plate

\[
\tau_{\text{max}} = \frac{3p a}{d}. \tag{88}
\]

13. Bending of Circular Plate by Forces Applied to Rigid Disk in Center

Let a circular plate fastened in an absolutely rigid disk be loaded by an axisymmetric system of normal forces applied to the disk. If the resultant equals \( P \), \( C_1 = -P/2\pi \) and, consequently, according to Eq. (70), with \( \varphi = 0 \), the bending moments and deformation are determined by the following expressions

\[
w = \frac{P (r^2 - a^2)}{4\pi (\lambda + 1) D_1} + \frac{P}{2\pi K_1} \ln \frac{r}{a} - \frac{C_1 (r^2 a - a^3)}{\lambda + 1} + \frac{C_1 (r^2 a - a^3)}{\lambda - 1} + \frac{C_1 (r^2 a - a^3)}{1 - \lambda};
\]

\[
\varphi = -\frac{P r}{2\pi (\lambda + 1) D_1} + C_2 \lambda + C_2 \lambda^{-1};
\]

\[
G_1 = D_1 \left[ \frac{P (1 + \varphi)}{2\pi (\lambda + 1) D_1} - C_2 (1 + \lambda \nu_1) \lambda^{-1} + C_2 (1 - \lambda \nu_1) \lambda^{-1} \right];
\]

\[
G_2 = D_1 \left[ \frac{P (1 + \varphi)}{2\pi (\lambda + 1) D_1} - C_2 (1 + \lambda \nu_1) \lambda^{-1} - C_2 (1 - \lambda \nu_1) \lambda^{-1} \right],
\]

\[
Q_1 = -\frac{P}{2\pi r}.
\]

a. Rim \( r = 0 \) hinge supported (Fig. 16).

\[
C_2 = \frac{P a (\lambda - 1)}{2\pi (\lambda + 1) D_1} \frac{1 + \nu_2 + (\lambda - \nu_3) q^{k+1}}{\lambda - \nu_3} q^{k+1} + \lambda + \nu_3; \tag{90}
\]

\[
C_3 = \frac{p b^{k+1}}{2\pi (\lambda + 1) D_1} \frac{\lambda + \nu_2 - (1 + \nu_3) q^{k+1}}{\lambda + \nu_3} q^{k+1} + \lambda + \nu_3.
\]
On the outer profile of the plate, bending moments

\[
G_1 = 0; \\
G_s = \frac{P \lambda (1 - \nu_i \nu_s)}{2n (\lambda^2 - 1)} \frac{(\lambda + 1) q^{2h} - 2q^{\lambda + 1} + \lambda - 1}{(\lambda - \nu_s) q^{2h} + \lambda + \nu_s}.
\]

Correspondingly, on the profile of the rigid disk, we obtain

\[
G_1 = -\frac{P}{2n (\lambda^2 - 1)} \times \\
\frac{(\lambda - 1) (1 - \nu_i \nu_s) q^{2h} + 2q (1 + \nu_s) q^{(\lambda - 1)} - (\lambda + 1) (1 + \nu_s)}{(\lambda - \nu_s) q^{2h} + \lambda + \nu_s} \times G_s/\nu_s G_1.
\]

b. Rim of plate is rigidly fastened (Fig. 19). In this case, the following can be obtained

\[
C_s = \frac{P a^{(\lambda - 1)}}{2n (\lambda^2 - 1) D_1} \frac{1 - q^{\lambda + 1}}{1 - q^{2h}}; \\
C_s = \frac{P a^{(\lambda + 1)}}{2n (\lambda^2 - 1) D_1} \frac{1 - q^{\lambda - 1}}{1 - q^{2h}}.
\]

The respective bending moments on the outer and inner profiles of the plate

\[
G_1 = -\frac{P}{2n (\lambda^2 - 1)} \frac{(\lambda + 1) q^{2h} - 2q^{\lambda + 1} + \lambda - 1}{1 - q^{2h}}; \\
G_s = \nu_i G_1
\]

\[
G_1 = -\frac{P}{2n (\lambda^2 - 1)} \frac{(\lambda - 1) q^{2h} - 2q^{\lambda + 1} + \lambda - 1}{1 - q^{2h}}; \\
G_s = \nu_i G_1.
\]

In both of the boundary condition cases considered, the greatest shearing stresses arise on the inner profile of the plate

\[
\tau_{\text{max}} = \frac{3P}{4n \pi \delta}.
\]

14. Bending of Annular Plate by Load Uniformly Distributed Over Inner Profile

Let an annular plate made of cylindrically orthotropic laminated plastic be loaded with load \( P = 2\pi r b q \) uniformly distributed over the inner profile (Fig. 20). With different fastenings of the outer profile of the plate, the
elastic forces and deformations are determined by general expressions (89).

a. Rim of plate \( r = a \) hinge supported. In this case, it is easy to obtain

\[
C_\alpha = \frac{p (1+p_\lambda)}{2\pi (\lambda^2-1) D_1 (\lambda+v_2)} \frac{(1-\alpha)^{\lambda+1}}{(1-\alpha^{2\lambda})};
\]

\[
C_\beta = -\frac{p (1+p_\lambda)}{2\pi (\lambda^2-1) D_1 (\lambda+v_2)} \frac{(1-\alpha^{2})^{\lambda+1}}{(1-\alpha^{2\lambda})}.
\]  

The bending moments on the inner and outer profiles of the plate are determined by the following respective expressions

\[
G_1 = 0;
\]

\[
G_\alpha = -\frac{p \lambda^1 (1-v_\lambda v_2)}{2\pi (\lambda^2-1) (\lambda^2-v_2^2)} \times \left\{ \begin{array}{l}
\frac{(\lambda+v_2) (\lambda+1) q^{2\lambda} - 2q (1+v_2) q^{\lambda+1} - (\lambda-1) \alpha}{1-\alpha^{2\lambda}}; \\
G_\beta = 0;
\end{array} \right.  
\]

\[
G_\beta = -\frac{p \lambda^1 (1-v_\lambda v_2)}{2\pi (\lambda^2-1) (\lambda^2-v_2^2)} \times \left\{ \begin{array}{l}
\frac{(\lambda-v_2) (\lambda-1) q^{2\lambda} + 2q (1+v_2) q^{\lambda+1} - (\lambda+1) \alpha}{1-\alpha^{2\lambda}}. \\
\end{array} \right.
\]

b. Rim of plate \( r = a \) rigidly fastened (Fig. 21). In this case, the following can be obtained

\[
C_\alpha = \frac{p a^{1-\lambda}}{2\pi (\lambda^2-1) D_1 (\lambda-v_\lambda) (\lambda+v_\lambda) q^{2\lambda}} \times \left\{ \begin{array}{l}
\frac{(\lambda-1) (1+v_\lambda) q^{2\lambda} - 2q (1+v_\lambda) q^{\lambda+1} - (\lambda-v_\lambda) (\lambda-1)}{\lambda-1 \alpha + (\lambda+v_\lambda) q^{2\lambda}}; \\
C_\beta = \frac{p b^{1-\lambda}}{2\pi (\lambda^2-1) D_1} \frac{(\lambda-v_\lambda) q^{2\lambda}}{\lambda-v_\lambda + (\lambda+v_\lambda) q^{2\lambda}}.
\end{array} \right.
\]

On the inner profile of plate \( r = a \), the bending moments

\[
G_1 = \frac{p}{2\pi (\lambda^2-1)} \times \left\{ \begin{array}{l}
\frac{(\lambda+1) (1+v_\lambda) q^{2\lambda} - 2q (1+v_\lambda) q^{\lambda+1} - (\lambda-v_\lambda) (\lambda-1)}{(\lambda-1) \alpha + (\lambda+v_\lambda) q^{2\lambda}}; \\
G_\beta = v_\beta G_1.
\end{array} \right.
\]

Correspondingly, on the inner profile, we obtain
In both cases considered, the greatest shearing stresses arise on the inner profile of the plate

\[ \tau_{\text{max}} = \frac{3p}{4\pi b^2}. \]  

It is easy to determine that it is advisable to make annular plates with a small opening in the center of laminated plastics, the modulus of elasticity in the radial direction of which is greater than the modulus of elasticity in the annular direction.

15. Bending of Circular Plates with Annular Fiber Reinforcing

A characteristic example of the practical use of circular cylindrically orthotropic laminated plates is circular plates made of synthetic polymers and reinforced in the annular direction with a fiber filler, fiberglass, for example. As has been noted, reinforcing of the plate only in the annular direction permits more efficient anisotropy of properties and, consequently, a more favorable distribution of stresses to be produced.

We consider a circular plate reinforced in the annular direction with uniformly placed fibers of circular cross section (Fig. 22).

If the reinforcing fibers are located at uniform distance \( \ell \) in each layer, the basic relationships which connect the stressed and deformed states of the plate can be presented in the form

\[
\begin{align*}
G_1 &= -D_1 \left( \psi' + \psi \frac{d}{d} \right); \\
G_2 &= -D_2 \left( \frac{d}{d} + \psi \psi' \right); \\
Q_1 &= -K_1 (\omega' + \varphi). 
\end{align*}
\]  

(104)

Flexural rigidities of the plate in the radial and annular directions \( D_1, D_2 \) and Poisson coefficients \( v_1, v_2 \) are determined by the following expressions.
where \( E_H, E_C \) are the moduli of elasticity of the fiber reinforcing and binder; \( v \) is the Poisson coefficient of the binder; \( \xi \) is the cross section radius of the reinforcing fiber; \( l \) is the distance between the reinforcing rings.

Such plates have a favorable elastic property anisotropy, since the following relationship occurs

\[
\lambda = \sqrt{\frac{D_1}{D_3}} = \sqrt{1 + k} \geq 1. \tag{106}
\]

Thus, for the calculation of laminated circular plates with uniform annular reinforcing, all the formulas obtained in the preceding sections are applicable. In the loading of such plates, both continuous and with an opening in the center, concentration of stresses and impermissible increase in deflection do not occur.

Simultaneous reinforcing of a plate in the annular and radial directions obviously is inadvisable with respect to stress distribution and complexity of production.

The basic difference of laminated circular plates reinforced only radially is that the anisotropy of their elastic properties changes radially.

16. Bending of Circular Plates with Radial Fiber Reinforcing

We consider a circular annular laminated plate obtained by bonding layers reinforced radially with a fiber filler (Fig. 23).

Evidently, the packing density of the fiber filler satisfies the relationship

\[
0 \leq \eta \leq \frac{\pi b}{2},
\]

where \( \xi \) is the cross section radius of the fiber filler.

The aperture angle of the fiber reinforcing \( \alpha = 2\pi \xi / b \). The basic relationships which connect the stressed and deformed state of such a cylindrically orthotropic plate can be presented in the form

\[
\begin{aligned}
G_1 & = -D_1(r) \psi' - D v \phi' / r; \\
G_2 & = -D \phi' / r - D v \psi'; \\
Q_1 & = -K_1(\omega' + \varphi),
\end{aligned}
\]

(107)
where $D = \frac{E_c \delta^3}{12(1-\nu^2)}$ is the flexural rigidity of the unreinforced plate; $E_c$, $\nu$ are the modulus of elasticity and Poisson coefficient of the binder; $K_1$ is the rigidity of the laminated plate with respect to interlayer shearing; $E_H$ is the modulus of elasticity of the fiber filler; $D_1$ is the radial flexural rigidity of the plate.

If the $E_c/E_H$ ratio is disregarded compared with unity, the following expression for rigidity can be obtained

$$D_1 = D \left(1 + \frac{k}{r}\right).$$

$$k' = \frac{n^2 (1-\nu^2) E_n}{2a E_c}.$$  \hspace{1cm} (109)

By substituting Eq. (107) in plate equilibrium Eq. (66), we obtain the following system of differential equations, which describe the bending of the circular plate with radial reinforcing

$$(r+k)\varphi'' + \varphi' - \frac{q}{k} = -\frac{Q_s}{D};$$

$$\omega' = -\frac{Q_s}{K_1} - \varphi;$$

$$Q_1 = -\frac{Pr}{r} + \frac{C_1}{r}.$$  \hspace{1cm} (110)

The following expressions also can be found for the bending moments and deformations

$$G_1 = -D_1 \left\{ \frac{C_1}{r} \left[ A \left(1 + \nu + \frac{k}{r}\right) - \frac{k(1+v)}{r} + \left(1+\nu+\frac{k}{r}\right) \ln \frac{r+k}{r} \right] + \frac{Pr^2}{8D} \times \right.$$

$$\left. \times \left[ \sum_{n=1}^{\infty} (-1)^n \frac{n+v}{n-1} \left(\frac{r}{k}\right)^{n-1} + \sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1} \left(\frac{r}{k}\right)^{n-2} \right] - \frac{C_1}{2D} \left[ \sum_{n=2}^{\infty} (-1)^n \frac{n+v}{n-1} \left(\frac{r}{k}\right)^{n-1} + \sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1} \left(\frac{r}{k}\right)^{n-2} \right] \right\}.$$  \hspace{1cm} (111)
Integration constants $A$, $B$, $C$, $C_1$ are determined from the boundary conditions on the inner and outer profiles of the plate.

17. General Relationships and Differential Equations of Asymmetric Bending of Circular Anisotropic Plates

Let a circular plate made of laminated cylindrically orthotropic material be bent by a transverse load, which is distributed symmetrically about the polar axis of the plate $x$. This is practically the most frequently encountered case of loading.

We place the origin of the $r$, $\theta$, $z$ cylindrical coordinate system at the pole of anisotropy, and we direct the $z$ axis along the axis of symmetry of elastic properties (Fig. 24).

A transverse load distributed symmetrically about the polar axis can be expanded in the trigonometric series

\begin{align}
P &= \sum_{n=0}^{\infty} P_n(r) \cos n \theta. \tag{112}
\end{align}

The general elasticity relationships are written in the form

\begin{align}

g_a &= -D \left[ \frac{C}{r^2} \left( A (1 + \nu) + (1 + \nu) \ln \frac{r+k}{r} - \frac{k (1 + \nu)(r+k)}{r (r+k)} \right) + \frac{E}{D} \sum_{n=1}^{\infty} (-1)^n \frac{r^n}{n-1} \left( \frac{r}{k} \right)^{n-1} - \frac{C}{2D} \sum_{n=2}^{\infty} (-1)^n \frac{r^n}{n-1} \left( \frac{r}{k} \right)^{n-1} \right];
\end{align}

\begin{align}

\varphi &= \frac{C}{r^2} \left[ Ar - k + r \ln \frac{r+k}{r} \right] + \frac{E}{D} \sum_{n=1}^{\infty} (-1)^n \frac{r^n}{n-1} \left( \frac{r}{k} \right)^{n-1} - \frac{C_1 k}{2D} \sum_{n=2}^{\infty} (-1)^n \frac{r^n}{n-1} \left( \frac{r}{k} \right)^{n-1} + B.
\end{align}
In accordance with Eq. (17), with $A_1 = 1$, $A_2 = r$, the plate equilibrium equations are written in the form

\begin{align}
\frac{\partial (r G_1)}{\partial r} + \frac{\partial Q_1}{\partial \theta} &= pr; \\
\frac{\partial (r G_2)}{\partial r} + \frac{\partial H}{\partial \theta} - G_3 &= Q_3 r; \\
\frac{\partial G_3}{\partial \theta} + \frac{\partial (r H)}{\partial r} + H &= Q_3 r.
\end{align}

(115)

Since bending of the plate will be symmetrical about the polar axis, the elastic forces and displacement can be sought in the trigonometric series

\begin{align}
\varphi &= \sum_{n=0}^{\infty} \varphi_n (r) \cos n \theta; \\
\psi &= \sum_{n=0}^{\infty} \psi_n (r) \sin n \theta; \\
w &= \sum_{n=0}^{\infty} w_n (r) \cos n \theta; \\
G_1 &= \sum_{n=0}^{\infty} G_1^n (r) \cos n \theta; \\
G_2 &= \sum_{n=0}^{\infty} G_2^n (r) \cos n \theta; \\
H &= \sum_{n=0}^{\infty} H^n (r) \sin n \theta; \\
Q_1 &= \sum_{n=0}^{\infty} Q_1^n (r) \cos n \theta; \\
Q_3 &= \sum_{n=0}^{\infty} Q_3^n (r) \sin n \theta.
\end{align}

(116)

In accordance with Eq. (113), (114), the coefficients of expansion of the forces are connected to the coefficients of expansion of displacements $\phi_n$, $\psi_n$, $w_n$ by the following relationships
\[ G_1''(r) = -D_1 \left[ \varphi_1 + \frac{r}{r^2} \left( n \psi_n + \varphi_n \right) \right]; \]

\[ G_2''(r) = -D_2 \left[ \frac{n}{r} \psi_n + \frac{r}{r^2} \psi_n \right]; \]

\[ l_1''(r) = -\frac{D_2}{2} \left[ \psi_n - \frac{r}{r^2} \psi_n \right]; \]

\[ Q_1''(r) = -K_1 (\psi_n + \varphi_n); \]

\[ Q_2''(r) = -K_1 \left( \psi_n - \frac{r}{r} \varphi_n \right). \]

(118)

By substituting Eq. (117) in the equilibrium equations with Eq. (118) taken into account, we obtain differential equations of asymmetric bending of circular plates:

\[ \psi_n + \frac{\psi_n}{r} - \left( \frac{n^2 \omega_1 + \lambda^2}{r^2} + k_1 \right) \psi_n + \]

\[ + \frac{n (\omega_1 + v_2)}{r} \psi_n - \frac{n (\omega_1 + \lambda^2)}{r^2} \psi_n = k_1 \omega_n; \]

\[ \omega_2 \psi_n + \frac{\omega_2}{r} \psi_n - \left( \frac{n^2 + \omega_2}{r^2} + k_2 \right) \psi_n - \]

\[ - \frac{n (\omega_2 + v_1)}{r} \psi_n - \frac{n (\omega_2 + 1)}{r^2} \psi_n = -nk_2 \omega_n; \]

\[ r \psi_n' + \frac{\psi_n'}{r} + g''n \psi_n = -\frac{p_{np} r}{K_1} - r \omega_n' - \]

\[- \omega_n' + \frac{n^2 g^2}{r} \omega_n. \]

(119)

where

\[ \frac{D_2}{D_1} = \lambda_1; \quad \frac{D}{D_1} = \frac{G (1 - v_1 v_2)}{E_1} = \omega_1; \]

\[ \frac{D}{D_2} = \frac{G (1 - v_1 v_2)}{E_2} = \omega_2; \]

\[ \frac{K_1}{D_1} = k_1; \quad \frac{K_2}{D_2} = k_2; \quad \frac{K_3}{K_1} = \frac{G_m}{G_1} = g^2. \]

(120)

If the load is distributed skew symmetrically about the polar axis, all the relationships are obtained by substitution of \( \cos n \theta \) by \( \sin n \theta \) and vice versa, with corresponding changes of signs of \( n \).

In the most general case of asymmetric loading, the solution is found by summing the asymmetric, symmetric and skew symmetric solutions. In conclusion, we note some identity relationships between the coefficients introduced above:

\[ g^2 \frac{k_1}{k_2} = \lambda^2; \quad \lambda \omega_2 = \omega_1; \]

\[ \lambda (1 + \lambda v_1) = \lambda + v_2. \]

(121)
CHAPTER 4. BENDING OF RECTANGULAR PLATES BY NORMAL LOAD

18. Differential Equation of Bending of Anisotropic Rectangular Plates

We consider a rectangular plate made of laminated anisotropic material and loaded by forces which are normal to the mean surface of the plate before deformation.

We select the x, y, z rectangular coordinate system as indicated in Fig. 25.

In accordance with Eq. (4), (5) and (11), (12),

\[ G_{1} = -D_{11} \frac{\partial^{2} \varphi}{\partial x^{2}} - D_{13} \frac{\partial^{2} \varphi}{\partial x \partial y} - D_{15} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \]
\[ G_{1} = -D_{13} \frac{\partial \varphi}{\partial x} - D_{15} \frac{\partial \varphi}{\partial y} - D_{13} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \]
\[ H = -D_{13} \frac{\partial \varphi}{\partial x} - D_{15} \frac{\partial \varphi}{\partial y} - D_{25} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \]

Fig. 25. Coordinate system and basic symbols.

\[ Q_{1} = -K_{1} \left( \varphi + \frac{\partial \omega}{\partial x} \right); \]
\[ Q_{1} = -K_{2} \left( \psi + \frac{\partial \omega}{\partial y} \right). \]

By substituting Eq. (122), (123) in the equilibrium equations

\[ \frac{\partial G_{1}}{\partial x} + \frac{\partial H}{\partial y} = Q_{1}; \]
\[ \frac{\partial G_{2}}{\partial y} + \frac{\partial H}{\partial x} = Q_{2}; \]
\[ \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y} = -p, \]

we obtain a system of differential equations of bending of rectangular anisotropic plates

\[ K_{1} \left( \varphi + \frac{\partial \omega}{\partial x} \right) = D_{11} \frac{\partial^{2} \varphi}{\partial x^{2}} + 2D_{13} \frac{\partial^{2} \varphi}{\partial x \partial y} + D_{15} \frac{\partial^{2} \varphi}{\partial y^{2}} + \]
\[ + D_{13} \frac{\partial^{2} \psi}{\partial x^{2}} + C \frac{\partial^{2} \psi}{\partial x \partial y} + D_{23} \frac{\partial^{2} \psi}{\partial y^{2}}; \]
\[ K_{2} \left( \psi + \frac{\partial \omega}{\partial y} \right) = D_{15} \frac{\partial^{2} \varphi}{\partial x^{2}} + 2D_{13} \frac{\partial^{2} \varphi}{\partial x \partial y} + D_{15} \frac{\partial^{2} \varphi}{\partial y^{2}} + \]
\[ + D_{23} \frac{\partial^{2} \psi}{\partial x^{2}} + 2D_{23} \frac{\partial^{2} \psi}{\partial x \partial y} + D_{23} \frac{\partial^{2} \psi}{\partial y^{2}}; \]
\[ K_{1} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x} \right) + K_{2} \left( \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial y} \right) = p. \]

Eq. (125) and (126) can be reduced to the following symmetrical form.

34
By multiplying Eq. (127) by operator \( L_4(\varphi) \) and with Eq. (128), (129) taken into account, a differential equation of bending of a laminar anisotropic rectangular plate can be obtained with interlayer shearing taken into account:

\[
L_4(\varphi) = D_{33}K_1 \frac{\partial^4 \varphi}{\partial x^4} + (2D_{13}K_1 - D_{12}K_2) \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \\
+ (D_{11}K_1 - CK_1) \frac{\partial^4 \varphi}{\partial y^4} - D_{33}K_1 \frac{\partial^4 \varphi}{\partial x^2} - K_1 K_3 \frac{\partial^4 \varphi}{\partial x \partial y^3};
\]

where

\[
L_4(\psi) = D_{33}K_2 \frac{\partial^4 \psi}{\partial x^4} + (2D_{13}K_2 - D_{12}K_1) \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \\
+ (D_{11}K_2 - CK_1) \frac{\partial^4 \psi}{\partial y^4} - D_{33}K_1 \frac{\partial^4 \psi}{\partial x^2} - K_1 K_3 \frac{\partial^4 \psi}{\partial x \partial y^3};
\]

(128)

(129)

\[
C = D_{11} + D_{22};
\]

(130)

(131)

\[
(132)
\]

For brevity, we introduce the following designations

\[
a_{40} = \frac{D_{11}D_{33} - D_{12}^2}{D_{33}}; \quad a_{41} = \frac{2(D_{11}D_{33} - D_{12}D_{13})}{D_{33}};
\]

\[
a_{31} = \frac{2(D_{11}D_{33} - D_{12}D_{13})}{D_{33}}; \quad a_{32} = \frac{D_{33}^2 + 2D_{13}D_{33} + D_{11}D_{33} - C^2}{D_{33}};
\]

\[
a_{12} = \frac{D_{33}^2 + 2D_{13}D_{33} + D_{11}D_{33} - C^2}{D_{33}}; \quad a_{13} = \frac{2(D_{12}D_{33} - D_{12}D_{13})}{D_{33}};
\]

\[
a_{11} = \frac{2(D_{13}K_3 + D_{33}K_1)}{D_{33}};
\]

(133)
Then, in dimensionless coordinates $\xi=x/a; \eta=y/b$, system of differential equations of bending of an anisotropic plate (132), (128), (129) is written in the form

\begin{equation}
L_6 (w) = L_4 (p);
\end{equation}

\begin{equation}
L_4 (\psi) = \left[ a_1 \frac{\partial^2 w}{\partial \xi^2} + a_2 \frac{\partial^2 w}{\partial \xi \partial \eta} + a_3 \frac{\partial^2 w}{\partial \eta^2} - \frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{\partial^2 w}{\partial \eta^2} \right];
\end{equation}

\begin{equation}
L_4 (\phi) = \left[ \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_2 \frac{\partial^2 w}{\partial \xi \partial \eta} + \beta_3 \frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 w}{\partial \xi \partial \eta} - \frac{\partial^2 w}{\partial \xi^2} \right],
\end{equation}

where the following differential operators are written through $L_6 (\cdot)$, $L_4 (\cdot)$

\begin{equation}
L_6 (\cdot) = a_{40} \frac{\partial^6 \psi}{\partial \xi^6} + a_{41} \frac{\partial^6 \psi}{\partial \xi^5 \partial \eta} + a_{42} \frac{\partial^6 \psi}{\partial \xi^4 \partial \eta^2} + a_{43} \frac{\partial^6 \psi}{\partial \xi^3 \partial \eta^3} + a_{44} \frac{\partial^6 \psi}{\partial \xi^2 \partial \eta^4} + a_{45} \frac{\partial^6 \psi}{\partial \xi \partial \eta^5} + a_{46} \frac{\partial^6 \psi}{\partial \eta^6} +
\end{equation}

\begin{equation}
+ a_{50} \frac{\partial^6 \phi}{\partial \xi^6} + a_{51} \frac{\partial^6 \phi}{\partial \xi^5 \partial \eta} + a_{52} \frac{\partial^6 \phi}{\partial \xi^4 \partial \eta^2} + a_{53} \frac{\partial^6 \phi}{\partial \xi^3 \partial \eta^3} + a_{54} \frac{\partial^6 \phi}{\partial \xi^2 \partial \eta^4} + a_{55} \frac{\partial^6 \phi}{\partial \xi \partial \eta^5} - a_{60}\left( \gamma_1 \frac{\partial^4 \phi}{\partial \xi^4} + \gamma_2 \frac{\partial^4 \phi}{\partial \xi^3 \partial \eta} + \gamma_3 \frac{\partial^4 \phi}{\partial \xi^2 \partial \eta^2} + \gamma_4 \frac{\partial^4 \phi}{\partial \xi \partial \eta^3} + \gamma_5 \frac{\partial^4 \phi}{\partial \eta^4} \right).
\end{equation}
19. General Equations of Bending of Orthotropic Rectangular Plates

If a laminated plate is made of an orthotropic material, the resulting general relationships and differential equations are simplified when the principal axes of anisotropy coincide with the coordinate axes, since \( D_{13} = D_{23} = 0 \).

By replacing \( D_{11} \) by \( D_1 \), \( D_{22} \) by \( D_2 \), \( D_{33} \) by \( D_3 \) and \( D_{12} \) by \( D_{12} \) or \( D_{21} \), for an orthotropic rectangular plate, we obtain

\[
G_1 = -D_1 \left( \frac{\partial \phi}{\partial x} + \nu_1 \frac{\partial \psi}{\partial y} \right);
\]

\[
C_1 = -D_1 \left( \frac{\partial \phi}{\partial y} + \nu_1 \frac{\partial \psi}{\partial x} \right);
\]

\[
H = -D_1 \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right);
\]

\[
Q_1 = -K_1 \left( \phi + \frac{\partial \psi}{\partial x} \right);
\]

\[
Q_1 = -K_1 \left( \psi + \frac{\partial \phi}{\partial y} \right).\]

System of differential Eq. (127)-(129) takes the form

\[
L_4(\psi) = D_3 K_1 \frac{\partial^4 \psi}{\partial z^2 \partial y^2} + \left( D_3 K_1 - C K_1 \right) \frac{\partial \psi}{\partial y} - K_1 K_2 \frac{\partial \psi}{\partial z} + \]

\[
L_4(\psi) = D_3 K_1 \frac{\partial^4 \psi}{\partial z^2 \partial y^2} + \left( D_3 K_1 - C K_1 \right) \frac{\partial \psi}{\partial y} - K_1 K_2 \frac{\partial \psi}{\partial z}; \]

(141)

\[
K_1 \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + K_1 \left( \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial y} \right) = p. \]

(142)

where \( L_4() \) is a differential operator in up to fourth order partial derivatives

\[
L_4() = D_4 D_4 \frac{\partial^4}{\partial z^2 \partial y^2} + (D_4 D_4 + D_4^* - C) \frac{\partial^4}{\partial z^2 \partial y^2} +
\]

\[
+ D_4 D_4 \frac{\partial^4}{\partial z^2 \partial y^2} - (D_4 K_1 + D_4 K_1) \frac{\partial^4}{\partial y^4} - \]

\[
- (D_4 K_1 + D_4 K_1) \frac{\partial^4}{\partial y^4} + K_1 K_1. \]

(143)
Differential equation of bending of an orthotropic plate (132) takes the form

\[
D_1 D_2 K_{,\nu} \frac{\partial^2 \nu}{\partial x^2} + [D_1 (D_2 K_1 + D_2 K_1) + K_1 (D_1^2 - C^2)] \frac{\partial^2 \nu}{\partial x^2 \partial y^2} + \\
+ (D_2 (D_1 K_1 + D_2 K_1) + K_2 (D_2^2 - C^2)) \frac{\partial^2 \nu}{\partial y^2} + D_2 D_2 K_{,\nu} \frac{\partial^2 \nu}{\partial y^2^2} - \\
- K_1 K_2 [D_1 \frac{\partial^2 \nu}{\partial x^2} + 2(C + D_1) \frac{\partial^2 \nu}{\partial x^2 \partial y^2} + D_1 \frac{\partial^2 \nu}{\partial y^2^2}] = F_4 (p).
\]

(144)

System of differential equations of bending of an orthotropic plate (135), (136) is simplified, since \(D_{13} = D_{23} = 0\) and, consequently,

\[
a_{20} = a_{21} = a_{41} = a_{11} = a_{13} = a_0 = a_4 = -
\]

\[
\beta_0 = \beta_4 = \gamma_0 = \gamma_4 = 0.
\]

(145)

In dimensionless coordinates \(\xi = x/a, \eta = y/b\), the system of equations has the form

\[
F_4 (\psi) = \alpha_1 \frac{\partial^2 \psi}{\partial \xi^2} + \alpha_3 \frac{\partial^2 \psi}{\partial \xi \partial \eta} - \alpha_5 \frac{\partial \psi}{\partial \xi}; \\
F_4 (\xi) = \beta_1 \frac{\partial^2 \psi}{\partial \eta^2} + \beta_3 \frac{\partial^2 \psi}{\partial \xi \partial \eta} - \beta_5 \frac{\partial \psi}{\partial \eta}; \\
F_4 (\psi) = F_4 (p),
\]

(146)  (147)

where \(F_6(), F_4()\) are the following differential operators in partial derivatives

\[
F_6 (\psi) = a_{60} \frac{\partial^4 \psi}{\partial \xi^4} + a_{24} \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + a_{24} \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + \\
+ \frac{\partial^4 \psi}{\partial \eta^4} - a_{00} \left( \gamma_1 \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + \gamma_3 \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + \gamma_5 \frac{\partial^4 \psi}{\partial \eta^4} \right);
\]

(148)

\[
F_4 (\psi) = a_{60} \frac{\partial^4 \psi}{\partial \xi^4} + a_{22} \frac{\partial^4 \psi}{\partial \xi^2 \partial \eta^2} + a_{04} \frac{\partial^4 \psi}{\partial \eta^4} - \\
- a_{20} \frac{\partial^2 \psi}{\partial \xi \partial \eta} - a_{02} \frac{\partial^2 \psi}{\partial \eta \partial \xi} + a_{00}.
\]

(149)

Coefficients \(a_1, \beta_1, \gamma_1\) (\(1=1, 3, 5\)) \(a_{13}\) are determined by Eq. (133), (134); with \(D_{13} = D_{23} = 0, D_{11} = D_1, D_{22} = D_2, D_{33} = D_3\), we have

\[
a_{40} = \frac{D_1 D_2}{a^4}; \quad a_{04} = \frac{D_1 D_2}{b^4}; \\
a_{28} = \frac{D_1 D_2^2 + D_3^2}{a^2 b^2} - C^2; \quad a_{20} = \frac{K_2 D_1 + K_1 D_2}{a^2}.
\]

(150)
Energy of Deformation of Anisotropic Rectangular Plate

In the majority of cases of solution of specific engineering problems, an exact solution cannot successfully be obtained. Therefore, various approximate methods of analysis must be used. In the theory of shells, variation methods based on the principle of the deformation energy minimum are most widespread. If an anisotropic plate is bent by normal load p, the potential energy of bending is determined by the well known expression

\[ U = \frac{1}{2} \int (G_{x}x_{1}^{2} + G_{y}x_{2}^{2} + 2Hx_{3} + Q_{1}y_{1} + Q_{2}y_{2}) \, dx \, dy. \] (152)

By using the Hooke's law relationships, we obtain

\[ U = \frac{1}{2} \int \left( D_{11}x_{1}^{2} + 2D_{12}x_{1}x_{2} + 3D_{22}x_{2}^{2} + + 3D_{33}x_{3}^{2} + 4D_{33}x_{3}^{2} + K_{1}y_{1} + K_{2}y_{2} \right) \, dx \, dy. \] (153)

By substituting \( \kappa_{1}^{e}, \kappa_{2}^{e}, \kappa_{3}^{e}, \gamma_{1}, \gamma_{2} \) from Eq. (11), (12) in Eq. (153), we obtain

\[ U = \frac{1}{2} \int \left[ D_{11} \left( \frac{\partial^{2} \varphi}{\partial x^{2}} \right) + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_{33} \left( \frac{\partial^{2} \psi}{\partial y^{2}} \right) + + D_{23} \left( \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + \frac{3}{2} D_{33} \frac{\partial \varphi}{\partial y} \left( \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + + \frac{3}{2} D_{33} \frac{\partial \psi}{\partial x} + K_{1} \left( \varphi + \frac{\partial w}{\partial x} \right) + + K_{2} \left( \psi + \frac{\partial w}{\partial y} \right) \right] \, dx \, dy. \] (154)

In this manner, while a possible deformed state of the plate is determined by functions \( \varphi, \psi, w \), the actual deformed state differs from all the kinematically possible states, i.e., those which satisfy the boundary conditions given, insofar as, for the actually deformed state, the functional
The integral is taken over the entire surface of the plate. If the plate is orthotropic and the directions of coordinate axes \(x, y\) coincide with the principal directions of anisotropy, Eq. (155) is significantly simplified and takes the form

\[
V = \frac{1}{2} \int \left[ D_{11} \left( \frac{\partial^2 \varphi}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_{22} \left( \frac{\partial \psi}{\partial y} \right)^2 + \frac{3}{2} D_{14} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right)^2 + \frac{3}{2} D_{44} \left( \frac{\partial \varphi}{\partial x} \right)^2 \right] \ dx \ dy
\]

(155)

has the minimum value.

The simplest alternate version of use of the principle of possible displacements, which is called the Ritz method, is as follows. Desired functions \(\varphi, \psi, \psi, w\) are assigned which satisfy the assigned boundary conditions at any values of random parameters \(C_1, C_2, \ldots, C_n\) and correspond as well as possible to the physical essence of the problem

\[
\begin{align*}
\varphi &= \varphi(x, y; C_1, C_2, C_3, \ldots, C_n); \\
\psi &= \psi(x, y; C_1, C_2, C_3, \ldots, C_n); \\
w &= w(x, y; C_1, C_2, C_3, \ldots, C_n).
\end{align*}
\]

(157)

By substituting these values in Eq. (155) for an anisotropic plate or in Eq. (156) for an orthotropic plate, after integration over \(x\) and \(y\) within limits which correspond to the entire surface of the plate, we obtain

\[
V = V(C_1, C_2, C_3, \ldots, C_n)
\]

(158)

We select constants \(C_1, C_2, \ldots, C_n\) in such a way that the energy of the system has the least value, i.e.,

\[
\frac{\delta V}{\delta C_1} = \frac{\delta V}{\delta C_2} = \ldots = \frac{\delta V}{\delta C_n} = 0.
\]

(159)
Values of random constants \( C_1, C_2, \ldots, C_n \) result from solution of
the system of \( n \) equations, which determine the desired solution of the
problem with the required degree of approximation. In the limit, as
\( n \to \infty \), an exact solution can be obtained. The accuracy of the solution
depends on how successfully functions \( \phi, \psi, w \) are selected.

21. Bending of Orthotropic Plate by Concentrated Force

Let a concentrated force applied at the point with coordinates \( x_o, y_o \) act on an orthotropic rectangular plate, the principal axes of aniso-
tropy of which coincide with the \( x, y \) coordinate axes (Fig. 26).

We will assume the sides of the plate to be supported and satisfy the
following boundary conditions

\[ w = g_1 = \psi = 0 \quad \text{at} \quad x = 0, x = a; \]
\[ w = g_2 = \phi = 0 \quad \text{at} \quad y = 0, y = b. \]

Such boundary conditions satisfy a possible deformed state, which is de-
termined by the expressions

\[ \begin{align*}
\varphi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b}; \\
\psi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b}; \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}.
\end{align*} \] (160)

In accordance with Eq. (156), the functional of the elastic energy of
the system

\[ V = \frac{1}{2} \int_0^a \int_0^b \left[ D_1 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{m \pi x}{a} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)^2 + \\
+ 2D_1 v_0 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{m \pi x}{a} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right) \times \\
\times \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \frac{n \pi y}{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right) + \\
+ D_2 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \frac{n \pi y}{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \right)^2 + \\
+ D_3 \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( A_{mn} \frac{n \pi y}{b} + B_{mn} \frac{m \pi x}{a} \right) \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \right)^2 \right] + (161) \]
Because the integrals of the product of the trigonometric functions discussed differ from zero only in the quadratic terms, we have

\[ V = \frac{ab}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ D_1 \left( \frac{m \pi}{a} \right)^3 A_{mn} + 2D_4 v_4 \frac{n \pi \sin \frac{n \pi x}{a}}{ab} A_{mn} + \right. \\
+ D_8 \left( \frac{n \pi}{b} \right)^2 B_{mn} + D_3 \left( A_{mn} \frac{n \pi}{b} + B_{mn} \frac{m \pi}{a} \right)^2 + \\
\left. + K_1 \left( A_{mn} + C_{mn} \frac{m \pi}{a} \right)^2 + K_2 \left( B_{mn} + C_{mn} \frac{n \pi}{b} \right)^2 \right] \\
- \frac{P}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}. \]  

The minimum of functional (162) is realized under the conditions

\[ A_{mn} \left[ D_1 \left( \frac{m \pi}{a} \right)^3 + D_3 \left( \frac{n \pi}{b} \right)^2 + K_1 \right] + \\
+ B_{mn} \left( D_4 v_4 + D_2 \frac{n \pi \sin \frac{n \pi x}{a}}{ab} + C_{mn} K_1 \frac{m \pi}{a} \right) = 0; \]

\[ A_{mn} \left( D_1 v_4 + D_2 \frac{n \pi \sin \frac{n \pi x}{a}}{ab} + B_{mn} \frac{n \pi}{b} \right) + \\
+ D_3 \left( \frac{m \pi}{a} \right)^3 + K_2 \right] + C_{mn} \frac{n \pi}{b} = 0; \]

\[ A_{mn} K_1 \frac{m \pi}{a} + B_{mn} K_2 \frac{n \pi}{b} + C_{mn} \left[ K_1 \left( \frac{m \pi}{a} \right)^2 + \\
+ K_2 \left( \frac{n \pi}{b} \right)^2 \right] = \frac{4P}{ab} \frac{n \pi \sin \frac{n \pi x}{a}}{a} \sin \frac{m \pi \sin \frac{n \pi y}{b}}{b}. \]  

for \( m, n = 1, 2, 3, \ldots \)

By solving system of Eq. (162), we find

\[ A_{mn} = - \frac{4P}{ab} \frac{A_{1m}}{\Lambda_3 m} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}; \]

\[ B_{mn} = - \frac{4P}{ab} \frac{A_{2m}}{\Lambda_3 m} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}; \]

\[ C_{mn} = \frac{4P}{ab} \frac{A_{3m}}{\Lambda_3 m} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}. \]  

\( (m, n = 1, 2, 3, \ldots) \)

where
\[ \Delta_{1m} = K_1D_b \left( \frac{m}{a} \right)^3 + (K_1D_1 - K_6C) \frac{m}{a} \left( \frac{n}{b} \right)^3 + \]
\[ + K_1K_6 \frac{m}{a} \frac{n}{b} ; \]
\[ \Delta_{2m} = K_2D_b \left( \frac{n}{b} \right)^3 + (K_2D_1 - K_6C) \left( \frac{m}{a} \right)^3 \frac{n}{b} + \]
\[ + K_2K_6 \frac{n}{b} ; \]
\[ \Delta_{3m} = D_1D_3 \left( \frac{m}{a} \right)^3 + (D_1D_1 + D_1^* - C^*) \left( \frac{m}{a} \right)^3 \left( \frac{n}{b} \right)^3 + \]
\[ + D_1D_3 \left( \frac{m}{a} \right)^4 + (D_1K_1 + D_1K_3) \left( \frac{m}{a} \right)^3 + \]
\[ + (D_1K_1 + D_1K_3) \left( \frac{n}{b} \right)^3 + K_1K_3. \]
\[ \Delta_{4m} = D_1D_2 \left( \frac{m}{a} \right)^4 + [D_1(D_1K_1 + D_1K_3)] + \]
\[ + K_1[D_1^* - C^*] \left( \frac{m}{a} \right)^3 \left( \frac{n}{b} \right)^3 + [D_1(D_1K_1 + D_1K_3) + \]
\[ + K_1K_3 \left( \frac{n}{b} \right)^3 \left( \frac{m}{a} \right)^3 + D_2D_2K_3 \left( \frac{n}{b} \right)^3 + \]
\[ + K_1K_3 \left[ D_1 \left( \frac{m}{a} \right)^4 + 2(C + D_3) \left( \frac{m}{a} \right)^3 \left( \frac{n}{b} \right)^3 + \]
\[ + D_3 \left( \frac{n}{b} \right)^3 \right] ; \]
\[ C = D_1v_3 + D_3 = D_3v_1 + D_3. \]

for \( m, n = 1, 2, 3, \ldots \)

The bending moments and cutting forces are determined by the expressions

\[ G_1 = \frac{4PD_1 \pi}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\Delta_{1m}}{m} + \frac{\Delta_{2m}}{m} \frac{n}{b} \right) v_1 \times \]
\[ \times \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ; \]
\[ G_2 = \frac{4PD_1 D_2}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\Delta_{3m}}{m} + \frac{\Delta_{4m}}{m} \frac{n}{b} \right) \times \]
\[ \times \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ; \]
\[ H = \frac{4PD_2}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\Delta_{5m}}{m} + \frac{\Delta_{6m}}{m} \frac{n}{b} \right) \times \]
\[ \times \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ; \]

(166)
22. Bending of Orthotropic Plate by Uniformly Distributed Load

Let an orthotropic hinge supported rectangular plate with sides \(a, b\) be bent by a uniformly distributed load of intensity \(p\). We select the coordinate system as indicated in Fig. 26.

We will seek a solution of Eq. (146), (147) in the binary trigonometric series

\[
\varphi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos m \pi \xi \sin n \pi \eta; \\
\psi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin m \pi \xi \cos n \pi \eta; \\
w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin m \pi \xi \sin n \pi \eta. 
\]

We represent load \(p\) in the form of the binary trigonometric series

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin m \pi \xi \sin n \pi \eta, 
\]

where

\[
P_{mn} = \frac{16p}{\pi^4 mn} (m, n = 1, 3, 5, \ldots). 
\]

By substituting Eq. (167), (168) in bending Eq. (146), (147), we obtain

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left[ \pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{30} m^3 + a_{03} n^3) + \pi a_{40} \cos m \pi \xi \sin n \pi \eta \right] \\
+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \left[ \pi^2 (a_{40} m^3 + a_{04} n^4) + \pi a_{40} \cos m \pi \xi \sin n \pi \eta \right] \\
= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \left[ \pi^2 (a_{30} m^2 + a_{03} n^3) + \pi a_{40} \cos m \pi \xi \sin n \pi \eta \right] \\
\times \cos m \pi \xi \sin n \pi \eta; 
\]
\[
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} B_{mn} \left[ \pi^4 (a_{w} m^4 + a_{e} m^2 n^2 + a_{w} n^4) + \\
+ \pi^2 (a_{w} m^2 + a_{e} n^4) + a_{mn} \sin m \pi \xi \cos n \pi \eta \right] = \\
- \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} \left[ \pi^2 (\beta_1 m^3 + \beta_2 m^2 n + \pi \beta_3 n) \times \\
x \sin m \pi \xi \cos n \pi \eta; \\
- \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{mn} \left[ \pi^2 (a_{w} m^2 + a_{e} m^2 n^2 + a_{w} m^4 n^2 + a_{w} n^4) + \\
+ \pi^2 a_{mn} (\gamma_1 m^6 + \gamma_2 m^4 n^2 + \gamma_3 n^4) \sin m \pi \xi \sin n \pi \eta \right] = \\
\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} P_m \left[ \pi^2 (a_{w} m^2 + a_{e} m^2 n^2 + a_{w} n^4) + \\
+ \pi^2 (a_{w} m^4 + a_{e} n^4) + a_{mn} \sin m \pi \xi \sin n \pi \eta. \right]
\]

According to Eq. (169), coefficients \( A_{mn}, B_{mn}, C_{mn} \) differ from zero only at odd values of indices \( m, n = 1, 5, 3, \ldots \) therefore, we will not subsequently stipulate this, and we will understand that summing is carried out only over the odd indices.

We introduce the following designations

\[
\begin{align*}
t^m_n &= \pi^4 (a_{w} m^4 + a_{e} m^2 n^2 + a_{w} n^4) + \\
&+ \pi^2 (a_{w} m^2 + a_{e} n^4) + a_{mn} (\gamma_1 m^6 + \gamma_2 m^4 n^2 + \gamma_3 n^4), \\
t^m_n &= \pi^4 (a_{w} m^4 + a_{e} m^4 n^2 + a_{w} n^4) + \\
&+ \pi^2 (a_{w} m^2 + a_{e} n^4) + a_{mn}; \\
t^m_n &= \pi^4 (a_{e} n^4) + a_{mn} (\gamma_1 m^6 + \gamma_2 m^4 n^2 + \gamma_3 n^4); \\
t^m_n &= \pi^4 (\beta_1 m^3 + \beta_2 m^2 n + \pi \beta_3 n) + a_{mn} (\gamma_1 m^6 + \gamma_2 m^4 n^2 + \gamma_3 n^4); \\
t^m_n &= \pi^4 (\beta_1 n^3 + \beta_2 m^3 n) + \pi \beta_3 n
\end{align*}
\]

for \( m, n = 1, 3, 5, \ldots \)

In accordance with Eq. (170), we obtain

\[
\begin{align*}
C_{mn} &= -P_{mn} t_{m,n}^m; \\
A_{mn} &= P_{mn} t_{m,n}^m; \\
B_{mn} &= P_{mn} t_{m,n}^m.
\end{align*}
\]

and, consequently, the solution of Eq. (146), (147) has the form
The maximum deflection in the center of the plate

$$|w|_{\text{max}} = \frac{16p}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m+n-2)^2}{m^2 n^2} \frac{c^{m,n}}{b_p^{m,n}}.$$  \hspace{1cm} (174)

The bending moments, torques and cutting forces are determined by the following expressions

$$G_1 = \frac{16p D_1}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} + v_1 c^{m,n}}{b_p^{m,n}} \sin m \pi \xi \sin n \eta;$$

$$G_2 = \frac{16p D_2}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} + v_1 c^{m,n}}{b_p^{m,n}} \cos m \pi \xi \sin n \eta;$$

$$H = \frac{16p D_3}{\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} + v_1 c^{m,n}}{b_p^{m,n}} \cos m \pi \xi \cos n \eta;$$

$$Q_1 = \frac{16p K_1}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} - v_2 c^{m,n}}{b_p^{m,n}} \sin m \pi \xi \sin n \eta;$$

$$Q_2 = \frac{16p K_2}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} - v_2 c^{m,n}}{b_p^{m,n}} \sin m \pi \xi \cos n \eta;$$

$$Q_3 = \frac{16p K_3}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} - v_2 c^{m,n}}{b_p^{m,n}} \cos m \pi \xi \sin n \eta;$$

$$Q_4 = \frac{16p K_4}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{c^{m,n} - v_2 c^{m,n}}{b_p^{m,n}} \cos m \pi \xi \cos n \eta;$$

The maximum shearing stresses on the edges of the plate \((x=0, x=a)\) and \((y=0, y=b)\) arise in the middles of the sides, and they are determined by the expressions

$$\tau_{1 \text{ max}} = \frac{24p K_1}{\pi b^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\zeta_1^{m,n} - \zeta_3^{m,n}}{b_p^{m,n}},$$

$$\tau_{2 \text{ max}} = \frac{24p K_2}{\pi b^4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\zeta_1^{m,n} - \zeta_3^{m,n}}{b_p^{m,n}},$$

where \(\zeta_0^{m,n}, \zeta_1^{m,n}, \zeta_2^{m,n}, \zeta_3^{m,n}\) are determined by Eq. (133), (134) and (171).

A solution in the form of binary trigonometric series is inconven-
ient for practical use. Therefore, it is advisable to consider a solution presented in single series.

23. Bending of Orthotropic Rectangular Plate with Two Supported Edges by Uniformly Distributed Load

Let the principal axes of anisotropy of the plate be parallel to the sides. We select the coordinate system as shown in Fig. 27.

We will assume the edges of the plate $x=0$, $x=a$ to be supported and to satisfy the following boundary conditions

$$w=g_1=\psi=0.$$ 

In this case, solution of plate bending system of differential Eq. (146), (147) can be sought in the form of single trigonometric series of the following type

$$\begin{align*}
\varphi &= \sum_{n=1}^{\infty} \varphi_n(\eta) \cos \lambda_n \xi; \\
\psi &= \sum_{n=1}^{\infty} \psi_n(\eta) \sin \lambda_n \xi; \\
w &= \sum_{n=1}^{\infty} w_n(\eta) \sin \lambda_n \xi,
\end{align*}$$

where $(\lambda_n = n \pi)$.

We expand the uniformly distributed load in the trigonometric series

$$p = \sum_{n=1}^{\infty} A_n \sin \lambda_n \xi,$$

where

$$A_n = \frac{4p}{\lambda_n} \text{ for } n = 1, 3, 5, \ldots$$

By substituting Eq. (177), (178) in bending Eq. (146), (147), we obtain

$$\begin{align*}
\sum_{n=1, 3}^{\infty} \left[ a_{04} \psi_n^{IV} - (\lambda_n^4 a_{22} + a_{22}) \psi_n + (\lambda_n^4 a_{40} + \\
+ \lambda_n^2 a_{20} + a_{00}) \varphi_n \right] \cos \lambda_n \xi &= \sum_{n=1, 3}^{\infty} \left[ \lambda_n a_{8} w_n^* - \\
- \lambda_n (\lambda_n a_1 + a_1) w_n \right] \cos \lambda_n \xi; \\
\sum_{n=1, 3}^{\infty} \left[ a_{04} \psi_n^{IV} - (\lambda_n^4 a_{22} + a_{22}) \psi_n + (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + \\
+ a_{00}) \psi_n \right] \sin \lambda_n \xi &= \sum_{n=1, 3}^{\infty} \left[ \beta_i w_n^{***} - (\lambda_n \beta_3 + \right]
\end{align*}$$

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Here and subsequently, summing is carried out only by odd indices \( n = 1, 3, 5, \ldots \).

System of Eq. (180) is satisfied if, for each \( n = 1, 3, 5, \ldots \), there is a solution of the following system of conventional differential equations

\[
\begin{align*}
a_04 \Psi_n^{IV} - (\lambda_n^0 a_{22} + a_{02}) \Psi_n + (\lambda_n^0 a_{40} + \lambda_n^0 a_{30} + a_{00}) \Psi_n &= \lambda_n a_2 \Psi_n - \lambda_n (\lambda_n a_1 + a_3) \Psi_n; \\
a_04 \Psi_n^{IV} - (\lambda_n^0 a_{22} + a_{02}) \Psi_n^* + (\lambda_n^0 a_{40} + \lambda_n^0 a_{30} + a_{00}) \Psi_n &= \beta_1 \Psi_n - (\lambda_n^0 \beta_3 + \beta_3) \Psi_n; \\
a_06 \Psi_n^{VI} - (\lambda_n^0 a_{24} + a_{02}) \Psi_n^{IV} + \lambda_n^0 (\lambda_n^0 a_{48} + a_{00}) \Psi_n^* &= -\lambda_n^0 (\lambda_n a_{46} + \lambda_n a_{30} + a_{00}) \Psi_n = \frac{4\rho}{\lambda_n^0} (\lambda_n^0 a_{40} + \lambda_n^0 a_{30} + a_{00}).
\end{align*}
\]

We present system of Eq. (181) in canonical form, doing them correspondingly by coefficients \( a_{04}, a_{06} \) which, according to Eq. (150), are different from zero

\[
\begin{align*}
\Psi_n^{IV} - e_n^{(1)} \Psi_n + e_n^{(2)} \Psi_n &= f_n^{(1)} \Psi_n^{*} - f_n^{(2)} \Psi_n; \\
\Psi_n^{IV} - e_n^{(1)} \Psi_n^* + e_n^{(2)} \Psi_n &= f_n^{(3)} \Psi_n^{**} - f_n^{(4)} \Psi_n^*; \\
\Psi_n^{VI} - \omega_n^{(1)} \Psi_n^{IV} + \omega_n^{(2)} \Psi_n^* - \omega_n^{(3)} \Psi_n &= \Omega_n
\end{align*}
\]

for \( n = 1, 3, 5, 7, \ldots \), where

\[
\begin{align*}
e_n^{(1)} &= \frac{\lambda_n^0 a_{22} + a_{02}}{a_{04}} = \frac{b_4}{D_4 D_3} \left[ \frac{\lambda_n^0}{a^2} (D_4 D_3 + D_3^3 - C^3) + \right. \\
&\left. + \lambda_1 D_3 + K_2 D_3 \right]; \\
e_n^{(2)} &= \frac{\lambda_n^0 a_{40} + \lambda_n^0 a_{30} + a_{00}}{a_{04}} = \frac{b_4}{D_4 D_3} \left[ \frac{\lambda_n^0}{a^2} D_4 D_3 + \frac{\lambda_n^0}{a^2} \times \\
&\times (K_2 D_4 + K_1 D_3) + K_1 K_2 \right];
\end{align*}
\]

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\[ f_n^{(1)} = \frac{\lambda_n a_2}{a_{44}} = \frac{b^4}{D_4 D_5} \frac{\lambda_n}{a} (K_1 D_4 - 2 CK_4); \]
\[ f_n^{(2)} = \frac{\lambda_n (\lambda_n^2 a_1 + a_3)}{a_{44}} = \frac{b^4}{D_4 D_5} \frac{\lambda_n}{a} \left( \frac{\lambda_n}{a^2} D_4 + K_1 \right); \]
\[ f_n^{(3)} = \frac{b}{a_{44}} = K_1 \frac{b}{D_6}; \]
\[ f_n^{(4)} = \frac{\lambda_n a_3 + b_n}{a_{44}} = \frac{b^4}{D_4 D_5} \left[ \frac{\lambda_n}{a^2} (K_2 D_1 - 2 CK_4) + K_1 K_3 \right]; \]
\[ \omega_n^{(1)} = \frac{\lambda_n^2 a_{44} + a_{44}}{a_{44}} = \]
\[ = \frac{b^4}{D_4 D_5 K_4} \left( \frac{\lambda_n}{a^2} \left[ D_4 D_5 K_4 + K_1 D_4 + D_5^2 - C^3 \right] \right) + \]
\[ + D_5 K_1 K_3 \right); \]
\[ \omega_n^{(2)} = \frac{\lambda_n (\lambda_n^2 a_{44} + a_{44})}{a_{44}} = \frac{b^4}{D_4 D_5 K_4} \times \]
\[ \times \lambda_n \left[ \frac{\lambda_n}{a^2} \left[ K_2 D_1 D_5 + K_1 D_4 + D_5^2 - C^3 \right] \right] + \]
\[ + 2 (C + D_5) K_1 K_3 \right); \]
\[ \omega_n^{(3)} = \frac{\lambda_n^2 (\lambda_n^2 a_{44} + a_{44})}{a_{44}} = \]
\[ = \frac{b^4 D_4 K_1}{D_4 D_5 K_4} \frac{\lambda_n}{a^2} \left( \frac{\lambda_n}{a^2} D_5 + K_4 \right); \]
\[ \Omega_n = \frac{b^4}{\lambda_n} \frac{b^4}{a_{44}} \left[ \frac{\lambda_n^2}{a^2} D_4 D_5 + \right. \]
\[ + \frac{\lambda_n}{a^2} (K_2 D_1 + K_1 D_2) + K_1 K_3 \right]\]

for \( n = 1, 3, 5, \ldots \)

After finding the general solution for plate deflection \( w_n \), for deformation functions \( \phi_n, \psi_n \), partial solutions of Eq. (182) are taken. In this manner, the solution of system of differential Eq. (182), (183) is determined by the roots of the characteristic equation

\[ k_4^2 - \omega_n^{(1)} k_4 + \omega_n^{(2)} k_4 - \omega_n^{(3)} = 0. \]  

Eq. (185) always has two real roots. The remaining roots are determined by coefficients \( \omega_n^{(j)} (j = 1, 2, 3) \).

We consider the most general case of complex roots, i.e., we will assume that the roots of characteristic Eq. (185) are

\[ \pm k_n; \pm (s_n \pm r_n), \]

(186)
The general solution of differential Eq. (183) can then be presented in the following form (for brevity, we will omit index n of coefficients $C_i, A_i, k_i, s_i, r_i$)

$$
\omega_n = -\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \text{ch} k \eta + C_2 \Phi_1(\eta) + C_3 \Phi_2(\eta) + \\
+ C_4 \text{sh} k \eta + C_5 \Phi_3(\eta) + C_6 \Phi_4(\eta),
$$

(187)

where the well known functions of V.Z. Vlasov are designated by

$$
\Phi_1(\eta) = \text{ch} s \eta \cos r \eta; \quad \Phi_2(\eta) = \text{sh} s \eta \cos r \eta; \\
\Phi_3(\eta) = \text{ch} s \eta \sin r \eta; \quad \Phi_4(\eta) = \text{sh} s \eta \sin r \eta.
$$

(188)

There are the following relationships for these functions

$$
\begin{align*}
\Phi_1' &= s \Phi_3 - r \Phi_4; & \Phi_2' &= s \Phi_1 - r \Phi_2; \\
\Phi_3' &= s \Phi_4 + r \Phi_3; & \Phi_4' &= s \Phi_2 + r \Phi_1; \\
\Phi_1'' &= (s^3 - r^3) \Phi_1 - 2rs \Phi_2; & \Phi_2'' &= (s^3 - r^3) \Phi_2 - 2rs \Phi_1; \\
\Phi_3'' &= (s^3 - r^3) \Phi_3 - 2rs \Phi_4; & \Phi_4'' &= (s^3 - r^3) \Phi_4 - 2rs \Phi_3; \\
\Phi_1''' &= s(s^3 - 3r^3) \Phi_3 + r(r^3 - 3s^3) \Phi_4; & \Phi_2''' &= s(s^3 - 3r^3) \Phi_2 + r(r^3 - 3s^3) \Phi_3; \\
\Phi_3''' &= s(s^3 - 3r^3) \Phi_1 + r(r^3 - 3s^3) \Phi_2; & \Phi_4''' &= s(s^3 - 3r^3) \Phi_4 + r(r^3 - 3s^3) \Phi_1; \\
\Phi_1^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_1 - 4rs(s^3 - r^3) \Phi_2; & \Phi_2^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_2 - 4rs(s^3 - r^3) \Phi_1; \\
\Phi_3^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_3 - 4rs(s^3 - r^3) \Phi_4; & \Phi_4^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_4 - 4rs(s^3 - r^3) \Phi_3.
\end{align*}
$$

(189)

A table of functions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ [15] is presented in the appendices.

In the coordinate system selected with a uniformly distributed load, solution of system of Eq. (182), (183) should be even relative to $\phi_n, \psi_n$ and odd relative to $\psi_n$, i.e., the solution must be sought in the form

$$
\begin{align*}
\varphi_n(\eta) &= \varphi_0 + A_1 \text{ch} k \eta + A_2 \Phi_1(\eta) + A_3 \Phi_2(\eta); \\
\psi_n(\eta) &= B_1 \text{sh} k \eta + B_2 \Phi_3(\eta) + B_3 \Phi_4(\eta); \\
w_n(\eta) &= -\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \text{ch} k \eta + C_4 \Phi_1(\eta) + C_5 \Phi_3(\eta).
\end{align*}
$$

(190)
By substituting Eq. (190) in differential Eq. (182), we obtain

\[
A_t k^4 \text{ch } k \eta + A_{\theta} [(s^4 - 6s'^2 + r^2) \Phi_1 (\eta) -
- 4rs (s^2 - r^2) \Phi_4 (\eta)] + A_{\phi} [(s^4 - 6s'^2 + r^2) \Phi_3 (\eta) +
+ 4rs (s^2 - r^2) \Phi_2 (\eta)] - c_n^{(1)} [A_t k^4 \text{ch } k \eta +
+ A_{\theta} [(s^4 - r^2) \Phi_1 (\eta) - 2rs \Phi_3 (\eta)] +
+ A_{\phi} [(s^4 - r^2) \Phi_2 (\eta) + 2rs \Phi_1 (\eta)] +
+ d_n^{(2)} (q_0 + A_{\theta} \text{ch } k \eta + A_{\phi} \Phi_1 (\eta) + A_{\phi} \Phi_2 (\eta)) =
= f_n^{(1)} [C_t k^3 \text{sh } k \eta + C_{\phi} [s (s^2 - 3r^2) \Phi_3 (\eta) -
- 2rs \Phi_4 (\eta)] + C_{\phi} [(s^2 - r^2) \Phi_3 (\eta) + 2rs \Phi_4 (\eta)] -
- f_n^{(2)} [q_0 + C_{\phi} \text{sh } k \eta + C_{\phi} \Phi_1 (\eta) + C_{\phi} \Phi_2 (\eta)];
B_t k^3 \text{sh } k \eta + B_{\theta} [(s^4 - 6s'^2 + r^2) \Phi_3 (\eta) +
+ 4rs (s^2 - r^2) \Phi_2 (\eta)] - c_n^{(1)} [B_t k^3 \text{sh } k \eta +
+ B_{\theta} [(s^4 - r^2) \Phi_2 (\eta) - 2rs \Phi_3 (\eta)] +
+ B_{\phi} [(s^4 - r^2) \Phi_2 (\eta) + 2rs \Phi_3 (\eta)] +
+ d_n^{(2)} [B_{\theta} \text{sh } k \eta + B_{\phi} \Phi_2 (\eta) + B_{\phi} \Phi_3 (\eta)] =
= f_n^{(3)} [C_t k^3 \text{sh } k \eta + C_{\phi} [s (s^2 - 3r^2) \Phi_3 (\eta) +
+ r (r^2 - 3s^2) \Phi_4 (\eta)] + C_{\phi} [s (s^2 - 3r^2) \Phi_4 (\eta) -
- r (r^2 - 3s^2) \Phi_3 (\eta)] - f_n^{(1)} [C_t k^3 \text{sh } k \eta +
+ C_{\phi} [s \Phi_3 (\eta) - r \Phi_4 (\eta)] + C_{\phi} [s \Phi_4 (\eta) + r \Phi_3 (\eta)].
\]

Since equality (191) occurs with any values of \( n \), on the assumption \( k^4 - e_n (1) k^2 + e_n (2) \neq 0 \), the following expressions can be obtained for coefficients \( A_1, B_1 (i=1, 2, 3) \)

\[
q_n = \frac{f_n^{(2)}}{e_n^{(2)}} \frac{\Omega_n}{w_n^{(3)}} ;
A_1 = C_1 \frac{f_n^{(1)} k^3 - f_n^{(2)}}{k^4 - e_n^{(1)} k^2 + e_n^{(2)}} ;
B_1 = C_1 k^4 \frac{f_n^{(3)} k^2 - f_n^{(4)}}{k^4 - e_n^{(1)} k^2 + e_n^{(2)}} ;
A_2 = \Delta_n^{(1)} C_3 + \Delta_n^{(2)} C_3 ;
B_2 = \Delta_n^{(3)} C_2 + \Delta_n^{(4)} C_3 ;
A_3 = -\Delta_n^{(2)} C_3 + \Delta_n^{(4)} C_3 ;
B_3 = -\Delta_n^{(3)} C_2 + \Delta_n^{(5)} C_3 ;
\]

where

\[
\Delta_n^{(1)} = \frac{[f_n^{(1)} (s^2 - r^2) - f_n^{(2)}] [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]}{[-s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]} ;
\]

\[
\Delta_n^{(2)} = 2rs \frac{[f_n^{(1)} (s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} (s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]}{[s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} (s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]} ;
\]

\[
\Delta_n^{(3)} = \frac{s [f_n^{(3)} (s^2 - 3r^2) - f_n^{(2)}] [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]}{[s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)] +
+ c_n^{(2)} [s^4 - 6s'^2 + r^4 - e_n^{(1)} (s^2 - r^2)]} ;
\]

51
\[
\Delta_n^{(4)} = r \frac{2r^8 \left[ e_n^{(1)} - 2 (r^8 - r^4) \right] \left[ f_n^{(3)} (r^8 - 3r^4) + f_n^{(4)} \right]}{\left[ r^8 - 6r^4 r^4 + r^4 - e_n^{(1)} (r^8 - r^4) \right]^2 + r_n^{(2)} \left[ e_n^{(1)} - r (r^8 - r^4) \right]^2}.
\]

Thus,
\[
\begin{align*}
\varphi &= \sum_{n=1,3}^{\infty} \left[ q_0 + A_{2n} \Phi_{2n}(\eta) + \lambda_n \Phi_{2n}(\eta) \right], \\
\psi &= \sum_{n=1,3}^{\infty} \left[ B_{2n} \Phi_{2n}(\eta) + B_{2n} \Phi_{4n}(\eta) \right], \\
\nu &= \sum_{n=1,3}^{\infty} \left[ C_{2n} \Phi_{2n}(\eta) + C_{2n} \Phi_{4n}(\eta) \right],
\end{align*}
\]

where coefficients \( A_{jn}, B_{jn} \) (\( j=1, 2, 3 \)) are expressed through random constants \( C_{jn} \) (\( j=1, 2, 3 \)) by Eq. (192), and \( \lambda_n \) is a real root of characteristic Eq. (185).

Correspondingly, the following expressions can be obtained for the bending moments and forces
\[
\begin{align*}
G_1 &= D_1 \sum_{n=1}^{\infty} \left[ \frac{1}{a} \left( q_0 + A_{2n} \Phi_{2n}(\eta) \right) + A_{2n} \Phi_{2n}(\eta) \right] + \\
&+ A_{2n} \Phi_{2n}(\eta) - \frac{v}{\nu} \left\{ k_n B_{2n} \chi k_n \eta + \\
&+ B_{2n} \Phi_{4n}(\eta) + B_{2n} \Phi_{4n}(\eta) \right\} \sin \lambda_n; \\
G_3 &= -D_3 \sum_{n=1}^{\infty} \left[ \frac{1}{b} \left( q_0 + A_{2n} \Phi_{2n}(\eta) \right) + A_{2n} \Phi_{2n}(\eta) \right] + \\
&+ A_{2n} \Phi_{2n}(\eta) - \frac{\lambda_n \nu}{\nu} \left\{ k_n B_{2n} \chi k_n \eta + \\
&+ B_{2n} \Phi_{4n}(\eta) + A_{2n} \Phi_{4n}(\eta) \right\} \sin \lambda_n; \\
H &= -D_3 \sum_{n=1}^{\infty} \left[ \frac{1}{b} \left( q_0 + A_{2n} \Phi_{2n}(\eta) \right) + A_{2n} \Phi_{2n}(\eta) \right] + \\
&+ B_{2n} \Phi_{4n}(\eta) + B_{2n} \Phi_{4n}(\eta) + B_{2n} \Phi_{4n}(\eta) \right\} \cos \lambda_n. \\
\end{align*}
\]

\[195\]
\[
Q_1 = -K_1 \sum_{n=1}^{\infty} \left\{ \left[ q_0 + A_{1n} \cosh k_n \eta + A_{2n} \Phi_{1n}(\eta) + \right. \right.
\left. + A_{3n} \Phi_{3n}(\eta) + \frac{1}{a} \left[ -\frac{\omega_n}{\omega_n(3)} + C_{1n} \cosh k_n \eta + \right. \right.
\left. + C_{2n} \Phi_{1n}(\eta) + C_{3n} \Phi_{3n}(\eta) \right] \cos \lambda_n \xi; \right. \\
\left. \left. + C_{2n} \Phi_{2n}(\eta) \right] \sin \lambda_n \xi \right\}; \\
Q_2 = -K_2 \sum_{n=1}^{\infty} \left\{ \left[ B_{1n} \sinh k_n \eta + B_{2n} \Phi_{1n}(\eta) + \right. \right.
\left. + B_{3n} \Phi_{3n}(\eta) + \frac{1}{b} \left[ C_{1n} \sinh k_n \eta + C_{2n} \Phi_{1n}(\eta) + \right. \right.
\left. + C_{3n} \Phi_{3n}(\eta) \right] \sin \lambda_n \xi \right\}. 
\]

For each number \(n=1, 3, 5\), random constants \(C_{1n}, C_{2n}\) are determined from the boundary conditions at the edges of the plate \(\eta=\pm1/2\).

In conformance with conditions (22), the boundary conditions for the edge \(\xi=\text{const}\) have the form

- unsupported edges: \(G_2=H=Q_2=0\); \\
- rigidly fastened: \(\psi=\phi=w=0\); \\
- loosely supported: \(w=G_2=H=0; \quad \psi=H=Q_2=0; \)
\(w=\phi=G_2=0; \quad \psi=w=H=0; \)
\(\psi=\phi=Q_2=0.\)

Besides these conditions, there can be different fastening of both edges, i.e., any pair combination of the boundary conditions written above. However, in these cases, the deformed and stressed states of the plate will not be symmetrical about the \(x\) axis and, consequently, all six random constant remain necessary in solution of (187).

24. **Case of Different Real Roots of Characteristic Equation**

Bicubic Eq. (185), by the known substitution

\[
k = k^* - \frac{\omega_n(1)}{3},
\]

can be reduced to canonical form

\[
k^3 + 3pk + 2q = 0,
\]

where
The number of real roots of Eq. (197) depends on the sign of discriminant $D = q^2 + p^2$

$$D = \frac{\omega_n^{(2)}}{27} + \frac{\omega_n^{(3)}}{4} + \frac{\omega_n^{(4)}}{27} - \frac{\omega_n^{(5)}}{27} - \frac{\omega_n^{(6)}}{8}.$$  

Consequently, a case of three different real roots can be visualized. Let these roots be $k_1^0, k_2^0, k_3^0$. The roots of bicubic Eq. (185) will then be in the form $+k_1^0, +k_2^0, +ik_3^0$. We will assume that $\pm k_1^{(i)}$ ($i=1, 2, 3$) are not solutions of uniform Eq. (182), i.e.,

$$k_1^{(i)} - k_2^{(i)} + \epsilon_n^{(2)} > 0$$

($i = 1, 2, 3$).

The solution of system of differential Eq. (182), (183) can then be presented in the following form

$$\varphi_n(\eta) = \varphi_0 + A_1 \cosh k_1 \eta + A_2 \cosh k_2 \eta + A_3 \cosh k_3 \eta + A_4 \sinh k_1 \eta + A_5 \sinh k_2 \eta + A_6 \sinh k_3 \eta;$$

$$\psi_n(\eta) = B_1 \cosh k_1 \eta + B_2 \cosh k_2 \eta + B_3 \cosh k_3 \eta + B_4 \sinh k_1 \eta + B_5 \sinh k_2 \eta + B_6 \sinh k_3 \eta;$$

$$\omega_n(\eta) = -\frac{\Omega_n}{\omega_n^{(2)}} + C_1 \cosh k_1 \eta + C_2 \cosh k_2 \eta + C_3 \cosh k_3 \eta + C_4 \sinh k_1 \eta + C_5 \sinh k_2 \eta + C_6 \sinh k_3 \eta.$$  

In this case, it is easy to obtain a solution for the general case of asymmetric boundary conditions.

By substituting Eq. (200) in system of Eq. (182), we obtain

$$A_1 k_1^0 \cosh k_1 \eta + A_2 k_2^0 \cosh k_2 \eta + A_3 k_3^0 \cos k_3 \eta +$$

$$+ A_4 k_1^0 \sinh k_1 \eta + A_5 k_2^0 \sinh k_2 \eta + A_6 k_3^0 \sin k_3 \eta -$$

$$- \epsilon_n^{(1)} [A_1 k_1^0 \cosh k_1 \eta + A_2 k_2^0 \cosh k_2 \eta - A_3 k_3^0 \cos k_3 \eta +$$

$$+ A_4 k_1^0 \sinh k_1 \eta + A_5 k_2^0 \sinh k_2 \eta - A_6 k_3^0 \sin k_3 \eta] +$$

(201)
for $n=1, 3, 5, \ldots$

Since Eq. (201) should be satisfied with any values of $n$, from Eq. (201) we obtain

\[
(202) \quad f_{n} = \frac{\pi}{2}\left(1 - \frac{1}{\pi}\right)^{n-1} \sqrt{\frac{\pi}{2}}.
\]
In this manner,

\[
\begin{align*}
\varphi &= \sum_{n=1,3,5,\cdots}^{\infty} \left[ q_{2n} + A_{2n} \cosh k_{2n} \eta + A_{2n} \sinh k_{2n} \eta + A_{2n} \cos k_{2n} \eta \cos \lambda_{n \xi} + A_{2n} \sin k_{2n} \eta \sin \lambda_{n \xi} \right] \\
\psi &= \sum_{n=1}^{\infty} \left[ B_{2n} \sinh k_{2n} \eta + B_{2n} \cosh k_{2n} \eta + B_{2n} \sin k_{2n} \eta \sin \lambda_{n \xi} + B_{2n} \cos k_{2n} \eta \cos \lambda_{n \xi} \right] \\
\omega &= \sum_{n=1}^{\infty} \left[-\frac{\Omega}{\omega^2_n} + C_{2n} \cosh k_{2n} \eta + C_{2n} \sinh k_{2n} \eta + C_{2n} \cos k_{2n} \eta \cos \lambda_{n \xi} + C_{2n} \sin k_{2n} \eta \sin \lambda_{n \xi} \right]
\end{align*}
\]

The following corresponding expressions can be obtained for the elastic forces and moments

\[
\begin{align*}
G_1 &= -D_1 \sum_{n=1}^{\infty} G_{1n}(\eta) \sin \lambda_{n \xi} \\
G_2 &= -D_2 \sum_{n=1}^{\infty} G_{2n}(\eta) \sin \lambda_{n \xi} \\
H &= -D_3 \sum_{n=1}^{\infty} H_n(\eta) \cos \lambda_{n \xi} \\
Q_1 &= -K_1 \sum_{n=1}^{\infty} Q_{1n}(\eta) \cos \lambda_{n \xi} \\
Q_2 &= -K_2 \sum_{n=1}^{\infty} Q_{2n}(\eta) \sin \lambda_{n \xi}
\end{align*}
\]

where

\[
\begin{align*}
G_{1n}(\eta) &= -\frac{\lambda_n}{a} \left[ q_{2n} + \sum_{j=1}^{2} A_{j+3} \cosh k_{j+3} \eta + \sum_{j=1}^{2} k_{j+3} B_{j+3} \cosh k_{j+3} \eta \right] \\
G_{2n}(\eta) &= -\frac{1}{b} \left[ \sum_{j=1}^{2} (k_{j} B_{j} \cosh k_{j} \eta + k_{j} B_{j+3} \cosh k_{j+3} \eta) + k_{2n} B_{2n} \cos k_{2n} \eta \right]
\end{align*}
\]
\[ H_n(\eta) = \frac{1}{b} \left[ \sum_{j=1}^{3} (A_{jn} \sin k_j \eta + A_{j+3} \cos k_j \eta) - A_{jn} \sin k_n \eta + A_{j+3} \cos k_n \eta \right] + \frac{k_n}{a} \left[ \sum_{j=1}^{2} (B_{jn} \sin k_j \eta + B_{j+3} \cos k_j \eta) - B_{jn} \sin k_n \eta + B_{j+3} \cos k_n \eta \right]; \]

\[ Q_{1n}(\eta) = \left[ \Phi_{0n} + \sum_{j=1}^{2} (A_{jn} \cosh k_j \eta + A_{j+3} \sinh k_j \eta) + A_{3n} \cos k_3 \eta + A_{6n} \sin k_3 \eta \right] + \frac{k_n}{a} \left[ \frac{-\Omega_n}{\omega_0^{(1)}} + \sum_{j=1}^{3} (C_{jn} \cosh k_j \eta + C_{j+3} \sinh k_j \eta) + C_{3n} \cos k_3 \eta + C_{6n} \sin k_3 \eta \right]; \]

\[ Q_{2n}(\eta) = \sum_{j=1}^{2} (B_{jn} \sin k_j \eta + B_{j+3} \cos k_j \eta) + B_{3n} \sin k_3 \eta + B_{6n} \cos k_3 \eta + \frac{1}{b} \left[ \sum_{j=1}^{2} (C_{jn} \sin k_j \eta + C_{j+3} \cos k_j \eta) - C_{3n} k_3 \eta + C_{6n} k_3 \eta \right]. \]

For \( n = 1, 3, 5, \ldots \)

Coefficients \( A_{jn}, B_{jn}, (j=1, 2, 3, 4, 5, 6) \) are determined through random constants \( C_{jn} \) by expressions (202).

For determination of constants \( C_{jn} (j=1, 2, 3, 4, 5, 6) \), boundary conditions (196) are used. In this manner, there can be 36 different combinations of possible support fastenings of the edges of the plate \( \eta = \pm 1/2 \).

If it turns out that any of functions \( e^{\pm k_{jn} \eta} \) is a solution of uniform Eq. (182), i.e., \( k_{jn}^4 e_n(1) k_{jn}^2 e_n(2) = 0 \), the solution corresponding to the solution \( w = C_1 \cosh k_{jn} \eta + C_2 \sinh k_{jn} \eta \) should be sought in the form
\[ \psi_n(\eta) = A_1 k_{jn} \eta \sin k_{jn} \eta + \frac{A_2 k_{jn}}{4} \cos k_{jn} \eta, \]
\[ \psi_n(\eta) = B_1 k_{jn} \eta \sin k_{jn} \eta + \frac{B_2 k_{jn}}{4} \cos k_{jn} \eta. \]

where

\[ A_1 = C_1 \frac{\beta_1^2 - \beta_2^2}{2k_{jn}(2k_{jn} - \varepsilon_1^{(1)})}; \]
\[ A_2 = C_2 \frac{\beta_3^2 - \beta_4^2}{2k_{jn}(2k_{jn} - \varepsilon_1^{(1)})}; \]
\[ B_1 = C_3 \frac{\beta_5^2 - \beta_6^2}{2k_{jn}(2k_{jn} - \varepsilon_1^{(1)})}; \]
\[ B_2 = C_4 \frac{\beta_7^2 - \beta_8^2}{2k_{jn}(2k_{jn} - \varepsilon_1^{(1)})}. \]
CHAPTER 5. STABILITY OF ANISOTROPIC RECTANGULAR PLATES

25. Formulation of Problem of Stability of Plates and General Methods of Determination of Critical Loads

The advent of high strength materials led to the extensive use of thin walled structures containing thin plates and shells as the basic elements in industry. Experience in the use of such structures shows that they, as a rule, turn out to be unsuccessful, not because the stresses which develop in them exceed permissible limits, but because of disturbance of the equilibrium of individual thin walled components.

Questions of the instability of equilibrium arise everywhere where there are thin walled structures. For thin walled structures made of laminated plastics, because of the low rigidity of the latter, assurance of stability is a particularly important problem in designing them.

If a parameter which characterizes the thin walled nature of a structural element, for example, the ratio of wall thickness to the characteristic plan dimension, is designated \( \varepsilon \), questions of stability assurance will be significant in the event the critical load is determined by the relationship \( p = A \varepsilon^q \), where \( q > 1 \) since, in this case, a reduction in wall thickness will significantly decrease the critical load, while stress will increase only in proportion to the decrease in thickness. The critical load in such thin walled structures proves to be one or two orders of magnitude less than the load at which failure of the material occurs. For plates, \( q = 2 \) and, consequently, if the bulging of units which consist of rectangular plates is an undesirable structure according to the operating conditions, proper selection of dimensions which ensure structural stability is an extremely important problem. Dimensions can be selected with the availability of calculation formulas or nomograms which define the critical load as a function of geometric dimensions and elastic constants.

The problem of stability of a flat plate subjected to forces applied in the plane of the plate can be formulated in the following manner. It is assumed that the magnitude and principle of distribution of extreme forces remain constant and that parameter \( \gamma \) characterizes the external load. The critical value of parameter \( \gamma \) is determined at the time of appearance of other forms of plate equilibrium accompanied by distortion of its mean plane.

The theory of elastic stability has been worked out extremely thoroughly, and a number of effective methods are available. One method of determination of the critical load is as follows. On the assumption that, at some value of load parameter \( \gamma \), the development of a distorted form of plate equilibrium is possible, differential bending equations are compiled with external forces \( T_1 = \gamma T_1^0, \gamma T_2 = T_2^0, S = \gamma S_0 \), which are applied in the mean plane of the plate and give bending component \( p \) normal to the mean plane of the plate, taken into account. The solution of such an equation, which contains \( \gamma \) as a parameter and which satisfies all boundary conditions, exists only with certain specific values of
parameter \( \gamma \), which are called the eigenvalues of the problem.

Each eigenvalue \( \gamma_k \) (\( k = 1, 2, 3, \ldots \)) defines a critical load which corresponds to a specific form of loss of stability. It is evident that only a load determined by the smallest eigenvalue of parameter \( \gamma \) is of practical importance.

Engineering practice usually is limited to obtaining approximate values, for which variation methods, based on general theorems of the equilibrium of mechanical systems and according to which the potential energy of the system has a minimum value in the equilibrium position, are extensively used.

If \( U_0 \) is the potential energy of a plate in planar equilibrium and \( U \) is the potential energy of the plate in the distorted state of equilibrium, the critical load is determined from the equation

\[
\begin{align*}
F(A, B, C, \gamma) &= \sum_m \sum_n (U_m(A, B, C) - \gamma V_m(C)) = 0.
\end{align*}
\]  

(210)

If a finite number of terms is taken in Eq. (209), Eq. (210) is not exactly satisfied. In this case, it is evident that the best approximation is obtained upon satisfaction of the conditions
By setting the determinant of uniform linear system (211) equal to zero, a characteristic equation can be obtained for determination of the critical value of load parameter $\gamma$.

Investigation of stability can be approached from more general standpoints of the stability of motion. Here, instability or stability of the planar shape of a plate exposed to forces applied in the mean plane of the plate should be indicated. Together with this unperturbed form of equilibrium of the plate, perturbations of the form of motion similar to it are considered. If the smallest perturbations desired cause finite deviations from unperturbed equilibrium over time, the latter are called unstable.

As applied to plates, this method is reduced to the following. A differential equation of transverse vibrations are compiled, with the longitudinal forces taken into account. Further, natural oscillation frequency $\omega_{mn}$ is determined. It depends on the plate dimensions, elastic constants of the material $c_{ij}$ and load parameter $\gamma$. At some values of parameter $\gamma$, the frequencies may turn out to be zero or imaginary, and their corresponding deflections will increase indefinitely. Such values of parameter $\gamma$ determine the critical load.

26. Differential and Variation Equations of Stability of Rectangular Plates

We consider a rectangular anisotropic plate with sides $a$, $b$. We select a coordinate system such that the $x$, $y$ axes are along the sides of the plate. Let the plate be loaded along the edges with forces $T_1^0$, $T_2^0$, $S^0$ in the mean plane of the plate (Fig. 28).

Let bulging of the plate occur at some combination of forces $T_1^0$, $T_2^0$, $S^0$. It is evident that, with as small a distortion of the mean plane of the plate as desired, the equilibrium equations in addition to the internal forces in the plane of the plate which arise in bending do not depend on initial forces $T_1^0$, $T_2^0$, $S^0$. More than that, these forces generally can be disregarded. The equations of equilibrium of the forces normal to the mean plane of the plate depend essentially on the initial forces, since the projections of these forces on the normal to the deformed mean plane are on the same order of smallness as the

$$\frac{\partial F}{\partial A_i} = \frac{\partial F}{\partial B_i} = \frac{\partial F}{\partial C_i} = 0,$$

$\ i=1, 2, 3, \ldots, n.$
cutting forces which arise upon bulging.

By projecting forces $T_1^0$, $T_2^0$, $S^0$ on the normal of the mean plane of the plate after bulging, we obtain

$$-p = T_1^0 \frac{\partial w}{\partial x} + 2S^0 \frac{\partial w}{\partial x} + T_2^0 \frac{\partial w}{\partial y},$$

(212)

i.e., the normal component of initial forces $T_1^0$, $T_2^0$, $S^0$ is equivalent to the distributed transverse load determined from Eq. (212). Since, because of the smallness of the bulge, all relationships obtained in study of the bending of a plate remain unchanged, the following system of differential equations can be obtained which describe the bulging of rectangular plates.

1. Differential equations of stability of anisotropic rectangular plate. For rectangular plates of anisotropic structure, the system of differential equations of stability has the following form

$$L_6(w) + \left( T_1^0 \frac{\partial^2 w}{\partial x^2} + T_2^0 \frac{\partial^2 w}{\partial y^2} + 2S^0 \frac{\partial^2 w}{\partial x \partial y} \right) L_4(w) = 0;$$

$$L_4(\psi) = \beta_1 \frac{\partial^2 w}{\partial x^2} + \beta_2 \frac{\partial^2 w}{\partial x \partial y} + \beta_3 \frac{\partial^2 w}{\partial y^2} - \beta_4 \frac{\partial w}{\partial x} - \beta_5 \frac{\partial w}{\partial y}.$$  

(213)

where differential operators in partial derivatives $L_6()$, $L_4()$ are determined from Eq. (137), (138).

2. Differential equations of stability of orthotropic rectangular plates. In the case of orthotropic plates, the principal axes of anisotropy of which are parallel to the sides of the plate, differential equations of stability (213), (214) are simplified somewhat and they take the form

$$F_6(w) + \left( T_1^0 \frac{\partial^2 w}{\partial x^2} + T_2^0 \frac{\partial^2 w}{\partial y^2} + 2S^0 \frac{\partial^2 w}{\partial x \partial y} \right) F_4(w) = 0;$$

$$F_4(\psi) = \beta_1 \frac{\partial^2 w}{\partial x^2} + \beta_2 \frac{\partial^2 w}{\partial x \partial y} + \beta_3 \frac{\partial^2 w}{\partial y^2} - \beta_4 \frac{\partial w}{\partial x} - \beta_5 \frac{\partial w}{\partial y}.$$  

(215)

(216)

Differential operators in partial derivatives $F_6()$, $F_4()$ are determined by expressions (148), (149).

Eq. (213)-(216), together with boundary conditions (22), permit
determination of the critical values of forces $T_1^0$, $T_2^0$, $S^0$, which are applied to the edges of the plate in the mean plane. It is particularly easy to obtain results in simple loading, i.e., when the external forces change in proportion to parameter $\gamma$.

It should be noted that, in the practical use of the method of direct integration of differential equations reported above, great difficulties arise in a number of cases, which are connected with satisfaction of the boundary conditions. Moreover, as a rule, the characteristic equations which can be obtained in determination of the critical load are transcendental, and they do not permit expression of the dependence of the critical load on the geometric dimensions of the plate in explicit form. Thus, it is highly advisable to have an approximate method for determination of the critical load. Approximate methods are based on consideration of the potential energy of bending of a plate upon bulging.

In derivation of the variation equation of stability, an expression must be obtained for the work of the external forces which is accomplished in bulging of the mean plane of the plate.

This work is determined by the expression [17]

$$\delta A = \frac{1}{2} \int_0^b \int_0^a \left[ T_1^0 \left( \frac{\partial \psi}{\partial x} \right)^* + T_2^0 \left( \frac{\partial \psi}{\partial y} \right)^* + 2S^0 \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial \psi}{\partial y} \right) \right] dx dy. \quad (217)$$

Based on the general theorems of mechanics, the equilibrium is stable if the potential energy of the system is at a minimum. Consequently, the magnitude of the critical load is determined from the condition that the increment of potential energy of bending of the plate upon bulging equals the work of the external forces.

Since the potential energy of bending of the plate is determined by Eq. (155), (156), we have the following variation equations of stability of anisotropic rectangular plates.

1. Variation equation of stability of anisotropic plate. For a rectangular plate made of anisotropic laminated plastic, the variation equation of stability has the following form

$$\int_0^b \int_0^a \left[ D_{11} \left( \frac{\partial \psi}{\partial x} \right)^4 + 2D_{12} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + D_{11} \left( \frac{\partial \psi}{\partial y} \right)^4 + D_{33} \left( \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial \psi}{\partial x} \right) \right] dx dy =$$

$$+ \frac{3}{2} D_{13} \frac{\partial \psi}{\partial x} \left( \frac{\partial \psi}{\partial y} \right)^3 + \frac{3}{2} D_{32} \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) + K_1 \left( \chi + \frac{\partial \psi}{\partial x} \right) +$$

$$+ K_2 \left( \psi + \frac{\partial \psi}{\partial y} \right) dx dy = \int_0^b \int_0^a \left[ T_1^0 \left( \frac{\partial \psi}{\partial x} \right)^4 + T_2^0 \left( \frac{\partial \psi}{\partial y} \right)^4 \right] dx dy +$$

$$+ 2S^0 \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial \psi}{\partial y} \right) dx dy. \quad (218)$$

2. Variation equation of stability of orthotropic rectangular
late. For an orthotropic plate, when the principal axes of anisotropy are parallel to the sides of the plate, Eq. (218) is somewhat simplified, and it takes the form

\[ \int_0^b \int_0^a \left[ D_1 \left( \frac{\partial^2 \psi}{\partial x^2} \right) + 2D_1 \left( \frac{\partial \psi}{\partial x} \right)^2 + D_2 \left( \frac{\partial^2 \psi}{\partial y^2} \right) + D_3 \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)^2 + 
\]

\[ + K_1 \left( \psi + \frac{\partial \psi}{\partial x} \right) + K_2 \left( \psi + \frac{\partial \psi}{\partial y} \right) \right] \, dx \, dy = 
\]

\[ - \int_0^b \int_0^a \left[ T_x \left( \frac{\partial \psi}{\partial x} \right)^2 + T_y \left( \frac{\partial \psi}{\partial y} \right)^2 + 2S \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} \right] \, dx \, dy. \]  

(219)

27. Stability of Orthotropic Rectangular Plate Compressed in One Principal Direction of Anisotropy

Let a rectangular orthotropic plate be compressed in one principal direction of anisotropy by forces \( T_1 \) uniformly distributed along the sides \( x=0, x=a \) (Fig. 29).

![Fig. 29. Diagram of plate compressed in one direction.](image-url)

We obtain a general expression for determination of the critical load by means of integration of the differential equation of stability. By increasing the intensity of load \( T_1 \), such a state can be reached in which the planar form of equilibrium of the plate becomes unstable and bulging of the plate occurs.

The system of differential equations of stability has the form

\[ F_6(w) = \frac{T_1}{a} \frac{\partial^2}{\partial x^2} F_6(w); \]  

\[ F_4(\psi) = a_1 \frac{\partial^2 \psi}{\partial x^2} + a_3 \frac{\partial^2 \psi}{\partial y^2} - a_5 \frac{\partial \psi}{\partial x}; \]

\[ F_4(\psi) = \beta_1 \frac{\partial^2 \psi}{\partial y^2} + \beta_3 \frac{\partial^2 \psi}{\partial x^2} - \beta_5 \frac{\partial \psi}{\partial y}. \]  

(220)

(221)

where functionals \( F_6() \) and \( F_4() \) are determined by Eq. (145) and (149).

a. Stability of plate hinge supported on profile. The solution of Eq. (220) which satisfies the boundary conditions of hinge support along the contour

at \( x=0, x=a \) \( w=G_1=\psi=0 \);

at \( y=0, y=b \) \( w=G_2=\phi=0 \),

can be sought in the form

64
\[ q = A \cos m \pi \xi \sin n \pi \eta; \]
\[ \psi = B \sin m \pi \xi \cos n \pi \eta; \]
\[ w = C \sin m \pi \xi \sin n \pi \eta. \]  
\( (222) \)

By substituting Eq. (222) in system of Eq. (220), (221), we obtain

\[ \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + \]
\[ + a_{60} \pi^4 (\gamma_1 m^4 + \gamma_3 m^2n^2 + \pi n^4) = \]
\[ = \frac{\pi^4}{a^2} m^2 n^2 \{ \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + \pi^4 (a_{20}m^2 + a_{6}m^4 + a_{6}n^4) \}; \]  
\( (223) \)

\[ A \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + \pi^4 (a_{20}m^2 + a_{6}m^4 + a_{6}n^4) + \]
\[ + a_{60} \pi^4 = \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + \]
\[ + a_{60} \pi^4. \]  
\( (224) \)

In conformance with Eq. (223), the critical load is determined by the following expression as a function of two integer parameters \( m, n, \) which determine the mode of wave formation

\[ T_1^0 = \frac{\pi^4 a^2}{m^2} \times \]
\[ \times \frac{\pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + a_{60} (\gamma_1 m^4 + \gamma_3 m^2n^2 + \gamma n^4)}{\pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4) + a_{60}}. \]  
\( (225) \)

We present Eq. (225) in a more convenient form for practical use. We introduce the following designations

\[ r_1 (q) = \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4); \]
\[ \xi_2 (q) = a_{60} (\gamma_1 m^4 + \gamma_3 m^2n^2 + \gamma n^4); \]
\[ \xi_3 (q) = \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4); \]
\[ \xi_4 (q) = \pi^4 (a_{40}m^4 + a_{22}m^2n^2 + a_{6}m^4 + a_{6}n^4). \]  
\( (226) \)

where

\[ q = \frac{m^3}{n^4}; \quad n^2 = u. \]  
\( (227) \)

The critical load is then determined by the simple expression

\[ T_1^0 = \frac{\pi^4 a^2}{q^2} \frac{u^2 \zeta_1 (q) + u \zeta_2 (q)}{u^2 \zeta_3 (q) + u \zeta_4 (q) + a_{60}}. \]  
\( (228) \)

From expression (228), we obtain

\[ \frac{\delta T_1}{\delta u} = \frac{\pi^4 a^2}{q^2} \frac{(\zeta_2 \zeta_4 - \zeta_2 \zeta_3) u^2 + 2a_{60} \zeta_1 u + a_{60} \zeta_3}{(\zeta_2 u^2 + \zeta_4 u + a_{60})^2}. \]  
\( (229) \)
It is easy to note that the sign of the right side of Eq. (229) is determined by the coefficient
\[
\zeta_1 \zeta_4 - \zeta_5 \zeta_3 = \pi^4 [ (a_{00}a_{20} - a_{00}a_{00} T_{12}) q^2 + a_{00}a_{00} - a_{00}a_{00} T_{12} + + (a_{20}a_{20} + a_{20}a_{00} - a_{20}a_{00} T_{12} - a_{20}a_{00} T_{12}) q^2 + + (a_{00}a_{00} + a_{00}a_{00} - a_{00}a_{00} T_{12} - a_{00}a_{00} T_{12}) q^2 + + (a_{00}a_{00} + a_{00}a_{00} - a_{00}a_{00} T_{12} - a_{00}a_{00} T_{12}) q^2].
\]

or, according to Eq. (150)
\[
\zeta_1 \zeta_4 - \zeta_5 \zeta_3 = \pi^4 \left[ \frac{D_i D_i K_i^*}{a_i^2} \right] q^2 + \frac{D_i D_i K_i^*}{a_i^2} \left[ (D_i D_i + 2D_i^* - C) (D_i K_i^* + D_i K_i^*) - 2 \left( D_i D_i + D_i^* - C \right) K_i^* \right] q^2 + \frac{D_i D_i K_i^*}{a_i^2} \left[ D_i K_i^* + + \left( 2D_i D_i + D_i^* - C \right) K_i^* \right. 2 CD_i K_i^* \left) q^2 + \frac{D_i D_i K_i^*}{a_i^2} \right].
\]

In this manner, over a wide range of change of plate rigidity parameters \( \frac{\partial T_0}{\partial u} \equiv 0 \) and, consequently, in bulging of a hinge supported plate compressed in one principal direction of anisotropy, one half wave forms transverse to the compression, i.e., \( n=1 \).

The critical load is determined by the smallest value of the expression
\[
\left( \gamma^0 \right)_{cr} = \frac{\pi^4 a_i^4}{q} \frac{\zeta_1 \left( \gamma \right) + \zeta_5 \left( \gamma \right)}{\zeta_3 \left( \gamma \right) + \zeta_4 \left( \gamma \right) + a_{00}},
\]

where \( q=1^2, 2^2, \ldots, m^2 \).

It now is advisable to consider the case frequently encountered in practice of the cylindrical shape of loss of stability of a rectangular plate upon compression in the direction of the unsupported edges.

b. Stability of plate with two supported and two unsupported edges. In this case, the approximate solution of system of Eq. (220), (221) can be sought in the form
\[
\varphi = A \cos m \pi \xi; \quad \psi = 0; \quad w = C \sin m \pi \xi.
\]

By substituting Eq. (233) in the system of differential equations of stability, we obtain
\[
\pi^4 m^4 a_{40} + \pi^4 m^4 a_{00} T_{12} = \frac{\pi^4}{\alpha^4} m^2 \pi^2 (\pi^4 m^4 a_{40} + \pi^2 m^2 a_{20} + a_{00})
\]
\[
A (m^4 \pi^4 a_{40} + m^2 \pi^2 a_{20} + a_{00}) = -B (m^2 \pi^2 a_1 + m \pi a_2),
\]

66
whence, for determination of the critical load, the following expression can be obtained

\[
T_1 = D_1K_1 \left( \frac{m \pi}{a} \right)^8 \frac{D_3 \left( \frac{m \pi}{a} \right)^4 + K_1}{D_1D_3 \left( \frac{m \pi}{a} \right)^4 + (D_3K_4 + D_5K_5) \left( \frac{m \pi}{a} \right)^4 + K_1K_5},
\]

or

\[
T_1^* = \frac{D_1K_1 \left( \frac{m \pi}{a} \right)^8}{D_1 \left( \frac{m \pi}{a} \right)^4 + K_1}.
\]

In accordance with Eq. (237), in cylindrical bulging of a laminated strip in the direction of compression, one half wave forms (m=1) and, consequently

\[
T_{cr} = \frac{\pi^2 D_1K_1}{\pi^2 D_1 + K_1 a^4}.
\]

The critical load for higher forms of loss of stability (m>1) does not tend toward infinity, as occurs in the case of uniform shells, but toward finite limit \(K_1\). The equivalent of this is that, with decrease in length of a rod, the critical load, upon increasing, asymptotically tends towards finite limit \(K_1\). This phenomenon should be taken into account in the use of reinforcing ribs made of laminated plastics to strengthen a cylindrical shell. The carrying capacity of such stiffening ribs can be limited by their rigidity in transverse shear.

28. Stability of Hinge Supported Rectangular Orthotropic Plate in Compression in Two Principal Directions of Anisotropy

We now consider the problem of the stability of a rectangular hinge supported plate with sides \(a, b\), in which the principal axes of anisotropy are parallel to the sides and which is compressed by uniformly distributed forces \(T_1^0, T_2^0\) (Fig. 30).

We obtain a solution by using variation equation of stability (219). We assign the deformed state of the plate after bulging in the form

\[
\begin{align*}
\phi &= A \cos \lambda x \sin \eta y; \\
\psi &= B \sin \lambda x \cos \eta y; \\
w &= E \sin \lambda x \sin \eta y,
\end{align*}
\]

where

\[
\lambda = \frac{m \pi}{a}; \quad \eta = \frac{n \pi}{b};
\]

Fig. 30. Diagram of plate compressed in principal directions of anisotropy.
the following can be obtained

\[
\begin{align*}
\frac{\partial \phi}{\partial x} &= -A \lambda \sin \lambda x \sin \eta y; & \frac{\partial \phi}{\partial y} &= -B \eta \sin \lambda x \sin \eta y; \\
\frac{\partial \psi}{\partial x} &= -A \sin \lambda x \cos \eta y; & \frac{\partial \psi}{\partial y} &= -B \lambda \cos \lambda x \sin \eta y; \\
\frac{\partial \xi}{\partial x} &= B \cos \lambda x \cos \eta y; & \frac{\partial \xi}{\partial y} &= E \lambda \sin \lambda x \cos \eta y.
\end{align*}
\] (241)

By substituting Eq. (239), (241) in variation equation of stability (219), we obtain

\[
U = \int \int_D \left( D_1 \lambda^2 \sin^2 \lambda x \sin^2 \eta y + 2C \lambda \eta AB \sin \lambda x \sin \eta y + \\
+ D_2 \lambda^2 \eta^2 \sin^2 \lambda x \sin^2 \eta y + D_3 (\eta A + \lambda B)^2 \cos \lambda x \cos \eta y + \\
- K_1 (A + \lambda E)^2 \cos \lambda x \sin \eta y + K_2 (B + \eta E)^2 \sin \lambda x \cos \eta y - \\
- T_1 \lambda^4 E^2 \cos \lambda x \sin \eta y - T_2 \eta^4 E^2 \sin \lambda x \cos \eta y \right) dx dy = \\
\frac{ab}{4} [D_1 \lambda^3 A^2 + 2C \lambda \eta AB + D_2 \eta^2 B^2 + D_3 (\eta A + \lambda B)^2 + \\
+ K_1 (A + \lambda E)^2 + K_2 (B + \eta E)^2 - (T_1 \lambda^4 + T_2 \eta^4) E^2].
\] (242)

where, as before, \( C = D_1 + D_3 \).

The minimum potential energy condition has the form

\[
\frac{\partial U}{\partial A} = \frac{\partial U}{\partial B} = \frac{\partial U}{\partial E} = 0,
\] (243)

or, after reduction,

\[
\begin{align*}
A (D_1 \lambda^3 + D_3 \eta^3 + K_1) + B (C + D_3) \lambda \eta + E \lambda K_1 &= 0; \\
A (C + D_2) \lambda \eta + B (D_3 \eta^3 + D_2 \lambda^2 + K_2) + E \eta K_2 &= 0; \\
A \lambda K_1 + B \eta K_2 + E (\lambda^2 K_1 + \eta^2 K_2 - T_1 \lambda^2 - T_2 \eta^2) &= 0.
\end{align*}
\] (244)

The condition of nontriviality of the solution of this system gives the necessary characteristic equation for determination of the critical load.
As a result, for determination of the critical load of a hinge supported orthotropic plate compressed in the principal directions of anisotropy, the following expression can be obtained

\[
T_i^* \left( \frac{m}{a} \right)^1 + T_i^* \left( \frac{n}{b} \right)^1 = \pi^2 \frac{\xi_1(m,n) + \xi_2(m,n)}{\xi_3(m,n) + \xi_4(m,n) + \omega}.
\]  

(247)

where, in accordance with Eq. (150), the following designations are introduced

\[
\begin{align*}
\xi_1(m,n) &= \pi^2 (a_4m^n + a_5m^2n^2 + a_6n^4); \\
\xi_2(m,n) &= \omega (\gamma_1 m + \gamma_2 m^2 + \gamma_3 n^4); \\
\xi_3(m,n) &= \pi^4 (a_1m^2 + a_2m^4 + a_3n^2); \\
\xi_4(m,n) &= \pi^4 (a_5m^4 + a_6n^2). 
\end{align*}
\]  

(248)

In simple loading, when the load along all edges increases in proportion to a single parameter, i.e., when \(T_1^0 = T, T_2^0 = \omega T\), the critical value of parameter \(T\) is determined from the expression

\[
T^* \left( \frac{m}{a} \right)^1 + \omega \left( \frac{n}{b} \right)^1 = \pi^2 \frac{\xi_1(m,n) + \xi_2(m,n)}{\xi_3(m,n) + \xi_4(m,n) + \omega}.
\]  

(249)

i.e., the problem is reduced to finding the smallest value of the right side of Eq. (249) as a function of integer parameters \(m, n\).

29. Stability of Infinitely Wide Orthotropic Plate in Compression

Along Short Sides

Let an extremely wide orthotropic plate of length \(a\) be compressed by forces \(T_1^0\), which are uniformly distributed along the wide edges. We will assume that the principal directions of anisotropy coincide with the sides (Fig. 31). In this case, a cylindrical form of loss of stability can be considered, i.e., it can be assumed that all components
of displacements and forces do not depend on coordinate \( y \). The system of differential equations of stability then takes the form

\[
\begin{align*}
\frac{d^2}{dx^2} a_{66} w^V - a_{66} \gamma_1 w^I = & \frac{T}{d^2} (a_{66} w^V - a_{66} w^I + a_{66} w'');
\frac{d^2}{dx^2} a_{46} q^V - a_{46} \varphi'' + a_{46} \psi = & a_1 w'' - a_6 w'.
\end{align*}
\]  

(250)

In accordance with Eq. (150), system of Eq. (250) can be presented in the following form:

\[
\begin{align*}
\left( D_3 \frac{d^4}{dx^4} - K_1 \right) D_1 K_1 \frac{d^4 w}{dx^4} = & T \left( D_3 \frac{d^4}{dx^4} - K_1 \right) \left( D_1 \frac{d^4}{dx^4} - K_1 \right) \frac{d^4 w}{dx^4};
\left( D_3 \frac{d^4}{dx^4} - K_1 \right) D_1 \frac{d^4 \varphi}{dx^4} - K_1 \varphi = & \left( D_3 \frac{d^4}{dx^4} - K_1 \right) K_1 \varphi - K_1 w'.
\end{align*}
\]  

(251)

System of Eq. (251) satisfies the solution of the system

\[
\begin{align*}
\frac{d^4}{dx^4} w^V + k^2 w'' = 0;
\frac{d^4}{dx^4} \varphi'' - p^2 \varphi = p^2 w'.
\end{align*}
\]  

(252)

where

\[
k^2 = \frac{T K_1}{D_1 (K_1 - T)}; \quad p^2 = \frac{K_1}{D_1}.
\]  

(253)

The general solution of system of differential Eq. (252) has the form

\[
\begin{align*}
w = & C_1 \cos kx + C_2 \sin kx + C_3 x + C_4;
\varphi = & \frac{p^2 k}{p^2 + k^2} (C_1 \sin kx - C_2 \cos kx) - C_3.
\end{align*}
\]  

(254)

It also is easy to find the bending moment and cutting force

\[
\begin{align*}
G_1 = & - \frac{D_1 p^2 k^2}{p^2 + k^2} (C_1 \cos kx + C_2 \sin kx);
Q_1 = & \frac{K_1 k^3}{p^2 + k^2} (C_1 \sin kx - C_2 \cos kx).
\end{align*}
\]  

(255)

Further, we consider some partial cases of fastening of the plate edges \( x = \pm a/2 \).

a. **Edge of plate hinge supported.** Because of symmetry, an even solution for \( x \) can be considered, i.e., it can be assumed that \( C_2 = C_3 = 0 \).
Consequently, the boundary conditions are satisfied if \( \cos \frac{ka}{2} = 0 \) or \( ka = \pi \), and the critical load is determined by the expression

\[
T_{cr} = \frac{n^2 \eta_1 k_1}{n^2 D_1 + k_1 a^2}.
\]  

(256)

b. Edge of plate rigidly fastened. The boundary conditions have the form

\[
\begin{align*}
C_1 \cos \frac{ka}{2} + C_2 \frac{p_{ik}}{p^2 + k^2} \sin \frac{ka}{2} &= 0; \\
C_1 \cos a + C_2 \sin a &= 0;
\end{align*}
\]

(257)

as a result, it follows that \( ka = 2\pi \) and, consequently,

\[
T_{cr} = \frac{4n^2 \eta_1 k_1}{4n^2 D_1 + k_1 a^2};
\]

(258)

c. Edge \( x = -a/2 \) rigidly fastened, edge \( x = a/2 \) unsupported. In this case, the boundary conditions have the form

\[
\begin{align*}
C_1 \cos a - C_2 \sin a - C_3 \frac{a}{2} + C_4 &= 0; \\
C_1 \sin a + C_2 \cos a + C_3 \frac{p_{ik}^2 + k^2}{p^2} &= 0; \\
C_1 \cos a + C_2 \sin a &= 0; \\
C_3 T &= 0,
\end{align*}
\]

(259)

where

\[
a = \frac{ka}{2} = \frac{a}{2} \sqrt{\frac{7K_1}{D_1(K_1 - T)}}.
\]

(260)

The characteristic equation for determination of the critical load has the form

\[
\begin{vmatrix}
\sin a & \cos a \\
\cos a & \sin a
\end{vmatrix} = 0,
\]

(261)

from which we have \( k^2 a^2 = \pi^2 / 4 \) and, consequently,

\[
T_{cr} = \frac{n^2 \eta_1 k_1}{n^2 D_1 + 4k_1 a^2}.
\]

(262)

d. Edge \( x = -a/2 \) rigidly fastened, edge \( x = a/2 \) hinge supported. For such boundary conditions

at \( x = -\frac{a}{2} \) \( w = \varphi = 0 \);

at \( x = \frac{a}{2} \) \( w = C_4 = 0 \).

71
from which we obtain the following system of linear equations

\[
\begin{align*}
C_1 \cos \alpha - C_2 \sin \alpha - C_3 \frac{x}{2} + C_4 &= 0; \\
C_1 \sin \alpha + C_2 \cos \alpha + C_3 \frac{x^2}{p^2 + k^2} &= 0; \\
C_1 \cos \alpha + C_2 \sin \alpha + C_3 \frac{x}{2} + C_4 &= 0; \\
C_1 \cos \alpha + C_2 \sin \alpha &= 0.
\end{align*}
\]  

(263)

From Eq. (263), it is easy to obtain the following transcendental equation for determination of the critical load

\[\tan k\alpha = \frac{p^2 k a}{p^2 + k^2} - \frac{k a}{1 + \gamma k^2 a^2},\]  

(264)

where parameter \(\gamma\) characterizes the effect of interlayer shearing on the critical load of the plate and is determined by the expression

\[\gamma = \frac{D_1}{K_1 \alpha^2}.\]  

(265)

If the least root of Eq. (264) is designated by \(\omega\), the critical load is determined by the following formula

\[T_{cr} = \frac{\omega^3 D_1 K_1}{\omega^3 U_1 + K_1 \alpha^2}.\]  

(266)

The least root of Eq. (266) as a function of \(\gamma\) is determined by the graph presented in Fig. 32.

Fig. 32. Graph for determination of least root of equation \(\tan x = \frac{x}{1 + \gamma x^2}\).
CHAPTER 6. TRANSVERSE OSCILLATIONS OF ANISOTROPIC LAMINATED PLATES

30. Formulation of Problem of Transverse Oscillations of Anisotropic Laminated Plates

We will consider small bending oscillations of uniform anisotropic plates of constant thickness bounded by a simple profile. We will assume the bending deformations which arise in the oscillations to be small elastic oscillations which are governed by the generalized Hooke's law. Such oscillations are described by differential equations which are similar to the differential bending equations. Their fundamental difference is the dependence of the external load and, consequently, deformation functions $\phi, \psi$ and plate deflection $w$ on time, as well as the presence of additional terms which define the inertial load.

Forced oscillations of the plate which arise as a result of variable transverse load $p(x, y; t)$ should be distinguished from the natural free oscillations. We will state that the plate accomplishes free transverse oscillations if any forces which impart deflections and velocities to the particles of the mean surface are instantaneously removed.

Thus, the system of differential equations of oscillations of an anisotropic plate can be written in the following form

\[ L_6(w) \cdot \rho \frac{\partial^2}{\partial t^2} L_6(w) + L_6(q); \]

\[ \begin{align*}
L_6(\phi) &= a_1 \frac{\partial^2 w}{\partial x^2} + a_2 \frac{\partial^2 w}{\partial y^2} + a_3 \frac{\partial^2 w}{\partial z^2} - a_4 \frac{\partial w}{\partial y} - a_5 \frac{\partial w}{\partial z}; \\
L_6(\psi) &= \beta_1 \frac{\partial^2 w}{\partial y^2} + \beta_2 \frac{\partial^2 w}{\partial z^2} - \beta_3 \frac{\partial w}{\partial y} - \beta_4 \frac{\partial w}{\partial z};
\end{align*} \]

where $\rho$ is the plate material density; $q$ is the variable transverse load applied to the plate.

Differential operators $L_6()$, $L_4()$ and coefficients $a_1, \beta_1$ are determined from Eq. (134), (137) and (138).

In the case of a rectangular orthotropic plate, the principal axes of anisotropy of which are parallel to the sides, the system of differential equations is simplified, and it takes the form

\[ F_6(w) = \rho \frac{\partial^2}{\partial t^2} F_4(w) + F_4(p); \]

\[ \begin{align*}
F_4(\phi) &= a_1 \frac{\partial^2 w}{\partial x^2} + a_2 \frac{\partial^2 w}{\partial y^2} + a_3 \frac{\partial^2 w}{\partial z^2} - a_4 \frac{\partial w}{\partial y} - a_5 \frac{\partial w}{\partial z}; \\
F_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial y^2} + \beta_2 \frac{\partial^2 w}{\partial z^2} - \beta_3 \frac{\partial w}{\partial y} - \beta_4 \frac{\partial w}{\partial z};
\end{align*} \]

Operators $F_6()$, $F_4()$ and coefficients $a_1, \beta_1$ are determined by
Eq. (148)-(151). The free oscillations of the plate are determined by solution of uniform system of Eq. (269), (270).

Deformation functions $\phi$ and $\psi$, as well as plate deflection $w$ should satisfy boundary conditions which depend on the fastening conditions of the bounded profile of the plate and the initial conditions which define the form and velocity of displacement of the particles of the mean surface at the initial moment of time, i.e., at $t=0$, the following conditions should be satisfied

$$\begin{align}
\varphi &= \varphi_0(\xi, \eta); \quad \psi = \psi_0(\xi, \eta); \quad w = w_0(\xi, \eta); \\
\varphi &= \psi = 0; \quad \frac{\partial w}{\partial t} = w_0(\xi, \eta).
\end{align}$$

The solution of the problem of free oscillations of the plate is reduced to determination of the form of the oscillations, which is determined by the mode of functions $\phi$, $\psi$, $w$ and the natural oscillation frequency. It should be noted that, in oscillation theory, the eigenfrequencies of an elastic system are of extremely great importance.

Following S.G. Lekhnitskiy [17], we reproduce the trend of the solution of the problem of free bending oscillations of an anisotropic plate by the Fourier method.

We represent the solution of system of Eq. (267), (268) in the form of the product of the periodic time functions by the amplitude of the corresponding functions, i.e., we set

$$\begin{align}
\varphi &= (C_1 \cos pt + C_2 \sin pt) \Phi(\xi, \eta); \\
\psi &= (C_1 \cos pt + C_2 \sin pt) \Psi(\xi, \eta); \\
w &= (C_1 \cos pt + C_2 \sin pt) W(\xi, \eta),
\end{align}$$

where $p$ is the frequency of the natural oscillations of the plate.

By substituting Eq. (272) in Eq. (267), (268) for determination of $\phi$, $\psi$, $W$, we obtain the system of differential equations

$$\begin{align}
L_4(W) + p^2 \phi \Phi L_4(W) &= 0; \\
L_4(\Phi) &= a_1 \frac{\partial W}{\partial t^2} + a_2 \frac{\partial W}{\partial \xi^2 \eta^2} + \\
+ a_3 \frac{\partial W}{\partial \xi \eta^2} - a_4 \frac{\partial W}{\partial \eta^2} + a_5 \frac{\partial W}{\partial \xi^2}; \\
L_4(\Psi) &= \beta_1 \frac{\partial W}{\partial \eta^2} + \beta_2 \frac{\partial W}{\partial \xi^2 \eta^2} + \\
+ \beta_3 \frac{\partial W}{\partial \xi \eta^2} - \beta_4 \frac{\partial W}{\partial t} - \beta_5 \frac{\partial W}{\partial \eta}.
\end{align}$$

By satisfying the assigned boundary conditions of the problem, as is done in determination of the critical load, we obtain a characteristic equation which determines the presence of the nontrivial solution.
A(p)=0, which gives an infinite spectrum of the eigenfrequencies of oscillation of the plate.

The eigenfrequencies of oscillation of the plate depend on two integer parameters m, n=1, 2, 3, ... The lowest frequency is called the eigenfrequency of the primary tone, and the remaining frequencies are called frequencies of the second, third, etc. order. Each eigenfrequency \( p_{mn} \) corresponds to the form of the natural oscillations \( \phi_{mn} \), \( \psi_{mn} \), \( \psi_{mn} \), which is determined to within an arbitrary factor.

Functions \( \phi_{mn} \), \( \psi_{mn} \), \( \psi_{mn} \) frequently are called eigenfunctions. They are used in solution of problems of oscillations of a plate.

If the deformed state of the plate must be determined at any moment of time, the following procedure is used: initial functions \( \phi_0 \), \( \psi_0 \), \( w_0 \) and initial velocity \( v_0(\xi, \eta) \) are expanded in series by the eigenfunctions

\[
\begin{align*}
\Psi_0(\xi, \eta) &= \sum_{m,n} a_{mn} \Phi_{mn}(\xi, \eta); \\
\Psi_0(\xi, \eta) &= \sum_{m,n} \beta_{mn} \Psi_{mn}(\xi, \eta); \\
w_0(\xi, \eta) &= \sum_{m,n} \gamma_{mn} W_{mn}(\xi, \eta); \\
v_0(\xi, \eta) &= \sum_{m,n} \delta_{mn} W_{mn}(\xi, \eta);
\end{align*}
\]

and the solution is found in the form of the analogous series

\[
\begin{align*}
\varphi(\xi, \eta; t) &= \sum_{m,n} (C_{1mn} \cos pt + C_{2mn} \sin pt) \Phi_{mn}(\xi, \eta); \\
\psi(\xi, \eta; t) &= \sum_{m,n} (C_{1mn} \cos pt + C_{2mn} \sin pt) \Psi_{mn}(\xi, \eta); \\
w(\xi, \eta; t) &= \sum_{m,n} (C_{1mn} \cos pt + C_{2mn} \sin pt) W_{mn}(\xi, \eta).
\end{align*}
\]

There is no difficulty in finding constants \( C_{1mn}, C_{2mn} \) and, consequently, the deformed state is determined by the sum of simple harmonic oscillations.

The possibility of expansion of the solution in series by eigenfunctions is based on the orthogonal nature of the latter. Actually, let the plate accomplish simple harmonic oscillations of frequency \( p \), when the inertial load acting on the plate is \( \rho \delta p^2 w(\chi, \eta) \). Since the
plate acquire disables deflection $w_1$ as a result of characteristic load $\rho \delta p_1^2 w_1$, and deflection $w_k$ as a result of characteristic load $\rho \delta p_k^2 w_k$, according to the reciprocity principle of the work of Beatty, we have

$$\eta \delta \int p_1^k w \, d\xi \, d\eta = \phi \delta \int p_k w \, d\xi \, d\eta,$$

from which it follows that

$$(p_1^k - p_k^k) \int \gamma w_1(\xi, \eta) w_k(\xi, \eta) \, d\xi \, d\eta = 0,$$

(277)

i.e., eigenfunctions $w_1$ are orthogonal. The orthogonal nature of eigenfunctions $\phi_i, \psi_i$ ($i=1, 2, 3, \ldots$) is proved similarly.

Thus, if deflection is presented in the form of an expansion by eigenfunctions

$$w(\xi, \eta) = \sum_{i}^{\infty} a_i w_i(\xi, \eta),$$

the coefficients of expansion are determined by the expressions

$$a_i = \frac{\int \gamma w(\xi, \eta) w_i(\xi, \eta) \, d\xi \, d\eta}{\int \gamma w^2_i(\xi, \eta) \, d\xi \, d\eta}.$$

(278)

If the eigenfunctions are normalized, i.e., if

$$\int \gamma w_i^2(\xi, \eta) \, d\xi \, d\eta = 1,$$

(279)

the coefficients of expansion by eigenfunctions are determined by the expressions

$$a_i = \int \gamma w(\xi, \eta) w_i(\xi, \eta) \, d\xi \, d\eta.$$

(280)

31. Variation Equation of Transverse Oscillations of Rectangular Plates

Exact determination of the form and frequency of oscillation of a plate, with the exception of the simplest cases of a hinge supported rectangular plate, involves the solution of extremely complicated systems of differential Eq. (267), (268) for anisotropic plates or Eq. (269), (270) for orthotropic plates. In the solution of specific engineering problems, approximate methods based on some general principles of mechanics are extremely effective. In theories of core systems, such methods permit rapid determination of the frequency of oscillation of the primary tones, which are of the greatest practical interest, without integration of differential equations. These methods can be generalized for the case of transverse oscillations of plates.
We consider the action functional of Ostrogradskiy–Hamilton

$$S = \int_{t_A}^{t_B} (T - U) dt,$$  \hspace{1cm} (281)

where $T$ and $U$ are, respectively, the kinetic and potential energies of a plate accomplishing transverse oscillations.

From the class of permissible functions which describe bending oscillations of a plate, we take the population of principal oscillations with frequency $\omega$.

By integrating over time for one period of oscillation $t_B - t_A = 2\pi/\omega$, we obtain the variation equation of transverse oscillations of the plate in the form

$$\delta (T_{\text{max}} - U_{\text{max}}) = 0;$$ \hspace{1cm} (282)

the natural primary oscillations satisfy this equation.

We now write Eq. (282) in expanded form. For this, we determine the maximum values of the kinetic and potential energies of the plate.

If the plate accomplishes transverse oscillations $\phi_0(x, y; t)$, $\psi_0(x, y; t)$, $w_0(x, y; t)$ the corresponding potential energies for a plate with a general type of anisotropy and an orthotropic plate, the principal axes of anisotropy of which are parallel to the coordinate axes, are determined by the expressions

$$U_A = \frac{1}{2} \iint \left[ D_{11} \left( \frac{\partial \phi_0}{\partial x} \right)^2 + 2 D_{12} \frac{\partial \phi_0}{\partial x} \frac{\partial \psi_0}{\partial y} + D_{22} \left( \frac{\partial \psi_0}{\partial y} \right)^2 \right. +$$

$$+ D_m \left( \frac{\partial \phi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right)^2 \left. + \frac{3}{2} D_{12} \frac{\partial \phi_0}{\partial x} \left( \frac{\partial \phi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right) \right] \, dx \, dy;$$

$$U_0 = \frac{1}{2} \iint \left[ D_1 \left( \frac{\partial \phi_0}{\partial x} \right)^2 + 2 D_{12} \frac{\partial \phi_0}{\partial x} \frac{\partial \psi_0}{\partial y} + D_{22} \left( \frac{\partial \psi_0}{\partial y} \right)^2 \right. +$$

$$+ D_3 \left( \frac{\partial \psi_0}{\partial y} + \frac{\partial \phi_0}{\partial x} \right)^2 + K_1 (\phi_0 + \frac{\partial w_0}{\partial x})^2 + K_2 \left( \psi_0 + \frac{\partial w_0}{\partial y} \right)^2 \left. \right] \, dx \, dy. \hspace{1cm} (283)$$

The kinetic energy of a plate accomplishing transverse oscillations is determined by the known expression

$$T = \frac{1}{2} \rho \phi \int \left( \frac{\partial w_0}{\partial t} \right)^2 \, dx \, dy. \hspace{1cm} (285)$$
When the plate accomplishes one of the primary oscillations, i.e., when

\[
\begin{align*}
\varphi_0(x, y; t) &= \psi(x, y) \sin (pt + a);
\psi_0(x, y; t) &= \psi(x, y) \sin (pt + a);
\omega_0(x, y; t) &= \omega(x, y) \sin (pt + a).
\end{align*}
\]  

(286)

the greatest potential energies are determined by Eq. (284), (285) where, instead of deformation components \(\phi_0, \psi_0, \omega_0\), their amplitude values \(\phi, \psi, \omega\) stand, and the greatest kinetic energy of the oscillating plate

\[
T_{\text{max}} = \frac{1}{2} \rho \alpha p^2 \int \int w^2(x, y) \, dx \, dy.
\]  

(287)

Consequently, the variation equations for the principal natural oscillations of the plate can be written in the following form:

a. variation equation of oscillation of anisotropic plate

\[
\delta' L = \delta' \left\{ \int \int \left[ D_{11} \left( \frac{\partial \phi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} + D_{22} \left( \frac{\partial \psi}{\partial y} \right)^2 \right] + \right.
\]

\[
+ D_{13} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + \frac{3}{2} D_{13} \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + K_1 \left( \varphi + \frac{\partial \omega}{\partial x} \right)^2 + K_1 \left( \psi + \frac{\partial \omega}{\partial y} \right)^2 \left. \right] \, dx \, dy -
\]

\[
- \rho \delta p^2 \int \int w^2(x, y) \, dx \, dy = 0;
\]  

(288)

b. variation equation of oscillation of orthotropic plate, the principal axes of anisotropy of which coincide with the coordinate axes

\[
\delta' L = \delta' \left\{ \int \int \left[ D_1 \left( \frac{\partial \phi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial y} + D_2 \left( \frac{\partial \psi}{\partial y} \right)^2 \right] + \right.
\]

\[
+ D_3 \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + K_1 \left( \varphi + \frac{\partial \omega}{\partial x} \right)^2 +
\]

\[
+ K_2 \left( \psi + \frac{\partial \omega}{\partial y} \right)^2 - \rho \delta p^2 \right] \, dx \, dy \right\}.
\]  

(289)

In Eq. (288), (289), \(\delta'\) designates the variation of the functional.

The solution of the variation problem of transverse oscillations of a plate, as in the case of static bending and stability, can be obtained, for example, by the Ritz method, namely, deformation components \(\phi, \psi, \omega\) are assigned in the form of an infinite sum with the indeterminate coefficients

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where $\phi_{mn}$, $\psi_{mn}$, $\omega_{mn}$ are complete systems of continuous functions dependent on two parameters and satisfying the conditions of fastening of the edges of the plate. After formulation of Eq. (290) in variation equations and integration, the problem is reduced to finding the minimum of the quadratic functions of arguments $A_{mn}$, $B_{mn}$, $C_{mn}$, i.e., to solution of a system of uniform linear algebraic equations for $A_{mn}$, $B_{mn}$, $C_{mn}$.

The condition of nontriviality of the solution leads to the characteristic equation for determination of the oscillation frequency

$$|\alpha_{\nu}(p)| = 0.$$ (291)

The smallest value of the root gives an approximate value of the oscillation frequency of the primary tone. The remaining roots are the frequencies of the higher tones.

32. Determination of Frequencies of Natural Oscillations of Orthotropic Rectangular Plate

We consider bending oscillations of a rectangular plate made of an orthotropic material. We will assume the principal axes of anisotropy to be parallel to the sides of the plate (Fig. 33).

\begin{align*}
F_0(w) + \varrho \delta \rho^2 F_0(w) &= 0; \\
F_{\psi}(\psi) &= \alpha_1 \frac{\partial^2 \psi}{\partial x^2} + \alpha_2 \frac{\partial^2 \psi}{\partial x \partial y} - \alpha_3 \frac{\partial \psi}{\partial x}; \\
F_{\omega}(\omega) &= \beta_1 \frac{\partial^2 \omega}{\partial y^2} + \beta_2 \frac{\partial^2 \omega}{\partial x \partial y} - \beta_3 \frac{\partial \omega}{\partial y},
\end{align*}

(293)

where operators $F_0()$, $F_{\psi}()$ and coefficients $\alpha_1$, $\beta_1$ are determined by

Fig. 33. Basic designations.
Eq. (148)-(151).

The solution of system of Eq. (292), (293) which satisfies the conditions of hinge support of the plate, i.e., the conditions

\[ \begin{align*}
    &\text{at } x=0, \ x=a \ w = \psi = G_1 = 0; \\
    &\text{at } y=0, \ y=b \ w = \phi = G_2 = 0,
\end{align*} \]

can be sought in the form

\[
\begin{align*}
    \varphi &= A \cos \pi \xi \sin \pi \eta; \\
    \psi &= B \sin \pi \xi \cos \pi \eta; \\
    w &= C \sin \pi \xi \sin \pi \eta,
\end{align*}
\]

where \( m, n \) are whole numbers.

The oscillation frequency is determined from Eq. (292). By substituting Eq. (294) in this equation, we obtain

\[
\varpi^2 \rho = \frac{\pi^4 (a_{m0} + a_{m1} a_{n1} + a_{m2} a_{n2} + a_{m3} a_{n3})}{\pi^4 (a_{m0} + a_{m1} a_{n1} + a_{m2} a_{n2} + a_{m3} a_{n3}) + \pi^2 (a_{m0}^2 + a_{m1}^2 a_{n1}^2 + a_{m2}^2 a_{n2}^2 + a_{m3}^2 a_{n3}^2)}.
\]

where coefficients \( a_{ij}, \gamma_1 \) are determined by expressions (150), (151).

The oscillation frequency of the primary tone at \( m=1, n=1 \)

\[
p_{11} = \frac{\pi^2}{\varpi^2 \rho} \frac{\pi^4 (a_{m0} + a_{m1} a_{n1} + a_{m2} a_{n2} + a_{m3} a_{n3})}{\pi^4 (a_{m0} + a_{m1} a_{n1} + a_{m2} a_{n2} + a_{m3} a_{n3}) + \pi^2 (a_{m0}^2 + a_{m1}^2 a_{n1}^2 + a_{m2}^2 a_{n2}^2 + a_{m3}^2 a_{n3}^2)}.
\]

b. Free oscillations of rigidly fastened rectangular plate. For a plate with rigidly fastened edges, it is extremely complicated to obtain a precise solution. Therefore, for determination of the frequency of the natural oscillations, we use variation Eq. (289), which we write in dimensionless coordinates \( \xi = x/a; \eta = y/b; \)

\[
\begin{align*}
    &\delta' \int_0^1 \int_0^1 \left[ D_1 \left( \frac{\partial^2 \varphi}{\partial \xi^2} \right)^2 + \frac{2D_{12}}{ab} \frac{\partial \varphi}{\partial \xi} \frac{\partial \varphi}{\partial \eta} + D_2 \left( \frac{\partial^2 \psi}{\partial \eta^2} \right)^2 + \\
    &+ D_3 \left( \frac{1}{b} \frac{\partial \psi}{\partial \eta} + \frac{1}{a} \frac{\partial \varphi}{\partial \xi} \right)^2 + K_1 \left( \varphi + \frac{1}{a} \frac{\partial \varphi}{\partial \xi} \right)^2 + \\
    &+ K_2 \left( \psi + \frac{1}{b} \frac{\partial \psi}{\partial \eta} \right)^2 - \varpi^2 \rho w^2 \right] d\xi d\eta = 0.
\end{align*}
\]

We will seek the form of oscillation of the primary tone in the form
\[
\begin{align*}
\psi &= A \sin \frac{2\pi x}{a} \left( 1 - \cos \frac{2\pi y}{b} \right); \\
\psi &= B \left( 1 - \cos \frac{2\pi x}{a} \right) \sin \frac{2\pi y}{b}; \\
w &= F \left( 1 - \cos \frac{2\pi x}{a} \right) \left( 1 - \cos \frac{2\pi y}{b} \right).
\end{align*}
\]  

By substituting Eq. (298) in the variation equation, the following can be obtained

\[
\delta' \left[ 3 \left( \frac{D_1}{a^4} + \frac{D_2}{b^4} + K_1 \right) A^2 + 3 \left( \frac{D_4}{b^4} + \frac{D_5}{3a^4} + K_2 \right) B^2 \right] + \\
\left( \frac{3K_1}{a^4} + \frac{3K_2}{b^4} - \frac{9\delta_0 \rho^3}{4\pi^2} \right) K^2 + \frac{2(D_4+D_5)}{ab} AB + \\
\frac{3K_1}{a^4} AF + \frac{3K_2}{b^4} BF = 0, 
\]  

where

\[
K_1 = \frac{K_1}{4a^4}; \quad K_2 = \frac{K_2}{4b^4}.
\]  

The condition of nontriviality of the system of linear equations

\[
\frac{\partial L}{\partial A} = \frac{\partial L}{\partial B} = \frac{\partial L}{\partial F} = 0
\]

has the form

\[
\begin{vmatrix}
3 \left( \frac{D_1}{a^4} + \frac{D_2}{b^4} + K_1 \right); & \frac{C}{ab} & \frac{3K_1}{2a^4} \\
\frac{C}{ab} & 3 \left( \frac{D_4}{b^4} + \frac{D_5}{3a^4} + K_2 \right) & \frac{3K_2}{2b^4} \\
\frac{3K_1}{a^4} & \frac{3K_2}{b^4} & \frac{3K_1}{a^4} + \frac{3K_2}{b^4} - \frac{9\delta_0 \rho^3}{4\pi^2}
\end{vmatrix} = 0;
\]

from which we find

\[
\frac{3\delta \rho^3}{4\pi^2} \left[ (\lambda_1 + K_1)(\lambda_2 + K_2) - \frac{C^2}{9a^4b^4} \right] = \frac{2C}{3a^4b^4} K_1 K_2 + \\
+ \left( \frac{K_1}{a^4} + \frac{K_2}{b^4} \right) \left[ (\lambda_1 + K_1)(\lambda_2 + K_2) - \frac{C^2}{9a^4b^4} \right] - \\
- \frac{K_1 K_2}{a^4}(\lambda_1 + K_1) - \frac{\lambda_1 K_2}{a^4}(\lambda_2 + K_2),
\]

where

\[
\lambda_1 = \frac{D_1}{a^4} + \frac{D_2}{3b^4}; \quad \lambda_2 = \frac{D_4}{b^4} + \frac{D_5}{3a^4}.
\]

Consequently, the frequency of natural oscillation of the primary tone

\[
\text{81}
\]
As \( K_1 \rightarrow \infty, K_2 \rightarrow \infty \), i.e., without accounting for interlayer shearing, the known formula for the frequency of natural oscillations of an orthotropic rectangular plate with rigidly fastened edges follows from Eq. (305):

\[
p_{11} = \frac{22.70}{\sqrt{\varepsilon}} \sqrt{\frac{D_1}{a_1^4} + \frac{2(C + D_1)}{a_1^2b_1^2} + \frac{D_3}{b_1^4}} \times \frac{K_1K_2}{(\lambda_1 + \bar{K}_1)(\lambda_2 + \bar{K}_2)} - \frac{C_1^2}{9a_1^6b_1^4}. \tag{305}
\]

### c. Transverse oscillations of laminated strips.

Free primary oscillations of laminated strips are described by the following system of differential equations

\[
\begin{align*}
\omega^4 + 2r^2\omega'' - s^4\omega &= 0; \\
\varphi'' &= \omega^2\varphi = \omega^2w',
\end{align*}
\tag{307}
\]

where

\[
\omega^2 = \frac{K_1}{D_1}; \quad 2r^2 = \frac{a^2b^2}{K_1}; \quad s^4 = \frac{a^4b^4}{D_1}. \tag{308}
\]

If

\[
k_1^2 = \sqrt{r^4 + s^4} - r^4; \quad k_2^4 = \sqrt{r^4 + s^4} + r^4, \tag{309}
\]

is designated, it is easy to obtain a solution of Eq. (307) in the form

\[
\begin{align*}
\omega &= C_1 \text{ch} k_1x + C_2 \text{sh} k_1x + C_3 \cos k_2x + C_4 \sin k_2x; \\
\varphi &= \frac{k_1\omega^4}{k_1^2 - \omega^2} (C_1 \text{sh} k_1x + C_2 \text{ch} k_1x) + \\
&+ \frac{k_1\omega^4}{k_1^2 + \omega^2} (C_3 \sin k_2x - C_4 \cos k_2x). \tag{310}
\end{align*}
\]

There also can be found

\[
G_1 = -\frac{D_1k_1^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_1^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x); \tag{311}
\]

\[
G_2 = -\frac{D_2k_1^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_1^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x); \tag{312}
\]

\[
G_3 = -\frac{D_1k_2^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_2^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x); \tag{313}
\]

\[
G_4 = -\frac{D_2k_2^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_2^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x); \tag{314}
\]

\[
G_5 = -\frac{D_1k_2^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_2^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x); \tag{315}
\]

\[
G_6 = -\frac{D_2k_2^2\omega^4}{k_1^2 - \omega^2} (C_1 \text{ch} k_1x + C_2 \text{sh} k_1x) - \frac{D_3k_2^2\omega^4}{k_1^2 + \omega^2} (C_3 \cos k_2x + C_4 \sin k_2x). \tag{316}
\]
We select a coordinate system as indicated in Fig. 34, and we determine the free oscillation frequency with various fastenings of the ends of the strip \( x = \pm a/2 \).

### a. Edge of strip hinge supported.
By satisfying the boundary conditions

\[ w = G_1 = 0 \text{ at } x = \pm a \]

the following can be obtained

\[
\begin{align*}
C_1 \sin a_1 + C_2 \sin a_2 &= 0; \\
C_1 \cos a_1 + C_2 \cos a_2 &= 0; \\
C_3 \cos a_1 + C_4 \cos a_2 &= 0; \\
C_3 \sin a_1 + C_4 \sin a_2 &= 0.
\end{align*}
\]

where

\[
\begin{align*}
a_1 &= \frac{k a}{2} = \frac{\alpha r}{2} \sqrt{1 + 2 \frac{v a^2}{v}} - 1; \\
a_2 &= \frac{k a}{2} = \frac{\alpha r}{2} \sqrt{1 + 2 \frac{v a^2}{v}} + 1.
\end{align*}
\]

The characteristic equation can be written in the form

\[
\cos a_2 = 0; \quad \sin a_2 = 0;
\]

and, consequently, the frequency spectrum of free oscillations of a laminated strip is determined by the expression

\[
p = \frac{\pi^2 m^2}{v} \sqrt{\frac{K_1 \omega_1}{\pi^2 m^2 + K_1 \omega^2}}.
\]

### b. Edge of strip \( x = 0 \) rigidly fastened, edge \( x = a \) unsupported (Fig. 35).

By satisfying the boundary conditions

\[ w = \phi = 0 \text{ at } x = 0, \]

\[ G_1 = Q_1 = 0 \text{ at } x = a, \]

we obtain the following system of linear equations for determination of the frequencies
Consequently, the frequency spectrum of the natural oscillations of a laminated bracket is determined by the equation

\[(316)\]

\[
\left(\frac{\omega}{r}\right)^2 + \left(1 + \frac{\omega}{r}\right) \text{ch} 2\alpha_1 \cos 2\alpha_2 - \frac{1}{2} \frac{\omega}{r} \text{sh} 2\alpha_1 \sin 2\alpha_2 = 0,
\]

where \(\alpha_1\) and \(\alpha_2\) are determined by Eq. (313).

As \(K_1 \to \infty\), there is the known equation for determination of the frequency of natural oscillations of the bracket

\[
\text{ch} k_1 a \cos k_1 a = 1.
\]

33. Axisymmetric Transverse Oscillations of Circular Plate

If a circular plate made of a cylindrically orthotropic laminated plastic executes axisymmetric transverse oscillations, the system of differential equations has the form [14]

\[(318)\]

\[
\begin{align*}
\frac{\partial (rG_1)}{\partial r} - G_1 &= Q_r, \\
\frac{\partial (rQ_1)}{\partial r} &= -q \delta r \frac{\partial \sigma_1}{\partial t} - q(r,t) r,
\end{align*}
\]

where \(\rho\) is the density of the laminated plastic; \(q\) is the external transverse load which changes over time.

By substituting basic relationships (64), (65) in Eq. (318), we obtain

\[(319)\]

\[
\begin{align*}
\frac{\partial^2 \varphi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_1}{\partial r} - \frac{\lambda}{r^2} \varphi_1 &= \frac{K_1}{D_1} \left( \frac{\partial \sigma_1}{\partial r} + \varphi_1 \right); \\
\frac{\partial^2 \sigma_1}{\partial r^2} + \frac{1}{r} \frac{\partial \sigma_1}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r \varphi_1) &= \frac{q \delta}{K_1} \frac{\partial^2 \sigma_1}{\partial t^2} + \frac{q(r,t)}{K_1}.
\end{align*}
\]

For the principal normal oscillations of the plate
\[ q_1(r, t) = \varphi(r)(C_1 \cos \omega t + C_2 \sin \omega t); \]
\[ w_1(r, t) = w(r)(C_1 \cos \omega t + C_2 \sin \omega t); \]

and, consequently, the oscillation amplitudes comply with the following system of differential equations

\[ \psi'' + \frac{\psi'}{r} - \left( \frac{L^2}{r^2} + k^2 \right) \psi = k^2 \psi'; \]  \hspace{1cm} (321)
\[ w'' + \frac{w'}{r} + p^2 w = - \left( \psi' + \frac{\psi}{r} \right); \]  \hspace{1cm} (322)

where

\[ k^2 = \frac{K_i}{D_i}; \quad \lambda^2 = \frac{E_i}{K_1}; \quad p^2 = \frac{\delta \omega^2}{K_1}. \]  \hspace{1cm} (323)

Eq. (321), (322) are equivalent to the following differential equations

\[ \varphi^{IV} + \frac{2}{r} \varphi''' - \left( \frac{L^2 + 2}{r^2} - p^2 \right) \varphi'' + \] 
\[ + \left( \frac{3L^2}{r^3} + \frac{p^2}{r} \right) \varphi' - \left( \frac{L^2}{r^2} + \frac{L^4}{r^4} + p^2 k^2 \right) \varphi = 0; \]  \hspace{1cm} (324)
\[ w = - \frac{1}{kp^2} \left( \varphi''' + \frac{2}{r} \varphi'' - \frac{L^2}{r^2} \varphi' + \frac{L^4}{r^4} \varphi \right). \]  \hspace{1cm} (325)

Eq. (324) is a fourth order Fuchs class differential equation. We will seek its solution in the form

\[ \varphi = r^a \sum_{m=0}^{\infty} A_m r^m. \]  \hspace{1cm} (326)

By substituting Eq. (326) in Eq. (324), the following can be obtained

\[ \sum_{m=0}^{\infty} \left[ a + bm + cm(m - 1) + em(m - 1)(m - 2) + \right. \] 
\[ \quad + \left. m(n - 1)(m - 2)(m - 3) \right] A_m r^m + \] 
\[ + \sum_{n=2}^{\infty} \left[ d + f(m - 2) + p^2(m - 2)(m - 3) \right] A_{m-2} r^m - k^2 p^2 \sum_{m=4}^{\infty} A_{m-4} r^m = 0, \]  \hspace{1cm} (327)

where

\[ a = q_0^4 - 4q_0 + (3 - \lambda^4) q_0^4 + 4\lambda^4 q_0 - 3\lambda^4; \]
\[ b = 4q_0^5 - 6q_0^5 - 2(\lambda^4 + 1) q_0 + 3\lambda^4; \]
\[ c = 6q_0^6 - (\lambda^4 + 2); \]
\[ d = p^2 (q_0^4 - \lambda^4); \]
\[ e = 2(2q_0 + 1); \]
\[ f = p^2 (2q_0 + 1). \]  \hspace{1cm} (327)
The following recurrent formulas for coefficients \( A_m' \) can be obtained from Eq. (327):

For \( m = 0, m = 1 \):

\[
(a + bm + cm (m - 1) + dm (m - 1) (m - 2)) A_m' = 0;
\]

\[
(329)
\]

For \( m = 2, m = 3 \):

\[
A_m' = \frac{d + f (m - 2) + p^2 (m - 2) (m - 3)}{a + bm + cm (m - 1) + dm (m - 1) (m - 2) + m (m - 1) (m - 2) (m - 3)} A_m';
\]

\[
(330)
\]

For \( m \geq 4 \):

\[
A_m' = \frac{k^2 p^2 A_{m-4} - [d + f (m - 2) + p^2 (m - 2) (m - 3)] A_{m-2}}{a + bm + cm (m - 1) + dm (m - 1) (m - 2) + m (m - 1) (m - 2) (m - 3)}.
\]

\[
(331)
\]

In accordance with Eq. (329), the characteristic equation for determination of \( \rho_0 \) has the form:

\[
\rho_0^4 - 4 \rho_0^3 + (3 - \lambda^4) \rho_0^2 + 4 \lambda^4 \rho_0 - 3 \lambda^4 = 0.
\]

\[
(332)
\]

The roots of this characteristic equation are:

\[
\rho_0^1 = 4; \quad \rho_0^2 = 3; \quad \rho_0^3 = \lambda; \quad \rho_0^4 = -\lambda.
\]

\[
(333)
\]

It is easy to note that \( A_{2m+1}' = 0 \) (\( m = 0, 1, 2, 3, \ldots \)).

In the case of a continuous plate, the solution which corresponds to root \( \rho_0^1 = \lambda \) should be set equal to zero, and roots \( \rho_0^0 = 1, \rho_0^2 = 3 \) gives linearly dependent solutions. The missing solution should be sought in the form:

\[
\varphi = \rho' \left( A_0 \varphi_1 + \sum_{m=0}^{\infty} A_m'' r^m \right).
\]

\[
(334)
\]

where \( \varphi_1 \) is the solution which corresponds to the root \( \rho_0^2 = 3 \).

By substituting Eq. (334) in Eq. (324), we obtain

\[
\sum_{m=0}^{\infty} \left[ 3 (16 - \lambda^4) m + (52 - \lambda^4) m (m - 1) + 14 m (m - 1) (m - 2) + \\
+ m (m - 1) (m - 2) (m - 3) \right] A_m'' r^m + \sum_{k=2}^{\infty} \left[ p^4 (9 - \lambda^4) + 7 p^2 (m - 2) + \\
+ p^2 (m - 2) (m - 3) \right] A_m'' r^m - k^2 p^2 \sum_{k=4}^{\infty} A_{m-k}'' r^m =
\]

\[
= A_0 [2 (9 - \lambda^4) + 2 (34 - \lambda^4) m + 36 m (m - 1) +
\]

\[
(335)
\]
In this manner, for coefficients $A_m''$, the following recurrent relationships can be obtained

\[ m = 2 \]

\[ A_m'' = -\frac{1}{8(25 - \lambda^3)} \left\{ p^8(0 - \lambda^3) A_0'' + \right. \]

\[ + A_0 \left[ 6k^8 (A_1')_1 - \frac{2k^8(113 - 3\lambda^3)(0 - \lambda^3)}{8(25 - \lambda^3)} (A_1')_1 \right] \right\}. \]  

(336)

where

\[ (A_1')_1 = -\frac{p^8(0 - \lambda^3)}{3(25 - \lambda^3)} (A_1')_1; \]  

(337)

for $m \geq 2n$ ($n = 1, 2, 3, \ldots$)

\[ A_m'' = \left\{ \frac{1}{3(16 - \lambda^3)m + (52 - \lambda^3)m(m - 1) + 14m(m - 1)(m - 2)} \right\} \times \]

\[ \times \left\{ [p^8(9 - \lambda^3) + 7p^8(m - 2) + p^8(m - 2)(m - 3)] A_m'' - \right. \]

\[ - k^8 p^8 A_{m-2} + A_0 \left[ 2(9 - \lambda^3) + 2(34 - \lambda^3)m + \right. \]

\[ + 36m(m - 1) - 4m(m - 1)(m - 2)] (A_m')_1 + \]

\[ + A_0 \left[ 6k^8 + 2k^8(m - 2) \right] (A_m')_1 \right\}. \]  

(338)

where coefficients $(A_m')_1$ are determined by the expression

\[ (A_m')_1 = \]

\[ = \frac{k^8 p^8 (A_m')_1 - [p^8(9 - \lambda^3) + 7p^8(m - 2)(m - 3)] (A_m')_1}{3(16 - \lambda^3) + (52 - \lambda^3)(m - 1) + 14(m - 1)(m - 2) + (m - 1)(m - 2)(m - 3)}. \]  

(339)

The odd coefficients again equal zero.
CHAPTER 7. MEMBRANE THEORY OF ANISOTROPIC LAMINATED SHELLS

34. Region of Applicability of Membrane Theory of Shells and Boundary Conditions

The simplest alternate version of the general theory of shells is the membrane theory, which is widely used for calculation of various engineering structures and buildings. The explanation of this is that the membrane theory quite satisfactorily describes the behavior of thin shells under various loads which have to be of concern in engineering. The simplicity and value of membrane theory is not only significant mathematical simplification of the basic differential equations of the theory of shells but also that, in many cases, the results of the basic stage of the theory, which consists of determination of the nature of transmission of forces from the equations of equilibrium, are valid for any thin shells regardless of their structure and nature of deformation. Structural inhomogeneity within the shell material appears in subsequent stages of solution of the problem, which are connected with determination of the deformed state and the nature of distribution of stresses through the shell.

As in the case of isotropic or anisotropic shells [1, 8], we will call membrane theory an approximate method of calculation, based on the assumption that bending stresses are small compared with the stresses uniformly distributed through the shell. This assumption is mathematically equivalent to the assumption that cutting forces \( Q_1, Q_2 \) can be disregarded in the first three equilibrium Eq. (17). With the intention that only shells of rotation will be considered subsequently, we write the basic equations of membrane theory for this partial case. Membrane theory of anisotropic shells is discussed in greater detail in the monograph of S.A. Ambartsumyan [1].

As curvilinear Gaussian coordinates which define the positions of points on the mean surface of a shell, we use arc length \( s \), reckoned from the initial parallel (point \( M_0 \) in Fig. 36) and angle \( \beta \) between two planes passing through the axis of rotation. One such plane was selected as the initial plane. We introduce two more coordinates: shell cross section radius \( r \) and angle \( \alpha \) between the normal to meridian \( n \) and the axis of rotation.

Principal radii of curvature \( R_1, R_2 \) are determined by the expressions [8]

\[
\begin{align*}
R_1 &= \frac{dx}{d\alpha}, \quad R_2 = \frac{r}{\sin \alpha}. \quad (340)
\end{align*}
\]

Two Gauss-Codacci relationships are satisfied identically, and the third has the form

\[
\frac{dr}{d\alpha} = \frac{d}{d\alpha} \left( R_1 \sin \alpha \right) = R_1 \cos \alpha. \quad (341)
\]

The last relationship can be obtained from geometric considera-
tions (Fig. 37)

\[
\frac{\Delta r}{H_1 \Delta \alpha} = \cos \alpha.
\]

If the components of the distributed surface load acting on the shell are \(X, Y, Z\), equilibrium Eq. (17), with \(A_1 = 1, A_2 = r\), take the form

\[
\begin{align*}
\frac{r \partial \sigma_1}{\partial r} - (T_3 - T_1) \cos \alpha + \frac{\partial S}{\partial \theta} &= -Xr; \\
\frac{\partial T_3}{\partial \theta} + r \frac{\partial S}{\partial \theta} + 2S \cos \alpha &= -Yr; \\
\frac{T_1}{H_1} + \frac{T_3}{H_2} &= Z. 
\end{align*}
\] (342)

In accordance with Eq. (10), the components of deformation of the mean surface are determined by the expressions

\[
\begin{align*}
e_1 &= \frac{\partial u}{\partial s} + \frac{\omega}{H_1} \\
\varepsilon_1 &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{u}{r} \cos \alpha + \frac{\omega}{H_1} \\
\omega &= \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial \nu}{\partial s} - \frac{\nu}{r} \cos \alpha.
\end{align*}
\] (343)

For shells with undetermined anisotropy of elastic properties and for shells made of orthotropic materials, the principal axes of anisotropy of which do not coincide with the coordinate axes, in accordance with Eq. (8), (343), Hooke's law has the form

\[
\begin{align*}
\frac{\delta u}{\delta s} + \frac{\omega}{H_1} &= \frac{1}{\delta} (a_{11} T_1 + a_{12} T_2 + a_{15} S); \\
\frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{u}{r} \cos \alpha + \frac{\omega}{H_1} &= \frac{1}{\delta} (a_{12} T_1 + a_{22} T_2 + a_{25} S); \\
\frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{\partial \nu}{\partial s} - \frac{\nu}{r} \cos \alpha &= \frac{1}{\delta} (a_{23} T_1 + a_{33} T_2 + a_{35} S). 
\end{align*}
\] (344)

Correspondingly, for orthotropic laminated shells, the principal axes of anisotropy of which coincide with the coordinate axes, in accordance with Eq. (28) and (343), the elasticity relationships have the form

\[
\begin{align*}
\frac{\partial u}{\partial s} + \frac{\omega}{H_1} &= \frac{T_1 - v_1 T_3}{B_1 (1 - v_1 v_3)}; \\
\frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{u}{r} \cos \alpha + \frac{\omega}{H_1} &= \frac{T_3 - v_3 T_1}{B_3 (1 - v_1 v_3)}; \\
\frac{1}{r} \frac{\partial \nu}{\partial \theta} + \frac{\partial \nu}{\partial s} - \frac{\nu}{r} \cos \alpha &= \frac{S}{B_3}. 
\end{align*}
\] (345)

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Thus, the systems of differential equations of membrane theory can be integrated in two successive stages:

**Stage 1.** By solution of system of Eq. (342), elastic forces $T_1$, $T_2$, $S$ of the shell are determined;

**Stage 2.** From system of Eq. (344) or (345), displacements $u$, $v$, $w$ are determined for anisotropic and orthotropic shells.

Since, by the definition of membrane theory, interlayer shears $\gamma_1$, $\gamma_2$ are absent or negligibly small under load, by Eq. (12) we obtain

$$\varphi = -\frac{\partial w}{\partial s} + \frac{u}{K_1}; \quad \psi = -\frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{v}{K_2}. \quad (346)$$

The components of effective change of curvature and torsion of the mean surface, in accordance with Eq. (11) and (346), are determined by the following expressions

$$\kappa^e_1 = -\frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right);$$

$$\kappa^e_2 = -\left[ \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{\cos \alpha}{r} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right];$$

$$2\kappa^e_3 = -\left[ r \frac{\partial}{\partial s} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{\cos \alpha}{r} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) \right] \quad (347)$$

Bending moments and torques $G_1$, $G_2$, $H$ can now be determined by Eq. (4) or (25).

For shells of rotation with random anisotropy of properties and for orthotropic shells, the principal axes of anisotropy of which do not coincide with the coordinate axes, there are the following expressions for determination of the bending moments and torque

$$G_1 = D_{11} \frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) + D_{12} \left( \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{\cos \alpha}{r} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right) \quad +$$

$$+ \frac{\cos \alpha}{r} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right] + D_{13} \left[ r \frac{\partial}{\partial s} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right];$$

$$G_2 = D_{22} \frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) + D_{23} \left[ r \frac{\partial}{\partial s} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right];$$

$$H = D_{13} \frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) + D_{23} \left[ r \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{v}{K_2} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial w}{\partial s} - \frac{u}{K_1} \right) \right]. \quad (348)$$

90

\[\hat{c} = 2\]
Correspondingly, for orthotropic shells of rotation, the principal axes of anisotropy of which coincide with the coordinate axes,

\[
G_1 = D_1 \left( \frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{H_1} \right) + \nu_s \left[ \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial w}{\partial \phi} - \frac{v}{H_1} \right) + \frac{\cos \alpha}{r} \left( \frac{\partial w}{\partial s} - \frac{u}{H_1} \right) \right] \right);
\]

\[
G_2 = D_1 \left( \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial w}{\partial \phi} - \frac{v}{H_1} \right) + \cos \alpha \left( \frac{\partial w}{\partial s} - \frac{u}{H_1} \right) + \nu_s \frac{\partial}{\partial \phi} \left( \frac{\partial w}{\partial \phi} - \frac{v}{H_1} \right) \right);
\]

\[
H = D_2 \left[ \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{\partial w}{\partial \phi} - \frac{v}{H_1} \right) + \frac{1}{r} \frac{\partial}{\partial s} \left( \frac{\partial w}{\partial s} - \frac{u}{H_1} \right) - \frac{\cos \alpha}{r} \left( \frac{\partial w}{\partial \phi} - \frac{v}{H_1} \right) \right].
\]

From the last two of equilibrium Eq. (17), cutting forces \(Q_1, Q_2\), which were eliminated in equilibrium Eq. (342), can be determined

\[
\begin{align*}
rQ_1 & = \frac{\partial (rG_1)}{\partial s} + \frac{\partial H}{\partial \phi} - G_1 \cos \alpha; \\
rQ_2 & = -\frac{\partial G_2}{\partial \phi} + \frac{\partial (rH)}{\partial s} + H \cos \alpha.
\end{align*}
\]

System of Eq. (348), (350) or (349), (350) are supplementary in membrane theory, and they are used only for checking the possibilities of its use, namely, if it turns out that the bending stresses actually are negligibly small compared with the membrane stresses, i.e., they are uniformly distributed in the thickness of the membrane, this is confirmation of the applicability of membrane theory.

In some cases, it can be foreseen that membrane theory cannot sufficiently well describe the axisymmetric stressed and deformed states of a shell of rotation. This will occur in those cases when there is a break in continuity of geometric dimensions \(\delta, R_1, R_2\), rigidity characteristics \(c_{ij}\), including rigid fastening or other kinematic connections or, finally, there are areas of discontinuity of external surface load \(X, Y, Z\).

The condition of the presence of areas of slight disturbance of the geometric, elastic or strength parameters can be replaced by more general ones, namely, for inapplicability of membrane theory, it is sufficient that the abovementioned parameters have a large index of variability.
In areas of a shell where there are such features, additional stresses can develop, which cause local bending of the mean surface of the shell. Exact solutions show that areas of bending stresses are extremely small and, consequently, at some distance from such areas, shell calculations can be carried out according to membrane theory.

Thus, membrane theory is described by systems of Eq. (342), (344) for shells with random anisotropy and for orthotropic shells, the principal axes of anisotropy of which do not coincide with the coordinate axes, and by system of Eq. (342), (345) for orthotropic shells, the principal axes of anisotropy of which coincide with the coordinate axes.

In accordance with Eq. (22), the boundary conditions of membrane theory have the form:

1. unsupported edge $T=S=0$; \hspace{1cm} (351)
2. rigidly fastened edge $u=v=0$; \hspace{1cm} (352)
3. hinge supported edge $T_1=v=0$ or $u=S=0$. \hspace{1cm} (353)

It follows from boundary conditions (351)-(353) that membrane theory is applicable in the event the shell is not loaded by cutting forces and moments on the edges, since end effects, i.e., local bending of the shell, will develop on the edges.

35. Membrane Theory of Symmetrically Loaded Shells of Rotation

If a shell of rotation is loaded symmetrically about the axis of rotation, the surface loading components should be functions of arc $s$ alone, i.e., they should not depend on angle $\beta$:

$$X = X(s); \quad Y = Y(s); \quad Z = Z(s).$$ \hspace{1cm} (354)

Since all geometric parameters $R_1, R_2, r$ of shells of rotation depend on arc $s$, the elastic forces and displacements also are functions of coordinate $s$ alone and, consequently, the equations of equilibrium of a symmetrically loaded shell of rotation have the form

$$\begin{align*}
    r \frac{dT_1}{ds} - (T_1 - T_1) \cos a &= -Xr; \\
    r \frac{ds}{ds} + 2S \cos a &= -rY; \\
    \frac{T_1}{R_1} + \frac{T_2}{R_2} &= Z.
\end{align*}$$ \hspace{1cm} (355)

If, following V.V. Novozhilov [21], stress functions are introduced,

$$\phi(s) = T_1 r \sin a; \quad \psi(s) = Sr^2;$$ \hspace{1cm} (356)

the following can be obtained from the first two of Eq. (355)
\[
\frac{d\phi}{ds} = r(Z\cos\alpha - X\sin\alpha); \quad \frac{d\psi}{ds} = -r^2Y. 
\]

Consequently, the stress functions are determined by the expressions

\[
\Phi(s) = \int_0^s (Z\cos\alpha - X\sin\alpha)\,ds + \Phi_0; \\
\Psi(s) = -\int_0^s r^2Y\,ds + \Psi_0.
\]

By determining force \(T_2\) from the last equation of equilibrium, in accordance with Eq. (356), (358), (359), we obtain

\[
T_1 = \frac{\Phi}{r\sin\alpha}; \quad T_2 = R_1\left(Z - \frac{T_1}{R_1}\right); \quad S = -\frac{\psi}{r^2}.
\]

According to membrane theory, the normal, tangential and shearing stresses are determined by the simple expressions

\[
\sigma_1 = \frac{T_1}{E}; \quad \sigma_2 = \frac{T_2}{E}; \quad \tau_0 = \frac{S}{G}; \\
\sigma_i = \frac{1}{E}(X^+ + X^-) + \frac{X^+ - X^-}{2}; \\
\tau_i = \frac{1}{E}(Y^+ + Y^-) + \frac{Y^+ - Y^-}{2},
\]

where \(X^+, X^-, Y^+, Y^-\) are components of the external surface load applied to the upper \((z=\delta/2)\) and lower \((z=-\delta/2)\) bounding surfaces of the shell, respectively.

As was noted in the preceding section, the expressions for the elastic forces of an anisotropic laminated shell coincide with the corresponding for an isotropic shell.

In accordance with Eq. (344), movements of a symmetrically loaded shell of rotation are determined by the following system of linear differential equations

\[
\frac{du}{ds} + \frac{u}{R_1} = \frac{1}{E}(a_{11}T_1 + a_{12}T_2 + a_{13}S); \\
\frac{u}{r}\cos\alpha + \frac{v}{R_1} = \frac{1}{E}(a_{11}T_1 + a_{12}T_2 + a_{13}S); \\
\frac{dv}{ds} - \frac{v}{r}\cos\alpha = \frac{1}{E}(a_{11}T_1 + a_{12}T_2 + a_{13}S).
\]

System of Eq. (362) is equivalent to the following:
\[
\begin{align*}
    w &= \frac{R_1}{\delta} \left[ (a_{11} - a_{15} q) T_1 + a_{15} S + a_{15} R_z Z \right] - R_1 \frac{du}{dz} ;
    \\
    \frac{du}{dz} - q \frac{\cos \alpha}{r} u &= (a_{11} - 2a_{15} q + a_{25} q^3) \frac{T_1}{\delta} + \\
    &+ (a_{13} - a_{25} q) S + (a_{13} - a_{25} q) \frac{R_z}{\delta} Z ;
    \\
    \frac{dv}{dz} - \frac{\cos \alpha}{r} v &= (a_{13} - a_{25} q) \frac{T_1}{\delta} + a_{25} S + a_{25} R_z \frac{Z}{\delta} ,
\end{align*}
\]

where

\[
q = \frac{R_1}{\delta} .
\]

If, as in determination of stresses, the deformation functions

\[
\xi = \frac{u}{\sin \alpha} ; \quad \eta = \frac{v}{r} ,
\]

are introduced, from Eq. (364), it is easy to obtain

\[
\begin{align*}
    \frac{d\xi}{ds} &= \frac{1}{\delta \sin \alpha} \left[ (a_{11} - 2a_{15} q + a_{25} q^3) T_1 + \\
    &+ (a_{13} - a_{25} q) S + (a_{13} - a_{25} q) R_z Z \right] \frac{ds}{\sin \alpha} + \psi_0 ;
    \\
    \eta &= \frac{1}{\delta} \int_{\alpha_0}^{\alpha} \left[ (a_{13} - a_{25} q) \frac{T_1}{r} + a_{25} S + a_{25} R_z \frac{Z}{r} \right] ds + \psi_0 ;
\end{align*}
\]

where \( \phi_0, \psi_0 \) are the values of the deformation functions at the edge of the shell.

In this manner, the components of movement of an anisotropic symmetrically loaded shell of rotation are determined by the following expressions

\[
\begin{align*}
    u &= \xi \sin \alpha ; \quad v = r \eta ;
    \\
    w &= -\xi \cos \alpha + \frac{R_1}{\delta} \left[ (a_{13} - a_{25} q) T_1 + a_{25} S + a_{25} R_z Z \right] .
\end{align*}
\]

As should be expected, in distinction from isotropic shells, with any boundary conditions, each movement depends on all three components of external surface load \( X, Y, Z \). The nature of the stressed state depends essentially on the boundary conditions, namely, only with static indeterminate boundary conditions will the forces depend on all three components of the external load.

Eq. (369) also determine movements of orthotropic shells of rotation, the principal directions of anisotropy of which do not coincide.
In the case of orthotropic shells of rotation, the principal directions of anisotropy of which coincide with the coordinate axes, the expressions for the elastic forces remain as before, and the movement functions are determined by the formulas

\[
\begin{align*}
\xi &= \frac{1}{2} \int \left[ \left( \frac{1}{E_1} + \frac{2\nu_1}{E_1} \right) T_1 - \left( \frac{1}{E_2} + \frac{2\nu_2}{E_2} \right) T_1 \right] \sin \alpha \, ds + \xi_0; \\
\eta &= \frac{1}{3} \int \frac{S}{r} \, ds + \eta_0; \\
\end{align*}
\]

\( (370) \)

\[
\begin{align*}
u &= \xi \sin \alpha; \\
w &= \nu r; \\
\end{align*}
\]

\( w = -\xi \cos \alpha + \frac{R_1}{3} \left[ \frac{R_2 Z}{E_2} - \left( \frac{\nu_1}{E_1} + \frac{\nu_2}{E_2} \right) T_1 \right] \)

\( (371) \)

In this case, movements \( u, w \) are caused by the radial and meridional components of the external load \( Z, X, \) and displacement \( v \) is caused by annular forces \( Y, \) i.e., the same as in isotropic shells of rotation.

In this manner, calculation of symmetrically loaded anisotropic and orthotropic shells of rotation is reduced to determination of four random integration constants \( \phi_0, \psi_0, \phi_1, \psi_1. \) Consequently, on each edge of the shell \( s=s_0; s=s_1, \) for an unambiguous solution, two boundary conditions each must be assigned. In this case, at least two of the boundary conditions should be kinematic. Otherwise, the existence of the stressed membrane state will be impossible, i.e., bending of the mean surface of the shell without stretching (compressing) or shearing will occur, or displacement of the shell as a solid will be possible.

We now consider some examples of calculation of symmetrically loaded shells of rotation according to membrane theory.

**36. Calculation of Closed Containers Operating under Constant Internal Pressure**

Shells of rotation in the form of cylindrical and conical shells closed by end plates of different geometric shape and of spherical and toroidal containers are exceptionally widely used in industry. Particularly in chemical equipment, these shells operate under uniform internal pressure. Such structures are calculated according to membrane theory, with the exception of small end effect areas, where more exact equations, which will be obtained later, must be used for the calculation. In such zones, special design measures must be used to moderate stress concentrations and more uniformly distribute the stress.
In shells of rotation subjected to uniform internal pressure, \( X=Y=0, Z=p \), where \( p=\text{const} \) is the intensity of the internal pressure.

In accordance with Eq. (358), the stress function

\[
\Phi = p \int_{0}^{a} R_{\alpha} \sin \alpha d(R_{\alpha} \sin \alpha) + \Phi_{0} = \frac{p R_{\alpha}^{2} \sin^{2} \alpha}{2} + C. \tag{372}
\]

Consequently, the elastic forces are determined by the expressions

\[
\begin{align*}
T_{1} &= \frac{p R_{1}}{2} - \frac{C}{R_{1} \sin^{2} \alpha}; \\
T_{3} &= \frac{p R_{3}}{2} \left( 2 - \frac{R_{3}}{R_{1}} \right) - \frac{C}{R_{1} \sin^{2} \alpha}.
\end{align*}
\tag{373}
\]

It is evident that, for shells of rotation closed at the top, \( C=0 \) must be set and, consequently, for such shells

\[
T_{1} = \frac{p R_{1}}{2}; \quad T_{3} = \frac{p R_{3}}{2} \left( 2 - \frac{R_{3}}{R_{1}} \right). \tag{374}
\]

The deformation functions for closed shells of rotation, according to Eq. (370),

\[
\xi = \frac{p}{2 E_{1} \delta} \int_{\alpha_{0}}^{\alpha} \left[ 1 - 2 v_{1} - \frac{2 v_{1} (1 - v_{3})}{v_{3}} Q + \frac{v_{1}}{v_{3}} Q^{3} \right] \frac{R_{1} R_{2} d\alpha}{\sin \alpha} + \xi_{0}. \tag{375}
\]

and the components of movement

\[
\begin{align*}
\mathbf{u} &= \frac{p \sin \alpha}{2 E_{1} \delta} \int_{\alpha_{0}}^{\alpha} \left[ 1 - 2 v_{1} - \frac{2 v_{1} (1 - v_{3})}{v_{3}} Q + \frac{v_{1}}{v_{3}} Q^{3} \right] \times \\
&\quad \times \frac{R_{1} R_{2} d\alpha}{\sin \alpha} + \xi_{0} \sin \alpha; \\
\mathbf{w} &= \frac{p \cos \alpha}{2 E_{1} \delta} \int_{\alpha_{0}}^{\alpha} \left[ 1 - 2 v_{1} - \frac{2 v_{1} (1 - v_{3})}{v_{3}} Q + \frac{v_{1}}{v_{3}} Q^{3} \right] \times \\
&\quad \times \frac{R_{1} R_{2} d\alpha}{\sin \alpha} - \xi_{0} \cos \alpha + \frac{p R_{1}^{2}}{2 E_{1} \delta} (2 - v_{3} - Q). \tag{376}
\end{align*}
\]

Consequently, radial movement of the shell, i.e., movement perpendicular to the axis of rotation, is determined by the expression

\[
\zeta = \Delta r = \mathbf{u} \cos \alpha + \mathbf{w} \sin \alpha = \frac{p R_{1}^{2}}{2 E_{1} \delta} (2 - v_{3} - Q). \tag{377}
\]
Correspondingly, the angle of rotation of a meridional element
\[
\phi_0 = \frac{PR_0 \cos \alpha}{2E_0 \delta \sin \alpha} \left[ (q^2 + 2(1 + v_t) e + 4 - \frac{v_t}{v_1}) - \frac{PR_0^2}{2E_0 \delta} \frac{d\phi}{ds} \right].
\] (378)

a. Spherical container or spherical bottom (Fig. 38). In the case of a spherical container

\[R_1 = R_2 = a\]

and, consequently, the forces generated in the shell

\[T_1 = T_2 = \frac{Pa}{2}.\] (379)

The radial deformation and angular displacement of an element of the meridian are determined by the respective expressions

\[
z_0 = \frac{Pa (1 - v_t)}{2E_0 \delta}; \quad \phi_0 = \frac{Pa \cos \alpha}{2E_0 \delta} \left( 7 + 2v_t - \frac{v_t}{v_1} \right).\] (380)

b. Closed cylindrical container (Fig. 39). In this case, \(R_1 = \infty\); \(p = 0\); \(R_2 = r = a\) and, consequently, according to Eq. (374)

\[T_1 = \frac{pa}{2}; \quad T_2 = pa.\] (381)

Correspondingly, the radial deformation and angular displacement

\[
z_0 = \frac{pa (2 - v_t)}{2E_0 \delta}; \quad \phi_0 = 0.\] (382)

c. Ellipsoidal bottom (Fig. 40). In this case

\[e = \frac{a^2}{b^2} - 1.\]

The radii of curvature of an ellipsoidal bottom are determined by the known expressions

\[
R_1 = \frac{sa \sqrt{1 + e}}{\sqrt{1 + e \sin^2 \alpha}}; \quad R_2 = \frac{sa \sqrt{1 + e}}{\sqrt{1 + e \sin^2 \alpha}}.
\] (383)
Consequently, the elastic forces and deformations

\[ T_1 = \frac{p a}{2} \sqrt{1 + e \sin a}; \]
\[ T_2 = \frac{p a}{2} (1 - e \sin a) \sqrt{\frac{1 + e}{1 + e \sin a}}; \]
\[ \xi = \frac{H \tan \alpha}{R_1} \cos \alpha; \]
\[ \eta = \frac{p a}{2 E \delta} \sqrt{1 + e \sin a} \sin a \times \left[ \frac{q^2 + 2(1 + \nu_3)q + 4 - \nu_3^2}{\nu_3} - 2e \sin^2 \alpha \right]. \]

(384)

**d. Conical bottom** (Fig. 41). In this case

\[ R_1 = 0; \quad R_2 = x \tan \gamma; \quad \rho = 0. \]

If distance \( x \) is reckoned along the generatrix of the cone from the top, it is easy to obtain

\[ T_1 = \frac{p a x \tan \gamma}{2}; \quad T_2 = p x \tan \gamma; \]
\[ \xi = \frac{p x^2 \tan^2 \gamma}{2 E \delta} (2 - \nu_3); \]
\[ \eta = \frac{p x^2 \tan^2 \gamma}{2 E \delta} \left( \frac{4 - \nu_3}{\nu_3} \right). \]

(385)

**e. Toroidal container** (Fig. 42). In this case

\[ R_1 = \frac{R + r \sin \alpha}{\sin \alpha}; \quad R_1 = r. \]

A toroidal reservoir is not closed at the top. Consequently, the forces are determined by Eq. (373), and the value of constant \( C \) must be determined.

From the conditions of equilibrium of an element of a torus cut along a plane passing through the curvilinear axis of the torus, and of a cylindrical surface passing through the curvilinear axis, we find

\[ (T_1)_{\rho = \frac{r}{2}} = \frac{p r (2R + r)}{R + r}; \]

(386)

and, consequently, the desired constant is \( C = -\frac{p R^2}{2} \). Thus,
The respective radial and angular displacements

\[ \zeta = \frac{pR^2}{24\pi \delta} (2 - \nu_1 - \nu_2); \]
\[ \chi = \frac{pR^2 \epsilon \sigma \alpha}{24\pi \delta} \left[ \alpha^2 + 2(1 - \nu_1) \alpha + \frac{\nu_1}{\nu_2} + \frac{R}{\alpha \sin \alpha} \right]. \]  

Further, we consider some problems in determination of efficient bottom parameters, which ensure strength with the least possible weight.

37. Some Problems Connected with Determination of Parameters of Least Weight Bottom

Let a cylindrical container of radius \( r \) be subjected to internal pressure of intensity \( p \). Spherical and conical bottoms of the least possible weight must be selected.

Since the problem is solved from the point of view of membrane theory of shells, some simplifying assumptions must be made. Namely, we will assume that, in uneven joining of the bottom with the cylindrical part of the container, the resulting thrust is taken up by a reinforcing ring which is mounted in the butt section. We designate the permissible yield strength of the material by \( \sigma \) and the specific weight of the material by \( \gamma \).

a. Spherical bottom (Fig. 43). In this case

\[ R = \frac{\sigma}{\sin \alpha}. \]

Since the stresses are uniform in a spherical shell, the bottom should be made of constant thickness and provide isotropic structure of the laminated plastic.

The weight of the bottom is determined by the formula

\[ G_{sh} = 2\pi \gamma (1 - \cos \alpha) R^2 \delta. \]

The required bottom thickness

\[ \delta = \frac{pR}{2\sigma}. \]

Consequently, with bottom aperture angle \( \alpha \), the weight of the bottom shell

\[ G_{sh} = \frac{\pi p \sigma \alpha^3}{\alpha \sin^2 \alpha}. \]  

(389)
The required weight of the reinforcing ring in the butt section

\[ G_r = \frac{\pi \rho \gamma a^2}{\sigma_{wn}} \cos \alpha \cdot \frac{\cos \alpha}{\sin^2 \alpha} \]  

(390)

The total weight of the spherical bottom shell with the reinforcing ring

\[ G = \frac{\pi \rho \gamma a^2}{c} \cdot \frac{1 + (k-1) \cos \alpha}{\sin^2 \alpha} \]  

(391)

where

\[ k = \left( \frac{\sigma}{a} \right)_w \]  

(392)

Fig. 44 presents the results of calculation of the weight of the bottom as a function of the relative specific strength of the shell and ring material \( k \) and of aperture angle \( \alpha \) of the spherical bottom.

Eq. (391) and the calculation results show that it is advisable to reinforce the butt section of the bottom with a ring of a material, the specific strength of which is considerably greater than the specific strength of the laminated plastic. The maximum possible weight advantage over a hemispherical bottom reaches 28%. In reinforcing with the same material, because of the unidirectional nature of the laminated plastic, the relative specific strength \( k = 0.5 \).

b. Constant thickness conical bottom (Fig. 45). The shell weight of a conical bottom of constant thickness

\[ G_{sh} = \frac{\pi \rho \gamma a^2}{c} \cdot \frac{2}{\sin 2\alpha} \]  

(393)

the required weight of the reinforcing ring

\[ G_r = \frac{\pi \rho \gamma a^2}{\sigma_r} \sin \alpha \]  

(394)

Consequently, the total weight of the conical bottom shell and reinforcing ring

100
Fig. 45. Conical bottom with reinforcing ring.

Fig. 46. Constant thickness conical bottom weight vs. half aperture angle and relative specific strength of plastic.

Key: a. Relative weight

\[ G = \frac{\pi p y s^2}{\sigma} \left[ \frac{2}{\sin \alpha} + k \beta \alpha \right]. \] \hspace{1cm} (395)

The results of calculation of the bottom weight by Eq. (395) are presented in Fig. 46 as a function of \( k \) and the half aperture angle of the cone.

The minimum weight advantage of the use of a conical bottom over a hemispherical bottom is 50%, if it is considered that the specific strength of a conical bottom made of uniform oriented laminated plastic is greater than the specific strength of a hemispherical bottom. Actually, if the yield strength of a unidirectional plastic is \( \sigma \), the yield strength of a full strength plastic is \( \sigma / 2 \), and the yield strength of the plastic of a conical bottom is \( 2/3 \sigma \).

c. Variable thickness conical bottom. In the use of laminated plastics for manufacture of bottoms, available technological methods of continuous winding permit a variable thickness bottom to be obtained without difficulty.

Since the stressed state of a conical bottom is variable along the generatrix, it is more advisable to make a variable thickness conical bottom. Evidently, the relationship of change in thickness of the bottom is the following

\[ \delta(x) = \frac{p x \beta \alpha}{\sigma}. \] \hspace{1cm} (396)

Consequently, the weight of the bottom shell

\[ G_{sh} = \frac{\pi p y s^2}{\sigma} \frac{4}{3 \sin 2 \alpha}. \] \hspace{1cm} (397)

Since the weight of the reinforcing ring remains the same, the total weight of a variable thickness conical bottom

\[ G = \frac{\pi p y s^2}{\sigma} \left( \frac{4}{3 \sin 2 \alpha} + k \beta \alpha \right). \] \hspace{1cm} (398)

The results of calculation of the bottom weight by Eq. (398) is presented in Fig. 47 as a function of cone half aperture angle \( \alpha \) and \( k \).
The minimum weight advantage of the use of a variable thickness conical bottom over a hemispherical bottom with $k=0$ is 33%. If it is considered that the specific strength of the conical bottom is greater than the specific strength of a hemispherical bottom, with $k=0$, the minimum possible weight of a variable thickness conical bottom equals the weight of the hemispherical bottom. This could have been expected beforehand, since both bottoms are full strength.

d. Box bottom (Fig. 48). We now consider a box bottom, obtained by even joining of the spherical part with the cylindrical part through a toroidal shell.

The weight of the spherical part of the bottom

$$G_{sph} = \frac{\pi\rho a^3}{3} \frac{(1-\varepsilon + \varepsilon \sin \alpha)^3}{(1+\varepsilon \cos \alpha)}.$$ (399)

The weight of the toroidal part of the bottom

$$G_{tor} = \frac{\pi\rho a^3}{3} \left\{ (2-\varepsilon) \left( \frac{\pi}{2} - a \right) (1-\varepsilon + \varepsilon \cos \alpha) \right\}. (400)$$

Thus, the total weight of the box bottom

$$G = \frac{\pi\rho a^3}{3} \left\{ \frac{(1-\varepsilon + \varepsilon \sin \alpha)^3}{(1+\varepsilon \cos \alpha)} + \\
+ \varepsilon (2-\varepsilon) \left( \frac{\pi}{2} - a \right) (1-\varepsilon + \varepsilon \cos \alpha) \right\}. (401)$$

It is easy to note that the weight of the bottom decreases with decrease in $\varepsilon$ and, consequently, the smallest joint radius based on design or other considerations must be used.

The maximum weight advantage will be at $\alpha=60^\circ$ and $\varepsilon=0$, i.e., the toroidal part of the bottom, by ensuring even joining, replaces the ring, as it were. The weight advantage is 23% over a hemispherical bottom.

The results of calculation of the box bottom weight as a function of $\varepsilon$ and $\alpha$ are presented in Fig. 49.
Fig. 49. Box bottom weight vs. geometric parameters.

Key: a. Relative weight
CHAPTER 8. OPTIMUM METHODS OF CONTINUOUS WINDING OF CYLINDRICAL FIBER-
GLASS REINFORCED PLASTIC SHELLS

38. Basic Concepts and Initial Hypotheses

One of the most convenient and widespread methods of production of laminated shells is continuous winding. There are various methods of continuous winding, which differ in the method of placement on the mandrel and type of filler, as well as the nature of impregnation of the filler. Rotation of the mandrel combined with forward motion of the carriage with the bobbin along the mandrel permits the most diverse filler orientation to be achieved. After winding, the shell together with the mandrel go through heat treatment, as a result of which hardening of the binder occurs. After heat treatment, the shell is removed from the mandrel. To make removal of the shell from the mandrel easier the latter is covered with a film before winding, which prevents adhesion of the filler.

For mass production of cylindrical shells and types, high capacity coil winders usually are used. One of them is shown in Fig. 50.

Fabric and nonfabric glass filler, in the form of threads, tapes, bands and fabrics are used for continuous winding of fiberglass reinforced plastic shells. Polyester, phenol, epoxy, organo-silicon resins and various modifications of them are used as the binders.

Fiberglass reinforced plastic cylindrical shells obtained by continuous winding are anisotropic laminated materials. In distinction from natural anisotropic materials, the nature of the anisotropy of the fiberglass reinforced plastics and other reinforcing plastics can be regulated by change in orientation and mutual location of the filler during production. It is expedient to call such anisotropy of the material controllable technological anisotropy, in distinction from structural anisotropy, which is produced by strengthening the shells with stiffening ribs.

The most efficient reinforced plastic structures are those in which anisotropy of the elastic properties most profitably corresponds to the stressed state of the shell or ensures its maximum rigidity with respect to a given load. Determination of the optimum structure of laminated plastics in various structures presents interesting new problems of the theory of elasticity and the theory of shells.

This chapter discusses the problem of the selection of optimum structure of a fiberglass reinforced plastic in a cylindrical shell.
which is exposed to axisymmetric loads, which produce a uniform state of stress in it, determined by the components of the normal forces $T_1$, $T_2$.

It is assumed that the binder has ideal elastic, strength and adhesive properties, which ensure compatibility of the deformation of individual glass filler elements all the way to failure. The glass filler is considered in the form of circular cross section fibers, which does not restrict the generality of the results obtained.

It is easy to show that, in elastic deformations, forces $T_1$, $T_2$ will be taken up by the filler and binder in proportion to their moduli of elasticity $E_H$, $E_C$ and their volumetric content in the material.

If the relative volumetric content of binder is designated $\zeta$, the relative fraction of the forces which are taken up by the binder is determined by the expression

$$ q = \frac{E_C \zeta}{E_H (1-\zeta)}. \quad (402) $$

The moduli of elasticity of available resins change between $3 \cdot 10^2$ and $7 \cdot 10^2 \text{ n/m}^2$, and the modulus of elasticity of glass $E_H = 7 \cdot 10^3 \text{ n/m}^2$. The optimum content of binder in fiberglass reinforced plastic is approximately $30\%$ and, consequently, by Eq. (402), $q = 2-4\%$.

Thus, the normal and shearing forces in fiberglass reinforced plastic shells are primarily taken up by the glass filler. This determines the carrying capacity of the structure. Based on this, we will assume that the effective load on the shell is taken up by the glass filler.

We will call continuous winding the optimum if it ensures equilibrium of the glass filler without the binder. It should be noted that, in nonlinear deformations in the binder and in plastic deformations of the material at the time of failure, the fraction of the load which is taken up by the binder decreases sharply. Therefore, selection of the optimum winding general speaking is of decisive importance for increasing the carrying capacity of a shell. We will call a shell composed of fibers alone the basic system.

Since actual resins which are used as binders in the manufacture of shells have various properties, they provide compatibility of deformation of the glass filler in different ways, and this explains the results of studies in which a significant effect of the binder on the elastic and strength properties of fiberglass reinforced plastics was found. Actually, with slight adhesion of the binder to the glass filler, the distribution of forces through the shell will be irregular. This leads to both premature destruction of the filler in the most stressed fibers and to over-stress and failure of the binder, i.e., an increase in irregularity of distribution of the forces and subsequent reduction of carrying capacity of the shell.
If the axial force in a cylindrical shell is $T_1$ and the annular force is $(1+a) T_1$, the normal forces which act on a surface which is located at angle $\theta$ to the generatrix of the shell is determined by the expression

$$T_\theta = T_1 (1 + a \cos^2 \phi). \quad (403)$$

Let glass fibers be wound on a cylindrical mandrel at angle $\phi$ to the generatrix (Fig. 51).

It is easy to determine that the same number of fibers passes through segments $AB$, $AD$ and $BC$ (Fig. 51). If the length of a segment of the generatrix $AB=a$, the length of a segment perpendicular to the fiber direction equals $a \sin \phi$. Consequently, $n a \sin \phi$ fibers pass through the segments indicated above, where $n$ is the glass fiber packing density, i.e., the number of fibers passing through a unit segment perpendicular to the fibers.

The angle between the normal to area $BC$ dropped from point $A$ ($AE\parallel BE$), and the length of segment $BC$

$$a = \frac{a}{2} \sin (\theta + \phi); \quad x = |BC| = \frac{a \sin \phi}{\sin (\theta + \phi)}. \quad (404)$$

Consequently, the normal force which arises in area $BC$ in stretching of the fibers by force $f$ is determined from the expression

$$T_\theta = \frac{fn \sin \phi \cos \phi}{x} = fn \sin^2 (\phi + \psi). \quad (405)$$

According to Eq. (405), the distribution of normal forces in a cylindrical shell depends essentially on the orientation of the glass filler during winding, i.e., on angle $\phi$.

According to Eq. (405), winding of the fibers at one constant angle $\phi$ does not ensure equilibrium of the basic system. We will subsequently assume Eq. (405) to be written for the limiting state of the shell, i.e., we will assume fiber tension $f$ to be equal to the breaking force of the fiber. For convenience in use, it is advisable to subsequently present Eq. (405) in the form

$$T_\theta = \frac{fn}{2} [1 + \cos 2\phi - 2 \cos 2\phi \cos^2 \theta + \sin 2\phi \sin 2\theta]. \quad (406)$$

39. Continuous Winding of Cylindrical Shells with Unidirectional Glass Fillers at Optimum Angles to Shell Generatrix

Since winding a cylindrical shell at one constant angle does not
ensure equilibrium of the basic system, we consider the case when the shell is wound with two glass fiber systems at angles \( \phi_1 \) and \( \phi_2 \) to the shell generatrix (Fig. 52).

We initially study the simplest case, when each layer of one system of fibers corresponds to one layer of the other system. According to Eq. (406), the force on the area \( \delta \) is presented in the form

\[
T = \frac{m}{2} \left[ 2 + \cos 2\phi_1 + \cos 2\phi_2 - 2(\cos 2\phi_1 + \cos 2\phi_2) \cos \theta + \right. \\
+ \left. (\sin 2\phi_1 + \sin 2\phi_2) \sin \theta \right].
\]  

(407)

The equilibrium conditions of the glass fibers have the form

\[
\begin{align*}
\cos 2\phi_1 + \cos 2\phi_2 &= -\frac{2a}{\alpha + a}; \\
\sin 2\phi_1 + \sin 2\phi_2 &= 0.
\end{align*}
\]  

(408)

From Eq. (408), it is easy to find

\[
\phi_2 = \phi_1 = -\phi_1 + m\pi \quad (m = 0, 1, 2, \ldots);
\]  

(409)

\[
\cos 2\theta = \frac{a}{\alpha + a},
\]  

(410)

where \( \alpha > 1 \), i.e., the winding should be crossed at angles \( \pm \phi \). For different stresses of the shell, the optimum cross winding angles are determined by Eq. (410).

The basic relationship which connects the carrying capacity of the shell with the strength of the glass fibers has the form

\[
T_1 = \frac{2}{\alpha + a} f_n.
\]  

(411)

We now consider a more general case of continuous winding, when \( N \) layers wound at angle \( \phi_2 \) to the generatrix are applied to each layer wound at angle \( \phi_1 \) to the generatrix.

The equilibrium conditions of the fibers have the form...
\[
\begin{align*}
\cos 2\varphi_1 - N \cos 2\varphi_2 &= \frac{(N+1)a}{2+a} \\
\sin 2\varphi_1 - N \sin 2\varphi_2 &= 0.
\end{align*}
\]

\section*{TABLE 1. OPTIMUM WINDING ANGLES OF CYLINDRICAL SHELL UNDER AXISYMMETRIC LOAD VS. \(T_2/T_1\)}

<table>
<thead>
<tr>
<th>(\frac{T_2}{T_1})</th>
<th>(\varphi^a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0°</td>
</tr>
<tr>
<td>1</td>
<td>4°</td>
</tr>
<tr>
<td>2</td>
<td>54°44'</td>
</tr>
<tr>
<td>3</td>
<td>60°35'</td>
</tr>
<tr>
<td>4</td>
<td>65°55'</td>
</tr>
<tr>
<td>5</td>
<td>71°35'</td>
</tr>
<tr>
<td>6</td>
<td>90°</td>
</tr>
</tbody>
</table>

Key:  
\begin{itemize}
  \item a. Type loading of shell
  \item b. Axial tension
  \item c. Internal pressure with axial force
\end{itemize}

Consequently, the optimum winding angles are determined by the following expressions

\[
\begin{align*}
\cos 2\varphi_1 &= \frac{2(a+1)N -(a^2+2a+2)}{(2+a)a} \\
\cos 2\varphi_2 &= \frac{(a^2+2a+2)N - 2(a+1)}{(2+a)aN}.
\end{align*}
\]

The carrying capacity of the shell

\[
T_1 = \frac{(N-1)}{(2+a)} \text{fn.}
\]

Determination of the optimum winding angles by Eq. (413) is not difficult in any axisymmetric stress of the shell. Further, we study the cases of loading with uniform internal pressure most frequently encountered in practice.

If a shell operates under uniform internal pressure, the optimum winding angles are determined by the formulas

\[
\begin{align*}
\cos 2\varphi_1 &= \frac{4N-5}{3}; \quad \cos 2\varphi_2 = \frac{4-5N}{3N}.
\end{align*}
\]

It follows from Eq. (415) that...
The results of calculation of the optimum winding angles by Eq. (415) are presented in Table 2.

\[ N-1 \text{ corresponds to bias cross winding; } N-2 \text{ corresponds to longitudinal-transverse winding, when two layers are wound in the annular direction } (\phi_2=90^\circ) \text{ on one longitudinally laid layer.} \]

The calculation results presented in Table 2 are illustrated in Fig. 54.

We now compare the yield strengths of shell materials obtained by continuous winding with the same glass fibers and different anisotropy of the strength properties. According to basic relationship (414), which connects the carrying capacity of the shell with the strength of the fibers, the following results can be obtained (Table 3).

### Table 2. Optimum Winding Angles of Cylindrical Shells Operating Under Internal Pressure vs. Layer Ratio N

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>54°44'</td>
<td>125°16'</td>
</tr>
<tr>
<td>1.1</td>
<td>50°46'</td>
<td>121°29'</td>
</tr>
<tr>
<td>1.2</td>
<td>48°54'</td>
<td>118°08'</td>
</tr>
<tr>
<td>1.3</td>
<td>43°06'</td>
<td>115°04'</td>
</tr>
<tr>
<td>1.4</td>
<td>40°14'</td>
<td>112°12'</td>
</tr>
<tr>
<td>1.5</td>
<td>38°10'</td>
<td>109°28'</td>
</tr>
<tr>
<td>1.6</td>
<td>31°08'</td>
<td>106°46'</td>
</tr>
<tr>
<td>1.7</td>
<td>25°34'</td>
<td>104°02'</td>
</tr>
<tr>
<td>1.8</td>
<td>21°28'</td>
<td>101°06'</td>
</tr>
<tr>
<td>1.9</td>
<td>14°58'</td>
<td>97°30'</td>
</tr>
<tr>
<td>2</td>
<td>0°</td>
<td>90°</td>
</tr>
</tbody>
</table>

Fig. 54. Optimum winding angles of cylindrical shells operating under internal pressure.

Fiberglass fabric is an aggregate of two mutually orthogonal glass fiber systems connected together with varied amounts of interweaving in textile processing. By type and amount of interweaving of the warp and woof fibers, card, satin and serge cloth are distinguished (Fig. 55).
### TABLE 3. FIBERGLASS REINFORCED PLASTIC STRENGTH VS. NATURE OF ANISOTROPY

<table>
<thead>
<tr>
<th>Type of Fiberglass</th>
<th>Character of Anisotropy</th>
<th>Yield Strength in Principal Directions of Anisotropy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\sigma_{10}$</td>
</tr>
<tr>
<td>d. Unidirectional</td>
<td>1:0</td>
<td>$\frac{m}{n}$</td>
</tr>
<tr>
<td>e. Quasi-unidirectional</td>
<td>1:1</td>
<td>$\frac{1}{2}$ $\frac{n}{m}$</td>
</tr>
<tr>
<td></td>
<td>2:1</td>
<td>$\frac{2}{3}$ $\frac{n}{m}$</td>
</tr>
<tr>
<td>m &gt; n</td>
<td>m:n</td>
<td>$\frac{m}{m+n}$ $\frac{n}{m+n}$</td>
</tr>
</tbody>
</table>

Key:
- a. Fiberglass plastic type
- b. Nature of anisotropy
- c. Yield strength in principal directions of anisotropy
- d. Unidirectional
- e. Full strength

![Fig. 55. Fiberglass fabric structure: a. card; b. satin; c. serge.](image)

Satin fabrics have the greatest flexibility, they ensure the highest quality packing in continuous winding, and they have the best capacity for impregnation with binders. Fiberglass reinforced plastics produced from fabrics usually are called fiberglass laminates.

For convenience, we introduce the following designations:

- $f_1$, $f_2$ are the breaking forces of the warp and weft fibers;
\( n_1, n_2 \) are the packing density of the warp and woof fibers;

\( k = \frac{f_2 n_2}{f_1 n_1} \) is the relative strength of the fabric \((k \leq 1)\).

As in winding with unidirectional glass fillers, winding of a cylindrical shell with fabrics at a constant angle does not ensure equilibrium of the basic system, except for straight winding of the warp in the annular or longitudinal directions. In this case, of course, the anisotropy of the fabric should be \( k = \frac{T_2}{T_1} \) or \( k = \frac{T_1}{T_2} \).

We consider the general case of cross winding of a cylindrical shell with two fabric systems, which are laid so that the direction of the warp fabric is at angles \( \phi_1, \phi_2 \) with the shell generatrix.

Both systems are composed of the same fabric of relative strength \( k \) in which, on each layer of the first system, there are \( N \) of the second system.

In accordance with Eq. (406), the normal force on area \( \theta \)

\[
T_0 = \frac{f_n}{2} \left[(N + 1)(k + 1) - (k - 1)(\cos 2\varphi_1 + N \cos 2\varphi_2) + 
+ 2(k - 1)(\cos 2\varphi_1 + N \cos 2\varphi_2) \cos \theta - 
- (k - 1)[\sin 2\varphi_1 + N \sin 2\varphi_2 \sin 2\theta]\right].
\]

Consequently, the equilibrium conditions of the basic system are written in the form

\[
\begin{align*}
\cos 2\varphi_1 + N \cos 2\varphi_2 &= \frac{(N + 1)(k + 1) \alpha}{(k - 1)(2 + \alpha)}; \\
\sin 2\varphi_1 + N \sin 2\varphi_2 &= 0.
\end{align*}
\] (419)

The relationship which connects the carrying capacity of the reinforced shell with the strength of the fibers has the form

\[
T_1 = \frac{f_n}{2 + \alpha} (N + 1)(k + 1).
\] (420)

The following expressions for determination of the optimum winding angles can be obtained from Eq. (419):

\[
\begin{align*}
\cos 2\varphi_1 &= -\frac{2 \alpha^2 (k + 1)^2 N^2 - (N + 1)^2 (k + 1)^2 \alpha^2 (N + 1)^2 - (k - 1)^2 (2 + \alpha)^2}{2(N + 1)(4^2 - 1) \alpha (2 + \alpha)}; \\
\cos 2\varphi_2 &= -\frac{(k - 1)^2 (2 + \alpha)^2 N^2 + (k + 1)^2 \alpha^2 (N + 1)^2 - (k - 1)^2 (2 + \alpha)^2}{2N(N + 1)(k^2 - 1) \alpha (2 + \alpha)}.
\end{align*}
\] (421)

Since it is quite complex to study Eq. (421) in the general case of loading a cylindrical shell, we consider the case of loading a shell with uniform internal pressure \((\alpha = 1)\) in greater detail, for cross wind-
In this partial case, \( \phi_1 = \phi_2 = \phi \)

\[
\cos 2\phi = \frac{(k+1)}{3(k-1)};
\]  

(422)

it follows from this formula that, for manufacture of fiberglass fabric cylindrical shells operating under uniform internal pressure, fiberglass fabrics must be used which satisfy the condition

\[ 0 < k \leq \frac{1}{2}; \]  

(423)

i.e., the fabric strength along the woof should not be greater than half the fabric strength along the warp. Otherwise, the excess strength of the fabric along the woof cannot be used, even in bias winding.

The results of the calculation by Eq. (422) are presented in Table 4 and are depicted in Fig. 56.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \theta )</th>
<th>( \alpha )</th>
<th>( \phi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>54° 44'</td>
<td>0.25</td>
<td>61° 48'</td>
</tr>
<tr>
<td>0.05</td>
<td>55° 48'</td>
<td>0.30</td>
<td>64° 07'</td>
</tr>
<tr>
<td>0.10</td>
<td>57° 07'</td>
<td>0.35</td>
<td>66° 50'</td>
</tr>
<tr>
<td>0.15</td>
<td>58° 24'</td>
<td>0.40</td>
<td>70° 32'</td>
</tr>
<tr>
<td>0.20</td>
<td>60°</td>
<td>0.5</td>
<td>90°</td>
</tr>
</tbody>
</table>

According to Eq. (420), the carrying capacity of a shell under internal pressure equals

\[
pR = \frac{4}{3} f_1 n_1 (k+1).\]  

(424)

It can be concluded from this that fiberglass laminate shells are considerably inferior in strength to shells made by winding unidirectional glass fillers, the carrying capacity of which equals

\[
pR = \frac{8}{3} f_1.\]  

(425)

Fig. 56. Optimum winding angle of cylindrical shell vs. relative warp and woof strength of fabric.
even in the most favorable case, when \( f_{1n_{1}}w_{1n} \), the carrying capacity of a fiberglass laminate shell is 25% less than the carrying capacity of a shell produced by winding unidirectional glass fillers.

It also follows from Eq. (424) that with a given warp strength of fiberglass fabric, the maximum carrying capacity of the shell is reached at \( k=0.5 \), i.e., in straight winding of fabrics, the warp strength of which is twice the woof strength.

It seems advisable to consider a still simpler method of manufacture of shells, when annular winding is accompanied by laying the fabric warp along the shell generatrix.

Let \( N \) layers be wound in the annular direction on each layer of fabric laid along the generatrix or vice versa. In this case, the normal force on area \( \theta \) is represented by the expressions

\[
T_{o} = f_{1n_{1}}(N_{1}k + 1) \left[ 1 - \frac{(1-k)(1-N_{2})}{N_{k+1}} \cos^{2} \theta \right] \tag{426}
\]

or

\[
T_{o} = f_{1n_{1}}(N_{2} + k) \left[ 1 + \frac{(1-k)(1-N_{1})}{N_{k+1}} \cos^{2} \theta \right]. \tag{427}
\]

The relative fabric strength is not limited by bias winding conditions, and it changes in the \( 0 \leq k \leq 1 \) interval.

Thus, the number of annular layers on one layer of fabric laid along the generatrix is determined by the formula

\[
N_{1} = \frac{1+a-k}{1-k-a} \quad \text{or} \quad N_{1} = \frac{1-k-k+a}{k+a-1}. \tag{428}
\]

Since \( N>0 \), the fiberglass fabric should satisfy the conditions

\[
k \leq \min \left( 1 + a; \frac{1}{1+a} \right) \leq 1; \tag{429}
\]

We consider several possible cases.

1. Axial tension of shell \( \alpha=1 \); the optimum fabrics which ensure the greatest strength of the shell should be unidirectional \((k=0)\). According to Eq. (428) \( N_{1}=0 \).

2. Uniform tension of shell \( \alpha=0 \); for the manufacture of a cylindrical shell, the annular stress of which equals the axial stress, it is advisable to use full strength fabrics \((k=1)\). According to Eq. (428), \( N_{1}=1 \).

3. Shell under internal pressure \( \alpha=1 \); in this case, the
optimum fabrics should have relative wof strength \( k = \frac{1}{2} \) and be wound only annularly. According to Eq. (428), \( N_2 = 0 \).

### 41. Optimum Methods of Combined Winding of Cylindrical Shells

There is still another method of continuous winding of shells with fiberglass fabrics alternately with some unidirectional glass fillers. This combined winding method permits the improvement of fabrics which do not satisfy the conditions of optimality. Moreover, in laying the fabric warp in the direction of the generatrix, the strength of the shell is increased correspondingly compared with fiberglass laminate shells.

We initially consider the combined continuous winding method as applied to the manufacture of shells which operate under uniform internal pressure. Let the warp of the fabric form angle \( \phi_1 \) with the generatrix of the shell and the unidirectional filler be wound at angle \( \phi_2 \) to the generatrix.

If \( N \) layers of unidirectional glass filler are wound on each layer of fabric, the optimum winding angles are determined by the following expressions

\[
\cos 2\phi_1 = \frac{4k^3 - 5k^2 - k + 8k - k_i - 5}{3(1 - k)(k_i + k + 1)}; \quad \cos 2\phi_2 = \frac{5k^3 - 4k^2 + k + 4k_i + 4}{3k_i (k_i + k + 1)}.
\]

where

\[
k_i = \frac{f_n}{I_1}, \quad N.
\]

\( f \) is the tensile strength of the fiber; \( n \) is the packing density of the unidirectional filler.

We also consider the case of winding of the greatest practical importance, when the fabric is laid with the warp in the axial direction \( (\phi_1 = 0) \), and the unidirectional glass filler is cross wound at angles \( \pm \phi \).

In this case, the normal force on area \( \theta \) is represented by the formula

\[
T_\theta = f_n \left[ 1 + \frac{k_1 + \cos 2\phi}{2} - (k_1 \cos 2\phi + 1 - k) \cos^2 \theta \right].
\]

The carrying capacity of the shell is determined by the expression

\[
pR = \frac{2}{3} (f_1 n_1 + f_2 n_2 + f n N),
\]

and the optimum cross winding angle of the unidirectional glass filler is found by the formula
where parameters $k_1$, $k$ should satisfy the condition

$$\xi = k_1 + k \geq 2.$$  \hspace{1cm} (434)

In the individual case when $k_1 + k = 2$, $\phi = 90^\circ$, i.e., the unidirectional glass filler should be wound in the annular direction.

**TABLE 5. OPTIMUM COMBINED WINDING ANGLES**

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\xi - k_1 + k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$70^\circ 11'$</td>
</tr>
<tr>
<td>0.2</td>
<td>$69^\circ 44'$</td>
</tr>
<tr>
<td>0.3</td>
<td>$69^\circ 26'$</td>
</tr>
<tr>
<td>0.4</td>
<td>$69^\circ 02'$</td>
</tr>
<tr>
<td>0.5</td>
<td>$68^\circ 35'$</td>
</tr>
<tr>
<td>0.6</td>
<td>$68^\circ 07'$</td>
</tr>
<tr>
<td>0.7</td>
<td>$67^\circ 37'$</td>
</tr>
<tr>
<td>0.8</td>
<td>$67^\circ 06'$</td>
</tr>
<tr>
<td>0.9</td>
<td>$66^\circ 32'$</td>
</tr>
<tr>
<td>1</td>
<td>$65^\circ 50'$</td>
</tr>
</tbody>
</table>

The results of calculation by Eq. (433) are presented in Table 5 and Fig. 57.

We now study the general case when it is necessary to determine the optimum combined winding of a cylindrical shell operating under axisymmetric load $T_2 / T_1 = 1 + a$. We consider the simplest longitudinal-transverse winding, when the fabric warp is laid along the axis of the shell or is wound with the warp in the annular direction, depending on the nature of the stress of the shell.

Similarly to the preceding, for determination of the optimum winding angles of the unidirectional glass filler, the following calculation formulas can be obtained

$$\cos 2\varphi = \frac{2k - k_1 a - 2(1 + a)}{(2 + a) k_1};$$  \hspace{1cm} (435)

$$\cos 2\varphi = \frac{2(1 - a) - 2k(1 + a) - k_1 a}{k_1 (2 + a)}.$$  \hspace{1cm} (436)

When the warp of the fabric is laid axially, according to Eq. (435), the filler parameters should satisfy the conditions.

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If the warp of the fabric is wound annularly, the filler parameters should satisfy the conditions

\[ k_1 + k > 1 + a; \quad 1 + k_1 \geq \frac{k}{1 + a}. \]  

(437)

42. Optimum Winding Angles of Bottoms of Varied Geometric Shape

We will consider a bottom manufactured by continuous winding, in the form of a shell of rotation and smoothly joined to a cylindrical body of radius a (Fig. 58).

If the \((r, x)\) coordinate origin is placed in the pole of the bottom, the principal radii of curvature of the bottom are determined by the known expressions

\[ R_1 = \frac{1}{\sqrt{(1+r^2)^3}}; \quad R_2 = r\sqrt{1+r^3}. \]  

(439)

Fig. 58. Coordinate system for shell of rotation.

The meridional and annular forces which are generated in the bottom as a result of uniform internal pressure are equal to

\[
\begin{align*}
T_1 &= \frac{pr^2}{r} \sqrt{1+r^3}; \\
T_2 &= \frac{pr^2}{r} \sqrt{1+r^3}(2 + \frac{r^2}{1+r^3}).
\end{align*}
\]  

(440)

Since a shell of rotation with positive Gaussian curvature \(1/R_1R_2>0\) is a nondevelopable surface, the bottom can be made only by continuous winding of glass fibers.

Let \(nc\) fibers pass through small segment \(c\) perpendicular to the fiber direction. We consider an element of the bottom cut by two axial planes and two conical surfaces, so that the condition \(ds_1 \sin \phi = c\), \(ds_2 \cos \phi = c\), where \(\phi\) is the winding angle, i.e., the angle between the fiber direction and the meridian of the surface, is satisfied (Fig. 59).

The same number of fibers, equal to the product of the packing density and the length of segment \(c\) (perpendicular to the fiber direction), i.e., equal to \(nc\), passes through segments \(ds_2, c\) and \(ds_1\). In distinction from a cylindrical shell, the fiber packing density on a
double curvature shell is not constant, but it changes direction toward
the pole of the shell, i.e., the fiber packing density in winding shells
of rotation is a function of the cross section radius or axial coordi-
nate \( x \).

The normal forces which are generated in area \( ds_1, ds_2 \), with the
fibers under tension of force \( f \),
\[
T_1 = fn \cos^2 \phi; \quad T_2 = fn \sin^2 \phi. \tag{441}
\]

According to Eq. (440), (441), a system of differential equations
which determines the optimum continuous winding of shells of rotation
has the form
\[
\begin{align*}
pr \sqrt{1 + r^2} &= 2/2n \cos \psi; \\
pr \frac{1 + r^2 (2 + \frac{rr'}{1 + r^2})}{2n} &= 2/n \sin \psi. \tag{442}
\end{align*}
\]

By dividing the second equation of (442) by the first, we obtain
an expression for the square of the tangent of the optimum winding
angle of the bottom as a function of the shape of the bottom
\[
\tan^2 \phi = 2 + \frac{rr'}{1 + r^2}. \tag{443}
\]

This basic relationship permits the pattern of change of winding angles
\( \phi \) in the manufacture of shells of rotation of arbitrary shape to be
found:

Shape of shell of rotation  Optimum Winding angle
Hemispherical  \( 45 \ \arctg \sqrt{2 - \frac{a^2}{a^2 - (a^2 + b^2)r^2}} \)
Ellipsoidal  \( 54°44' \ \arctg \sqrt{\frac{2(a^2 - b^2)^2}{2 - b^2}} \)
Ellipsoidal \( b = a/\sqrt{2} \)  \( \arctg \sqrt{\frac{2}{k + r - a}} \)
Conical  \( \arctg \sqrt{\frac{r}{r - a}} \)
Ogival  \( \arctg \sqrt{\frac{2}{r}} \)
Box

However, in continuous winding of shells with a nonzero Gaussian
curvature, it must be kept in mind that, besides satisfying the condition
of equilibrium of the basic system, i.e., equilibrium of the glass filler
without the binder, still another no less important condition must be met
which specifies no slipping of the glass filler from the shell surface,
and it is geometrically reduced to winding along the geodetic lines of \( 1/136 \)
the surface. According to the Klero theorem for a surface of rotation,
this condition can be written in the form

\[ r \sin \phi = h, \tag{444} \]

where \( h \) is the geodetic line parameter.

With an opening at the pole and continuous winding of the opening, the geodetic line parameter equals the cross section radius of the opening.

According to Eq. (443), (444), the differential equation which determines the shape of the bottom and the optimum winding can be written in the form

\[ \frac{r''}{1 + \frac{r'}{r}} + \frac{2r' - \frac{2h^2}{r}}{1 + \frac{h^2}{r}} = 0. \tag{445} \]

By replacement of the variables

\[ \frac{r}{h} = \zeta; \quad r' = U(\zeta), \tag{446} \]

differential Eq. (445) can be reduced to the form

\[ \frac{U'd'}{1 + U^2} = \frac{(3 - 2\zeta) d\zeta}{\zeta (\zeta^2 - 1)}, \tag{447} \]

from which, after integration, the desired shape of the bottom is determined in the form of a simple quadrature as a function of parameter \( h \)

\[ x = h \int \frac{\zeta^2 d\zeta}{\zeta (\zeta^2 - 1) - \lambda^2} + x_0. \tag{448} \]

where

\[ \lambda = \frac{\left(\frac{a}{h}\right)^2}{\left(\frac{a}{h}\right)^2 - 1}. \tag{449} \]

According to Eq. (448), the optimum shape only exists in regions where the subintegral expression is positive.

Optimum shape of bottom with longitudinal-transverse winding.

There is interest in determination of the optimum shape of a bottom with longitudinal-transverse winding of the cylindrical shell.

In longitudinal-transverse winding, the glass fibers will go in the direction of the meridian of the shell. Consequently, to ensure equilibrium of the basic system in the annular direction, a bottom shape must be selected in which the annular stresses equal zero.

According to Eq. (440), the desired bottom shape is determined by the following differential equation

\[ rr'' + 2r' + 2 = 0. \tag{450} \]
The solution of Eq. (450) is represented by the elliptical integral
\[ x = \int_0^1 \frac{t^2 \, dt}{\sqrt{1-t^4}} , \] (451)

which can be expressed through elliptical functions.

In the dimensionless coordinates \( \eta = x/a, \tau = r/a \), the shape of the bottom is determined by the equation
\[ \eta = \Phi(\tau) , \] (452)

where function \( \Phi(\tau) \) and its first derivative are presented in Table 6.

### TABLE 6. OPTIMUM SHAPE OF BOTTOM WITH LONGITUDINAL-TRANSVERSE WINDING

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( \Phi(\tau) \cdot 10^\alpha )</th>
<th>( \Phi'(\tau) )</th>
<th>( \tau )</th>
<th>( \Phi(\tau) \cdot 10^\alpha )</th>
<th>( \Phi'(\tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.52</td>
<td>4.7629</td>
<td>0.2669</td>
</tr>
<tr>
<td>0.02</td>
<td>0.3 \cdot 10^{-3}</td>
<td>0.0004</td>
<td>0.54</td>
<td>5.3484</td>
<td>0.3049</td>
</tr>
<tr>
<td>0.04</td>
<td>0.213 \cdot 10^{-3}</td>
<td>0.0016</td>
<td>0.56</td>
<td>5.9834</td>
<td>0.3303</td>
</tr>
<tr>
<td>0.06</td>
<td>0.720 \cdot 10^{-3}</td>
<td>0.0036</td>
<td>0.58</td>
<td>6.6702</td>
<td>0.3572</td>
</tr>
<tr>
<td>0.08</td>
<td>0.0177</td>
<td>0.0004</td>
<td>0.60</td>
<td>7.4131</td>
<td>0.3850</td>
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<td>0.10</td>
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<td>0.0100</td>
<td>0.62</td>
<td>8.2152</td>
<td>0.4164</td>
</tr>
<tr>
<td>0.12</td>
<td>0.0376</td>
<td>0.0144</td>
<td>0.64</td>
<td>9.0707</td>
<td>0.4490</td>
</tr>
<tr>
<td>0.14</td>
<td>0.0815</td>
<td>0.0196</td>
<td>0.66</td>
<td>10.013</td>
<td>0.4839</td>
</tr>
<tr>
<td>0.16</td>
<td>0.1365</td>
<td>0.0256</td>
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<td>0.5215</td>
</tr>
<tr>
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<td>0.0324</td>
<td>0.70</td>
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</tr>
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<td>0.72</td>
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<td>0.6062</td>
</tr>
<tr>
<td>0.22</td>
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<td>0.0485</td>
<td>0.74</td>
<td>14.521</td>
<td>0.6514</td>
</tr>
<tr>
<td>0.24</td>
<td>0.4613</td>
<td>0.0577</td>
<td>0.76</td>
<td>15.880</td>
<td>0.7076</td>
</tr>
<tr>
<td>0.26</td>
<td>0.5865</td>
<td>0.0678</td>
<td>0.78</td>
<td>17.344</td>
<td>0.7666</td>
</tr>
<tr>
<td>0.28</td>
<td>0.7327</td>
<td>0.0786</td>
<td>0.80</td>
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<td>0.8329</td>
</tr>
<tr>
<td>0.30</td>
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<td>0.0894</td>
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</tr>
<tr>
<td>0.32</td>
<td>1.0947</td>
<td>0.1029</td>
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</tr>
<tr>
<td>0.34</td>
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<td>0.1164</td>
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<td>0.8716</td>
</tr>
<tr>
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<td>0.1307</td>
<td>0.88</td>
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<td>0.8170</td>
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<tr>
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<td>0.90</td>
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<td>0.7240</td>
</tr>
<tr>
<td>0.40</td>
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<td>0.1621</td>
<td>0.92</td>
<td>31.435</td>
<td>0.6292</td>
</tr>
<tr>
<td>0.42</td>
<td>2.4861</td>
<td>0.1792</td>
<td>0.94</td>
<td>34.682</td>
<td>0.5294</td>
</tr>
<tr>
<td>0.44</td>
<td>2.8421</td>
<td>0.1973</td>
<td>0.96</td>
<td>38.702</td>
<td>0.4213</td>
</tr>
<tr>
<td>0.46</td>
<td>3.2764</td>
<td>0.2165</td>
<td>0.98</td>
<td>44.184</td>
<td>0.2901</td>
</tr>
<tr>
<td>0.48</td>
<td>3.7294</td>
<td>0.2368</td>
<td>1.00</td>
<td>58.115</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

The optimum bottom shape is presented in Fig. 60, where an ellipsoidal bottom with the same semiaxes as the optimum bottom is noted by the dashed line.

Fig. 60. Optimum shape of bottom with longitudinal-transverse winding.
CHAPTER 9. END EFFECTS IN AXISYMMETRICALLY LOADED CYLINDRICAL SHELLS

43. Differential Equation of Axisymmetrical Deformation of Cylindrical Shell

We consider an orthotropic cylindrical shell, the principal axes of anisotropy of which coincide with coordinate axes \( x, \beta \), which characterize the distance along the shell generatrix and the circumferential angle (Fig. 61).

According to Eq. (10)-(12) and (24)-(26), in axisymmetric loading of an orthotropic cylindrical shell, the basic elasticity relationships are presented in the form

\[
\begin{align*}
T_1 &= B_1 \left( \dot{u}' + v_1 \frac{w}{R} \right), \\
T_2 &= B_2 \left( \frac{w}{R} + v_1 u' \right), \\
Q_1 &= -K_1 (\varphi + \dot{\varphi}'), \\
G_1 &= -D_1 \dot{\varphi}; \quad G_2 = v G_1.
\end{align*}
\]

The equilibrium equations of a shell subjected to a surface load and end forces can be written in the form

\[
\begin{align*}
T_1' + X(z) &= 0; \\
Q_1 + \frac{T_2}{R} &= Z(x); \\
G_1' &= Q_1.
\end{align*}
\]

The axial force generated in the shell is determined by the expression

\[
T_1 = - \int_{x_0}^{x} X(t) \, dt + T_0,
\]

where \( T_0 \) is the axial force applied to the end of shell \( x=x_0 \).

From the first elasticity relationship, with Eq. (455) taken into account, the following can be obtained

\[
u' = - \frac{1}{B_1} \int_{x_0}^{x} X(t) \, dt + \frac{T_2}{B_1} - v_1 \frac{w}{R}.
\]

By substituting Eq. (456) in the second elasticity relationship, we obtain
If stress function $\Phi = D \phi$ is introduced, the elastic forces, moments and movements of the shell are determined by the following expressions

\[ G_i = \Phi_i', \quad Q_i = \Phi_i''; \]
\[ T_s = RZ(x) - R \Phi''; \]
\[ w = \frac{R^4}{E_s \delta} \left[ Z(x) - v_s \frac{T_s}{R} + \frac{v_s}{R} \int X(\xi) d\xi - \Phi'' \right]; \]
\[ u = -\frac{1}{B_1} \int_{z_s}^{z} dz \int_{z_s}^{z} X(\xi) d\xi + \frac{(z-z_0) T_s}{B_1} - \frac{v_s}{R} \int w dx + u_1, \]

where $u_1$ is the axial movement of the end of shell $x=x_0$.

The differential equation which describes the axisymmetric deformation of an orthotropic cylindrical shell made of laminated plastics has the form

\[ \Phi^{IV} - 2\phi^2 \Phi'' + k^4 \Phi = -v_s \frac{X}{R} + Z'(x), \]

where

\[ 2\phi^2 = \frac{E_s \delta}{K_1 R^4}; \quad k^4 = \frac{12(1-v_1 v_2) E_s}{R^4 \delta^3 E_1}. \]

If the partial solution of Eq. (459), which is found by conventional methods, is designated $\Phi_0(x)$, the forces, moments and movements which correspond to this partial solution are written in the form

\[ G_i = \Phi_i'; \quad Q_i = \Phi_i''; \quad T_s^0 = R (Z - \Phi_0'''); \]
\[ \psi_0 = -\frac{\Phi_0'}{B_1}, \]
\[ w_0 = \frac{v_s}{E_s \delta} \int_{z_s}^{z} X(\xi) d\xi - \frac{v_s}{E_s \delta} T_0 - \frac{R^4 \Phi_0'''}{E_s \delta} + \frac{R^4 Z}{E_s \delta}; \]
\[ u_0 = -\frac{1}{B_1} \int_{z_s}^{z} dz \int_{z_s}^{z} X(\xi) d\xi - \frac{(z-z_0) T_0}{B_1} - \frac{v_s}{R} \int w_0 dx + u_2. \]

The solution of homogeneous differential Eq. (459), which determines the end effects generated near zones of abrupt change of geometric and
rigidity parameters or external load, has the form, for long cylindrical shells,

\[ \Phi(x) = C_1 \Phi_1(x) + C_2 \Phi_2(x), \quad (462) \]

where

\[ \Phi_1(x) = e^{-\alpha x} \cos r x; \quad \Phi_2(x) = e^{-\alpha x} \sin r x; \quad (463) \]

\[ s = \sqrt{\frac{k^2 + q^2}{2}}; \quad r = \sqrt{\frac{k^2 - q^2}{2}}; \quad (464) \]

The following known relationships satisfy functions \( \Phi_1(x), \Phi_2(x) \)

\[
\begin{align*}
\Phi_1' &= -(r \Phi_3 + s \Phi_4); \\
\Phi_2' &= r \Phi_3 - s \Phi_4; \\
\Phi_1 &= (s' - r') \Phi_1 + 2rs \Phi_2; \\
\Phi_2 &= (s' - r') \Phi_2 - 2rs \Phi_1; \\
\Phi_1'' &= -s(s' - 3r') \Phi_1 + r(r' - 3s') \Phi_2; \\
\Phi_2'' &= -s(s' - 3r') \Phi_2 - r(r' - 3s') \Phi_1; \\
\end{align*}
\]

for short cylindrical shells,

\[ \Phi(x) = C_1 \Phi_1 + C_2 \Phi_2 + C_3 \Phi_3 + C_4 \Phi_4; \quad (466) \]

where \( \Phi_1(x), \Phi_2(x), \Phi_3(x), \Phi_4(x) \) are the functions of V.Z. Vlasov, defined by Eq. (188), satisfied by Eq. (189).

In short cylindrical shells, there is a general bent state of the shell, and, consequently, the "end effect" concept itself loses meaning.

It follows from Eq. (462)-(464) that, in laminated orthotropic shells, the end effect damping zone is determined not only by the geometric dimensions, but depends essentially on the nature of the anisotropy and rigidity with respect to interlayer shear.

The end effect zone decreases with increase in parameters \( E_2/E_1, \) \( E_2/G_{13}, \) i.e., the presence of interlayer shearing contributes to damping the stressed and deformed state of the shell. The concepts of "short" and "infinitely long" is not purely geometrical for laminated orthotropic shells, and it is determined by the value of parameter \( s. \)

It is assumed that, in actual laminated shells, the relationship \( k^2 > p^2 \) or the relationship

\[ \frac{G_{13}R}{E_1} > \frac{1}{2 \sqrt{3(1-\nu_1\nu_2)}}, \quad (467) \]

where \( G_{13} \) is the interlayer shear modulus along the shell axis, is satisfied.
Long cylindrical shells. We select the distance from section \( x = x_0 \) as the axial coordinate in which the end effect is studied (Fig. 62).

With this coordinate system, in order to maintain the validity of the expressions obtained above for the elastic forces and movements, the change of sign in differentiation in region I, located to the left of the section, should be taken into account. In the expressions presented below, the upper sign of the double sign concerns region II, where \( x > x_0 \) and the lower sign, region I, where \( x < x_0 \):

![Fig. 62. Hypothetical regions of propagation of end effects.](image)

The two random integration constants are determined from the boundary conditions in section \( x = 0 \).

Short cylindrical shells. In short cylindrical shells, for convenience in practical calculations, it is more convenient to express the solution of Eq. (466) in initial parameters \( \phi_0, \omega_0, G_0, Q_0 \), which designate the angle of rotation of the normal, deflection, bending moment and cutting force on the end of the shell (\( x = 0 \)). Since two constants will be assigned on one end, in specific calculations, the problem is reduced to determination of the two remaining constants from the boundary conditions on the other end of the shell.

According to Eq. (461), for short shells,

\[
\varphi = -\frac{\Phi}{R_1}; \quad G = \Phi'; \quad Q = \Phi''; \quad w = -\frac{R^2 \varphi^2}{R_1 \delta}.
\] (469)

The following expressions for the basic components of the forces and deformations can be obtained in the initial parameters

\[
\begin{align*}
\varphi &= A_{\phi\phi}(x)\varphi_0 + A_{\phi\omega}(x)\omega + A_{\phi Q}(x)Q_0 + A_{\phi G}(x)G_0; \\
\omega &= A_{\omega\phi}(x)\varphi_0 + A_{\omega\omega}(x)\omega + A_{\omega Q}(x)Q_0 + A_{\omega G}(x)G_0; \\
Q &= A_{Q\phi}(x)\varphi_0 + A_{Q\omega}(x)\omega + A_{Q Q}(x)Q_0 + A_{Q G}(x)G_0; \\
G &= A_{G\phi}(x)\varphi_0 + A_{G\omega}(x)\omega + A_{G Q}(x)Q_0 + A_{G G}(x)G_0;
\end{align*}
\] (470)
Influence coefficients $A_{ij}$ are determined through hyperbolo-trigonometric Vlasov functions (188), by the expressions

$$
A_{ww}(x) = \Phi_1 \frac{\Delta}{2\pi} \frac{1}{r^2} (r \Phi_4 - r \Phi_2);
$$

$$
A_{wq}(x) = -\frac{\Delta}{2\pi} \Phi_2;
$$

$$
A_{wG}(x) = \frac{1}{2\pi} \left[ r(r^2 - 3s^2) \Phi_4 + 3 (s^2 - 3r^2) \Phi_2 \right];
$$

$$
A_{wq}(x) = -\frac{1}{2\pi} (s \Phi_4 + r \Phi_2);
$$

$$
A_{wG}(x) = \Phi_1 + \frac{s^2 - r^2}{2\pi} \Phi_3;
$$

$$
A_{wq}(x) = \frac{D_1 (s^2 + r^2)}{2\pi} \Phi_3;
$$

$$
A_{wG}(x) = -\frac{D_1 (s^2 + r^2)}{2\pi} (s \Phi_4 - r \Phi_2);
$$

$$
A_{Gw}(x) = -\frac{\Delta}{2\pi} \Phi_4;
$$

$$
A_{Gq}(x) = \frac{1}{2\pi} (s \Phi_4 + r \Phi_2);
$$

$$
A_{GG}(x) = \Phi_1 - \frac{s^2 - r^2}{2\pi} \Phi_3.
$$

In finding the initial parameters from the boundary conditions to solution (470), the corresponding solution obtained from membrane theory must be added.

44. Stressed and Deformed States of Cylindrical Shell Generated by Annular Concentrated Forces

We consider a long cylindrical shell compressed by a concentrated annular force of intensity $q$ (Fig. 63).

In view of the symmetry, only one half of the shell can be considered, for example, that located to the left of the section where the pressure is applied.

Since there are no surface loads or axial forces on the end of
the shell \((X=Z=T_0=0)\), the initial forces, moments and movements equal zero. Consequently,

\[
Q_i = C_1 [(s^2 - r^2) \Phi_1 + 2r s \Phi_2] + C_s [(s^2 - r^2) \Phi_3 - 2r s \Phi_1]; \\
G_i = - [C_1 (s \Phi_1 + r \Phi_2) - C_s (r \Phi_1 - s \Phi_0)]; \\
u = \frac{q \nu s R}{E_s \delta} [Q_i (x) - Q_i (x_0)] + u_0; \\
\psi = - \frac{1}{E_s} (C_2 \Phi_1 + C_s \Phi_3); \\
w = \frac{R_s^2}{E_s \delta} [C_1 (s (s^2 - 3r^2) \Phi_1 - r (r^2 - 3s^2) \Phi_2) + C_s (s (s^2 - 3r^2) \Phi_1 + s (s^2 - 3r^2) \Phi_2)].
\]

Integration constants \(C_1, C_2\) are determined from the boundary conditions

\[
\begin{align*}
\text{at } x = 0 & \quad \psi = 0; \quad Q_i = - \frac{q}{2}; \\
C_1 = 0; \quad C_2 = \frac{q}{4r s}
\end{align*}
\]

Consequently, the stressed and deformed states of a laminated cylindrical shell compressed by a concentrated annular force are determined by the following expressions

\[
Q_i = - \frac{q}{4r s} [2r s \Phi_1 - (s^2 - r^2) \Phi_3]; \\
G_i = \frac{q}{4r s} (r \Phi_1 - s \Phi_3); \\
u = \frac{q \nu s R}{2E_s \delta} \left[(1 - \Phi_1) + \frac{s^2 + r^2}{2r s} \Phi_2\right]; \\
\psi = - \frac{q}{4r s D_1} \Phi_3; \\
w = \frac{q R^2}{4r s E_s \delta} [r (r^2 - 3s^2) \Phi_1 + s (s^2 - 3r^2) \Phi_3].
\]

The maximum deflection in the section where the annular force is applied

\[
w_{\text{max}} = - \frac{(3s^2 - r^2) q R^2}{4s^2 E_s \delta}.
\]

The maximum flexural stress and the greatest shearing stress between the layers are generated in section \(x=0\):

\[
\sigma_{\text{max}} = \frac{3q}{2r \delta}, \quad \tau_{\text{max}} = \frac{3q}{4b}.
\]
We investigate the stressed and deformed states of a laminated cylindrical shell subjected to uniform internal pressure, which is strengthened by equally spaced reinforcing rings (Fig. 64).

We will assume that a segment of the shell of length \( l \) located between two rings can be considered infinitely long in the sense indicated above, i.e., the stressed and deformed shell states described by Eq. (474) are completely damped in distance \( l/2 \). In actual shells, these conditions usually are always satisfied, since the reinforcing rings must be larger to preserve the circular shape of the shell than to increase the strength.

Let the rigidity of the reinforcing ring under tension equal \( R_F \). The end forces and surface load components

\[
T_o = \frac{R}{2}; \quad Z = p; \quad X = 0.
\]

Consequently, the solution of the inhomogeneous equation has the form

\[
G_o = Q_o = q_o = 0; \\
(478)
\]

The intensity of the force of interaction between the reinforcing ring and the shell \( q \) is determined from the condition of compatibility of deformation of the ring and shell

\[
\frac{j h^2}{l F} = \frac{(2 - \nu_2) \rho R^2}{2 E_s b} - \frac{(3 \alpha^2 - \beta^2) \sigma R^2}{4 E_s b}.
\]

Consequently, the intensity of the force of interaction

\[
q = (2 - \nu_2) \rho \cdot \frac{1}{2} \frac{E_s b}{E_F} \frac{1}{\left[ \frac{E_s b}{E_F} + \frac{3 \alpha^2 - \beta^2}{4 s} \right]}. \quad (480)
\]

The normal stress generated in the reinforcing ring

\[
\sigma_{max} = \frac{(2 - \nu_3) \rho R}{2 s} \cdot \frac{1}{\left[ \frac{E_s b}{E_F} + \frac{(3 \alpha^2 - \beta^2) F}{4 s \delta} \right]}. \quad (481)
\]

The stressed and deformed states of the shell are the sum of the momentless state generated by the internal pressure and the stressed and deformed states generated by the resistance of the ring to deformation.
of the shell. The latter is determined by Eq. (474), where \( q \) is found by Eq. (480).

Thus, the greatest normal stresses and maximum shearing stresses between the layers develop in the reinforced sections of the shell, and they are determined by the expressions

\[
\sigma_{1\ max} = \frac{pR}{2\delta} + \frac{3(2-\nu_2)R}{4\delta} \left[ \frac{1}{EF + \frac{3A^2 - R^2}{4\delta}} \right]; \\
\sigma_{2\ max} = \frac{pR}{2\delta} + \frac{3(2-\nu_2)\nu_2 R}{4\delta} \left[ \frac{1}{EF + \frac{3A^2 - R^2}{4\delta}} \right]; \\
\tau_{\ max} = \frac{3(2-\nu_2)R}{8\delta} \left[ \frac{1}{EF + \frac{3A^2 - R^2}{4\delta}} \right].
\]  

(482)

45. Calculation of Laminated Cylindrical Shell with Variable Wall Thickness Changing by Steps

We determine the stresses generated in a closed laminated cylindrical container, which is supported by the base on a circular support and is filled with liquid of specific weight \( \gamma \) and is under low pressurization \( p_0 \), which is required to give the container the necessary rigidity (Fig. 65).

The lateral hydrostatic pressure of the liquid on the container wall is proportional to the height of the liquid column to the section under consideration and the specific weight of the liquid. If the coordinate origin is selected on the open surface of the liquid,

\[
X = 0; \quad T_x = \frac{p_0 R}{2}; \quad Z(x) = (\gamma x + p_0) .
\]  

(483)

In accordance with Eq. (461), the momentless state of the shell is determined by the expressions

\[
G_o = 0; \quad T^9_x = (p_0 + \gamma x)R; \\
q_o = -\frac{\gamma R^3}{E_0 \delta}; \quad \nu_o = \frac{(2-\nu_2)p_0 R^3}{2E_0 \delta (x)} + \frac{\gamma x R^3}{E_0 \delta (x)} .
\]  

(484)

Fig. 65. Variable thickness chemical container. Consequently, it is inadvisable to make the walls of such containers of constant thickness. Let the wall thickness be increased toward the base, changing by steps. We investigate the end effects which arise in the sections of the container where there is wall thickness discontinuity \( \delta (x) \).
Let the thickness of adjacent sections of the shell be $\delta_1$, $\delta_2$ ($\delta_2 > \delta_1$). We consider two sections of the tank separately, on the assumption that they are sufficiently long: $l > \pi/s$ (Fig. 66).

For section I

$Q_1 = C_1 \left( (s_1^2 - r_1^2) \Phi_1 + 2r_1 \Phi_1 \right) + C_2 \left( (s_1^2 - r_1^2) \Phi_2 - 2r_1 \Phi_2 \right)$;

$G_1 = C_1 (s_1 \Phi_1 + r_1 \Phi_2) - C_2 (r_1 \Phi_1 - s_1 \Phi_2)$;

$\omega_i = \omega^0_i - \frac{R_1^2}{E_i \delta_i} \frac{C_1 \left( s_1 \Phi_1^2 - 3r_1 \Phi_1 \right) \Phi_4 - r_1 \left( \Phi_1^2 - 3s_1 \Phi_1 \right) \Phi_5 + s_1 \left( s_1 \Phi_1^2 - 3r_1 \Phi_1 \right) \Phi_6}{r_1^2 \left( s_1^2 + r_1^2 \right)}$;

$\varphi = -\frac{12 \left( 1 - \nu_1 \nu_2 \right)}{E_1 \delta_1^3 \left( s_1^2 + r_1^2 \right)} (C_1 \Phi_1 + C_2 \Phi_2)$.

By satisfying the boundary conditions, the following can be obtained

$$C_1 = \frac{2\pi \Gamma \omega_0 - Q_0}{s_1^2 + r_1^2}; \quad C_2 = \frac{s_1^2 - r_1^2}{r_1^2 \left( s_1^2 + r_1^2 \right)}.$$

Consequently, in the initial parameters, Eq. (485) take the form

$$Q_1 = G_0 \frac{s_1^2 + r_1^2}{r_1^2} \Phi_4 + Q_0 \left( \Phi_1 - \frac{s_1^2}{r_1^2} \Phi_2 \right);$$

$$G_1 = G_0 \left( \Phi_1 + \frac{s_1^2}{r_1^2} \Phi_2 \right) - Q_0 \Phi_3;$$

$$\omega_i = \omega^0_i + \frac{R_1^2}{E_i \delta_i} \left[ G_0 \left( r_1^2 + s_1^2 \right) \left( \Phi_1 - \frac{s_1^2}{r_1^2} \Phi_2 \right) - Q_0 \left( 2s_1 \Phi_1 - \frac{s_1^2}{r_1^2} \Phi_5 \right) \right];$$

$$\varphi = -\frac{12 \left( 1 - \nu_1 \nu_2 \right)}{E_1 \delta_1^3 \left( s_1^2 + r_1^2 \right)} \left[ G_0 \left( 2s_1 \Phi_1 + \frac{s_1^2}{r_1^2} \Phi_2 \right) - Q_0 \left( \Phi_1 + \frac{s_1^2}{r_1^2} \Phi_3 \right) \right].$$

The movement and angle of rotation of the end of the shell $x=0$, respectively,

$$u_r = \frac{\left( 2 - \nu_2 \right) r_2 R_2 \nu_2}{2 E_2 \delta_2}, \quad \frac{\gamma R_2}{E_i \delta_i} + \frac{R_1^2}{E_i \delta_i} \left[ G_0 \left( r_2^2 + s_1^2 \right) - 2s_1 Q_0 \right];$$

$$\varphi_1 = -\frac{12 \left( 1 - \nu_1 \nu_2 \right)}{E_1 \delta_1^3 \left( s_1^2 - r_1^2 \right)} \left( 2s_1 G_0 - Q_0 \right) - \frac{\gamma R_1}{E_1 \delta_1}. \quad \{488\}$$

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For section II

\[ Q_1 = C_1 \left[ (s_0^2 - r_s^2) \Phi_1 + 2s_s \Phi_s \right] + C_1' \left[ (s_0^2 - r_s^2) \Phi_1 \right] - 2s_s \Phi_1; \]
\[ G_1 = -C_1 (s_s \Phi_1 + r_s \Phi_s) + C_1' (r_s \Phi_1 - s_s \Phi_s); \]
\[ \varphi = \frac{12 (1 - \nu \sigma)}{E \delta_s} \left( C_1' \Phi_1 + C_1' \Phi_s \right); \]
\[ w = w_{11} + \frac{R_s}{E_s \delta_s} \left[ \varsigma_1 \left( s_s (s_0^2 - 3r_s^2) \Phi_1 - r_s (r_s^2 - 3s_s^2) \Phi_1 + s_s (s_0^2 - 3r_s^2) \Phi_1 + \right) + s_s (s_0^2 - 3r_s^2) \Phi_1 \right]. \]

We again find constants \( C_1', C_2' \) from the boundary conditions

\[
\begin{align*}
C_1'(s_0^2 - r_s^2) - C_2' r_s^2 &= 0; \\
-C_1's_s + C_1'r_s &= G_0.
\end{align*}
\]

from which the following values of the constants can be obtained

\[ C_1' = -\frac{2s_s G_0 + Q_0}{s_0^2 + r_s^2}; \quad C_1' = -\frac{\left( s_0^2 - r_s^2 \right) G_0 + s_s Q_0}{r_s \left( s_0^2 + r_s^2 \right)}. \]

In initial parameters \( G_0, Q_0 \), Eq. (489) are presented in the form

\[ Q_1 = -G_0 \frac{s_0^2 - r_s^2}{r_s} \Phi_s + Q_0 \left( \Phi_1 + \frac{s_s}{r_s} \Phi_s \right); \]
\[ G_1 = G_0 \left( \Phi_1 + \frac{s_s}{r_s} \Phi_s \right) + Q_0 \frac{s_s}{r_s}; \]
\[ w = w_{11} + \frac{R_s}{E_s \delta_s} \left[ G_0 \left( r_s^2 + s_s^2 \right) \left( \Phi_1 - \frac{s_s}{r_s} \Phi_s \right) + \right] + Q_0 \left( 2s_s \Phi_1 + \frac{s_s^2 - r_s^2}{r_s} \Phi_s \right); \]
\[ \varphi = \frac{12 (1 - \nu \sigma)}{E_s \delta_s \left( s_0^2 + r_s^2 \right)} \left[ G_0 \left( 2s_s \Phi_1 + \frac{s_s^2 - r_s^2}{r_s} \Phi_s \right) + \right] + Q_0 \left( \Phi_1 + \frac{s_s}{r_s} \Phi_s \right). \]

The deflection and angle of rotation of the end of the shell \( x = 0 \)

\[ w_{11} = \frac{(2 - \nu \sigma) P_s R_s}{2E_s \delta_s} + \frac{\nu \sigma R_s}{E_s \delta_s} + \frac{R_s}{E_s \delta_s} \left[ G_0 (r_s^2 + s_s^2) + 2s_s Q_0 \right]; \]
\[ \varphi_{11} = \frac{12 (1 - \nu \sigma)}{E_s \delta_s \left( s_0^2 + r_s^2 \right)} \left( 2s_s G_0 + Q_0 \right) - \frac{\nu R_s}{E_s \delta_s}. \]

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The conditions of compatibility of the deformations in the butt sections of the shell can be reduced to the form

\[
G_o \left[ \delta_1 (s_1^2 + r_1^2) - \delta_2 (s_2^2 + r_2^2) \right] + 2Q_o \left( s_2 \delta_1 + s_1 \delta_2 \right) = -\left( \frac{2-v_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1);
\]
\[
2G_o \left[ s_2 (s_1^2 + r_1^2) \delta_1^* + s_1 (s_2^2 + r_2^2) \delta_2^* \right] + Q_o \left[ (s_1^2 + r_1^2) \delta_1^* - (s_2^2 + r_2^2) \delta_2^* \right] = \frac{\nu R}{2 (1-v_2 \nu_2) s^2} (s_1^2 + r_1^2) (s_2^2 + r_2^2);
\]

From this, for determination of end forces \(G_o, Q_o\), we obtain the following expressions

\[
G_o = \left( \frac{2-v_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1) f_1 + \frac{\nu R}{6 \lambda} (\delta_2 - \delta_1) f_2; \quad Q_o = -2 \left( \frac{2-v_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1) f_3 - \frac{\nu R}{6 \lambda^2} (\delta_2 - \delta_1) f_4.
\]

\[
f_1 = \frac{(s_1^2 + r_1^2) \delta_1^* - (s_2^2 + r_2^2) \delta_2^*}{\Delta}; \quad f_2 = \frac{(s_1^2 + r_1^2) (s_2^2 + r_2^2) (s_1^2 \delta_1 + s_2^2 \delta_2)}{\Delta}; \quad f_3 = \frac{s_1 (s_1^2 + r_1^2) \delta_1^* + s_1 (s_2^2 + r_2^2) \delta_2^*}{\Delta}; \quad f_4 = \frac{(s_1^2 + r_1^2) (s_2^2 + r_2^2) \delta_1 - (s_1^2 + r_1^2) \delta_2}{\Delta}
\]

\[
\Delta = [ (s_1^2 + r_1^2) \delta_1 - (s_2^2 + r_2^2) \delta_2 ] [ (s_1^2 + r_1^2) \delta_1^* - (s_2^2 + r_2^2) \delta_2^* ] - 4 (s_1 \delta_1 + s_2 \delta_2) [ s_1 (s_1^2 + r_1^2) \delta_1^* + s_2 (s_2^2 + r_2^2) \delta_2^* ] + s_1 (s_1^2 + r_1^2) \delta_2^*.
\]

After determination of the end forces, the stressed state in sections in which the thickness changes by steps is determined by the expressions obtained above.

We note that the manufacture of cylindrical shells made of laminated plastics with variable wall thickness presents no technical difficulties.

46. Calculation of Laminated Orthotropic Cylindrical Shell Subjected to Axial Eccentrically Applied Forces

In actual shells, axial forces may be applied to the side surface only with some eccentricity with respect to the mean surface (Fig. 67).
We will assume that the contact of the shell with the reinforcing ring is linear. The axial eccentrically applied load can be reduced to axial forces applied to the mean surface of the shell and to reactive bending moments uniformly distributed around the circumference of the shell cross section. The strength of the reactive bending moments is determined from the conditions of compatibility of the deformations of the ring and shell.

By separating the reinforcing ring from the shell and replacing the force of interaction between them by reactive moments of unknown intensity \( m \), we have the calculation scheme shown in Fig. 68. \( M \) is the external torque generated by the eccentricity of the axial load.

The calculation formulas for determination of the deformed and stressed states of an orthotropic laminated cylindrical shell, subjected to uniformly distributed bending moments \( m \), are obtained by differentiation of the solution obtained for the case of annular pressure with simultaneous substitution of \( m \) for \( q \) [18]. However, in taking account of interlayer shearing, this method [15] can lead to errors, since the effect of shearing in the limiting transition is not taken into account exactly. Therefore, we use general solution (468).

For the upper and lower sections of the shell, respectively

\[
Q_1 = C_1 \left[ (s^2 - r^2) \Phi_1 + 2rs \Phi_2 \right] + C_3 \left[ (s^2 - r^2) \Phi_3 - 2rs \Phi_4 \right]; \\
Q_2 = C_1 (s \Phi_1 + r \Phi_3) - C_3 (r \Phi_3 - s \Phi_2); \\
w = \frac{M}{E_i \delta} \left[ C_1 \left[ (s^2 - 3r^2) \Phi_1 + r (r^2 - 3s^2) \Phi_4 \right] + C_3 (r^2 - 3s^2) \Phi_1 + s (s^2 - 3r^2) \Phi_4 \right] + \\
+ C_1 \left( \frac{r (r^2 - 3s^2) \Phi_4 + s (s^2 - 3r^2) \Phi_3}{} \right) \right]; \\
q = - \frac{1}{E_i} \left( C_1 \Phi_1 + C_3 \Phi_3 \right); \\
u = u_0 + \frac{\nu \pi R}{E_i \delta} \left[ Q_1(x) - Q_1(x) \right].
\]

In section \( x=0 \), the bending moment has a discontinuity of \( m \). Consequently, integration constants \( C_1, C_2 \) can be obtained from the following boundary conditions.
By substituting Eq. (497) in Eq. (498), we obtain
\[ G_1(s^3 - 3r^2) - C_s r(3s^2 - r^2) = 0; \]
\[ C_s - C_s r = -\frac{m}{2}. \]

The following can be obtained from Eq. (499)
\[ C_1 = \frac{m(3s^2 - r^2)}{4s(r^2 + r^2)}; \quad C_2 = \frac{m(s^2 - 3r^2)}{4s(r^2 + r^2)}. \]

In this manner, there are the following calculation formulas for a shell subjected to bending moments of intensity \( m \) uniformly distributed around the perimeter of the section
\[ Q_1 = -\frac{m(s^2 + r^2)}{4r^2} (r \Phi_1 + s \Phi_s); \]
\[ G_1 = \mp \frac{m}{4r^2} (2rs \Phi_1 + (s^2 - r^2) \Phi_s); \]
\[ \Phi = \frac{m}{4rs(r^2 + r^2)} \Phi_s; \]
\[ w = \frac{mR^2}{4sE_s} \frac{(s^2 + r^2) + r^2}{r^2} \Phi_s; \]
\[ u = \frac{m}{E_s} \frac{[Q_1(x) - Q_1(x_0)]}{r^2}. \]

In Eq. (501), the upper sign refers to upper region I of the shell and the lower sign, to lower region II of the shell (see Fig. 68).

For shells made of laminated plastics, reinforcing rings with a continuous rectangular cross section are most acceptable. The resistance of such a ring to axisymmetrical torsion is determined by the flexural rigidity of the ring \( E_{y}r \), i.e., the rigidity of the ring when it is bent out of the plane of curvature.

From the compatibility condition of deformations of the shell and reinforcing ring \( \Phi_t = \Phi_s \), it can be found that
\[ m = \frac{M}{1 + \zeta}, \]
where the geometric rigidity parameter is designated \( \zeta \)
\[ \zeta = \frac{E_{y}r(3s^2 - r^2)}{4sE_s R^2 (r^2 + s^2)}. \]
In accordance with Eq. (502), an external uniformly distributed bending moment which is transmitted to the shell through an elastic reinforcing ring is taken up by both the shell and the reinforcing ring. The fractions taken up by the shell and reinforcing ring are determined by parameter $t$. An increase of this parameter entails a decrease in load which is transmitted to the shell. Thus, the stress and deformation of the shell can be regulated by increasing the flexural rigidity of the ring. Since this rigidity is characterized by moment of inertia $J_y$, for transmission of eccentrically applied axial forces to the laminated cylindrical shell, it is advisable to use wide reinforcing rings.

It should be noted that, with large shell dimensions, the effect of the reinforcing ring decreases sharply in proportion to $R^{3/2}$, i.e., in thin cylindrical shells of large diameter, the external bending moment is almost entirely absorbed by the shell.

The stresses and deformations due to uniform compression should be added in the lower part of the shell to the stresses and deformations which are determined by Eq. (501).
CHAPTER 10. AXISYMMETRIC DEFORMATION OF ORTHOTROPIC SHELLS OF ROTATION

47. Initial Relationships and Basic Differential Equations

We consider laminated shells of rotation loaded symmetrically relative to the axis, including the reactions of the support connections. We select circumferential angle $\beta$ and angle $\alpha$, formed by the normal to the mean surface with the axis of rotation as the curvilinear Gaussian coordinates on the mean surface of the shell (Fig. 69).

We will assume that the principal directions of anisotropy of an orthotropic laminated plastic are coincident with the coordinate directions, i.e., with the lines of curvatures of the mean surface of the shell.

Under axisymmetric loading, a shell of rotation will be deformed symmetrically relative to the axis. Consequently, the forces, bending moments and movements of the mean surface of the shell will be functions of angular coordinate $\alpha$ alone.

From the elastic forces in the shell, only normal forces $T_1, T_2$, transverse cutting force $Q_1$ and bending moments $G_1, G_2$ will result. Movements of points of the mean surface are completely determined by two components $u, w$, i.e., by movement along the meridian and deflection of the shell.

In accordance with basic Eq. (24)-(26) and Eq. (10)-(12), in axisymmetric deformation of a shell of rotation,

\[
\begin{align*}
T_1 &= B_1 \left( \frac{u' + w}{R_1} + v_1 \frac{u' \csc \alpha + w}{R_1} \right); \\
T_2 &= B_1 \left( \frac{u' \csc \alpha + w}{R_1} + v_1 \frac{u' + w}{R_1} \right); \\
Q_1 &= -K_1 \left( \varphi + \frac{u' - u}{R_1} \right); \\
G_1 &= -D_1 \left( \frac{q'}{R_1} + v_1 \frac{q' \csc \alpha}{R_1} \right); \\
G_2 &= -D_1 \left( \frac{u' \csc \alpha + w}{R_1} + v_1 \frac{q'}{R_1} \right),
\end{align*}
\]

where $R_1, R_2$ are the principal radii of curvature of the mean surface, and the primes designated differentiation over angular coordinate $\alpha$.

For symmetrically loaded shells of rotation, equilibrium Eq. (17) take the form
As the solution of nonhomogeneous system of Eq. (506), we will use the solution which corresponds to the membrane theory of shells. Further, we obtain the solution of homogeneous system of Eq. (506) on the assumption that \( X=0 \).

Following Meissner, we introduce the stress function
\[
V = R_2 Q_1.
\]

Then, from the first two equilibrium Eq. (506), we obtain
\[
T_1 = - \frac{V}{R_1} \cot \alpha; \quad T_2 = - \frac{V'}{R_1}.
\]

For axisymmetric loading of shells of rotation, relationships (8) take the form
\[
\begin{align*}
\dot{u} + w &= \frac{1}{E_1} \left( \frac{R_1}{E_1} V - \frac{R_1}{E_2} \frac{1}{R_1} V \cot \alpha \right); \\
\dot{u} \cot \alpha + w &= \frac{1}{E_2} \left( \frac{R_1}{E_1} V - \frac{R_1}{E_2} V' \right) \\
\end{align*}
\]

By substituting them in the obvious identity which expresses the condition of deformation compatibility
\[
\dot{w} - u = (\dot{u} \cot \alpha + w - \dot{u}' - w) \cot \alpha + (\dot{u} \cot \alpha + w)' = 0,
\]
we obtain one differential equation
\[
V'' \frac{R_1}{R_1} + V' \left[ \left( \frac{R_1}{R_1} \right)' + \frac{R_1}{R_1} \cot \alpha \right] - V \left[ \lambda \frac{R_1}{R_1} \cot^2 \alpha + \frac{E_2}{5G_1 \frac{R_1}{R_1}} - \nu_2 \right] = E_2 \frac{R_1}{R_1} \phi.
\]

By substituting Eq. (505) in the remaining unused equilibrium equation, a second equation can be obtained for stress and deformation functions \( V \) and \( \phi \)
\[
\phi'' \frac{R_2}{R_1} + \phi' \left[ \left( \frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \cot \alpha \right] - \phi \left[ \nu_2 + \frac{R_2}{R_1} \lambda \cot^2 \alpha \right] = - \frac{R_1}{R_1} \dot{V}.
\]
Thus, the axisymmetric deformation of shells of rotation is described by system of Eq. (511), (512), which it is convenient to present in the following symmetrical form

\[ L(V) + \left( \frac{\nu_v}{R_1} - \frac{k_1}{R_1} \right) V = E_1 \delta \varphi, \]  
\[ L(\varphi) - \frac{\nu_v}{R_1} \varphi = - \frac{V}{\delta_t}, \]  

where \( L() \) designates the differential operator in second order partial derivatives

\[ L() = R_1 \frac{d^2}{d \xi^2} + \left[ \left( \frac{R_1}{R_1} \right)^2 + \frac{R_1}{R_1} \cotg \alpha \right] \frac{d}{d \xi} - \frac{\lambda^2}{R_1} \cotg^2 \alpha (\xi). \]  

where \( k_1 = \frac{B_s (1 - \nu_v \nu_s)}{K_1} = \frac{\delta E_1}{5G_{12}} \),

here \( \xi = R_1 \alpha \) is the length of the arc of the meridian of the mean surface of the shell.

Consideration of shells of rotation of variable thickness \( \delta = \delta (\alpha) \) is of practical interest, since the manufacture of such shells from laminated plastics involves no fundamental difficulties compared with shells of constant thickness.

Without repeating the calculation, for axisymmetrically loaded shells of rotation of variable thickness the following system of differential equations can be obtained

\[ L(V) + \left( \frac{\nu_v}{R_1} - \frac{k_1}{R_1} \right) V - \frac{\delta}{\delta} (R_1 \dot{V} - \nu_v V \cotg \alpha) = E_1 \delta \varphi; \]
\[ L(\varphi) - \frac{\nu_v}{R_1} \varphi + 3 \frac{\delta}{\delta} (R_1 \dot{\varphi} + \nu_v \varphi \cotg \alpha) = - \frac{V}{\delta_t}, \]

where the points designate differentiation over the arc of the meridian \( \xi \).

Eq. (513), (514) and (516) are valid for laminated anisotropic shells of rotation of constant and variable thickness, which are made of laminated plastics of unchanging elastic constants along the meridian. However, if a convex shell of rotation is made by continuous winding of fiber fillers, as was pointed out above, the elastic constants of the material will change along the meridian, since the fiber packing density increases toward the pole of the shell.

The structure of the equations and asymptotic analysis of the solutions show that system of Eq. (513), (514) or (516) describe the simple end effects phenomenon, i.e., for not very "short" shells, the solutions contain terms which are rapidly damped with increase in the argument of \( \xi \) [1]. Consequently, the equations obtained will be valid
for shells of rotation, where there are factors which cause local flexural deformations.

If the indices of variability of the external loads, elastic constants and geometric dimensions of the shells are not very large, Eq. (513), (514) for constant thickness shells of rotation and Eq. (516) for variable thickness shells of rotation remain valid. However, further simplifications can be made in these equations.

48. Differential Equations of Technical Theory of Axisymmetrically Loaded Shells of Rotation

Exact solutions of Eq. (513), (514) present great mathematical difficulties, and they are obtained only with some simplifying hypotheses for individual cases. At the same time, for engineering applications in many cases, it is quite sufficient to be limited to the first approximation of asymptotic integration [7]. The error of the simplified equations which correspond to the first approximation is \( \lambda \frac{\delta}{2} \) compared with unity, i.e., it is completely sufficient for "thin" shells.

The technical theory equations can be obtained, if it is assumed that the index of variability of displacement functions \( \phi \) and stress functions \( \psi \) are considerably greater than the indices of variability of geometric dimensions \( R_1, R_2, \delta \), elastic constants \( c_{ij}, a_{ij} \) and the external load applied to the shell.

By multiplying Eq. (514) by random complex constant \( z=x+iy \) and summing with Eq. (513), we obtain

\[
z \left[ L(\psi) - \frac{\nu_2}{R_1} \psi - \frac{\nu_2 \delta}{z} \delta \right] + \left[ L(\nu) - \left( \frac{k_i}{R_i} - \frac{\nu_2}{R_1} \right) \nu + \frac{\nu}{D_i} z \right] = 0.
\]

Following A.I. Lur'ye [18], we set

\[
\begin{aligned}
\frac{\nu_2}{R_1} - \frac{k_i}{R_i} + \frac{1}{D_i} &= a + bi; \\
\frac{\nu_2}{R_1} + \frac{E \delta}{z} &= -(a + bi).
\end{aligned}
\]

By substituting \( z=x+iy \) in Eq. (518) and separating the real and imaginary parts, we obtain

\[
\begin{aligned}
\frac{\nu_2}{R_1} - \frac{k_i}{R_i} + \frac{x}{D_i} &= a; \\
y &= D_i b; \\
\frac{\nu_2}{R_1} + \frac{E \delta x}{x^2 + y^2} &= -a; \\
\frac{\nu E \delta}{x^2 + y^2} &= b.
\end{aligned}
\]

By disregarding the small terms, the following values of the constants introduced can be obtained:
\[
dx \approx \frac{k_1 D_1}{2 R_s} \approx \omega^4 D_1 R_s; \\
y = D_1 R_s \sqrt{\frac{12 k_1^2 (1-v_1 v_4)}{\rho s^4} - \frac{k_1^2}{\rho s^4}} \approx -D_1 D_4 V k^4 - \omega^4; \\
a = -\frac{k_1}{2 R_s} = -\omega^4 R_s; \\
b = R_s \sqrt{\frac{12 k_1^2 (1-v_1 v_4)}{\rho s^4} - \frac{k_1^2}{\rho s^4}} = -R_s V k^4 - \omega^4.
\]

where

\[
\omega^4 = \frac{k_1}{2 R_s} = \frac{3 v (1-v_1 v_4)}{2 K_s R_s^2}; \\
k_s = \frac{\lambda \sqrt{2 (1-v_1 v_4)}}{R_s \rho s^4}.
\]

Consequently, Eq. (517) takes the form

\[
L(U) + (a+bi)U = 0,
\]

where \( U \) is the complex function of stresses and deformations determined from the expression

\[
U + V + (x+iy)\phi.
\]

Because of the assumptions made as to the nature of variability of the quantities, operator \( L() \), which can be determined by Eq. (515), can be simplified

\[
L() = R_s \frac{d^2}{dx^2} + \left( \left( \frac{R_s}{R_1} \right)' + \frac{R_s}{R_1} \cot \alpha \right) \frac{d}{dc} - \frac{\lambda s}{R_s} \cot \alpha.
\]

Far from the pole of the shell (\( \alpha = 0 \)), the last terms in this operator can be disregarded and, consequently, the deformed and stressed states of the axisymmetrically loaded shell of rotation in zones quite far from the pole are described by the following second order differential equation

\[
\frac{d^2 U}{dx^2} - \left[ \omega^4 - i \right] k^4 - \omega^4] U = 0.
\]

The elastic forces, bending moments and radial movement of the shell far from the pole are determined by the formulas

\[
Q_1 = \frac{V}{R_1}; \quad T_1 = -\frac{V \cot \alpha}{R_1}; \quad T_s = -\frac{dV}{dc}; \\
G_1 = -D_1 \frac{dq}{dc}; \quad G_s = v_5 G_1; \\
\xi = \Delta r = -\frac{R_s}{h_2} \frac{dV}{dc} \sin \alpha.
\]
Correspondingly, the stressed and deformed states of the shell of rotation near the pole \((\alpha=0)\) are described by the following differential equation

\[
\frac{dU}{d\xi} + \frac{1}{\zeta} \frac{dU}{d\zeta} - \left[ \omega^2 + \frac{c^2}{\zeta^2} - i V k e^{-\omega^2} \right] U = 0,
\]

where

\[
\omega^2 = \frac{\lambda^2 R_1^2}{R_0^2}.
\]

The elastic forces, bending moments and radial movements of the shell in the area of the pole are determined through the stress and deformation function by the formulas

\[
Q_1 = \frac{V}{R_0}; \quad T_1 = -\frac{V R_1}{\zeta R_0}; \quad T_3 = -\frac{dV}{d\zeta};
\]

\[
G_1 = -D_1 \left( \frac{d\phi}{d\zeta} + \nu \frac{R_1}{\zeta} \right); \quad G_2 = -D_2 \left( \frac{\phi R_1}{\zeta} + \nu \frac{d\phi}{d\zeta} \right);
\]

\[
\xi = Dr = \frac{R \sin \alpha}{E \delta} \left( \frac{dV}{d\zeta} - \nu \frac{V R_1}{\zeta} \right).
\]

In accordance with Eq. (523), if a solution of differential Eq. (525) or (527) is found, i.e., if the following complex function of stresses and deformations is found

\[
U(\xi) = \Re U(\xi) + i \Im U(\xi),
\]

functions \(\phi\) and \(V\) are determined by the following expressions

\[
\phi = \frac{\Im U(\xi)}{\nu} = -\frac{\Im U(\xi)}{2\pi D_1 R_0};
\]

\[
V = \Re U(\xi) + \frac{1-s^2}{2s} \Im U(\xi),
\]

where

\[
s = \sqrt{\frac{k^2 + \omega^2}{2}}; \quad r = \sqrt{\frac{k^2 - \omega^2}{2}}.
\]

49. Calculation of Axisymmetrically Loaded Shells of Rotation in Their Coupling Zones

The examples of axisymmetrically loaded shells of rotation most often found in practice are the bottoms of cylindrical containers operating under internal pressure. In chemical containers, a bottom made up of smoothly joined spherical, conical and toroidal shells is used. At the joint sites in such shells, local bending stresses and deformations appear which are described by differential Eq. (525).
The solution of Eq. (525) which disappears at infinity can be written in the following form

\[ U = (c_1 + iC_1) e^{-i\tau} (\sin rs + i \cos rs). \]  
\( (533) \)

Consequently, the real and imaginary parts

\[ \begin{align*}
\text{Re} U(\xi) &= C'_1 \Phi_1 - C'_2 \Phi_2; \\
\text{Im} U(\xi) &= C'_1 \Phi_1 + C'_2 \Phi_2.
\end{align*} \]  
\( (534) \)

where \( \Phi_1(\xi), \Phi_2(\xi) \) are exponential trigonometric functions which disappear at infinity and are determined by Eq. (463). They are satisfied by Eq. (465), which are very convenient for practical strength calculations.

In accordance with Eq. (531), we obtain

\[ \begin{align*}
\varphi &= -\frac{1}{h_i} (C'_1 \Phi_1 + C'_2 \Phi_2); \\
\frac{V}{H_i} &= C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + C_4 [(s^2 - r^2) \Phi_3 - 2rs \Phi_4].
\end{align*} \]  
\( (535) \)

The elastic forces, bending moments and radial movements of an axisymmetrically loaded shell of rotation are determined by the following formulas

\[ \begin{align*}
Q_1 &= Q_0 + C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + C_4 [(s^2 - r^2) \Phi_3 - 2rs \Phi_4]; \\
G_1 &= G_0 \pm |C_1 (s \Phi_1 + r \Phi_2) - C_2 (r \Phi_1 - s \Phi_2)]; \\
T_1 &= |C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + C_4 [(s^2 - r^2) \Phi_3 - 2rs \Phi_4]| \cot \alpha + T_i^*; \\
T_2 &= \pm |C_1 [r (s^2 - s^2) \Phi_1 - r (3s^2 - r^2) \Phi_4] + C_4 [r (3s^2 - r^2) \Phi_3 + s (3s^2 - s^2) \Phi_4] R_i + T_i^*; \\
\varepsilon &= \pm \frac{H_i}{2 \pi \delta} |C_1 [s (s^2 - s^2) \Phi_1 - r (3s^2 - r^2) \Phi_4] + C_4 [r (3s^2 - r^2) \Phi_3 + s (3s^2 - s^2) \Phi_4]| \sin \alpha + \varepsilon_0.
\end{align*} \]  
\( (536) \)

where quantities obtained according to membrane theory are noted by the symbol "o." In Eq. (535), (536), angular coordinate \( \beta \) is reckoned from the section where the local stresses are investigated. The upper and lower signs concern the lower (I) and upper (II) regions of the shell, respectively (Fig. 70).

We consider a quite "long" shell of rotation, at one end of which \( \beta = 0 \) bending moments \( G_0 \) and thrusts \( P_0 \) are applied (Fig. 71).

For anisotropic shells of rotation in study of the deformed and stressed states under axisymmetrical loading, the concepts "long" and
"short," as was pointed out earlier, are not purely geometrical but, rather, they are determined by elastic constants of the shell \( k_1 \) or by parameter \( s_k \).

With given thrust \( P_0 \), cutting force \( Q_0 = P_0 \sin \alpha \) and, consequently, integration constants \( C_1, C_2 \) are determined through initial parameters \( G_0, Q_0 \) by the following expressions

\[
C_1 = \frac{2sG_e - Q_e}{\pi^2 + r^2}; \quad C_2 = \frac{(s^2-r^2)G_e - rQ_e}{r(\pi^2 + r^2)}. \tag{537}
\]

Thus, in expressions of the elastic forces, bending moments and deformations through the initial parameters, there are the following formulas

\[
Q_1 = G_0 \frac{s^2 + r^2}{r} \Phi_1 + Q_0 (\Phi_1 + \frac{s}{r} \Phi_2) + Q_0; \\
G_1 = G_0 \left( \Phi_1 + \frac{s}{r} \Phi_2 \right) - Q_0 \frac{\Phi_2}{r} + G_0; \\
\xi = \xi_0 + \frac{R_0}{E_0} \left[ G_0 \left( r^2 + s^2 \right) \left( \Phi_1 - \frac{s}{r} \Phi_2 \right) - Q_0 \left( 2s \Phi_1 - \frac{s^2 - r^2}{r} \Phi_2 \right) \right] \sin \alpha; \\
T_2 = R_0 \left[ G_0 \left( r^2 + s^2 \right) \left( \Phi_1 - \frac{s}{r} \Phi_2 \right) - Q_0 \left( 2s \Phi_1 - \frac{s^2 - r^2}{r} \Phi_2 \right) \right] + T_0; \\
T_1 = -\left( G_0 \frac{s^2 + r^2}{r} \Phi_1 + Q_0 \left( \Phi_1 + \frac{s}{r} \Phi_2 \right) \right) \cot \alpha + T_0; \\
\varphi = -\frac{1}{(r^2 + s^2)D_1} \left[ G_0 \left( 2s \Phi_1 + \frac{s^2 - r^2}{r} \Phi_2 \right) - Q_0 \left( \Phi_1 + \frac{s}{r} \Phi_2 \right) \right]. \tag{538}
\]

As an example, we consider the problem of calculation of joining a cylindrical tank to a spherical bottom. The stressed and deformed states in the butt zone of the section are determined: in the bottom, by Eq. (538); in the cylindrical part of the container, by Eq. (492), with \( \gamma = 0 \).

We designate all parameters of the cylindrical part of the container by exponent I and all quantities which relate to the bottom by exponent II.

From the deformation compatibility condition, the following values of initial parameters \( G_0, P_0 \) can be found, which determine the forces
of interaction of the cylindrical part of the container with the bottom:

\[
P_s = P \left[ \frac{1 - \nu_1}{\sin \alpha} - (2 - \nu_1) \right] \left( \frac{1}{\sin \alpha} - (2 - \nu_1) \right) e^{-\frac{2\alpha\cos \alpha}{\cos \alpha}} + \left( \frac{k_{II}^0 - k_1^0}{2} \right) \frac{\alpha \cos \alpha}{\frac{1}{\sin \alpha} - (2 - \nu_1)(d - \sin \alpha) - k_{II}^1(d - \sin \alpha)} \right]
\]

\[
P_s = P \left[ \frac{1 - \nu_1}{\sin \alpha} - (2 - \nu_1) \right] \left( \frac{1}{\sin \alpha} - (2 - \nu_1) \right) e^{-\frac{2\alpha\cos \alpha}{\cos \alpha}} + \left( \frac{k_{II}^0 - k_1^0}{2} \right) \frac{\alpha \cos \alpha}{\frac{1}{\sin \alpha} - (2 - \nu_1)(d - \sin \alpha) - k_{II}^1(d - \sin \alpha)} \right]
\]

where

\[
\frac{E_{II}^0}{E_1^0} = \epsilon; \quad \frac{D_{II}^0}{D_1^0} = d.
\]

In order to decrease the end effects which arise in the butt sections of the containers, it is advisable to make them as smooth and continuous as possible. Smoothness and continuity are particularly important in the butt section of a container where the cylindrical part is joined to the bottom. However, continuity of the curvature of the meridian cannot be preserved while ensuring smoothness of the contours in this section.

In the smoothest and most continuous coupling of the bottom to the cylindrical part of the container, except for the bending moment in the butt section, the initial thrusting forces disappear. With sharp breaks in coupling, they can lead to local collapse of the butt section of the container as a result of loss of stability. Examples of smooth nonthrust bottoms are ellipsoidal, box and hemispherical bottoms. If, for some reason, a smooth coupling cannot be achieved, the sections where there is a break in continuity should be reinforced with rings.

50. Flattened Laminated Spherical Shell Subjected to Concentrated Forces Applied at the Poles

We consider the problem of calculation of a thin laminated spherical shell loaded with concentrated forces at the poles (Fig. 72).

For isotropic spherical shells, such studies have been conducted in [7, 18]. It is evident that laminated spherical shells which are subjected to concentrated forces do not need to be anisotropic. More than that, if the point of application of the concentrated forces is not fixed, the most nearly optimum structure of the material is a full strength structure. Just such a case will be considered.

Parameter \(c^2\) in Eq. (527), which describe the stressed and deformed states of axisymmetrically loaded shells of rotation near the pole, equals unity. Consequently, the basic differential equation of the
The problem under consideration has the form
\[ \frac{d^2U}{dt^2} + \frac{1}{t} \frac{dU}{dt} - \left[ \omega^2 + \frac{1}{t^2} - i \sqrt{k^2 - \omega^2} \right] U = 0. \quad (541) \]

The elastic forces, bending moments and radial movements of a full strength spherical shell are determined through the stress and deformation functions by the following expressions:

\[ \begin{align*}
Q_t &= \frac{V}{k} ; \quad T_t = -\frac{V}{t} ; \quad T_r = -\frac{dV}{dt} ; \\
G_t &= -D \left( \frac{d\varphi}{dt} + \frac{V}{k} \right) ; \quad G_r = -D \left( \frac{\varphi}{t} + \frac{d\varphi}{dt} \right) ; \\
\Delta r &= -\frac{t}{k} \left( \frac{dV}{dt} - \frac{V}{t} \right).
\end{align*} \quad (524) \]

We make the following substitution of variables
\[ \zeta = \frac{t}{\sqrt{\omega^2 + i \sqrt{k^2 - \omega^2}}} \quad (543) \]

We then obtain
\[ \begin{align*}
\frac{dU}{dt} &= \frac{\sqrt{\omega^2 + i \sqrt{k^2 - \omega^2}}}{t} \frac{dU}{dt} ; \\
\frac{d^2U}{dt^2} &= -\left[ \omega^2 - i \sqrt{k^2 - \omega^2} \right] \frac{dU}{dt}.
\end{align*} \quad (544) \]

Differential Eq. (541) takes the canonical form
\[ \frac{d^2U}{dt^2} + \frac{1}{t} \frac{dU}{dt} + \left( 1 - \frac{1}{t^2} \right) U = 0. \quad (545) \]

The solution of Eq. (545) is written in first order Bessel functions of the first and second kinds
\[ U = C_1 J_1(\xi) + C_2 Y_1(\xi). \quad (546) \]

The deformed and stressed states of the shell near the pole are described by the second term, since they should decrease with increase in the argument. Consequently, \( C_1 = 0 \) must be set.

By switching to initial argument \( \zeta \), which is the length of an arc along the meridian, we obtain
\[ U = C_2 Y_1[(r + is)\xi], \quad (547) \]
where \( C_2 \) is the complex integration constant.

By separating the real and imaginary parts, we obtain
\[ U = (\bar{A} + B i) (\text{Re} Y_1 + i \text{Im} Y_1), \quad (548) \]

and, consequently
\[
\begin{align*}
\text{Re} U &= \bar{A} \text{Re} Y_1 - B \text{Im} Y_1; \\
\text{Im} U &= B \text{Re} Y_1 + \bar{A} \text{Im} Y_1.
\end{align*}
\quad (549)
\]

In accordance with Eq. (531), the deformation and stress functions are determined by the expressions

\[
\begin{align*}
\varphi &= -\frac{\bar{A} \text{Im} Y_1 + B \text{Re} Y_1}{2 rs \sqrt{D}}; \\
V &= \bar{A} \left( \text{Re} Y_1 + \frac{s^2 - r^2}{2 r} \text{Im} Y_1 \right) - B \left( \text{Im} Y_1 - \frac{s^2 - r^2}{2 r} \text{Re} Y_1 \right).
\end{align*}
\quad (550)
\]

or, after introduction of new constants

\[
\begin{align*}
\varphi &= -\frac{A \text{Im} Y_1 + B \text{Re} Y_1}{2 \pi \sqrt{D}}; \\
V &= A \left[ 2 rs \text{Re} Y_1 + \frac{s^2 - r^2}{2 r} \text{Im} Y_1 \right] - B \left[ 2 rs \text{Im} Y_1 - \frac{s^2 - r^2}{2 r} \text{Re} Y_1 \right].
\end{align*}
\quad (551)
\]

We find integration constants A, B from the boundary conditions at the pole of the shell, namely, in the limit as \(\zeta \to 0\), the following conditions should be satisfied

\[ \phi = 0; \quad 2 \pi \zeta V = -PR. \quad (552) \]

Since \( Y_1(\zeta) \zeta = 0 \approx -\frac{2}{\pi} \frac{1}{\zeta} \), we obtain

\[ \text{Re} Y_{1r} \zeta = 0 = -\frac{2r}{\pi k^2 c^2}; \quad \text{Im} Y_{1r} \zeta = 0 = \frac{2s}{\pi k^2 c^2}. \quad (553) \]

By substituting Eq. (553) in boundary conditions (552), we obtain

\[ A = \frac{PR}{8s}; \quad B = \frac{PR}{8r}. \quad (554) \]

Thus, the deformation and stress functions take the following final form

\[
\begin{align*}
\varphi &= -\frac{P}{8 rs \sqrt{D}} (s \text{Re} Y_1 + r \text{Im} Y_1); \\
V &= \frac{PKR}{8 rs} \left( s \text{Re} Y_1 - r \text{Im} Y_1 \right).
\end{align*}
\quad (555)
For a full strength spherical shell near the pole, according to Eq. (509),

\[
\frac{du}{da} + w = \frac{1}{E \delta} \left( \psi \frac{dV}{da} - \frac{V}{a} \right); \\
u + w = \frac{1}{E \delta} \left( \psi \frac{V}{a} - \frac{dV}{da} \right).
\]

(556)

By subtracting the second equation from the first, we obtain

\[
- \frac{du}{da} - \frac{u}{a} = \frac{1 + \nu}{E \delta} \left( \frac{dV}{da} - \frac{V}{a} \right).
\]

(557)

Consequently, the movement of points along the meridian and the deflection of the shell are determined by the following formulas

\[
\begin{align*}
\frac{du}{da} & = \frac{1 + \nu}{E \delta} V; \\
\frac{u}{a} & = -\frac{H}{E \delta} \left( \frac{dV}{da} - \frac{V}{a} \right).
\end{align*}
\]

(558)

The following rule of differentiation of a Bessel function must be taken into account in determining the forces and movements:

\[
\begin{align*}
\frac{d}{\alpha^2} (\text{Re } Y_1) & = r \text{ Re } Y_0 - s \text{ Im } Y_0 - \frac{\text{Re } Y_1}{\zeta}; \\
\frac{d}{\alpha^2} (\text{Im } Y_1) & = s \text{ Re } Y_0 + r \text{ Im } Y_0 - \frac{\text{Im } Y_1}{\zeta}.
\end{align*}
\]

(559)

The deflection of the shell at the point of application of the concentrated force

\[
w = \frac{P k^4 R^4}{4 E \delta \gamma k^4 - \omega^4} \text{ Im } Y_0.
\]

(560)

The elastic forces and moments which arise in the shell are determined by the expressions

\[
\begin{align*}
Q_1 & = \frac{P k^2}{8 \iota} (s \text{ Re } Y_1 - r \text{ Im } Y_1); \\
G_1 & = \frac{P}{8 \iota} \left( 2 s \text{ Re } Y_0 - (s^2 - r^2) \text{ Im } Y_0 - \frac{1 - \nu}{\zeta} s \text{ Re } Y_1 + r \text{ Im } Y_1 \right); \\
T_1 & = -\frac{P k^2 R}{8 \iota} \left( s \text{ Re } Y_1 - r \text{ Im } Y_1 \right) \frac{1}{\zeta} - \frac{P}{2 \pi \iota} \frac{1}{\sin^2 \alpha}; \\
T_2 & = \frac{P k^2 R}{8 \iota} \left[ k^2 \text{ Im } Y_0 + \frac{1}{\zeta} (s \text{ Re } Y_1 - r \text{ Im } Y_1) + \frac{P}{2 \pi \iota} \frac{1}{\sin^2 \alpha} \right].
\end{align*}
\]

(561)
For an axisymmetrically loaded orthotropic shell of rotation, differential Eq. (527) near the pole can be reduced to canonical form by the same substitution as for a spherical shell

\[
\frac{dU}{d\xi} + \frac{1}{\xi} \frac{dU}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right) U = 0. \tag{562}
\]

The solution of this equation is expressed through Bessel functions of the first and second kinds of order \(n \mp c\), which is determined by the type of pole of the shell and the nature of the anisotropy.

The solution of differential Eq. (562) which decreases with increase in the argument of \(\xi\) has the form

\[
U = c_2 Y_n[(r+s_1)\xi]. \tag{563}
\]

51. Calculation of Temperature Compensated Pipe

For compensation of temperature deformation of long cylindrical pipes which operate under uniform internal or external pressure, small annular toroidal corrugations are made very often (Fig. 73).

Within one corrugation, \(0 < a < a_0\). We will assume the radius of curvature of the corrugation to be smaller than the pipe radius, and shell thickness \(s\) to be smaller than corrugation radius \(p\).

The cross section radius of the corrugated portion

\[
r = R + p(s_0 - s_0). \tag{564}
\]

According to Eq. (504), (505), in axisymmetric loading, the elastic forces and moments are connected with the deformed state of the shell by the following expressions

\[
\begin{align*}
T_1 &= B_1 \left( \frac{u'+w'}{q} + v_1 \frac{u \cos a + w \sin a}{r} \right); \\
T_2 &= B_2 \left( \frac{u \cos a + w \sin a}{r} + v_1 \frac{u'+w'}{q} \right); \\
G_1 &= -D_1 \left( \frac{\varphi'}{q} + v_1 \frac{\varphi \cos a}{r} \right); \\
G_2 &= -D_1 \left( \frac{\varphi \cos a}{r} + v_1 \frac{\varphi}{q} \right); \\
Q_1 &= -K_1 \left( \varphi + \frac{u'-u}{q} \right). \tag{565}
\end{align*}
\]
We present the forces, bending moments and movements in the form of expansions by small parameter $\rho/R$

\[ T_1 = \sum_{n=0}^{\infty} T_1^{(n)} \left( \frac{\eta}{R} \right)^n; \]
\[ T_2 = \sum_{n=0}^{\infty} T_2^{(n)} \left( \frac{\eta}{R} \right)^n; \]
\[ Q_1 = \sum_{n=0}^{\infty} Q_n \left( \frac{\eta}{R} \right)^n; \]
\[ G_1 = \sum_{n=0}^{\infty} G_1^{(n)} \left( \frac{\eta}{R} \right)^n; \]
\[ G_2 = \sum_{n=0}^{\infty} G_2^{(n)} \left( \frac{\eta}{R} \right)^n; \]
\[ u = \sum_{n=0}^{\infty} u_n \left( \frac{\eta}{R} \right)^n; \]
\[ q = \sum_{n=0}^{\infty} q_n \left( \frac{\eta}{R} \right)^n; \]
\[ w = \sum_{n=0}^{\infty} w_n \left( \frac{\eta}{R} \right)^n. \]

In accordance with Eq. (565), the coefficients of the expansion are determined by the following expressions

\[ T_1^{(n)} = \frac{B_1}{v} \left[ u_n' + w_n + \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \times \right. \]
\[ \times (u_j \cos a + w_j \sin a) \left. \right]; \]
\[ T_2^{(n)} = \frac{B_2}{v} \left[ v_n' (u_n + w_n) + \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \times \right. \]
\[ \times (u_j \cos a + w_j \sin a) \left. \right]; \]
\[ Q_n = -\frac{K_1}{\rho} (q_n' + w_n' - u_n); \]
\[ G_1^{(n)} = -\frac{D_1}{v^2} \left[ q_n' + v_2 \cos a \sum_{j=0}^{n-1} (-\eta)^{n-j-1} q_j \right]; \]
\[ G_2^{(n)} = -\frac{D_2}{v^2} \left[ v_1 q_n' + \cos a \sum_{j=0}^{n-1} (-\eta)^{n-j-1} q_j \right]; \]

for $n=0, 1, 2, 3, \ldots$, where

\[ n = \sin \alpha - \sin \alpha_0. \]
By substituting Eq. (567) in the equilibrium equations, we obtain

\[
\begin{align*}
(rT_1') - T_1 q \cos \alpha - Q_1 r &= 0; \\
(rQ_1)' + T_1 r + T_1 q \sin \alpha &= p r q; \\
(rG_1') - G_1 q \cos \alpha - Q_1 q r &= 0;
\end{align*}
\]

(569)

a recurrent sequence of systems of differential equations can be obtained for solution of the problem considered.

For \( n=0 \),

\[
\begin{align*}
\frac{dT_0}{da} - Q_0 &= 0; \\
\frac{dQ_0}{da} + T_0 = p Q; \\
\frac{dG_0}{da} - q Q_0 &= 0.
\end{align*}
\]

(570)

For \( n \geq 1 \),

\[
\begin{align*}
\frac{dT_n}{da} - Q_n &= -\cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (T_1^{(j)} - T_2^{(j)}); \\
\frac{dQ_n}{da} + T_1^{(n)} &= -\sum_{j=0}^{n-1} (-\eta)^{n-j-1} (Q_1 \cos \alpha + T_2^{(j)} \sin \alpha); \\
\frac{dG_1^{(n)}}{da} - q Q_n &= -\cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (G_1^{(j)} - G_2^{(j)}).
\end{align*}
\]

(571)

The deformed state of the shell is determined by solutions of the following differential equations

\[
\begin{align*}
q_n' &= -\frac{a^2}{D_1} G_1^{(n)} - v_2 \cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \varphi_j; \\
\frac{1}{\rho_1} T_1^{(n)} - \frac{v_2}{\rho_1} Q_1 - \frac{\varphi_n'}{\rho_1} - w_n' &= -v_2 \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (u_j \cos \alpha + w_j \sin \alpha); \\
\frac{1}{\rho_1} T_1^{(n)} - u'_n - v_2 \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (u_j \cos \alpha + w_j \sin \alpha) &= 0
\end{align*}
\]

(572)

(573)

(574)

for \( n=0, 1, 2, \ldots \).

There are differential equations of the (573) type for determination of forces \( T_1^{(n)}, Q_n \) and, since integration of Eq. (572), (574) presents no fundamental difficulties, the problem of calculation of cylindrical pipes with transverse corrugations is reduced to the solu-
tion of second order differential equations of the type

\[ y'' + y = f(a). \]  

If the right side of Eq. (575) is presented in the form of the expansion in trigonometric series

\[ f(a) = \sum_{m=0}^{\infty} (A_m \cos ma + B_m \sin ma), \]  

where

\[ A_m = \frac{2}{\pi} \int_0^{\pi} f(a) \cos ma \, da; \quad B_m = \frac{2}{\pi} \int_0^{\pi} f(a) \sin ma \, da; \]  

the general solution of differential Eq. (575) is in the form

\[ y(a) = C_1 \cos a + C_2 \sin a + A_0 + a (A_1 \sin a - B_1 \cos a) - \sum_{m=2}^{\infty} \left( \frac{A_m}{m^2-1} \cos ma + \frac{B_m}{m^2-1} \sin ma \right). \]  

In the general case, the solution of Eq. (575) can be written in the form

\[ y(a) = C_1 \cos a + C_2 \sin a + \sin a \int_0^a f(\xi) \cos \xi \, d\xi - \cos a \int_0^a f(\xi) \sin \xi \, d\xi. \]  

An advantage of the proposed method of calculation is that all the relationships obtained remain valid for negative curvature corrugations (Fig. 74), if the sign of \( \rho \) is changed.

In particular, compensators with corrugation of alternating sign curvature which are continuously and smoothly joined together can be considered. In this case, it is not strictly possible to speak of pipe with corrugations, since such shells are more like bellows. However, in view of the absence of annular or conical springs, the rigidity of such degenerate bellows will be very substantial, approximately \((R/\rho)^3\) times greater than the rigidity of conventional noncompacted bellows.

52. Thermoelastic Stresses Generated in Orthotropic Shells of Rotation by Axisymmetric Heating

In the use of shells made of laminated plastics, it must be kept
in mind that, at comparatively low temperatures, the change in thermo-
physical properties of the material must be taken into account. This
is the basic difference of heat problems for laminated anisotropic
shells from the corresponding problems for isotropic metal shells, which
usually are solved on the assumption that the elastic constants are in-
dependent of temperature.

We suppose that axisymmetric heating of a shell of rotation
occurs to a temperature at which the Duhamel-Neumann thermoelastic
hypothesis remains valid, and that the creep of the material can be
disregarded. With such assumptions, problems were solved in [10] with-
out taking interlayer shearing into account.

We will assume that the shell material is orthotropic and that the
principal axes of anisotropy coincide with the lines of curvature of
the mean surface of the shell, i.e., they coincide with the coordinate
directions at each point of the surface. Since, in an orthotropic
shell under tension and compression in the principal directions of
anisotropy, the latter remain the principal axes of anisotropy, we will
consider that heating does not distort the angles between the axes of
elastic symmetry of the material.

If the standard thermoelastic hypotheses which established the
connection of the thermoelastic stresses with deformations are retain-
ed, the relationship of elasticity in heating to temperature $t$ can be
written in the form

$$
\begin{align*}
\epsilon_1 &= \frac{\alpha_1}{E_1} - \frac{\nu_1}{E_1} \sigma_1 + \alpha_1 t; \\
\epsilon_2 &= \frac{\alpha_2}{E_2} - \frac{\nu_2}{E_2} \sigma_2 + \alpha_2 t,
\end{align*}
$$

(580)

where $\epsilon_1, \epsilon_2$ are the components of the total deformation; $\alpha_1, \alpha_2$ are
the coefficients of linear expansion of the material in the axial and
annular directions.

Elasticity relationships (580) can be presented for the stress
components in the following form

$$
\begin{align*}
\sigma_1 &= c_{11} \epsilon_1 + c_{12} \epsilon_2 - \beta_1 t; \\
\sigma_2 &= c_{21} \epsilon_1 + c_{22} \epsilon_2 - \beta_2 t,
\end{align*}
$$

(581)

where

$$
\begin{align*}
c_{11} &= \bar{E}_1; & c_{12} &= \nu_1 \bar{E}_1 = \nu_1 \bar{E}_2; & c_{21} &= \nu_1 \bar{E}_2; & c_{22} &= \bar{E}_2; \\
\beta_1 &= c_{11}(\alpha_1 + \nu_2 \alpha_2); & \beta_2 &= c_{22}(\alpha_2 + \nu_1 \alpha_1).
\end{align*}
$$

(582)

In Eq. (580), (581), thermoelastic "constants" $E_1, E_2, \nu_1, \nu_2,$
c_{11}, c_{12}, c_{22}, \alpha_1, \alpha_2, \beta_1, \beta_2$ depend on shell temperature $t=t(\xi, \zeta),$ i.e., in the general case, they are functions of coordinates $(\xi, \zeta).$

By virtue of the rectilinear elements hypothesis, according to
Eq. (1),
\[
\sigma_1 = c_{11} e_1 + c_{12} e_2 + z \left( e_{11}' x_1' + c_{13} x_3' \right) - \beta_1 t; \\
\sigma_2 = c_{22} e_2 + c_{23} e_3 + z \left( e_{13}' x_1' + c_{22} x_2' \right) - \beta_2 t.
\]  
(583)

By reducing stresses (583) to the statically equivalent system of elastic forces and moments, we obtain the basic relationships which connect the deformed, stressed state of a laminated orthotropic shell

\[
T_1 = B_{11}(\zeta) e_1 + B_{12}(\zeta) e_2 + A_{11}(\zeta) x_1' + A_{13}(\zeta) x_3' - N_1(\zeta); \\
T_2 = B_{12}(\zeta) e_1 + B_{22}(\zeta) e_2 + A_{13}(\zeta) x_1' + A_{33}(\zeta) x_3' - N_3(\zeta);
\]  
(584)

\[
G_1 = - \left[ A_{11}(\zeta) e_1 + A_{12}(\zeta) e_2 + D_{11}(\zeta) x_1' + \right. \\
\left. + D_{13}(\zeta) x_3' - M_1(\zeta) \right]; \\
G_2 = - \left[ A_{12}(\zeta) e_1 + A_{33}(\zeta) e_2 + D_{22}(\zeta) x_3' + \right. \\
\left. + D_{33}(\zeta) x_3' - M_3(\zeta) \right];
\]  
(586)

where

\[
B_{ij}(\zeta) = \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) dz; \\
A_{ij}(\zeta) = \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) z dz; \\
D_{ij}(\zeta) = \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) z^2 dz
\]  
(586)

for \( i, j = 1, 2; \)

\[
N_j = \int_{-\delta/2}^{\delta/2} \beta_1(\zeta, z) t(\zeta, z) dz; \\
M_j = \int_{-\delta/2}^{\delta/2} \beta_1(\zeta, z) t(\zeta, z) z dz
\]  
(586)

for \( j = 1, 2. \)

By solving Eq. (584) for the deformation components and by substituting the resulting values in Eq. (585), we obtain

\[
e_1 = \frac{B_{11}}{\Delta} \frac{dV}{dz} - \frac{B_{11}}{\Delta} \frac{V c_{12}}{R_1} - \frac{F_{11}}{\Delta} x_1' - \frac{F_{13}}{\Delta} x_3' + \\
\quad + \frac{B_{12} N_1 - B_{13} N_1}{\Delta}; \\
e_2 = \frac{B_{11}}{\Delta} \frac{V c_{12}}{R_3} - \frac{B_{11}}{\Delta} \frac{dV}{dz} - \frac{F_{13}}{\Delta} x_3' - \frac{F_{13}}{\Delta} x_3' + \\
\quad + \frac{B_{12} N_3 - B_{13} N_3}{\Delta}; \\
G_1 = - \bar{D}_{11} x_1' - \bar{D}_{13} x_3' + F_{11} \frac{V c_{12}}{R_1} + F_{13} \frac{dV}{dz} + \bar{M}_1(\zeta); \\
G_2 = - \bar{D}_{12} x_1' - \bar{D}_{33} x_3' + F_{12} \frac{V c_{12}}{R_3} + F_{13} \frac{dV}{dz} + \bar{M}_3(\zeta),
\]  
(587)

where
For determination of the stress and deformation functions, two equations remained unused: the continuity of deformation equation and one equilibrium Eq. (506):

\[
\begin{align*}
\Delta &= B_{11}B_{11} - B_{11}\; ; \\
F_{11} &= \frac{A_{11}B_{11} - A_{11}B_{11}}{\Delta} \; ; \\
D_{11} &= D_{11} - \frac{A_{11}B_{11} + A_{11}B_{11}}{\Delta} + 2A_{11}A_{11}B_{11} \; ; \\
D_{11} &= D_{11} - \frac{A_{11}B_{11} + A_{11}B_{11}}{\Delta} - 2A_{11}A_{11}B_{11} \; ; \\
D_{11} &= D_{11} - \frac{A_{11}(A_{11}B_{11} + A_{11}B_{11}) - B_{11}(A_{11}A_{11} + A_{11})}{\Delta} \; ; \\
M_1(\zeta) &= M_1(\zeta) - F_{11}N_1 - F_{11}N_1; \\
M_1(\zeta) &= M_1(\zeta) - F_{11}N_1 - F_{11}N_1.
\end{align*}
\] (588)

Since the system of equations which is obtained after substituting Eq. (587) in Eq. (589) hardly seems suitable for practical calculations, it is more advisable to find a partial solution by one of the available approximate methods. In this case, terms dependent on $N_i, M_i (i=1, 2)$ appear as the heat load. Solution of the homogeneous system of equations can be obtained approximately, based on the technical equations of axisymmetrically loaded shells of rotation [14].

We consider the case of axisymmetric quite smoothly changing heat with length, in which the thermoelastic constants change considerably more slowly than the stress and deformation functions. Then, by obtaining the equations, the coefficients dependent on the thermoelastic constants and shell geometry can be considered constant and corresponding to the shell section under consideration. Besides, lower order derivatives can be disregarded compared with higher order derivatives.

With such assumptions, system of Eq. (589) takes the form

\[
\begin{align*}
\frac{d\varepsilon_1}{d_r} &= \frac{V}{K_1R_s} - \frac{\varphi}{R_s}; \\
\frac{dG_1}{d_r} &= \frac{V}{R_s},
\end{align*}
\] (590)

and Eq. (587), (588) are simplified:

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\[
\begin{align*}
\varepsilon_z &= -\frac{B_{1u}}{\Delta} \frac{dV}{d\zeta} - F_{1u} \frac{d\varphi}{d\zeta}; \\
G_z &= -\bar{D}_{1u} \frac{d\varphi}{d\zeta} + F_{1u} \frac{dV}{d\zeta}.
\end{align*}
\] (591)

By substituting the latter expressions in Eq. (590), the following can be obtained

\[
\begin{align*}
V'''' - 2q^4 V'' + k^4 V &= 0; \\
\varphi &= \frac{F_{1u}}{\bar{D}_{1u}} V,
\end{align*}
\] (592)

where

\[
\begin{align*}
2q^4 &= \frac{\bar{D}_{1u}}{K_1 R_z^2 (B_{1u} \bar{D}_{1u} + F_{1u} \Delta)}; \\
k^4 &= \frac{\Delta}{(B_{1u} \bar{D}_{1u} + F_{1u} \Delta) R_z^2}.
\end{align*}
\] (593)

Differential Eq. (592) is analogous to Eq. (459), which describes the end effects in a laminated cylindrical shell and, consequently, its solution, which disappears at infinity,

\[
V = C_1 \phi_1(\zeta) + C_2 \phi_2(\zeta),
\] (594)

where \( \phi_1, \phi_2 \) are degenerate Vlasov functions satisfied by Eq. (465).

For each section where the thermal end effect is investigated, the corresponding values of integration constants \( C_1, C_2 \) are used. The boundary conditions are written for a general solution, i.e., for the solution of Eq. (594) and some partial solution of nonhomogeneous system of Eq. (587) and (589).

The problem of calculation of the temperature stresses in axisymmetrically heated shells of rotation is significantly simplified in two cases frequently encountered in practice.

1. The thermoelastic constants do not change upon heating. In this case, basic relationships (584) and (585) take the form

\[
\begin{align*}
T_z &= B_{1u} \varepsilon_1 + B_{2u} \varepsilon_2 - N_i; \\
\dot{T}_z &= B_{1u} \varepsilon_1 + B_{2u} \varepsilon_2 - N_i; \\
G_z &= -D_{1u} x'_1 - D_{2u} x'_2 + M_i; \\
G_z &= -D_{1u} x'_1 - D_{2u} x'_2 + M_i.
\end{align*}
\] (595)

i.e., we have the conventional problem of an orthotropic shell in the presence of an axisymmetric heat load, which is characterized by parameters \( N_i, M_i \) (i=1, 2).
2. Uniform heating of shell. In uniform axisymmetric heating of a shell of rotation, the elastic constants and coefficients of linear expansion of the material change. However, they remain constant at the corresponding shell temperature. In this case, the basic elasticity relationships take the following form:

\[
\begin{align*}
T_1 &= B_{11}^{(11)} e_1 + B_{11}^{(11)} e_2 - \beta_i \delta t; \\
T_2 &= B_{11}^{(11)} e_1 + B_{11}^{(11)} e_2 - \beta_i \delta t; \\
G_i &= -D_{11}^{(11)} x_i - D_{11}^{(11)} x_i; \\
G_i &= -D_{11}^{(11)} x_i - D_{11}^{(11)} x_i.
\end{align*}
\]

(596)

where exponent t indicates that the rigidity coefficients for temperature t are used, i.e., we again approach the conventional problem for an orthotropic shell with an extremely simple axisymmetric heat load.
CHAPTER 11. CIRCULAR ANISOTROPIC CYLINDRICAL SHELLS

53. Basic Relationships and Differential Equations of Anisotropic Cylindrical Shell

We consider a circular cylindrical shell made of a laminated anisotropic plastic with elastic constants $c_{ij}$, $a_{ij}$.

Let $R$ be the cross section radius of the shell and $t$ be the wall thickness. We select dimensionless orthogonal coordinates $\alpha$, $\beta$ as the curvilinear Gaussian coordinates on the mean surface of the shell, of which $\alpha$ defines the relative distance along the generatrix of the shell and $\beta$ defines the circumferential angle (Fig. 75).

For convenience, we choose the Lamé parameters to be the same and equal to the shell radius, i.e., $A_1 = A_2 = R$. The principal radii of curvature of the mean surface of the circular cylindrical shell are $R_1 = R_2 = R$.

In the coordinate system selected, equilibrium Eq. (17) of a circular cylindrical shell are presented in the form

$$
\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} = -RX;
$$

$$
\frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} = -QX;
$$

$$
\frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} + T = RZ;
$$

$$
\frac{\partial G_1}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} = RQ_1;
$$

$$
\frac{\partial G_2}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} = RQ_2.
$$

The positive directions of the forces and moments in normal sections of the shell are shown in Fig. 1.

The components of the relative deformation and shear of the mean surface of a circular cylindrical shell, the effective changes in curvature and torsion, as well as the interlayer shearing along the coordinate axes, in accordance with Eq. (10)-(12), are determined by the following expressions

$$
\varepsilon_1 = \frac{1}{R} \frac{\partial u}{\partial \alpha}; \quad \varepsilon_2 = \frac{1}{R} \left( \frac{\partial v}{\partial \beta} + w \right);
$$

$$
\omega = \frac{1}{R} \left( \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right);
$$

$$
\chi_1 = \frac{1}{R} \frac{\partial \phi}{\partial \alpha}; \quad \chi_2 = \frac{1}{R} \frac{\partial \phi}{\partial \beta};
$$

$$
2\chi_3 = \frac{1}{R} \left( \frac{\partial \phi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right);
$$

$$
\gamma_1 = \psi + \frac{1}{R} \frac{\partial w}{\partial \alpha}; \quad \gamma_2 = \psi + \frac{1}{R} \frac{\partial w}{\partial \beta} - w.
$$

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By substituting the values of $\varepsilon_1, \varepsilon_2, \omega, \kappa_1, \kappa_2, \kappa_3, \gamma_1, \gamma_2$ in Eq. (3)-(5), we obtain the basic elasticity relationships of a circular anisotropic cylindrical shell

$$
T = \frac{1}{R^2} \left[ B_{11} \left( \frac{\partial u}{\partial a} + B_{13} \left( \frac{\partial v}{\partial p} + \omega \right) + B_{15} \left( \frac{\partial w}{\partial p} + \psi \right) \right) \right];
$$

$$
T_2 = \frac{1}{R^2} \left[ B_{11} \left( \frac{\partial u}{\partial a} + B_{23} \left( \frac{\partial v}{\partial p} + \omega \right) + B_{25} \left( \frac{\partial w}{\partial p} + \psi \right) \right) \right];
$$

$$
S = \frac{1}{R^2} \left[ B_{11} \left( \frac{\partial u}{\partial a} + B_{33} \left( \frac{\partial v}{\partial p} + \omega \right) + B_{35} \left( \frac{\partial w}{\partial p} + \psi \right) \right) \right];
$$

$$
Q = \left( \frac{K_1}{R^2} \left( R \varphi + \frac{\partial \varphi}{\partial a} \right) \right);
$$

$$
Q_2 = \left( \frac{K_2}{R^2} \left( R \psi + \frac{\partial \psi}{\partial a} \right) \right).
$$

Shell rigidity parameters $B_{1j}, D_{1j} (i, j=1, 2, 3), K_1, K_2$ are determined through the elastic constants of the material and the thickness of the shell by Eq. (6), (7).

By substituting elasticity relationships (599) in equilibrium Eq. (597), we obtain the following system of differential equations of a cylindrical anisotropic shell in movements

$$
\begin{align*}
\delta_{11} u + \delta_{12} v + \delta_{13} w + \delta_{14} \varphi + \delta_{15} \psi &= - R^2 X; \\
\delta_{21} u + \delta_{22} v + \delta_{23} w + \delta_{24} \varphi + \delta_{25} \psi &= - R^2 Y; \\
\delta_{31} u + \delta_{32} v + \delta_{33} w + \delta_{34} \varphi + \delta_{35} \psi &= - R^2 Z; \\
\delta_{41} u + \delta_{42} v + \delta_{43} w + \delta_{44} \varphi + \delta_{45} \psi &= 0; \\
\delta_{51} u + \delta_{52} v + \delta_{53} w + \delta_{54} \varphi + \delta_{55} \psi &= 0,
\end{align*}
$$

(600)

where designations are introduced for linear differential operators $\delta_{ij}$ in partial derivatives up to the second order:

$$
\begin{align*}
\delta_{11} &= B_{11} \frac{\partial^2}{\partial a^2} + 2B_{13} \frac{\partial^2}{\partial a \partial p} + B_{15} \frac{\partial^2}{\partial p^2}; \\
\delta_{12} &= B_{12} \frac{\partial^2}{\partial a \partial p}; \\
\delta_{13} &= B_{13} \frac{\partial^2}{\partial a^2} + (B_{11} + B_{23}) \frac{\partial^2}{\partial a \partial p} + B_{15} \frac{\partial^2}{\partial p^2}; \\
\delta_{22} &= B_{22} \frac{\partial^2}{\partial a^2} + 2B_{23} \frac{\partial^2}{\partial a \partial p} + B_{25} \frac{\partial^2}{\partial p^2} - K_5; \\
\delta_{23} &= B_{23} \frac{\partial^2}{\partial a \partial p}; \\
\delta_{33} &= - \left( K_1 \frac{\partial^2}{\partial a^2} + K_5 \frac{\partial^2}{\partial p^2} - B_{22} \right); \\
\delta_{41} &= - (K_1 \frac{\partial^2}{\partial a^2} + K_5 \frac{\partial^2}{\partial p^2} - B_{22}); \\
\delta_{42} &= - (B_{11} + B_{23}) \frac{\partial^2}{\partial a \partial p} + 2B_{13} \frac{\partial^2}{\partial a \partial p} + B_{15} \frac{\partial^2}{\partial p^2}; \\
\delta_{43} &= B_{23} \frac{\partial^2}{\partial a \partial p}; \\
\delta_{51} &= 0; \\
\delta_{52} &= 0;
\end{align*}
$$

(601)
System of differential Eq. (600) relative to linear differential operators \( \delta_{ij} \) is symmetrical. This property of symmetry of the differential equations in movements of isotropic and anisotropic shells was noted in [1, 4].

System of Eq. (600) can be reduced to one resolving differential equation of the tenth order relative to stress functions \( \phi(\alpha, \beta) \), through which all quantities which define the stressed and deformed states of the shell can be expressed.

We will consider system of differential Eq. (600) as a complete system of algebraic equations relative to movement components \( u, v, w \) and deformation functions \( \phi, \psi \), with constant coefficients \( \delta_{ij} \). The determinant of this system of equations

\[
\Delta = \begin{vmatrix}
\delta_{11} & \delta_{12} & \delta_{13} & 0 & 0 \\
\delta_{12} & \delta_{22} & \delta_{23} & 0 & \delta_{24} \\
\delta_{13} & \delta_{23} & \delta_{33} & \delta_{34} & \delta_{35} \\
0 & 0 & \delta_{34} & \delta_{44} & \delta_{45} \\
0 & \delta_{35} & \delta_{45} & \delta_{44} & \delta_{44}
\end{vmatrix} = \delta_{11}\Delta_{11} + \delta_{12}\Delta_{12} + \delta_{13}\Delta_{13} \tag{602}
\]

is a differential operator in partial derivatives up to the tenth order, with coefficients dependent on \( B_{ij}, D_{ij}, K_1, K_2 \).

We designate the minors of the determinant, which also are differential operators in partial derivatives by \( \Delta_{ij} (i, j=1, 2, 3) \).

Thus, for example

\[
\Delta_{12} = \begin{vmatrix}
\delta_{22} & \delta_{23} & 0 & \delta_{24} \\
\delta_{23} & \delta_{33} & \delta_{34} & \delta_{35} \\
0 & \delta_{34} & \delta_{44} & \delta_{45} \\
\delta_{25} & \delta_{35} & \delta_{45} & \delta_{44}
\end{vmatrix} = \delta_{22}\delta_{33}\delta_{44}\delta_{45} + 2\delta_{22}\delta_{34}\delta_{45}\delta_{45} \tag{603}
\]
be reduced to the following canonical form

\begin{align*}
\text{and the like.}
\end{align*}

We will seek a solution of system of Eq. (600) in the form of the following linear combination of functions \( F_1, F_2, F_3, F_4, F_5 \):

\begin{align*}
u = & \Delta_{12} F_1 + \Delta_{12} F_2 + \Delta_{12} F_3 + \Delta_{12} F_4 + \Delta_{12} F_5; \\
v = & \Delta_{12} F_1 + \Delta_{22} F_2 + \Delta_{22} F_3 + \Delta_{22} F_4 + \Delta_{22} F_5; \\
w = & \Delta_{12} F_1 + \Delta_{22} F_2 + \Delta_{22} F_3 + \Delta_{22} F_4 + \Delta_{22} F_5; \\
\varphi = & \Delta_{24} F_1 + \Delta_{24} F_2 + \Delta_{24} F_3 + \Delta_{24} F_4 + \Delta_{24} F_5; \\
\psi = & \Delta_{25} F_1 + \Delta_{25} F_2 + \Delta_{25} F_3 + \Delta_{25} F_4 + \Delta_{25} F_5.
\end{align*}

By substituting Eq. (608) in system of Eq. (600), the latter can be reduced to the following canonical form
Thus, the problem of calculation of a circular cylindrical shell was reduced to solution of a nonhomogeneous differential equation in partial derivatives of the tenth order.

If any partial solution of system of Eq. (609) is designated $F_1^*, F_2^*, F_3^*$, the partial solution of system of Eq. (600) can be written in the form

$$
\begin{align*}
\Delta F_1 &= -R^2X; \\
\Delta F_2 &= -R^2Y; \\
\Delta F_3 &= R^2Z; \\
\Delta F_4 &= \Delta F_5 = 0.
\end{align*}
$$

(609)

For the types of external surface load most frequently encountered in engineering practice, it is simpler to determine a partial solution directly from the solution of system of Eq. (600).  

The general solution of system of Eq. (609) equals the sum of partial solution (610) or a solution found by another method and a solution of the homogeneous equation

$$
\Delta F = 0.
$$

(611)

If it is set that $F_1 = F$ and $F_2 = F_3 = F_4 = F_5 = 0$, the general solution of system of Eq. (600) is presented in the form

$$
\begin{align*}
u &= v_0 + \Delta_{14}F; \\
v &= v_0 + \Delta_{14}F; \\
w &= w_0 + \Delta_{15}F; \\
\psi &= \psi_0 + \Delta_{14}F; \\
\end{align*}
$$

(612)

54. Equations of Technical Theory of Orthotropic Cylindrical Shell

The theory of flattened shells developed by V.Z. Vlasov, which is used for calculation of strength, stability and vibrations has become widespread in engineering practice [14]. This theory rests on certain
assumptions which permit significant simplification of the differential equations of general moment theory.

Following Vlasov, we will assume:

1. annular interlayer shearing $\gamma_2$ depends negligibly on annular movements $v$;

2. in the equations of equilibrium of annular forces, cutting force $Q_2$ can be disregarded.

Based on these hypotheses, for a circular orthotropic cylindrical shell, the principal axes of anisotropy of which coincide with the $a, B$ coordinate axes (see Fig. 75), we obtain the equilibrium equations

\begin{align}
\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= -RX; \\
\frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= -RY; \\
\frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} + T_s &= RZ; \\
\frac{\partial G_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} &= RQ_1; \\
\frac{\partial G_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} &= RQ_4. 
\end{align}

Geometric relationships:

\begin{align}
\varepsilon_1 &= \frac{1}{H} \frac{\partial u}{\partial \alpha}; \\
\varepsilon_2 &= \frac{1}{H} \left( \frac{\partial v}{\partial \beta} + \omega \right); \\
\chi_1 &= \frac{1}{H} \frac{\partial q}{\partial \alpha}; \\
\chi_2 &= \frac{1}{H} \frac{\partial q}{\partial \beta}; \\
\gamma_1 &= \phi + \frac{1}{H} \frac{\partial u}{\partial \alpha}; \\
\gamma_2 &= \psi + \frac{1}{H} \frac{\partial v}{\partial \beta}. 
\end{align}

Elasticity relationships:

\begin{align}
T_1 &= \frac{B_1}{H} \left[ \frac{\partial u}{\partial \alpha} + \varepsilon_2 \left( \frac{\partial v}{\partial \beta} + \omega \right) \right]; \\
T_2 &= \frac{B_2}{H} \left[ \frac{\partial v}{\partial \beta} + \omega + \varepsilon_1 \frac{\partial u}{\partial \alpha} \right]; \\
S &= \frac{B_3}{H} \left( \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right); \\
Q_1 &= -K_1 \left( \phi + \frac{1}{H} \frac{\partial u}{\partial \alpha} \right); \\
Q_2 &= -K_2 \left( \psi + \frac{1}{H} \frac{\partial v}{\partial \beta} \right); \\
G_1 &= -\frac{D_1}{H} \left( \phi + \varepsilon_2 \frac{\partial v}{\partial \alpha} \right); \\
G_2 &= -\frac{D_2}{H} \left( \psi + \varepsilon_1 \frac{\partial u}{\partial \beta} \right). 
\end{align}
where rigidity parameters $B_1, B_2, B_3, D_1, D_2, D_3, K_1, K_2$ are determined through elastic constants $c_{ij}$ of the material and shell thickness $\delta$ by Eq. (7), (27).

Homogeneous Eq. (613) are identically satisfied, if stress function $\phi(\alpha, \beta)$ is introduced in accordance with the expressions

$$T_1 = \frac{\partial^2 \Phi}{\partial \alpha^2}, \quad T_2 = \frac{\partial^2 \Phi}{\partial \alpha \partial \beta}, \quad S = -\frac{\partial^2 \Phi}{\partial \alpha^2 \partial \beta}. \quad (617)$$

By substituting Eq. (616) in equilibrium Eq. (614), we obtain

$$\begin{align*}
D_1 \frac{\partial^2 \Phi}{\partial \alpha^2} + D_3 \frac{\partial^2 \Phi}{\partial \beta^2} + C \frac{\partial^2 \psi}{\partial \alpha \partial \beta} &= K_1 R \left( \psi + \frac{1}{H} \frac{\partial \omega}{\partial \alpha} \right); \\
D_3 \frac{\partial^2 \psi}{\partial \alpha^2} + D_1 \frac{\partial^2 \psi}{\partial \beta^2} + C \frac{\partial^2 \psi}{\partial \alpha \partial \beta} &= K_2 R \left( \psi + \frac{1}{H} \frac{\partial \omega}{\partial \beta} \right); \\
\frac{\partial^2 \Phi}{\partial \alpha \partial \beta} &= K_1 \frac{\partial \psi}{\partial \alpha} + K_2 \frac{\partial \psi}{\partial \beta} + \frac{1}{H} \left( K_1 \frac{\partial \omega}{\partial \alpha^2} + K_2 \frac{\partial \omega}{\partial \beta^2} \right) + RZ.
\end{align*} \quad (618)$$

The last missing equation gives the deformation compatibility condition

$$\frac{\partial^4 \psi}{\partial \beta^2} - \frac{\partial^4 \omega}{\partial \alpha \partial \beta} + \frac{\partial^4 \zeta}{\partial \alpha^2} = \frac{1}{H} \frac{\partial^4 \omega}{\partial \alpha^2}. \quad (620)$$

System of Eq. (618) is equivalent to the following system of differential equations

$$\begin{align*}
m_2(\psi) &= \frac{D_1 K_1}{H^2} \frac{\partial^4 \omega}{\partial \alpha^2} + \left( D_3 K_1 - C K_3 \right) \frac{\partial^4 \psi}{\partial \alpha^2 \partial \beta^2} - \frac{K_1 K_2}{H} \frac{\partial \omega}{\partial \alpha}; \\
m_1(\psi) &= \frac{D_3 K_1}{H^2} \frac{\partial^4 \psi}{\partial \alpha^2 \partial \beta^2} + \left( D_3 K_1 - C K_3 \right) \frac{\partial^4 \psi}{\partial \alpha^2 \partial \beta^2} - \frac{K_1 K_2}{H} \frac{\partial \omega}{\partial \beta}. \quad (621)
\end{align*}$$

where there is introduced the differential operator in partial derivatives

$$m_2(\psi) = \frac{D_1 D_3}{H^4} \frac{\partial^6 \omega}{\partial \alpha^2} + \frac{(D_1 D_3 + D_1^3 - C^2)}{H^4} \frac{\partial^4 \psi}{\partial \alpha^2 \partial \beta^2} + \frac{D_1 D_2}{H^4} \frac{\partial^4 \omega}{\partial \beta^2} - \frac{D_3 K_1 + D_3 K_3}{H^2} \frac{\partial^4 \psi}{\partial \alpha^2 \partial \beta^2} + \frac{D_3 D_2}{H^4} \frac{\partial^2 \psi}{\partial \alpha^2} + \frac{K_1 K_2}{H} \frac{\partial \omega}{\partial \beta}. \quad (622)$$
From Eq. (620), with Eq. (617) taken into account, an expression for deflection of the shell can be obtained through the stress function

\[ \frac{B_s(B_s - B_{1s})}{R} w = B_s B_s \frac{\partial^4 \Phi}{\partial u^4} + \]
\[ + (B_s B_s + B_s - A^3) \frac{\partial^4 \Phi}{\partial u \partial \theta} + B_s \frac{\partial^4 \Phi}{\partial \theta^4}, \quad (623) \]

where
\[ A = B_{1s} + B_s, \quad (624) \]

or, in operator form
\[ w = m_1(\Phi), \quad (625) \]

and here,

\[ m_1(\Phi) = \frac{R}{B_s(B_s - B_{1s})} \times \]
\[ \times [B_s B_s \frac{\partial^4 \Phi}{\partial u^4} + (B_s B_s + B_s - A^3) \frac{\partial^4 \Phi}{\partial u \partial \theta} + B_s \frac{\partial^4 \Phi}{\partial \theta^4}] . \quad (626) \]

In this manner, the system of differential equations of the laminated orthotropic cylindrical shell has the form

\[ m_2(\psi) = \left( \frac{D_2 K_2}{R^2} \frac{\partial^2 \psi}{\partial u^2} + \frac{D_2 K_2 - K_2}{R^2} \frac{\partial^2 \psi}{\partial u \partial \theta} - \frac{K_2}{R^2} \frac{\partial \psi}{\partial u} \right); \]
\[ m_2(\psi) = \left( \frac{D_2 K_2}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{D_2 K_2 - K_2}{R^2} \frac{\partial^2 \psi}{\partial u \partial \theta} - \frac{K_2}{R^2} \frac{\partial \psi}{\partial \theta} \right); \]
\[ \frac{\partial^4 \phi}{\partial u^4} = \left( \frac{K_1}{R} \frac{\partial^2 \phi}{\partial u^2} + \frac{K_2}{R} \frac{\partial^2 \phi}{\partial \theta^2} \right) m_1(\Phi) + K_1 \frac{\partial \phi}{\partial u} + K_2 \frac{\partial \phi}{\partial \theta} + RZ; \quad (627) \]
\[ w = m_1(\Phi). \]

System of Eq. (627) can be reduced to one resolving equation relative to stress functions \( \Phi(\alpha, \beta) \)

\[ \frac{\partial^4 \phi}{\partial u^4} m_2(\Phi) = \frac{1}{R} \left( \frac{K_1}{R} \frac{\partial^2 \phi}{\partial u^2} + \frac{K_2}{R} \frac{\partial^2 \phi}{\partial \theta^2} \right) m_1 m_2(\Phi) + \]
\[ + K_1 \frac{\partial \phi}{\partial u} + K_2 \frac{\partial \phi}{\partial \theta} + R m_2(Z). \quad (628) \]

We represent operator \( m_2 \) in the form of the sum of operators

\[ m_2(\Phi) = \frac{1}{R^2} \times \]
\[ \times \left[ D_2 D_3 \frac{\partial^4 \Phi}{\partial u^4} + (D_2 D_3 + D_2 - C^2) \frac{\partial^2 \psi}{\partial u \partial \theta} + D_2 D_3 \frac{\partial^4 \Phi}{\partial \theta^4} \right]. \quad (629) \]

where
\[ m_2^{(1)}(\Phi) = \frac{1}{R^2} \times \]
\[ \times \left[ (D_2 K_2 + D_2 K_2) \frac{\partial \phi}{\partial u} + (D_2 K_2 + D_2 K_2) \frac{\partial \phi}{\partial \theta} \right]. \quad (630) \]
Consequently, Eq. (628) is presented in the form
\[
\begin{align*}
\frac{1}{H} \left( K_1 \frac{\partial^4}{\partial t^4} + K_2 \frac{\partial^4}{\partial p^4} \right) m_1 m_1^{(1)}(\Phi) - \\
- \frac{1}{H} \left( K_4 \frac{\partial^4}{\partial t^4} + K_2 \frac{\partial^4}{\partial p^4} \right) m_1 m_1^{(2)}(\Phi) + \\
+ \frac{K_1 K_2}{H} \left( K_1 \frac{\partial^4}{\partial t^4} + K_3 \frac{\partial^4}{\partial p^4} \right) m_1(\Phi) - \frac{\partial^4}{\partial t^4} m_1^{(1)}(\Phi) + \\
+ \frac{\partial^4}{\partial u^4} m_1^{(2)}(\Phi) - K_1 K_2 \frac{\partial^4 \Phi}{\partial u^4} + \\
+ \frac{K_1}{H} \left[ D_2 K_1 \frac{\partial^4}{\partial t^4} + \left( D_2 K_1 - C K_2 \right) \frac{\partial^4}{\partial u^4} \frac{\partial^4}{\partial p^4} \right] m_1(\Phi) - \\
- \frac{K_1 K_4}{H} \frac{\partial^4}{\partial t^4} m_1(\Phi) + \frac{K_1}{H} \left[ D_2 K_3 \frac{\partial^4}{\partial p^4} + \left( D_1 K_3 - C K_1 \right) \frac{\partial^4}{\partial u^4} \frac{\partial^4}{\partial p^4} \right] \times \\
\times m_1(\Phi) - \frac{K_1 K_4}{H} \frac{\partial^4}{\partial u^4} m_1(\Phi) = - R m_2(Z), \tag{631}
\end{align*}
\]

or, after reduction,
\[
\begin{align*}
\frac{1}{H} \left( K_1 \frac{\partial^4}{\partial t^4} + K_2 \frac{\partial^4}{\partial p^4} \right) m_1 m_1^{(1)}(\Phi) - \\
- \frac{\partial^4}{\partial t^4} m_1^{(1)}(\Phi) - \frac{K_1 K_2}{H} m_1 m_1^{(2)}(\Phi) + \\
+ \frac{\partial^4}{\partial u^4} m_1^{(2)}(\Phi) - K_1 K_2 \frac{\partial^4 \Phi}{\partial u^4} = - R m_2(Z), \tag{632}
\end{align*}
\]

where
\[
m_1^{(3)} = \left[ D_1 \frac{\partial^4}{\partial t^4} + 2(C + D) \frac{\partial^4}{\partial u^4} \frac{\partial^4}{\partial t^4} + D_2 \frac{\partial^4}{\partial p^4} \right]. \tag{633}
\]

In developed form, differential Eq. (632) has the form
\[
\begin{align*}
A_{10.0} \frac{\partial^4 \Phi}{\partial t^4} + A_{28.2} \frac{\partial^4 \Phi}{\partial p^4} + A_{6.4} \frac{\partial^4 \Phi}{\partial u^4} + \\
+ A_{4.6} \frac{\partial^4 \Phi}{\partial u^4} \frac{\partial^4 \Phi}{\partial p^4} + A_{2.8} \frac{\partial^4 \Phi}{\partial u^4} \frac{\partial^4 \Phi}{\partial p^4} + A_{0.10} \frac{\partial^4 \Phi}{\partial t^4} \frac{\partial^4 \Phi}{\partial u^4} - \\
- \left( A_{8.0} \frac{\partial^4 \Phi}{\partial u^4} + A_{6.2} \frac{\partial^4 \Phi}{\partial u^4} \frac{\partial^4 \Phi}{\partial p^4} + A_{4.4} \frac{\partial^4 \Phi}{\partial t^4} \frac{\partial^4 \Phi}{\partial p^4} + \\
+ A_{2.8} \frac{\partial^4 \Phi}{\partial u^4} \frac{\partial^4 \Phi}{\partial p^4} + A_{0.8} \frac{\partial^4 \Phi}{\partial t^4} \right) + A_{2.0} \frac{\partial^4 \Phi}{\partial u^4} + \\
+ A_{0.2} \frac{\partial^4 \Phi}{\partial u^4} \frac{\partial^4 \Phi}{\partial p^4} - A_{0.0} \frac{\partial^4 \Phi}{\partial u^4} = - R m_2(Z), \tag{634}
\end{align*}
\]
where the coefficients are determined through the rigidity parameters of the shell by the following expressions:

\[
\begin{align*}
A_{11,0} &= B_1 B_3 D_1 D_2 K_1; \\
A_{8,2} &= K_2 B_1 B_3 D_1 D_3 + K_4 [B_1 B_3 (D_1 D_2 + D_3^2 - C^2)] + D_1 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
A_{6,4} &= K_1 [B_1 B_5 D_2 D_3 + B_2 B_3 D_1 D_3 + (D_1 D_2 + D_3^2 - C^2) \times (B_1 B_2 + B_3^2 - A^2)] + K_2 [B_1 B_3 (D_1 D_2 + D_3^2 - C^2)] + D_1 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
A_{4,6} &= K_2 [B_1 B_3 D_2 D_3 + B_2 B_3 D_1 D_3 + (D_1 D_2 + D_3^2 - C^2) (B_1 B_2 + B_3^2 - A^2)] + \\
+ K_1 [B_2 B_3 (D_1 D_2 + D_3^2 - C^2) + D_2 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
A_{2,8} &= K_4 B_2 B_3 D_2 D_3 + K_2 [B_2 B_3 (D_1 D_2 + D_3^2 - C^2)] + D_2 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
A_{0,10} &= B_2 B_3 D_2 D_3 K_2; \\
A_{6,0} &= B_1 B_3 D_1 K_2 R^2 + B_3 D_1 D_3 (B_1 B_2 - B_3^2); \\
A_{6,2} &= K_2 K_4 R^2 [D_1 (B_1 B_2 + B_3^2 - A^2) + 2B_1 B_3 (C + D_3)] + \\
+ K_2 (B_1 B_2 - B_3^2) (D_1 D_2 + D_3^2 - C^2); \\
A_{4,4} &= K_1 K_2 R^2 [B_1 B_3 D_2 + B_2 B_3 D_1 + \\
+ 2(C + D_3) (B_1 B_2 + B_3^2 - A^2)] + B_3 D_2 D_3 (B_1 B_2 - B_3^2); \\
A_{2,6} &= K_1 K_2 R^2 [D_2 (B_1 B_2 + B_3^2 - A^2) + \\
+ 2B_2 B_3 (C + D_3)]; \\
A_{0,8} &= B_2 B_3 D_2 K_1 K_2 R^2; \\
A_{20} &= B_3 (B_1 B_2 - B_3^2) R^2 (D_1 K_2 + D_3 K_1); \\
A_{02} &= B_3 (B_1 B_2 - B_3^2) R^2 (D_2 K_1 + D_3 K_2); \\
A_{00} &= B_3 (B_1 B_2 - B_3^2) K_1 K_2 R^4; \\
M &= B_3 (B_1 B_2 - B_3^2) R^2.
\end{align*}
\]

55. Equations of Technical Theory of Orthotropic Shell in Movements

The equations of the technical theory of a cylindrical shell can be presented in movements, as was done in Section 53 for the general case of anisotropy.

For an orthotropic shell, the principal axes of anisotropy of which coincide with the coordinate axes, the equations in movements can be presented in the form
\[\begin{align*}
\delta_{11} u + \delta_{12} v + \delta_{13} w + \delta_{14} \phi + \delta_{15} \psi &= - R_1^2 X; \\
\delta_{12} u + \delta_{22} v + \delta_{14} w - \delta_{24} \phi + \delta_{25} \psi &= - R_2 Y; \\
\delta_{13} u + \delta_{23} v + \delta_{33} w + \delta_{34} \phi + \delta_{35} \psi &= R_3 Z; \\
\delta_{14} u + \delta_{24} v + \delta_{34} w + \delta_{44} \phi + \delta_{45} \psi &= 0; \\
\delta_{15} u + \delta_{25} v + \delta_{35} w + \delta_{45} \phi + \delta_{55} \psi &= 0.
\end{align*}\]  

(636)

where the linear differential operators are determined by the expressions

\[\begin{align*}
\delta_{11} &= B_1 \frac{\partial^2}{\partial r^2} + B_3 \frac{\partial^2}{\partial \theta^2}; \\
\delta_{12} &= (B_1 + B_2) \frac{\partial^2}{\partial r \partial \theta}; \\
\delta_{22} &= B_3 \frac{\partial^2}{\partial \theta^2} + B_5 \frac{\partial^2}{\partial \phi^2}; \\
\delta_{13} &= B_2 \frac{\partial}{\partial r}; \\
\delta_{23} &= B_3 \frac{\partial}{\partial \theta}; \\
\delta_{22} &= -(K_1 \frac{\partial^2}{\partial r^2} + K_2 \frac{\partial^2}{\partial \theta^2} - B_1); \\
\delta_{24} &= -K_3 \frac{\partial}{\partial r}; \\
\delta_{25} &= -K_2 \frac{\partial}{\partial \theta}; \\
\delta_{44} &= D_1 \frac{\partial^2}{\partial r^2} + D_3 \frac{\partial^2}{\partial \phi^2} - K_1 R^2; \\
\delta_{45} &= (D_1 + D_3) \frac{\partial}{\partial r}; \\
\delta_{55} &= D_3 \frac{\partial^2}{\partial \phi^2} + D_4 \frac{\partial^2}{\partial \theta^2} - K_3 R^2; \\
\delta_{13} &= \delta_{15} = \delta_{24} = \delta_{25} = 0.
\end{align*}\]  

(637)

If any partial solution of system of Eq. (636) is designated by \(u_\#), v_\#, w_\#, \phi_\#, \psi_\#, the general solution can be presented in the following form:

\[\begin{align*}
u &= u_\# + \Delta_{11} \Phi; \\
v &= v_\# + \Delta_{14} \Phi; \\
w &= w_\# + \Delta_{15} \Phi; \\
\phi &= \phi_\# + \Delta_{14} \Phi; \\
\psi &= \psi_\# + \Delta_{15} \Phi;
\end{align*}\]  

(638)

where \(\Delta_{1j}\) (\(j=1, 2, 3, 4, 5\)) is determined by the expressions /187
\[
\Delta_{11} = -B_2 B_1 D_2 D_3 K_1 \frac{\partial}{\partial a_1} - [K_1 (B_3 (D_1 D_4 - C^3 + D_4^3) + B_4 D_1 D_4] +
+ K_4 R_5 D_1 D_3 \frac{\partial}{\partial a_1 \partial \phi^3} - [K_1 (B_5 D_1 D_5 + B_6 (D_1 D_6 + D_6^3 - C^3))] +
+ K_5 [B_7 D_3 D_5 + B_8 (D_1 D_7 + D_7^3 - C^3)]) \frac{\partial}{\partial a_1 \partial \phi^5} - [K_1 B_4 D_4 D_5 D_7 +
+ K_9 [B_2 D_4 D_5 + B_3 (D_1 D_8 + D_8^3 - C^3)]) \frac{\partial}{\partial a_1 \partial \phi^3} - B_4 D_3 D_5 K_4 \frac{\partial}{\partial a_1 \partial \phi^3} +
+ B_2 D_1 (B_4 D_5 + K_1 K_4 R^4) \frac{\partial}{\partial a_1 \partial \phi^3} + B_4 B_5 (D_1 D_4 + D_4^3 - C^3) +
+ K_1 K_4 R^4 (B_4 D_1 + 2B_5 (C + D_3)) \frac{\partial}{\partial a_1 \partial \phi^3} - (B_8 B_5 D_4 D_8 +
+ K_2 D_3 D_5 + 2B_6 (C + D_3)) \frac{\partial}{\partial a_1 \partial \phi^3} + B_7 D_4 K_1 K_4 R^4 \frac{\partial}{\partial \phi^3} -
- B_4 B_3 R^4 (D_1 D_4 + D_4^3 + K_3 D_4 + K_1 D_1) \frac{\partial}{\partial a_1 \partial \phi^3} -
+ B_2 B_3 R^4 (K_4 D_1 + K_1 D_2) \frac{\partial}{\partial \phi^3} + B_4 B_3 R^4 (K_4 D_4 + K_1 D_4) \frac{\partial}{\partial \phi^3} +
+ B_2 B_3 K_1 K_4 R^4 \frac{\partial}{\partial \phi^3} ;
\]
\[
\Delta_{11} = A D_1 D_3 K_1 \frac{\partial}{\partial a^3} + A [K_1 (D_1 D_5^3 + D_5^3 - C^3)] + K_4 D_1 D_3 \frac{\partial}{\partial a^3 \partial \phi^3} +
+ A [K_1 D_1 D_5 + K_4 (D_1 D_7 + C^3)] \frac{\partial}{\partial a^3 \partial \phi^5} + A D_5 D_4 K_1 \frac{\partial}{\partial a^3 \partial \phi^5} -
- (B_4 B_5 D_3 + A D_1 K_1 K_4 R^4) \frac{\partial}{\partial a^3 \partial \phi^5} - [B_8 B_5 D_4 D_8 +
+ 2A (C + D_3) K_1 K_4 R^4 \frac{\partial}{\partial a^3 \partial \phi^5} - (B_8 B_5 D_4 D_8 + A D_1 K_1 K_4 R^4) \frac{\partial}{\partial \phi^5} +
+ B_2 B_3 R^4 (K_4 D_2 + K_1 D_2) \frac{\partial}{\partial \phi^5} + B_4 B_3 R^4 (K_4 D_5 + K_1 D_4) \frac{\partial}{\partial \phi^5} +
+ B_2 B_3 K_1 K_4 R^4 \frac{\partial}{\partial \phi^5} ;
\]
\[
\Delta_{11} = -B_2 B_1 D_1 D_2 K_1 \frac{\partial}{\partial a_1} - [B_1 B_2 (B_1 D_3 + D_3^3 - C^3)] -
- B_4 B_3 D_3 K_1 \frac{\partial}{\partial a_1 \partial \phi^3} - [B_2 B_4 (D_1 D_4 + D_4^3 - C^3)] +
+ B_1 B_2 D_4 D_3 \frac{\partial}{\partial a_1 \partial \phi^3} + B_2 B_3 D_4 D_3 \frac{\partial}{\partial a_1 \partial \phi^3} + B_1 B_2 (K_4 D_1 + K_1 D_1) \frac{\partial}{\partial a_1 \partial \phi^3} \times
\times \frac{\partial}{\partial \phi^3} - B_3 R^4 (K_4 (B_1 D_3 - B_1 D_3) + K_4 (B_1 D_3 - B_1 D_3)) \frac{\partial}{\partial a_1 \partial \phi^3} -
- B_2 B_3 R^4 (K_4 D_2 + K_1 D_2) \frac{\partial}{\partial \phi^3} + B_3 B_4 K_1 K_4 R^4 \frac{\partial}{\partial \phi^3} +
+ B_2 B_3 K_1 K_4 R^4 \frac{\partial}{\partial \phi^3} ;
\]
\[
\Delta_{11} = -B_1 B_3 D_5 K_1 \frac{\partial}{\partial a_1} - B_2 R (K_1 (B_2 D_2 - B_1 D_2) + B_3 C K_2) \times
\times \frac{\partial}{\partial a_1} + B_2 B_3 R (D_3 K_1 - C K_2) \frac{\partial}{\partial a_1 \partial \phi^3} + B_2 B_3 K_2 R^3 \frac{\partial}{\partial a_1 \partial \phi^3} -
- B_2 B_3 K_1 K_4 R^3 \frac{\partial}{\partial a_1 \partial \phi^3} ;
\]
\[ \Delta_{11} = B_{11}B_3(K_1C - K_2D_1)R \frac{\partial^5}{\partial a^5 \partial \beta} - B_2R(K_1B_2C - K_s \times \]
\[ \times (B_2D_1 - B_{12}D_3) \frac{\partial^5}{\partial a^5 \partial \beta} + B_{12}B_3K_1K_3R^2 \frac{\partial^5}{\partial a^5 \partial \beta}, \]
\[ - B_s B_3K_1K_3R^2 \frac{\partial^5}{\partial a^5 \partial \beta} , \]
(643)

and function \( \phi \) is the solution of the homogeneous differential equation of the tenth order \( \Delta \phi = 0 \).

The elastic forces and moments are determined by Eq. (616). Ten random integration constants \( C_i \) (i=1, 2, ..., 10) are determined from boundary conditions of the type of (22), (23), of which five are at each end of the shell. If a partial solution must be found, in accordance with Eq. (609), the nonhomogeneous equation has the form

\[ \Delta \phi = R^2 Z. \]
(644)

56. A Few Words on Integration of Equations of Technical Theory of Orthotropic Cylindrical Shell

The differential equations obtained in the preceding section can be used for solution of various engineering problems associated with the calculation of orthotropic laminated cylindrical shells.

It is convenient to present the integrals of resolving homogeneous Eq. (644) in the form of trigonometric series by coordinates \( \alpha \) or \( \beta \), depending on whether the shell is open or closed.

We begin with consideration of a closed cylindrical shell. We will seek resolving function \( \phi \) in the form

\[ \Phi(\alpha, \beta) = \sum_{n=0}^{\infty} (\Phi_n^{(1)} \cos n \beta + \Phi_n^{(2)} \sin n \beta), \]
(645)

where coefficients \( \Phi_n^{(1)}, \Phi_n^{(2)} \) are functions of longitudinal coordinate \( \alpha \) alone.

By substituting Eq. (645) in homogeneous differential Eq. (634), we obtain

\[ \sum_{n=0}^{\infty} [\Lambda_n(\Phi_n^{(1)}) \cos n \beta + \Lambda_n(\Phi_n^{(2)}) \sin n \beta] = 0, \]
(646)

where differential operator \( \Lambda_n \) is determined from the expression
\[ \Delta_n = A_{10.0} \frac{d^2\phi}{da^2} - (A_{8.0}n^2 + A_{6.0}) \frac{d\phi}{da} + 
\]
\[ + (A_{4.0}n^4 + A_{2.0}n^2 + A_{20}) \frac{d\phi}{da^2} - 
\]
\[ - (A_{4.0}n^4 + A_{4.0}n^2 + A_{10}) \frac{d\phi}{da^2} + 
\]
\[ + n^6 (A_{2.0}n^2 + A_{2.0}) \frac{d\phi}{da^2} - n^6 (A_{0.0}n^2 + A_{0.0}) \phi. \]

(647)

Coefficients \( A_{ij} \) are determined by the geometric dimensions of the shell and the elastic constants of the material by Eq. (635).

In order to satisfy Eq. (646), the solution of ordinary differential equation with constant coefficients

\[ \Delta_n(\phi) = 0 \]

(648)

must be used as functions \( \phi_n^1 \), \( \phi_n^2 \).

After finding resolving function \( \phi \), components of movement \( u, v, w \) and deformation functions \( \phi, \psi \) are determined by Eq. (638), and the elastic forces and moments by Eq. (616).

It is easy to determine that all the quantities listed are determined by equations of the type of (646), i.e., they are the sum of two states, one of which is symmetrical relative to initial generatrix \( \beta = 0 \) and the other is skew symmetrical. We agree to call the stressed and deformed states of the cylindrical shell which are described by function \( \phi_n^1 \) symmetrical and the stressed and deformed states described by functions \( \phi_n^2 \) skew symmetrical.

In the symmetrical state,

\[ \Phi(a, \beta) = \sum_{n=0}^{\infty} \Phi_n^{(1)}(a) \cos n \beta, \]

(649)

and, consequently, the following expressions can be obtained:

\[
\begin{align*}
\mathcal{U} &= \sum_{n=0}^{\infty} U_n(a) \cos n \beta; \\
\mathcal{V} &= \sum_{n=1}^{\infty} V_n(a) \sin n \beta; \\
\mathcal{W} &= \sum_{n=0}^{\infty} W_n(a) \cos n \beta; \\
\mathcal{\Psi} &= \sum_{n=1}^{\infty} \Psi_n(a) \sin n \beta; \\
\end{align*}
\]

(650)
In the skew symmetrical stressed, deformed state of the shell,

\[ \Phi(\alpha, \beta) = \sum_{n=1}^{\infty} \Phi_n^{(2)}(\alpha) \sin n \beta, \] (651)

and, correspondingly, for the elastic forces, moments and deformations, we obtain

\[\begin{align*}
\varphi &= \sum_{n=0}^{\infty} \varphi_n(\alpha) \cos n \beta; & S &= \sum_{n=1}^{\infty} S_n(\alpha) \sin n \beta; \\
T_1 &= \sum_{n=0}^{\infty} T_{1n}(\alpha) \cos n \beta; & H &= \sum_{n=0}^{\infty} H_n(\alpha) \sin n \beta; \\
T_2 &= \sum_{n=0}^{\infty} T_{2n}(\alpha) \cos n \beta; & Q_2 &= \sum_{n=0}^{\infty} Q_{2n}(\alpha) \sin n \beta; \\
G_1 &= \sum_{n=0}^{\infty} G_{1n}(\alpha) \cos n \beta; & G_2 &= \sum_{n=0}^{\infty} G_{2n}(\alpha) \cos n \beta; \\
Q_1 &= \sum_{n=1}^{\infty} Q_{1n}(\alpha) \cos n \beta.
\end{align*}\] (650)

The engineering theory form of solution of Eq. (645) is used in calculation of closed cylindrical shells, since the condition of periodicity of coordinate \(\beta\) is automatically fulfilled in this case. The random integration constants of Eq. (648) are determined from the boundary conditions at the curvilinear ends \(\alpha=0, \alpha=\alpha_1\) (Fig. 76).

It was assumed that the boundary conditions, as in many engineering problems, are simple, i.e., of canonical form. In this case, resolving Eq. (648) can be integrated and the integration constants found independently for each harmonic \(n\) and separately for the symmetrical and skew symmetrical states of the shell.
We now consider another method of integration of resolving Eq. (648), which is used in calculations of open cylindrical hinge supported panels with curvilinear ends \( a=0, a=a_1 \) (Fig. 77).

In this case, the stress function can be presented in the form

\[
\psi(a, \beta) = \sum_{m=1}^{\infty} \psi_m(\beta) \sin \lambda_m a, \tag{653}
\]

where

\[
\lambda_m = \frac{m \pi}{a}. \tag{654}
\]

After substitution of Eq. (653) in Eq. (648), we obtain the following differential equation for coefficients of expansion \( \psi_m(\beta) \)

\[
A_{m,0} \frac{d^2 \psi_m}{d \beta^2} + \left( A_{2,0} \lambda_m^2 + A_{0,2} \right) \frac{d^2 \psi_m}{d \beta^2} - \lambda_m^4 \left( A_{0,4} \lambda_m^2 + A_{4,0} \right) \frac{d \psi_m}{d \beta} + \lambda_m^4 \left( A_{2,2} \lambda_m^2 + A_{0,2} \right) \frac{d^2 \psi_m}{d \beta^2} - \chi_m \left( A_{10,0} \lambda_m^6 + A_{0,10} \lambda_m^6 + A_{0,2} \lambda_m^2 + A_{0,0} \right) \psi_m = 0 \tag{655}
\]

for \( m=1, 2, \ldots \).

The components of deformation, elastic forces and moments can be obtained in trigonometric series of the type of (653)

\[
\begin{align*}
v &= \sum_{m=1}^{\infty} V_m(\beta) \sin \lambda_m a; & u &= \sum_{m=0}^{\infty} U_m(\beta) \cos \lambda_m a; \\
w &= \sum_{m=1}^{\infty} W_m(\beta) \sin \lambda_m a; & q &= \sum_{m=0}^{\infty} q_m(\beta) \cos \lambda_m a; \\
\psi &= \sum_{m=1}^{\infty} \psi_m(\beta) \sin \lambda_m a; & S &= \sum_{m=0}^{\infty} S_m(\beta) \cos \lambda_m a; \\
T_1 &= \sum_{m=1}^{\infty} T_{1m}(\beta) \sin \lambda_m a; & H &= \sum_{m=0}^{\infty} H_m(\beta) \cos \lambda_m a; \\
T_2 &= \sum_{m=1}^{\infty} T_{2m}(\beta) \sin \lambda_m a; & Q_1 &= \sum_{m=0}^{\infty} Q_{1m}(\beta) \cos \lambda_m a; \\
G_1 &= \sum_{m=1}^{\infty} G_{1m}(\beta) \sin \lambda_m a; & Q_2 &= \sum_{m=1}^{\infty} Q_{2m}(\beta) \sin \lambda_m a; \\
G_2 &= \sum_{m=1}^{\infty} G_{2m}(\beta) \sin \lambda_m a.
\end{align*} \tag{656}
\]
It is inexpedient to present the expressions for the coefficients of expansion because of bulk.

The boundary conditions at the curvilinear edges of the panel \( \sigma = 0, \alpha = \alpha_1 \) are satisfied. As was done earlier, it can be shown that the boundary conditions on the rectilinear ends \( \xi = \pm B_1 \) are satisfied separately for each term of the expansion, if the boundary conditions are simple, as is the case in the majority of practical problems.

Methods which use expansions in trigonometric series are widely used in engineering practice. They are presented in the monograph of A.L. Gol'denveyzer [8].

Thus, the problem of calculation of closed and open cylindrical shells made of laminated orthotropic materials was reduced to solution of ordinary tenth order differential equations with constant coefficients.

Consequently, the diversity of the solutions is determined by the roots of the characteristic equations

\[
A_{10,0} z^{10} - (A_{4,2} n^2 + A_{6,0}) z^8 + (A_{4,4} n^4 + A_{6,2} n^2 + A_{8,0}) z^6 - (A_{4,8} n^8 + A_{6,4} n^4 + A_{8,2} n^2 + A_{10,0}) z^4 - (A_{4,16} n^{16} + A_{6,8} n^{12} + A_{8,4} n^8 + A_{10,2} n^4 + A_{12,0}) z^2 - (A_{4,32} n^{32} + A_{6,16} n^{24} + A_{8,8} n^{16} + A_{10,4} n^{12} + A_{12,2} n^8 + A_{14,0} n^4 + A_{16,2}) = 0; 
\]

\[
A_{10,10} z^{10} - (A_{4,8} \lambda_m^2 + A_{6,6} \lambda_m^4 + A_{8,4} \lambda_m^2 + A_{10,2} \lambda_m^0) z^8 - (A_{4,16} \lambda_m^8 + A_{6,12} \lambda_m^6 + A_{8,8} \lambda_m^4 + A_{10,4} \lambda_m^2 + A_{12,2} \lambda_m^0) z^6 - (A_{4,32} \lambda_m^{16} + A_{6,24} \lambda_m^{12} + A_{8,16} \lambda_m^8 + A_{10,8} \lambda_m^4 + A_{12,4} \lambda_m^2 + A_{14,2} \lambda_m^0 + A_{16,0}) z^4 - (A_{4,64} \lambda_m^{32} + A_{6,48} \lambda_m^{24} + A_{8,32} \lambda_m^{16} + A_{10,16} \lambda_m^{12} + A_{12,8} \lambda_m^8 + A_{14,4} \lambda_m^4 + A_{16,2} \lambda_m^2 + A_{18,0} \lambda_m^0 + A_{20,0}) = 0, 
\]

where coefficients \( A_{ij} \) are determined through the geometric dimensions of the shell and the elastic constants of the material by Eq. (635). It is not expedient to carry out analysis of the roots of Eq. (657) and (658) for an orthotropic shell in general form. It is more efficient to study them for specific problems.

### 57. Transverse Vibrations of Orthotropic Cylindrical Shell

As an example, we consider the natural transverse vibrations of a laminated cylindrical shell made of an orthotropic material. We will assume that the principal axes of anisotropy coincide with the coordinate axes.

In dynamic problems of the theory of shells, the components of the external load equal the corresponding components of the inertial forces, i.e.,

\[
X = -\rho \frac{\partial^2 u}{\partial t^2}; \quad Y = -\rho \frac{\partial^2 w}{\partial t^2}; \quad Z = -\rho \frac{\partial^2 w}{\partial t^2}. 
\]

(659)
Since the flexural rigidity of a cylindrical shell is considerably less than the rigidity during its deformation in the mean plane and shearing deformations are small, we will disregard the tangential and shearing components of the inertial forces in study of transverse vibrations. In accordance with Eq. (634), the problem of transverse vibrations of a laminated orthotropic cylindrical shell is reduced to solution of the following differential equation

$$\begin{align*}
A_{10,0} \frac{\partial^{10} w}{\partial t^{10}} + A_{5,2} \frac{\partial^{10} w}{\partial t^{10}} + A_{4,4} \frac{\partial^{10} w}{\partial t^{10}} + A_{4,5} \frac{\partial^{10} w}{\partial t^{10}} + \\
+ A_{2,8} \frac{\partial^{8} w}{\partial t^{8}} + A_{0,10} \frac{\partial^{10} w}{\partial t^{10}} - \left( A_{8,0} \frac{\partial^8 w}{\partial t^8} + A_{6,3} \frac{\partial^{10} w}{\partial t^{10}} + \\
+ A_{4,1} \frac{\partial^{10} w}{\partial t^{10}} + A_{2,6} \frac{\partial^{10} w}{\partial t^{10}} + A_{0,8} \frac{\partial^{10} w}{\partial t^{10}} \right) + \\
+ \frac{\partial^4}{\partial t^4} \left( A_{20} \frac{\partial^{10} w}{\partial t^{10}} + A_{02} \frac{\partial^{10} w}{\partial t^{10}} \right) - A_{40} \frac{\partial^{4} w}{\partial t^{4}} - M \frac{\partial^2}{\partial t^2} m_2 m_3 (\omega),
\end{align*}$$

(660)

where

$$M_0 = (B_1 B_2 - B_1^2) B_3 q \delta R^3,$$

(661)

and coefficients $A_{ij}$ are determined by Eq. (635).

If it is assumed that the natural vibrations of a cylindrical shell are harmonic with frequency $\omega_{mn}$, with hinge support of the ends of the shell, the form of the vibrations can be assigned in the form

$$w(\alpha, \beta; t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_m \sin \lambda_m \cos n \beta \sin \omega_{mn} t,$$

(662)

where

$$\lambda_m = \frac{\pi R_m}{t}.$$

(663)

By substituting Eq. (662) in differential Eq. (660), we obtain the following expression for determination of the frequency of natural transverse vibrations of an orthotropic cylindrical shell:

$$\partial^2 R^3 \omega_{mn}^2 = \frac{P_1 (\lambda_m, n) + P_2 (\lambda_m, n) + P_3 (\lambda_m, n)}{Q_1 (\lambda_m, n) [Q_2 (\lambda_m, n) + Q_3 (\lambda_m, n)]}$$

(664)

for $m = 1, 2, 3, \ldots$, where the following designations are introduced for polynomials $P_1 (\lambda_m, n)$, $Q_1 (\lambda_m, n)$

$$\begin{align*}
P_1 (\lambda_m, n) &= A_{10,0} \lambda_m^5 + A_{5,2} \lambda_m^3 n^2 + A_{4,4} \lambda_m n^4 + \\
&+ A_{4,5} \lambda_m n^4 + A_{2,8} \lambda_m n^4 + A_{0,10} n^{10}; \\
P_2 (\lambda_m, n) &= A_{8,0} \lambda_m^5 + A_{6,3} \lambda_m^3 n^2 + A_{4,4} \lambda_m n^4 + \\
&+ A_{2,6} \lambda_m n^4 + A_{0,8} n^{10};
\end{align*}$$

(665)
\[ P_q(\lambda_m, n) = \lambda_m^2 (A_m \lambda_m^2 + A_0 n^2 + A_w); \]
\[ Q_1(\lambda_m, n) = B_1 B_3 \lambda_m + B_2 (B_1 B_3 + B_4 - A_0) \lambda_m^2 + B_5 B_6 n^2; \]
\[ Q_3(\lambda_m, n) = \frac{1}{D_1 D_3} [D_1 D_3 \lambda_m^2 + (D_1 D_3 + D_2 D_4) \lambda_m^2 + D_2 D_4 n^2]; \]
\[ Q_5(\lambda_m, n) = \frac{1}{D_1 D_3} [(K_1 D_1 + K_1 D_3) \lambda_m^2 + (K_1 D_1 + K_1 D_3) n^2] + K_1 K_3. \]

If the shell is stretched by internal forces \( T_1^0, T_2^0 \), it is easy to show that the frequency of the natural vibrations increases and that the frequency of the natural oscillations decreases under precompression.

We now consider the problem of forced vibrations of a cylindrical shell as a result of a radial load which changes harmonically over time with frequency \( \omega \), i.e., load

\[ Z(\alpha, \beta, t) = p(\alpha, \beta) \sin \omega t. \quad (666) \]

We will consider regular loads \( p(\alpha, \beta) \), which can be presented in the form of the uniformly converging trigonometric series

\[ p = (\alpha, \beta) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \lambda_m \alpha \cos n \beta, \]

where

\[ a_{mn} = \frac{2R}{\pi} \int_0^\pi \int_0^{2\pi} p(\alpha, \beta) \sin \lambda_m \alpha \cos n \beta \, d\alpha \, d\beta. \quad (668) \]

In the presence of perturbing forces, the equation of the transverse vibrations of an orthotropic cylindrical shell can be presented in the form

\[ A_{10,0} \frac{\partial^{10} u}{\partial t^{10}} + A_{8,2} \frac{\partial^{10} u}{\partial t^{8} \partial \beta^2} + A_{6,4} \frac{\partial^{10} u}{\partial t^{6} \partial \beta^4} + A_{4,6} \frac{\partial^{10} u}{\partial t^{4} \partial \beta^6} + A_{2,8} \frac{\partial^{10} u}{\partial t^{2} \partial \beta^8} + A_{0,10} \frac{\partial^{10} u}{\partial t^{0} \partial \beta^{10}} - (A_{8,0} \frac{\partial u}{\partial t^8} + A_{6,2} \frac{\partial u}{\partial t^6} \partial \beta^2) + (A_{4,4} \frac{\partial u}{\partial t^4} \partial \beta^4 + A_{2,6} \frac{\partial u}{\partial t^2} \partial \beta^6) + A_{0,8} \frac{\partial u}{\partial t^0} \partial \beta^8 + A_{0,2} \frac{\partial u}{\partial t^2} \partial \beta^2) = M \frac{\partial^2 u}{\partial t^2} m_1 m_2 (w) - \beta_3 (B_1 B_3 - B_2^3) R^3 m_1 m_2 (Z). \quad (669) \]

We will seek the solution of Eq. (669) in the double trigonometric series

\[ u = \sin \omega t \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m \alpha \cos n \beta. \quad (670) \]
By substituting this expression in Eq. (669), we obtain

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \left[ P_1 + P_2 + P_3 - q \delta R^4 \omega^2 Q_1(Q_3 + Q_3) \right] \sin \lambda_m \alpha \cos n \beta =$$

$$- R^4 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} Q_1(Q_3 + Q_3) \sin \lambda_m \alpha \cos n \beta,$$

from which

$$[P_1 + P_2 + P_3 - q \delta R^4 \omega^2 Q_1(Q_3 + Q_3)] C_{mn} = Q_1(Q_3 + Q_3) a_{mn}$$

(671)

for $m, n = 1, 2, 3, \ldots$

According to Eq. (664),

$$P_1 + P_2 + P_3 = q \delta R^4 \omega_{mn}.$$  

(672)

Consequently, the following expression can be obtained for the coefficients of expansion

$$C_{mn} = \frac{1}{q \delta^2} \frac{a_{mn}}{\omega_{mn} - \omega},$$

(673)

for $m, n = 1, 2, 3, \ldots$

Thus, forced transverse vibrations of a laminated orthotropic cylindrical shell occur in the form

$$w(\alpha, \beta, x, t) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m \alpha \cos n \beta \sin \omega_{mn} t.$$

(674)

Upon coincidence of forced vibration frequency $\omega$ with frequency of natural oscillations of some tone $\omega_{mn} (m, n = 1, 2, \ldots)$, resonance vibrations arise.

We consider forced viorations of a shell, on the assumption that the natural vibrations disappear. The general solution of Eq. (669) will equal the sum of solutions (662), (670), i.e., the sum of the natural and forced vibrations

$$w = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m \alpha \cos n \beta \sin (\omega_{mn} t - \omega_0) +$$

$$+ \sin \omega t \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m \alpha \cos n \beta,$$

(675)

Fig. 78. Graph of beat with period $\pi/\omega_{mn}$.  

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where $C_{mn} (m, n=1, 2, 3, \ldots)$ is determined by Eq. (673) and parameters $C_{mn}, \omega_0$ are determined from the initial conditions.

Thus, for example, if the initial conditions are uniform, i.e., if $w=dw/dt=0$ at $t=0$, the following form of vibration of the shell can be obtained

$$
\dot{w} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left( \sin \omega t - \frac{\omega}{\omega_{mn}} \sin \omega_{mn} t \right) \frac{a_{mn} \sin \lambda \cos \alpha}{\omega_{mn} - \omega^2}.
$$

(677)

If the forced vibration frequency is close to the natural vibration frequency of some tone $\omega_{mn} (m, n=1, 2, 3, \ldots)$, so called beats occur, with oscillation period $T=2\pi/\omega_{mn}$ (Fig. 78).
CHAPTER 12. CALCULATION OF ORTHOTROPIC CYLINDRICAL SHELL SUBJECTED TO LOCALLY DISTRIBUTED AXIAL FORCES

58. Initial Hypotheses and Basic Differential Equations

Many works of domestic and foreign authors [11, 24-26] have dealt with study of the strength of isotropic cylindrical shells which are subjected to locally distributed loads. Among them, the work of V.M. Darevskiy should be distinguished, in which a partial solution of the most exact cylindrical shear equations were obtained in Fourier integrals, and the convergence of the series and characteristics of the solution were studied.

A significant contribution to the development of methods of calculation of cylindrical shells for concentrated loads was made by V.Z. Vlasov [4-6]. The semimembrane theory of a cylindrical shell he developed has been widely used in engineering practice, and it has shown satisfactory correspondence with experimental results. This chapter deals with generalization of this theory for laminated orthotropic shells.

Experimental studies of thin quite long cylindrical shells show a characteristic feature of their deformation, which is that significant annular deformation of the shell occurs compared with deformation of the generatrix as a result of concentrated radial loads. An orthogonal grid applied to the lateral surface of the shell remains nearly orthogonal after deformation but the annular lines, which bend sharply, remain almost incompressible. These characteristics of deformation, together with the results of other experimental studies, were the basis for the semimembrane theory of a cylindrical shell, two alternate versions of which were presented in the works of V.Z. Vlasov, which differed from each other in the number of initial simplifying hypotheses.

With the comparatively low resistance to shearing deformations of laminated plastics taken into account, since the shear modulus of laminated plastics is at least an order of magnitude less than the shear modulus of metals, shearing deformations of the mean surface of the shell cannot be disregarded, as was done in the last alternate version of semimembrane theory. Some other simplifying assumptions can also be dropped.

Following V.Z. Vlasov, for the calculation scheme of the cylindrical shell in its calculation as a result of locally distributed axial forces, we use a three dimensional elastic system consisting of rings which are connected together by vanishingly short connecting rods which ensure the transmission of axial forces and shearing forces (Fig. 79).

Each ring in the cross section plane of the shell works both by tension or compression and by bending as well as shearing. In the longitudinal direction of the basic calculation scheme, only momentless tangential forces \(N, S\) can appear. This calculation scheme corresponds to the assumption of the membrane structure of the shell in the axial direction, i.e., the small magnitude of bending moments \(G_1\) in cross sections and the insignificant effect of torques \(H\).
In this manner, according to the hypotheses used, the stressed state of a laminated cylindrical shell subjected to axial locally distributed forces is determined by the following parameters:

- longitudinal normal force $N=N(\alpha, \beta)$;
- shearing force $S=S(\alpha, \beta)$;
- cutting force in axial section $Q=Q$;
- annular bending moment $G=G(\alpha, \beta)$;
- normal annular force $T=T(\alpha, \beta)$.

For convenience in writing, we use symbols here which differ somewhat from the previous symbols for the elastic forces and moments. This permits subsequent avoidance of excess piling up of symbols.

The deformed state of the shell also is described by five functions:

- axial movement $u=u(\alpha, \beta)$;
- annular movement $v=v(\alpha, \beta)$;
- positive shell deflection toward outer normal $w=w(\alpha, \beta)$;
- deformation functions $\phi(\alpha, \beta)$, $\psi(\alpha, \beta)$, which characterize the interlayer shearing of the laminated shell.

The equilibrium equations have the form

$$
\begin{align*}
\frac{\partial u}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0; \\
\frac{\partial T}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= 0; \\
\frac{\partial Q}{\partial \beta} + T &= 0; \\
\frac{\partial G}{\partial \beta} &= RQ.
\end{align*}
$$

We present the elasticity relationships in the form

$$
\begin{align*}
e_1 &= \frac{N - \nu_1 T}{E_1 \delta}; \\
e_2 &= \frac{T - \nu_2 N}{E_2 \delta}; \\
\omega &= \frac{S}{G_{11} \delta};
\end{align*}
$$

$$
G = -\frac{D_{12}}{H} \left( \frac{\partial \phi}{\partial \eta} + \nu_2 \frac{\partial \psi}{\partial \eta} \right); \\
Q = -K_{12} \left( \psi + \frac{1}{H} \frac{\partial \omega}{\partial \eta} - \frac{\nu}{H} \right).
$$
It is easy to verify that equilibrium equations and elasticity relationships (679), (680) are identically satisfied, if the following stress function $F = F(u, \beta)$ is introduced

$\begin{align*}
\frac{\partial^3 F}{\partial u^3} &- \nabla \cdot \left( \frac{\partial^2 F}{\partial u^2} \right) + \frac{\partial^2 F}{\partial u^2} \left( \frac{\partial^3 F}{\partial u^3} \right) + (g^2 - u_3) \frac{\partial F}{\partial u_1} + \\
&+ \frac{\partial^2 F}{\partial u^2} - \nabla \cdot \left( \frac{\partial^2 F}{\partial u^2} \right) + k^2 \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial u^2} \left( \frac{\partial^3 F}{\partial u^3} \right) + (g^2 \frac{\partial^2 F}{\partial u^2} + \\
&+ \frac{\partial^2 F}{\partial u^2} = 0,
\end{align*}$

where the basic elasticity parameters

$\begin{align*}
\lambda^2 &= \frac{E_2}{E_1} ; \quad g^2 = \frac{E_3}{G_3} ; \\
k^2 &= \frac{6E_1}{5G_1} ; \quad c^2 = \frac{12R^2}{3} .
\end{align*}$

Eq. (683) is a differential equation in partial derivatives with constant coefficients, which depend on elastic parameters $\lambda^2$, $g^2$, $k^2$.
and \( v_2 \) and one geometric parameter \( c^2 \). In the case of a thin isotropic shell, only two independent parameters \( c^2 \) and \( v \) remain, since \( \lambda^2=1 \), \( k^2=0 \) and \( E_2/(E_1^2=2 \ (1+v)) \).

Because of relationships (682), differential Eq. (683) is satisfied both by stress functions \( F(a, \beta) \) and any of the ten functions which define the stressed and deformed states of a laminated cylindrical shell subjected to axial locally distributed load.

59. Integration of Eq. (683) in Single Trigonometric Series and Boundary Conditions

In the case of a closed cylindrical shell, the solution of Eq. (683) can be sought in the form of the single trigonometric series

\[ F(a, \beta) = \sum_{n=0}^{\infty} F_n(a) \cos n \beta. \tag{685} \]

Coefficients of expansion \( F_n(a) \) are satisfied by the following ordinary differential equations with constant coefficients

\[
\frac{d^4 F_n}{da^4} - 2\rho_n \frac{d^2 F_n}{da^2} + \rho_n^2 F_n = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots
\tag{686}
\]

where

\[
\rho_n^4 = \frac{\lambda_1 n^4 (n^2 - 1)^2 + 2\nu_2 n^4 (n^2 - 1)}{n^4 + k^2 n^2 + c^2}.
\tag{687}
\]

The solutions of differential Eq. (686) are written in V.Z. Vlasov functions, which are determined by Eq. (188) and are satisfied by Eq. (189),

\[
\Phi_1(a) = \cosh s_n a \cos r_n a; \quad \Phi_2(a) = \sinh s_n a \cos r_n a;
\]

\[
\Phi_3(a) = \sinh s_n a \sin r_n a; \quad \Phi_4(a) = \cosh s_n a \sin r_n a,
\tag{688}
\]

where

\[
s_n = \sqrt{\frac{\rho_n^4 + \rho_n^2}{2}}; \quad r_n = \sqrt{\frac{\rho_n^4 - \rho_n^2}{2}}.
\tag{689}
\]

If the elastic constants of a laminated plastic are such that the inequality

\[ G_{12} < \frac{1}{2} \frac{E_1 E_2}{2}, \tag{690} \]

is satisfied, with sufficiently large values of \( n \gg R/\delta \), parameter \( r_n \) becomes imaginary. In this case, instead of the V.Z. Vlasov functions,
hyperbolic functions should be used which are determined by the expressions

\[
\begin{align*}
\theta_1 (a) &= \cosh s_n a \cosh r_n a; & \theta_3 (a) &= \sinh s_n a \cosh r_n a; \\
\theta_2 (a) &= \sinh s_n a \sinh r_n a; & \theta_4 (a) &= \cosh s_n a \sinh r_n a,
\end{align*}
\]

(691)

where

\[
\begin{align*}
s_n &= \sqrt{\frac{\rho_n^2 + \rho_n^2}{2}}; & r_n &= \sqrt{\frac{\rho_n^2 - \rho_n^2}{2}}.
\end{align*}
\]

(692)

We also represent the components of the stressed and deformed states of the shell by the single trigonometric series

\[
\begin{align*}
N (a, \beta) &= \sum_{n=0}^{\infty} N_n (a) \cos n \beta; \\
T (a, \beta) &= \sum_{n=0}^{\infty} T_n (a) \cos n \beta; \\
S (a, \beta) &= \sum_{n=0}^{\infty} S_n (a) \sin n \beta; \\
Q (a, \beta) &= \sum_{n=0}^{\infty} Q_n (a) \sin n \beta; \\
G (a, \beta) &= \sum_{n=0}^{\infty} G_n (a) \cos n \beta; \\
u (a, \beta) &= \sum_{n=0}^{\infty} U_n (a) \cos n \beta; \\
v (a, \beta) &= \sum_{n=0}^{\infty} V_n (a) \sin n \beta; \\
w (a, \beta) &= \sum_{n=0}^{\infty} W_n (a) \cos n \beta; \\
\varphi (a, \beta) &= \sum_{n=0}^{\infty} \Phi_n (a) \cos n \beta; \\
\psi (a, \beta) &= \sum_{n=0}^{\infty} \Psi_n (a) \sin n \beta.
\end{align*}
\]

(693)

In accordance with Eq. (682), the coefficients of expansion of the elastic forces, moments, movements and deformation functions in the trigonometric series are determined by the following expressions

\[
\begin{align*}
U_n (a) &= \frac{R_n^2}{E_1 \delta} \left[ \nu_2 F_n^r + \lambda^2 (n^3 - 1) F_n \right]; \\
V_n (a) &= \frac{R_n^2}{E_1 \delta} \left[ \nu_2 (n^3 - 1) - \nu_2 n^3 \right] F_n^r - \lambda^2 n^3 (n^3 - 1) F_n.
\end{align*}
\]

(694)
\[ w_n(a) = -\frac{\dot{E}_n}{E_0^2} \left( [2p_n^2 - \varepsilon^2(n^2 - 1) + \nu_n(2n^2 - 1)] F_n^3 + \frac{\dot{E}_n}{E_0^2} n_{F_n} \right) \]

\[ \psi_n(a) = -\frac{e^2}{E_0^2} n_{F_n} \]

\[ N_n(a) = n^2(n^2 - 1) F_n^3; \]

\[ S_n(a) = n^4(n^2 - 1) F_n^3; \]

\[ Q_n(a) = -n^4(2p_n^2 F_n^3 - q_n F_n) \]

\[ G_n(a) = -n^3(2p_n^2 F_n^3 - q_n F_n). \]

In accordance with Eq. (688) and relationships (189), we obtain

\[ F_n(a) = C_1 \Phi_1(a) + C_2 \Phi_2(a) + C_3 \Phi_3(a) + C_4 \Phi_4(a); \]

\[ F_n'(a) = C_1 [s_n \Phi_2 - r_n \Phi_4] + C_2 [s_n \Phi_4 + r_n \Phi_2] + C_3 [s_n \Phi_4 + r_n \Phi_4]; \]

\[ F_n''(a) = C_1 \left[ (s_n^2 - r_n^2) \Phi_4 - 2r_n s_n \Phi_2 \right] + C_2 \left[ (s_n^2 - r_n^2) \Phi_2 + 2r_n s_n \Phi_4 \right] + C_3 \left[ (s_n^2 - r_n^2) \Phi_4 + 2r_n s_n \Phi_2 \right] - 2r_n s_n \Phi_2 + C_4 \left[ (s_n^2 - r_n^2) \Phi_2 + 2r_n s_n \Phi_4 \right]; \]

Because of the momentless structure of the shell in the axial direction at ends \( \alpha = 0, \alpha = a_1 \), only two boundary conditions each can be assigned:

a. static relative to elastic forces \( N_n, S_n \);

b. kinematic relative to movement \( U_n, V_n \);

c. or mixed static-kinematic relative to \( N_n, V_n \) or \( S_n, U_n \).

The canonical form of the uniform boundary conditions can be written thus:

a. hinge supported edges, at \( \alpha = 0, V_n(0) = 0; N_n(0) = 0; \) at \( \alpha = a_1, V_n(a_1) = 0; N_n(a_1) = 0; \)

b. rigidly fastened ends, at \( \alpha = 0, U_n(0) = 0; V_n(0) = 0; \) at \( \alpha = a_1, U_n(a_1) = 0; V_n(a_1) = 0; \)

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c. unsupported ends, at \( a=0 \), \( N_n(0)=0 \); \( S_n(0)=0 \); at \( a=a_1 \),
\( N_n(a_1)=0 \); \( S_n(a_1)=0 \).

For convenience in practical use, it is advisable to express integration constants \( C_1, C_2, C_3, C_4 \) by initial parameters \( U_n(0), V_n(0), N_n(0), S_n(0) \), i.e., by the values of functions \( U_n, V_n, N_n, S_n \) at the coordinate origin \( (a=0) \). After this, the problem of calculation of a laminated orthotropic shell subjected to concentrated and locally distributed forces is reduced to the solution of a system of two algebraic equations for two unknown initial functions, since two of them must be assigned.

It is easy to obtain the following expressions by Eq. (694):

\[
\frac{E_s b}{H} U_n(a) = n^2[C_1 (f_1 \Phi_1 + f_2 \Phi_2) + C_3 (f_1 \Phi_1 - f_2 \Phi_2) +
+ C_5 (f_1 \Phi_1 + f_2 \Phi_2) + C_6 (f_1 \Phi_1 - f_2 \Phi_2)] +
\frac{E_s b}{H} V_n(a) = n^2[C_1 (h_1 \Phi_1 - h_2 \Phi_2) + C_3 (h_1 \Phi_1 + h_2 \Phi_2) +
+ C_5 (h_1 \Phi_1 - h_2 \Phi_2) + C_6 (h_1 \Phi_1 + h_2 \Phi_2)];
\]

\[
N_n(a) = n^2 [(s_n^2 - t_n^2) \Phi_1 - 2r_n s_n \Phi_2] +
+ C_3 [(s_n^2 - t_n^2) \Phi_2 + 2r_n s_n \Phi_1] + C_4 [(s_n^2 - t_n^2) \Phi_2 - 2r_n s_n \Phi_1] +
+ C_6 [(s_n^2 - t_n^2) \Phi_2 + 2r_n s_n \Phi_1];
\]

\[
S_n(a) = n^2 [(t_n^2 - s_n^2) \Phi_1 + r_n (t_n^2 - s_n^2) \Phi_2] +
+ C_3 [(t_n^2 - s_n^2) \Phi_2 + r_n (t_n^2 - s_n^2) \Phi_1] +
+ C_4 [(t_n^2 - s_n^2) \Phi_2 - r_n (t_n^2 - s_n^2) \Phi_1];
\]

where

\[
f_1 = s_n [v_n (s_n^2 - 3r_n^2) + \lambda t_n^2 (n^2 - 1)];
\]

\[
f_2 = r_n [v_n (r_n^2 - 3s_n^2) - \lambda s_n^2 (n^2 - 1)];
\]

\[
h_1 = (s_n^2 - r_n^2) [g_n^2 (n^2 - 1) - n^2 v_n] - \lambda s_n^2 (n^2 - 1);
\]

\[
h_2 = 2r_n s_n [g_n^2 (n^2 - 1) - n^2 v_n];
\]

\[
C_1 = \frac{1}{\lambda s_n^2 (n^2 - 1)} \left[ \frac{n^2 (n^2 - 1) - n^2 v_n}{n^2 (n^2 - 1)} N_n(0) - \frac{E_s b}{H n^2} V_n(0) \right];
\]

\[
C_2 = \frac{1}{2 \lambda s_n^2 n^2 (n^2 - 1) \times}
\left[ \frac{s_n^2 - r_n^2}{s_n^2 (n^2 - 1)} \right] N_n(0) -
\frac{E_s b}{H n^2} V_n(0); \]

\[
C_3 = \frac{1}{2 \lambda s_n^2 n^2 (n^2 - 1) \times}
\left[ \frac{s_n^2 - r_n^2}{s_n^2 (n^2 - 1)} \right] U_n(0) +
+ \frac{\lambda (n^2 - 1) - v_n \left( r_n^2 - 3s_n^2 \right)}{n^2 (n^2 - 1)} S_n(0); \]

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60. Girder Analogies and Initial Parameters Method

Basic differential Eq. (686) of the semimembrane theory of an orthotropic laminated cylindrical shell and boundary conditions (696) are similar to the corresponding equations and boundary conditions for girders lying on a solid elastic base.

The analogy is that static quantities $N_n(\alpha)$ and $S_n(\alpha)$ in the bending theory of girders correspond to the bending moment and cutting force, and movement components $V_n(\alpha), U_n(\alpha)$ correspond to the deflection of the elastic axis of the girder and the angle of rotation of an element of this axis. The analogy goes still further, namely, at $n=0$ and $r=1$, differential Eq. (686) of the semimembrane theory changes to a differential equation of bending of the girder, i.e., it describes the deformed state which corresponds to the flat section principle, and members $n>2$ describe the deformed state which develops as a result of self-balancing loads, when there is warping of the cross sections of the shell.

Thus, if a locally distributed axial load applied to the cylindrical shell in section $\alpha=\xi$ (Fig. 80) is represented in the form of the trigonometric series

$$P = \frac{P}{2\pi R} + \frac{P}{\pi R} \cos \beta + \sum_{n=2}^{\infty} a_n \cos n\beta,$$

the solution which corresponds to the first two terms $n=0$ and $n=1$ can be found, by considering the shell as a girder with the corresponding support fastenings. The term $n=0$ represents an axisymmetric load uniformly distributed in section $\alpha=\xi$. The second term ($n=1$) represents an axial load distributed over section $\alpha=\xi$ by the $\cos \beta$ law, i.e., a bending moment applied in this section of the shell. Both of these cases of loading apply to the simplest problems of the strength of materials.

The remaining terms of the series $a_n \cos n\beta$ define self-balancing axial loads applied in section $\alpha=\xi$. We will call such self-balancing loads $n$-th order harmonic forces. They cause deformation of the outline and warping of the cross section of the shell.

For $n>2$, the basic forces and generalized movements can be presented in the following form (subscript $n$ is omitted for convenience):
In Eq. (700), generalized movements \( U(\alpha), V(\alpha), U(0), V(0) \) have the dimensionality of forces. These are components of movement multiplied by rigidity parameter \( E_2 \delta / R \).

Coefficients \( K_{NN}(\alpha), K_{NS}(\alpha), \ldots, K_{UU}(\alpha) \) are effect functions, which can be determined by the following expressions

\[
\begin{align*}
K_{NN}(\alpha) &= \Phi_1 - \frac{1}{2k^2 \alpha \pi n (n^2 - 1)} \left[ (s^2 + r^2)(s^2)(n^2 - 1) - n^2 v_3 \right] - \\
&\quad - \lambda^2 n^2 (n^2 - 1)(s^2 - r^2) \Phi_3; \\
K_{NS}(\alpha) &= \frac{n}{2k^2 \alpha \pi (n^2 - 1)} \left[ s \left[ (s^2 + r^2) \lambda (n^2 - 1) - n^2 v_3 \right] \Phi_3 + \\
&\quad + \frac{1}{r} \left[ \lambda^2 (n^2 - 1) + (s^2 - r^2) v_3 \right] \Phi_1 \right]; \\
K_{NV}(\alpha) &= \frac{(s^2 + r^2)^2}{2k^2 \alpha \pi n} \Phi_1; \\
K_{NU}(\alpha) &= \frac{s^2 + r^2}{2k^2 \alpha \pi} \left( r \Phi_3 - s \Phi_4 \right); \\
K_{SV}(\alpha) &= \frac{n}{2k^2 \alpha \pi (n^2 - 1)} \left[ (s^2 - r^2) \lambda (n^2 - 1) + \\
&\quad + (s^2 + r^2) v_3 \right] \Phi_1; \\
K_{SS}(\alpha) &= \frac{(s^2 + r^2)^2}{2k^2 \alpha \pi} \left( s \Phi_3 - r \Phi_4 \right); \\
K_{SV}(\alpha) &= \frac{n}{2k^2 \alpha \pi (n^2 - 1)} \left[ (s^2 - r^2) \lambda (n^2 - 1) + \\
&\quad + (s^2 + r^2) v_3 \right] \Phi_1; \\
K_{SU}(\alpha) &= \frac{n}{2k^2 \alpha \pi (n^2 - 1)} \left[ (s^2 - r^2) \lambda (n^2 - 1) + \\
&\quad + (s^2 + r^2) v_3 \right] \Phi_1.
\end{align*}
\]
\[ K_{yy}(a) = \frac{1}{2k^{s}(n^{3} - 1)(s^{3} + r^{3})} \left[ |r| \left( (s^{2} + r^{2}) \left( s^{2} (n^{3} - 1) - n^{2} \nu_{b} \right) \times \left( (s^{2} - 3s^{3}) \nu_{b} \right) \right) \right. \]
\[ + \lambda_{s}^{2} n^{2} (n^{3} - 1) (s^{2} - r^{2}) \nu_{b} \left. \right| \Phi_{b} \right); \]
\[ K_{yv}(a) = \frac{1}{2k^{s}(n^{3} - 1)(s^{3} + r^{3})} \left[ |r| \left( (s^{2} + r^{2}) \left( s^{2} (n^{3} - 1) - n^{2} \nu_{b} \right) \times \left( (s^{2} - 3s^{3}) \nu_{b} \right) \right) \right. \]
\[ + \lambda_{s}^{2} n^{2} (n^{3} - 1) (s^{2} - r^{2}) \nu_{b} \left. \right| \Phi_{b} \right); \]
\[ K_{uv}(a) = \frac{1}{2k^{s}(n^{3} - 1)(s^{3} + r^{3})} \left[ |r| \left( (s^{2} + r^{2}) \left( s^{2} (n^{3} - 1) - n^{2} \nu_{b} \right) \times \left( (s^{2} - 3s^{3}) \nu_{b} \right) \right) \right. \]
\[ + \lambda_{s}^{2} n^{2} (n^{3} - 1) (s^{2} - r^{2}) \nu_{b} \left. \right| \Phi_{b} \right); \]
\[ K_{uu}(a) = \frac{1}{2k^{s}(n^{3} - 1)(s^{3} + r^{3})} \left[ |r| \left( (s^{2} + r^{2}) \left( s^{2} (n^{3} - 1) - n^{2} \nu_{b} \right) \times \left( (s^{2} - 3s^{3}) \nu_{b} \right) \right) \right. \]
\[ + \lambda_{s}^{2} n^{2} (n^{3} - 1) (s^{2} - r^{2}) \nu_{b} \left. \right| \Phi_{b} \right); \]

where \( \Phi_{1}, \Phi_{2}, \Phi_{3}, \Phi_{4} \) are Vlasov functions, determined by Eq. (188) and which are satisfied by differential Eq. (189).

Vlasov function tables are presented in the Appendices.

Parameters \( r_{n}, s_{n} \), which are included in the values of the arguments and appear during differentiation, are determined by Eq. (639).

With a negative value of the argument, functions \( K_{1j}(a) \) revert to zero. Initial parameters \( U(0), V(0), N(0), S(0) \) play the part of integration constant, and are determined from the boundary conditions.

Laminated orthotropic cylindrical shells can be divided into three classes by the nature of transmission of axial locally distributed forces: long shells; medium length shells; short shells.

We will understand long cylindrical shells to be cylindrical shells for which the semimembrane theory presented above is valid, and an axial load applied to one end is transmitted uniformly distributed through the other end.
We classify laminated cylindrical shells, for which semimembrane theory remains applicable, but the stressed and deformed states which result from axial locally distributed loads depend significantly on the boundary conditions at both ends of the shell, as medium length shells. Finally, we will call short laminated cylindrical shells to which semimembrane theory is inapplicable.

Damping of the stressed and deformed states over the length of a laminated cylindrical shell, which results from axial locally distributed forces, is defined by the exponent $s_n^\alpha$ and, consequently, the class of long cylindrical shells includes shells in which the following inequality is satisfied

$$\frac{L^2}{h^2} \left[ \sqrt{\frac{E_2}{E_1}} + \frac{E_1}{G} \frac{(n^2-1)}{\sqrt{n^4 + k^2n^2 + c^2}} \right] \frac{n^2(n^2-1)}{\sqrt{n^4 + k^2n^2 + c^2}} > 50.$$  \hspace{1cm} (702)

In accordance with inequality (702), the concept of length of a shell is not purely geometrical, but it depends on both the geometric dimensions and on the nature of the loading and the elastic properties of the material.

We note that, with increase in shear modulus $G$, increase in annular modulus of elasticity $E_2$ and decrease in axial modulus of elasticity $E_1$, the rate of smoothing out the stressed and deformed states over the length of the shell increases. Interlayer shearing contributes to stress concentration.

The nature of loading of a shell shows up in that, beginning with some number $n$, inequality (702) is satisfied and, consequently, with respect to harmonic forces of sufficiently high order, all shells can be considered long. This situation is important in calculations, since it permits significant simplification of calculation formulas, beginning with a specific harmonic.

It also follows from this that, compared with the effect of low order forces, higher order harmonic forces are damped considerably more rapidly along the length of a cylindrical shell, i.e., the transmission of axial locally distributed loads is determined primarily by the first terms of the expansion in trigonometric series ($n=1, 2, 3, 4$). Calculations show that harmonic forces up to the fourth order go through a shell of elongation $L/R\approx 2-3$, and that the effect of higher order harmonic forces is damped without reaching the other end of the shell.

The nature of damping of harmonic forces of various orders as a result of axial concentrated loads is shown in Fig. 81.

Since the rate of damping of harmonic forces determines the capacity of the shell to resist the effect of locally distributed axial forces, for more efficient design of such systems, it can be recommended that shell elastic parameters $\lambda^2$ and $\sigma^2$ be increased if its dimensions...
cannot be decreased. With decrease in shell diameter, damping of the stressed and deformed states increases sharply.

The abovementioned effect of the elastic characteristics of a shell correctly reflects the pattern of transmission of axial concentrated forces in cylindrical shells. Actually, if a shell is visualized in which elastic parameter 
\[ \lambda^2 = \frac{E_2}{E_1} \approx 0, \]
namely, a shell consisting of longitudinal ribs covered by a thin film, it is clear that stress damping along such a shell is extremely slight, since the forces are transmitted along the ribs.

It is also follows from this that the installation of reinforcing rings is advisable for cylindrical shells subjected to axial loads which are not uniformly distributed over the perimeter.

As an example, we consider the calculation of a laminated cylindrical shell subjected to an axial load applied to one end.

61. Cylindrical Orthotropic Shell Subjected to Axial Locally Distributed Forces Applied to End

Let locally distributed axial forces applied to the upper end act on a cylindrical shell made of a laminated orthotropic material. We will assume that the principal axes of anisotropy coincide with the coordinate axes. We will assume the area over which each force is distributed is uniform and determined by central angle \( \gamma \) (Fig. 82).

Solution of the problem is reduced to determination of two initial parameters from the boundary conditions on shell ends \( \alpha = 0, \alpha = \alpha_1 \).

For example, let end \( \alpha = 0 \) where the forces are applied be free of connections and end \( \alpha = \alpha_1 \) be rigidly fastened.

We expand load \( P = p_1 + p_2 + p_3 \) in trigonometric series

\[
P = - \frac{P}{2n H} - \frac{P}{n H} \cos \beta - \sum_{n=2, 4} a_n \cos n \beta. \quad (703)
\]

From the boundary conditions for \( n \geq 2 \),

\[
N_n(0) = - a_n; \quad S_n(0) = 0; \\
U_n(\alpha_1) = 0; \quad V_n(\alpha_1) = 0; \quad (704)
\]
we determine initial parameters \( U_n(0), V_n(0) \) and, further, by Eq. (694), for which \( n \geq 2 \), we calculate the coefficients of expansion of the elastic forces, moments and deformations.

The solution for \( n=0 \) corresponds to uniform compression of the shell and, consequently,

\[
N_0(a) = -\frac{P}{2\pi R}; \quad S_0 = Q_0 = G_0 = 0. \tag{705}
\]

The \( n=1 \) case corresponds to bending of the cylindrical shell as a bracket, by bending moment \( M=PR \) applied to the \( a=0 \) end, i.e.,

\[
N_1(a) = \frac{P}{\pi R} \cos \beta; \quad S_1 = Q_1 = G_1 = 0. \tag{706}
\]

Further, by summing the forces, moments and movements in accordance with Eq. (693), we obtain the distribution of the stressed and deformed states over the surface of the shell.

We give the expansion in trigonometric series of the locally distributed axial loads most often encountered in engineering practice:

<table>
<thead>
<tr>
<th>Type of load</th>
<th>Trigonometric series ( P(\beta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( \frac{P}{2\pi R} ) + ( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
<tr>
<td>( 2 \times P )</td>
<td>( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
<tr>
<td>( 4 \times P )</td>
<td>( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
<tr>
<td>( 6 \times P )</td>
<td>( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
<tr>
<td>( 8 \times P )</td>
<td>( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
<tr>
<td>( 10 \times P )</td>
<td>( \frac{P}{2\pi R} \sum_{n=1}^{\infty} \frac{\sin n \gamma}{n \gamma} \cos n \beta )</td>
</tr>
</tbody>
</table>

\[
\frac{P}{\pi R} \sum_{n=1}^{\infty} \frac{1}{2} \frac{\sin n \gamma}{n \gamma} \left[ \cos n \beta + (-1)^{n+1} \frac{\sin n \beta}{2} \right]
\]
In the limit, as $\gamma \to 0$, we obtain an expansion of the system of concentrated axial forces in diverging trigonometric series.

We now consider transmission of an axial load to a cylindrical shell through an elastic ring (Fig. 83).

Because of low rigidity, it is advisable to make the laminated plastic rings quite massive, i.e., solid. Therefore, we will disregard warping of the cross section of the ring.

If the force of interaction of the ring and shell is designated

$$q = \sum_{n=2}^{\infty} q_n \cos n\beta.$$  \hspace{1cm} (707)

and the ring will be deformed as a result of force $P \mathbf{q}$ perpendicular to the plane of the ring

$$P = q \sum_{n=2}^{\infty} (P_n - q_n) \cos n\beta.$$  \hspace{1cm} (708)

and the shell will be compressed by distributed forces $q$.

Thus, the deflection of the ring from the plane of curvature as a result of periodic loading normal to the plane of the ring must be determined. Following Grammel [3], we consider a ring with a solid cross section, one of the principal axes of which lies in the plane of curvature. We will define the position of any section of the ring by angle $\beta$ (Fig. 84).

For a ring, one of the principal axes of inertia of the cross section of which lies in the plane of curvature, deformation in the plane of the ring and bending from the plane of curvature of the ring can be considered independently.

The stressed state of the ring in bending from the plane is defined by bending moment $M$, torque $H$ and cutting force $Q$. The deformed state of the ring is determined by deflection $w$ and cross section warp angle $\theta$. Rotation of an element of the elastic line of the ring around cross section radius $\psi$ is connected with the deflection of the ring.

If $P_n \cos n\beta$ transverse harmonic load $P_n \cos n\beta$ distributed around the circumference of the ring, the equilibrium equations have the form
The elasticity relationships which connect the stressed and deformed states of the ring can be presented in the form

\[
\frac{d\phi}{dp} + \psi = \frac{HR}{C}; \quad \frac{d\psi}{dp} - \Phi = \frac{MR}{D_x},
\]

where \( C \) is the torsional rigidity of the ring; \( D_x \) is the rigidity of the ring in bending from the plane of curvature.

From Eq. (709) and (710), it is easy to obtain differential equations for determination of bending moments \( M(\beta) \) and deflection angles \( \psi(\beta) \):

\[
\frac{d^2M}{d\beta^2} + M \cdot P_n R^2 \cos n \beta;
\]

\[
\frac{d^2\psi}{dp^2} + \Psi = \frac{HR}{D_x} \cdot \frac{dM}{dp} + \frac{RH}{C}.
\]

By integrating differential Eq. (709), (711) and (712), the following expressions can be obtained for the elastic forces, moments, deflection angles and deflection of the ring

\[
M(\beta) = C_1 \cos \beta + C_2 \sin \beta - \frac{P_n R^2}{n^2 - 1} \cos n \beta;
\]

\[
Q(\beta) = - \frac{P_n R}{n} \sin n \beta + C_3;
\]

\[
H(\beta) = - C_4 \sin \beta + C_5 \cos \beta + \frac{P_n R^2}{n (n^2 - 1)} \sin n \beta + C_6 R;
\]

\[
\psi(\beta) = C_4 \cos \beta + C_5 \sin \beta + \frac{R}{n^2} \frac{(C + D_x)}{D_x} \beta (C_1 \cos \beta + C_2 \sin \beta) + \frac{C_6 R^2}{C} + \frac{C_7 R^2}{C} \sin n \beta;
\]

\[
y(\beta) = R (C_1 \sin \beta - C_2 \cos \beta) + \frac{C_6 R^2}{C} + C_8 + \frac{n^2 C + D_x}{n^2 (n^2 - 1)} \sin n \beta.
\]

Because of the periodicity of the functions and the conditions that the ring be subjected to self balancing harmonic loads, \( C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0 \) and, consequently, the elastic forces and deformations of the ring are determined by the following equations
Transmission of the axial forces to the cylindrical shell through the elastic ring can now be calculated.

Deflection of the ring for each number \( n \geq 2 \) as a result of forces

\[
y = \frac{D_t + n^2 C}{C D_t} \cdot \left( \frac{P_n - q_n}{n^2(a^2 - 1)^{1.4}} \right) R^4 \cos n \beta.
\]  

Axial movement of the \( \alpha=0 \) end of the ring as a result of distributed axial load \( q \) equals \( U_n(0) \cos n \beta \).

From the condition of compatibility of the deformations of the ring and shell

\[
y_n = U_n(0)
\]  

the following expression can be obtained for the initial parameter

\[
U_n(0) = \frac{D_r + n^2 C}{C D_t} \cdot \frac{R^n}{n^2(a^2 - 1)^{1.4}} (P_n - q_n).
\]  

The remaining initial parameters are found from the boundary conditions:

- on the unsupported \( \alpha=0 \) end, \( S(0)=0, N(0)=q_n \);
- at the fixed \( \alpha=\alpha_1 \) end, \( U(\alpha_1)=0, V(\alpha_1)=0 \).
CHAPTER 13. SOME PROBLEMS OF SELECTION OF OPTIMUM STRUCTURE OF LAMINATED PLASTIC OF CYLINDRICAL SHELL

62. Initial Hypotheses, Assumptions and Relationships

The extensive use of laminated plastics in the most diverse fields of the national economy is explained by the exceptionally great diversity of their properties. Laminated plastics can have high unit strength, high chemical and biological stability, good electrical and sound insulating qualities, nonmagnetism, radiotransparency and other valuable properties, which structures of the most diverse technical or everyday purposes require.

The technology of production and processing of laminated plastics products does not require subsequent extremely laborious mechanical working, as a result of which, moreover, there are great losses of material. Laminated plastics are easily extruded, molded at low pressures and cast. Laminated plastics products can be manufactured directly in the process of producing the material itself. The production of reinforced plastics with a given orientation of the reinforcing material can be considered the origin of the extensive use of plastics as structural materials.

Cylindrical shells produced by continuous winding of various types of reinforcing fillers are laminated elastic anisotropic systems. The nature of the anisotropy of the elastic properties of a shell depends essentially on the mutual placement and orientation of the reinforcing filler, and it can be easily regulated during manufacture. This new property of laminated plastics, controllable anisotropy, favorably distinguishes them from traditional building and structural materials. As it were, they connect laminated plastics with the structures and, consequently, this permits the creation of that structure of the material which ensures the maximum resistance to given external loads during manufacture of the shells.

Some problems on selection of the optimum structure of fiberglass reinforced plastics were considered in Chapter 8, where the performance of a shell in the momentless stressed state was considered.

More complex cases are studied in this chapter, when maximum shell rigidity must be ensured [14].

Since the results obtained below are generally speaking of a qualitative nature, because of difficulties associated with obtaining initial data on the elastic characteristics, we will disregard the effects of interlayer shearing on the assumption that they are the same for the entire spectrum of elastic systems considered.

We will assume that the unit layers of which a laminated shell is composed are orthotropic and uniform. We will assume the elastic properties of such a layer to be known, for example, from mechanical testing of strips unwound from the shell after manufacture.

For brevity, we will call a unit layer of the shell the "fabric,"
and its principal directions of anisotropy the warp and woof directions, with the understanding that the warp is the principal direction of anisotropy with the greater modulus of elasticity. The elastic properties of the fabric are defined by four independent parameters: moduli of elasticity in the warp and woof directions $E_1, E_2$; shear modulus $G$; Poisson coefficients $v_1, v_2$, which are connected by the known relationship

$$E_1 v_2 = E_2 v_1.$$ 

If the coordinate axes do not coincide with the warp and woof directions of the fabric and are rotated relative to them by angle $\phi$ (Fig. 85), elastic constants $c_{ij}$ of the material are determined by Eq. (29) and (31).

In such a coordinate system, the basic elasticity relationships which connect the stressed and deformed states of the shell have the form

$$
\begin{align*}
\sigma_x &= c_{11} e_y + c_{12} e_x + c_{13} \omega_{xy} + c_{14} \omega_{x\tau} + c_{15} \omega_{y\tau} + c_{16} \omega_{y\tau}, \\
\sigma_y &= c_{21} e_x + c_{22} e_y + c_{23} \omega_{xy} + c_{24} \omega_{x\tau} + c_{25} \omega_{y\tau} + c_{26} \omega_{y\tau}, \\
\tau &= c_{32} e_x + c_{33} e_y + c_{34} \omega_{xy} + c_{35} \omega_{x\tau} + c_{36} \omega_{y\tau} + c_{37} \omega_{y\tau}, \\
\\varepsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \tau; \\
\varepsilon_y &= a_{12} \sigma_x + a_{22} \sigma_y + a_{23} \tau; \\
\omega_{xy} &= a_{13} \sigma_x + a_{23} \sigma_y + a_{33} \tau.
\end{align*}
$$

We note for subsequent use that, according to Eq. (29) and (31), elastic constants $a_{13}, a_{23}, c_{13}, c_{23}$, in distinction from the remaining constants, change sign with change in sign of $\phi$.

We consider a laminated shell a uniform anisotropic elastic system. Since the elastic properties of the shell are determined by the properties of the fabric and their mutual placement and orientation, after determination of the elastic properties of the shell through the elastic constants of the fabric and winding angle $\phi$, that orientation direction can be selected in which the structure of the laminated plastic becomes the optimum. In addition, problems can be solved which are connected with the selection of the best initial materials for manufacture of the shells.

It is evident that, if the shell is wound so that the warp and woof directions of adjacent layers either coincide or or mutually orthogonal, the elastic properties of the shell will be orthotropic. However, because the principal axes of anisotropy do not coincide with the coordinate axes, the elasticity relationships have the form...
If a laminated shell with a sufficiently large number of layers is made by cross bias winding of the reinforcing filler (at angle $+\phi$), it has an orthotropic structure which is symmetrical relative to the generatrix, i.e., the principal axes of anisotropy will coincide with the coordinate axes. In this case, the elasticity relationships are simplified, and they take the form

\[
\begin{align*}
T_1 &= B_{11} \varepsilon_1 + B_{13} \varepsilon_3 + B_{15} \omega; \\
T_3 &= B_{31} \varepsilon_1 + B_{33} \varepsilon_3 + B_{35} \omega; \\
S  &= B_{51} \varepsilon_1 + B_{53} \varepsilon_3 + B_{55} \omega; \\
G_1 &= -D_{11} x_1 - D_{13} x_3 - 2D_{15} x_5; \\
G_3 &= -D_{31} x_1 - D_{33} x_3 - 2D_{35} x_5; \\
H  &= -D_{51} x_1 - D_{53} x_3 - 2D_{55} x_5; \\
\varepsilon_1 &= A_{11} T_1 + A_{13} T_3 + A_{15} S; \\
\varepsilon_3 &= A_{31} T_1 + A_{33} T_3 + A_{35} S; \\
\omega &= A_{51} T_1 + A_{53} T_3 + A_{55} S.
\end{align*}
\] 

(720)

(721)

(722)

where shell rigidity parameters $A_{ij}$, $B_{ij}$, $D_{ij}$ are determined by the expressions

\[
A_{ij} = \frac{a_{ij}}{\delta}; \quad B_{ij} = c_{ij} \delta; \quad D_{ij} = \frac{c_{ij} \delta^3}{12}
\]

(725)

for $i, j = 1, 2, 3, \ldots$

Elastic constant $a_{ij}$, $c_{ij}$ of the material depend on winding angle $\phi$ and elastic constants $E_1$, $E_2$, $G$, $\nu_1$, $\nu_2$ of the fabric, and they are determined by Eq. (29)-(31).

The changes in curvature and warping of the mean surface of a tapered cylindrical shell are determined by the known expressions

\[
\begin{align*}
x_1 &= -\frac{1}{R^2} \frac{\partial^2 \omega}{\partial \alpha^2}; \\
x_3 &= -\frac{1}{R^2} \frac{\partial^2 \omega}{\partial \alpha \partial \beta}; \\
x_5 &= -\frac{1}{R^2} \frac{\partial^2 \omega}{\partial \beta^2}.
\end{align*}
\] 

(726)
63. Differential Equation of Cylindrical Shell Stability

We obtain the differential equation of stability of a laminated cylindrical shell made of an orthotropic material by single thread bias winding, i.e., for the general case of anisotropy when the principal axes of anisotropy do not coincide with the coordinate axes. It is evident that the equation of stability of a shell produced by straight winding will be a partial case, with

\[ a_{13} = a_{23} = c_{13} = c_{23} = 0. \]

We again use the orthogonal \( \alpha, \beta \) system of dimensionless coordinates (Fig. 86) as the curvilinear Gaussian coordinates on the surface of the shell.

By substituting Eq. (720), (721) in the equations of neutral equilibrium of the shell

\[
\begin{align*}
\frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0; \\
\frac{\partial T_1}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= 0;
\end{align*}
\]

(727)

\[
\frac{\partial^2 C}{\partial \alpha^2} + 2 \frac{\partial^2 H}{\partial \alpha \partial \beta} + \frac{\partial^2 G}{\partial \beta^2} + T_1^0 R = T_1^0 \frac{\partial^4 w}{\partial \alpha^4} + T_2^0 \frac{\partial^4 w}{\partial \beta^4} + 2 S^0 \frac{\partial^4 w}{\partial \alpha \partial \beta},
\]

(728)

where \( T_1^0, T_2^0, S^0 \) are components of the elastic membrane forces which act on the shell before buckling, we obtain

\[
B_{11} \frac{\partial^4 w}{\partial \alpha^4} + 2B_{13} \frac{\partial^4 w}{\partial \alpha \partial \beta^2} + B_{22} \frac{\partial^4 w}{\partial \beta^4} + B_{13} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + A \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} +
\]

\[
+ B_{23} \frac{\partial^4 w}{\partial \beta^4} = -B_{13} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2};
\]

\[
B_{13} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + A \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + 2B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + B_{23} \frac{\partial^4 w}{\partial \beta^4} =
\]

\[
= -B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} - B_{23} \frac{\partial^4 w}{\partial \beta^4};
\]

\[
D_{11} \frac{\partial^4 w}{\partial \alpha^4} + 4D_{13} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + 2(C + D_{23}) \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + 4D_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} +
\]

\[
+ D_{23} \frac{\partial^4 w}{\partial \beta^4} + R^1 \left( \delta^4 w + B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + B_{23} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + B_{23} \frac{\partial^4 w}{\partial \beta^4} \right) =
\]

\[
= R^1 \left( T_1^0 \frac{\partial^4 w}{\partial \alpha^4} + T_2^0 \frac{\partial^4 w}{\partial \beta^4} + 2S^0 \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} \right),
\]

(730)

where

\[ A = B_{13} + B_{23}; \quad C = D_{11} + D_{23}. \]

(731)
We introduce the following differential operators in second order partial derivatives

\[
\begin{align*}
  \nabla_1^t &= B_{11} \frac{\partial^t}{\partial a^t} + A \frac{\partial^s}{\partial a^s} + B_{33} \frac{\partial^t}{\partial \phi^t} ; \\
  \nabla_1^s &= B_{11} \frac{\partial^t}{\partial a^t} + 2B_{11} \frac{\partial^s}{\partial a^s} + B_{33} \frac{\partial^s}{\partial \phi^t} ; \\
  \nabla_1^s' &= B_{33} \frac{\partial^t}{\partial a^t} + 2B_{33} \frac{\partial^s}{\partial a^s} + B_{33} \frac{\partial^t}{\partial \phi^t} .
\end{align*}
\]

(732)

System of Eq. (729) is then presented in the following form

\[
\begin{align*}
  \nabla_1^t u + \nabla_1^s v &= -B_{11} \frac{\partial u}{\partial a} - B_{11} \frac{\partial v}{\partial \phi} ; \\
  \nabla_1^s u + \nabla_1^s' v &= -B_{33} \frac{\partial u}{\partial a} - B_{33} \frac{\partial v}{\partial \phi} .
\end{align*}
\]

(733)

System of differential Eq. (733) is equivalent to the following system of differential equations

\[
\begin{align*}
  \nabla_1^t u &= (B_{11}B_{33} - B_{11}B_{33}^t) \frac{\partial^w}{\partial a^w} + (B_{11}B_{33} - B_{11}B_{33}^s) \frac{\partial^w}{\partial \phi^w} + \\
  &\quad + (B_{11}B_{33} - B_{33}^s) \frac{\partial^w}{\partial a^s} ; \\
  \nabla_1^s v &= (B_{11}B_{33} - B_{11}B_{33}^s) \frac{\partial^w}{\partial a^s} + (B_{11}A - B_{11}B_{33} - B_{11}B_{33}) \frac{\partial^w}{\partial \phi^s} + \\
  &\quad + 2(B_{11}B_{33} - B_{11}B_{33}) \frac{\partial^w}{\partial a^s} ;
\end{align*}
\]

(734)

where

\[
\nabla_1^t = \nabla_1^t \nabla_1^t = (\nabla_1^s)^t .
\]

(735)

By expanding operator \( \nabla_1^t \) according to Eq. (732), we obtain

\[
\begin{align*}
  \nabla_1^t &= (B_{11}B_{33} - B_{11}B_{33}^t) \frac{\partial^t}{\partial a^t} + 2(B_{11}B_{33} - B_{11}B_{33}^s) \frac{\partial^t}{\partial \phi^t} + \\
  &\quad + (B_{11}B_{33} - B_{11}B_{33}^s) \frac{\partial^t}{\partial a^s} + 2B_{11}B_{33} - 2B_{11}B_{33}^s \frac{\partial^t}{\partial \phi^t} + \\
  &\quad + 2(B_{11}B_{33} - B_{11}B_{33}^s) \frac{\partial^t}{\partial a^s} + (B_{33}B_{33} - B_{33}) \frac{\partial^t}{\partial \phi^t} .
\end{align*}
\]

(736)

If the following new differential operator now is introduced

\[
\begin{align*}
  \nabla_1^t &= D_{11} \frac{\partial^t}{\partial a^t} + 4D_{13} \frac{\partial^t}{\partial \phi^t} + 2(C + D_{33}) \frac{\partial^t}{\partial a^s} + \\
  &\quad + 4D_{33} \frac{\partial^t}{\partial \phi^t} + D_{33} \frac{\partial^t}{\partial \phi^t} ,
\end{align*}
\]

(737)

differential Eq. (730) can be presented in the form
By multiplying Eq. (738) by operator $\nabla^4_1$, the following differential equation can be obtained for the stability of a laminated cylindrical shell made of a single thread bias wound orthotropic material

$$
\nabla^4_1 w + R^2 \left( \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial \phi^2} + 2 \frac{\partial^2 w}{\partial t \partial \phi} \right) = R^4 \left( T_{11}^* \frac{\partial^2 \delta w}{\partial t^2} + T_{11}^* \frac{\partial^2 \delta w}{\partial \phi^2} + 2 \frac{\partial^2 \delta w}{\partial t \partial \phi} \right).
$$

(739)

where

$$
B = B_{11} B_{16} B_{66} + 2 B_{16} B_{16} B_{66} - B_{11} B_{16} B_{66} - B_{16} B_{16} B_{66}.
$$

(740)

Based on Eq. (9) and (725), differential operator $\nabla^4_1$ can be presented in the form

$$
\nabla^4_1 = \nabla^4 = B \left[ a_{11} \frac{\partial^4 w}{\partial t^4} - 2 a_{16} \frac{\partial^4 w}{\partial t \partial \phi^3} + (2 a_{16} + a_{33}) \frac{\partial^4 w}{\partial \phi^4} - 2 a_{16} \frac{\partial^4 w}{\partial t^2 \partial \phi^2} + a_{11} \frac{\partial^4 w}{\partial \phi^4} \right].
$$

(741)

and, consequently, the differential equation of stability of a laminated cylindrical shell finally takes the form

$$
\nabla^4_1 w + R^2 \frac{\partial^2 w}{\partial t^2} = R^4 \left( T_{11}^* \frac{\partial^2 \delta w}{\partial t^2} + T_{11}^* \frac{\partial^2 \delta w}{\partial \phi^2} + 2 \frac{\partial^2 \delta w}{\partial t \partial \phi} \right) \nabla^4 w.
$$

(742)

where operators $\nabla^4_0$, $\nabla^4_1$ are determined from Eq. (737) and (741), respectively.

It appears to be extremely complicated to find an exact solution of Eq. (742). However, since the magnitude of the critical load of a cylindrical shell of medium length depends little on the boundary conditions, an approximate solution can be selected in the form

$$
\psi = w \sin (\lambda \phi \pm n \beta),
$$

(743)

where

$$
\lambda = \frac{m n R}{L}.
$$

(744)

Since the loss of stability of a cylindrical shell under external pressure and uniform axial compression is accompanied by the formation of a large number of annular waves, Eq. (743) approximately satisfies the hinge support boundary conditions at a quite large number of points ($n \beta = k \pi$).
By substituting Eq. (743) in differential equation of stability (742), the following general expression, which depends on two arbitrary parameters \( \lambda, n \) can be obtained for determination of the critical load:

\[
-R'(T_o^2 + T_o^2 n^2 + 25^2 \lambda n)_{eq}^n \\
D_{11} \lambda^4 + 4D_{12} \lambda^3 n + 2(C + D_{12}) \lambda^2 n^2 + 4D_{13} \lambda n^3 + D_{11} n^4 + \\
\frac{R^4 \lambda^4}{\varepsilon_{11} \lambda^4 - 2\varepsilon_{12} \lambda^2 n + (2\varepsilon_{12} + \varepsilon_{66}) \lambda n^3 - 2\varepsilon_{11} \lambda^2 n^2 + \varepsilon_{11} n^4}.
\]  

(745)

64. Optimum Structure of Laminated Plastic in Cylindrical Shell Operating under Uniform External Pressure

Let a laminated cylindrical shell be subjected to uniform external pressure. In this case, on the assumption that \( T_1^0 = S_1^0 = 0 \) and \( T_2^0 = pR \), for determination of the critical pressure, the following calculation formulas can be obtained by Eq. (745):

a. shell made by single thread bias winding

\[
P_{cr} = \frac{\partial}{\partial \theta} \left[ \psi_1(n, \varphi) + \frac{\lambda^4 R^4}{\partial \psi_2(n, \varphi)} \right],
\]  

(746)

where

\[
\psi_1(n, \varphi) = \left[ c_{11} \lambda^4 + 4c_{12} \lambda^3 n + 2(c_{12} + 2c_{66}) \lambda^2 n^2 + \\
+ 4c_{66} \lambda n^3 + c_{66} n^4 \right] \frac{1}{12};
\]

\[
\psi_2(n, \varphi) = \frac{1}{4} \left[ c_{11} \lambda^4 + 2(c_{12} + 2c_{66}) \lambda^2 n^2 + c_{66} n^4 \right];
\]

(747)

elastic constants \( c_{ij}, a_{ij} \) depend on bias winding angle \( \phi \);

b. shell made by cross bias winding

\[
P_{cr} = \frac{\partial}{\partial \theta} \left[ \theta_1(n, \varphi) + \frac{\lambda^4 R^4}{\partial \theta_2(n, \varphi)} \right],
\]  

(748)

where

\[
\theta_1(n, \varphi) = \frac{1}{12} \left[ c_{11} \lambda^4 + 2(c_{12} + 2c_{66}) \lambda^2 n^2 + c_{66} n^4 \right];
\]

\[
\theta_2(n, \varphi) = \frac{1}{4} \left[ c_{11} \lambda^4 + 2(a_{12} + a_{66}) \lambda^2 n^2 + a_{66} n^4 \right].
\]

(749)

In the event \( n^2 >> \lambda^2 \), the following approximate formula can be obtained

\[
P_{cr} = \frac{1.78 R^4}{I R^4} \sqrt{(1-v', v') B_{11} D_{11}^3}.
\]  

(750)

In accordance with Eq. (746), (748), selection of the optimum structure of the laminated plastic for a given fabric is reduced to the following simple procedure: for each value of winding angle \( \phi \), which
changes in the $0^\circ<\phi<90^\circ$ interval, the lowest pressure $p$ is determined as a function of whole number parameter $m$, $(m=1)$. This value determines the critical external pressure of a cylindrical shell made by single thread bias or cross bias winding, i.e.,

$$p_K=p(\phi).$$  \hspace{1cm} (751)

It is evident that the optimum structure of the laminated plastic is determined from the condition that the critical pressure is the highest: $p_K=(p_{cr})_{\text{max}}$ (Fig. 87).

The maximum value of function $p=p(\phi)$ determines the optimum winding angle and the maximum pressure which a cylindrical shell made by continuous winding with a given fabric can stand.

As calculations show, over a wide range of change of elastic constants $E_1$, $E_2$, $G$, $v$, of the fabric and geometric dimensions $R$, $l$, $\delta$ of the shell, the most stable shells made by cross bias winding are, as a rule, shells produced by right angle winding ($\phi=0^\circ$ and $\phi=90^\circ$). Consequently, calculations by Eq. (746) will determine the optimum structure of the laminated plastic. It should be stated that the optimum winding of long cylindrical shells is straight annular winding of the warp, since such shells lose stability in the form of collapse of the cross section and, consequently, the maximum annular rigidity of the shell must be ensured.

The results of calculation to determine the optimum single thread winding angles for some fabric elastic constants and geometric dimensions of shells of medium length are presented in Table 7 and Fig. 88.

It is evident from Table 7 and Fig. 88 that the winding angle in manufacture of shells by single thread bias winding significantly affects the critical external uniform pressure. The optimum winding angle evidently is determined by the elastic constants of the fabric and the geometric dimensions of the shell.

Thus, for the manufacture of cylindrical shells of medium length which operate under uniform external pressure, single thread bias winding may prove to be more expedient. The explanation of this is that single thread bias winding of medium length shells produces anisotropy of the laminated plastic which disturbs the symmetrical nature of the wave formation and forces it to buckle with wave formation at higher pressure. It also follows from this that this conclusion is only valid
for not very long shells.

Calculation formulas (746), (748) can be used for selection of the most nearly optimum fabrics, i.e., fabrics with elastic constants which provide the greatest critical pressure.

### TABLE 7. CRITICAL EXTERNAL PRESSURE OF CYLINDRICAL SHELL VS. WINDING ANGLE

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\frac{a}{R}$</th>
<th>$\frac{r}{a}$</th>
<th>$\frac{t}{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.314</td>
<td>0.504</td>
<td>3.83</td>
</tr>
<tr>
<td>5</td>
<td>0.308</td>
<td>0.514</td>
<td>3.94</td>
</tr>
<tr>
<td>10</td>
<td>0.311</td>
<td>0.513</td>
<td>3.95</td>
</tr>
<tr>
<td>15</td>
<td>0.322</td>
<td>0.507</td>
<td>3.95</td>
</tr>
<tr>
<td>20</td>
<td>0.333</td>
<td>0.496</td>
<td>3.65</td>
</tr>
<tr>
<td>25</td>
<td>0.347</td>
<td>0.483</td>
<td>3.83</td>
</tr>
<tr>
<td>30</td>
<td>0.365</td>
<td>0.472</td>
<td>4.09</td>
</tr>
<tr>
<td>35</td>
<td>0.389</td>
<td>0.467</td>
<td>4.42</td>
</tr>
<tr>
<td>40</td>
<td>0.418</td>
<td>0.470</td>
<td>4.80</td>
</tr>
<tr>
<td>45</td>
<td>0.452</td>
<td>0.482</td>
<td>4.98</td>
</tr>
<tr>
<td>50</td>
<td>0.460</td>
<td>0.472</td>
<td>4.48</td>
</tr>
<tr>
<td>55</td>
<td>0.481</td>
<td>0.477</td>
<td>4.07</td>
</tr>
<tr>
<td>60</td>
<td>0.502</td>
<td>0.481</td>
<td>3.83</td>
</tr>
<tr>
<td>65</td>
<td>0.527</td>
<td>0.487</td>
<td>3.50</td>
</tr>
<tr>
<td>70</td>
<td>0.542</td>
<td>0.492</td>
<td>3.28</td>
</tr>
<tr>
<td>75</td>
<td>0.563</td>
<td>0.502</td>
<td>3.03</td>
</tr>
<tr>
<td>80</td>
<td>0.581</td>
<td>0.511</td>
<td>2.70</td>
</tr>
<tr>
<td>85</td>
<td>0.602</td>
<td>0.520</td>
<td>2.36</td>
</tr>
<tr>
<td>90</td>
<td>0.623</td>
<td>0.530</td>
<td>1.97</td>
</tr>
</tbody>
</table>

Key: a. Curve 1, 2, 3, 4 (Fig. 88)

65. Most Stable Laminated Cylindrical Shell under Axial Uniform Compression

Let a cylindrical shell made of laminated plastic be subjected to uniform axial compression (Fig. 89). In this case, we study the question of the selection of the optimum structure of the laminated plastic which realizes the greatest carrying capacity of the shell at a given weight. Two possible types of elastic property symmetry of the laminated plastic which correspond to single thread bias and cross bias winding, should also be considered here.

According to Eq. (745), with $T_{2}^{\omega} = T_{1}^{\omega} = 0$, the critical load of a laminated cylindrical shell under uniform axial compression is determined by the following expression.
(T_c^*)^{R \frac{d}{R}} = \frac{1}{12} \left[ c_{11} \lambda^4 + 4c_{10} \lambda^3 n + 2(c_{18} + 2c_{m})\lambda^2 n^2 + 4c_{m} \lambda n^3 + c_{10} n^4 \right] + \frac{\lambda^4 (R \frac{d}{R})^4}{16} + \frac{\sqrt{c_{11} + 4c_{10} \mu + 2(c_{18} + 2c_{m})\mu^2 + 4c_{m} \mu^3 + c_{10} \mu^4}}{2c_{m} \mu + (2c_{18} + c_{m}) \mu^2 - 2c_{10} \mu^3 + c_{11} \mu^4} \right] \tag{752}

where $\lambda$, $n$ are arbitrary parameters which define the form of wave formation upon loss of stability, and $c_{ij}$, $a_{ij}$ ($i, j = 1, 2, 3$) are the elasticity constants of the laminated plastic, which depend on the elastic characteristics of the fabric and winding angle in continuous winding with fabrics and which are determined by Eq. (29)-(31).

According to Eq. (752), for calculation of the critical axial load of a laminated cylindrical shell, the following calculation formulas can be obtained, which depend only on random wave formation parameter $\mu = n / \lambda$:

a. shell made by single thread bias winding

\[
(T_c^*)^{R \frac{d}{R}} = \frac{c_{11} + 2(c_{18} + 2c_{m})\mu + 4c_{m} \mu^2 + c_{10} \mu^3}{2c_{m} \mu + (2c_{18} + c_{m}) \mu^2 - 2c_{10} \mu^3 + c_{11} \mu^4} \tag{753}
\]

b. shell made by cross bias winding

\[
(T_c^*)^{R \frac{d}{R}} = \frac{c_{11} + 2(c_{18} + 2c_{m})\mu^2 + c_{m} \mu^3}{c_{11} + 2(c_{18} + 2c_{m})\mu^2 + c_{m} \mu^3} \tag{754}
\]

Determination of the critical load by Eq. (753), (754) is again reduced to finding the minimum of the right side relative to parameter $\mu$ for a given winding angle $\phi$.

The maximum critical load determines the optimum winding angle and, consequently, the optimum structure of the laminated plastic and the upper limit of the carrying capacity of the shell which can be reached by a change in winding angle.

It should be noted that, in uniform axial compression of isotropic cylindrical shells, the critical load obtained by linear theory is in poor agreement with experimental results. For laminated shells, the correspondence of experimental data with the results of calculation of the critical load by linear theory is more satisfactory, and the scatter of the experimental data is not so great as in the case of thin isotropic shells.

Some numerical calculation results. It is known that the elastic properties of fiberglass reinforced plastic are determined primarily by the properties and orientation of the glass filler. The moduli of elasticity in the principal directions of anisotropy depend on the number of
glass fibers oriented in these directions of anisotropy. Consequently, with a given total warp and woof fabric density, the sum of the moduli of elasticity in the principal directions of anisotropy $E_1 + E_2$ remains nearly constant. Based on this, in order to investigate the effect of the shear modulus of the fabric, the initial data given in Table 8 were selected for comparative calculations.

**TABLE 8. INITIAL DATA**

<table>
<thead>
<tr>
<th></th>
<th>$E_i \cdot 10^{-8}$ kN/m$^2$</th>
<th>$E_s \cdot 10^{-8}$ kN/m$^2$</th>
<th>$G \cdot 10^{-8}$ kN/m$^2$</th>
<th>$\nu_i$</th>
<th>$\nu_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>0.4</td>
<td>0.2</td>
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<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>3.5</td>
<td>3.5</td>
<td>0.5</td>
<td>0.2</td>
</tr>
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<td></td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Key: a. Elastic constants of fabric  
  b. Variants  
  c. n/m$^2$

**TABLE 9. CRITICAL LOAD OF CYLINDRICAL SHELL MADE BY CROSS BIAS WINDING**

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
<th>$\epsilon$</th>
<th>$\zeta$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.70</td>
<td>1.22</td>
<td>1.41</td>
<td>1.57</td>
<td>1.72</td>
<td>1.86</td>
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<td>1.74</td>
<td>1.87</td>
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</tr>
<tr>
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<td>1.30</td>
<td>1.48</td>
<td>1.64</td>
<td>1.78</td>
<td>1.91</td>
<td>1.14</td>
</tr>
<tr>
<td>3</td>
<td>0.85</td>
<td>1.39</td>
<td>1.57</td>
<td>1.72</td>
<td>1.85</td>
<td>1.96</td>
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</tr>
<tr>
<td>4</td>
<td>0.96</td>
<td>1.50</td>
<td>1.66</td>
<td>1.80</td>
<td>1.95</td>
<td>2.05</td>
<td>1.22</td>
</tr>
<tr>
<td>5</td>
<td>1.19</td>
<td>1.70</td>
<td>1.84</td>
<td>2.00</td>
<td>2.04</td>
<td>2.16</td>
<td>2.19</td>
</tr>
<tr>
<td>6</td>
<td>1.42</td>
<td>1.87</td>
<td>1.98</td>
<td>2.04</td>
<td>2.13</td>
<td>2.19</td>
<td>1.43</td>
</tr>
</tbody>
</table>

Key: a. Variant

The results of calculation to determine the critical load, obtained by computer, are presented in Table 9.

For shells made by cross bias winding, critical load vs. winding angle is presented in Fig. 90.

As should have been expected, single thread bias winding extremely insignificantly increases the axial critical load of a cylindrical shell.
The explanation of this is that the most nearly optimum ratio of the moduli of elasticity in the axial and annular directions, and the shear modulus which can be obtained by change in winding angle and which should lead to a significant increase in the critical load, is associated with an increase in the degree of freedom, which is expressed by the possibility of the appearance of oblique forms of wave formation upon loss of stability. This leads to a decrease in the critical load.

It should be noted that the conduct of calculations in order to obtain recommendations on the most diverse cases of fabric elasticity property ratios and geometric shell dimensions requires a great amount of work and is hardly advisable. Evidently, it is more reasonable to carry out the calculations for given geometric dimensions and a limited number of fabrics from which the most nearly optimum must be selected, i.e., obtaining the greatest critical load for the weight of the shell.

General Eq. \((745)\) permits calculation to determine critical loads in various types of uniform and simple combined loading of a shell, as well as selection of the optimum fabrics. It should be stated that the form of wave formation depends, in each case, extremely appreciably on the nature of anisotropy of the reinforced plastic. Therefore, the simplifications usually made with respect to order of magnitude in the theory of stability of thin isotropic shells should be made with great caution.

**TABLE 10. CRITICAL LOAD OF CYLINDRICAL SHELLS OF OPTIMUM AND NONOPTIMUM STRUCTURE VS. SHEAR MODULUS OF FABRIC**

<table>
<thead>
<tr>
<th>a Намотка</th>
<th>b (G \cdot 10^{-9} \text{ N/m}^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Однозаходная</td>
<td>1.40</td>
</tr>
<tr>
<td>Перекрестная</td>
<td>2.47</td>
</tr>
<tr>
<td>Пряная</td>
<td>0.7</td>
</tr>
</tbody>
</table>


The critical axial load of a cylindrical shell of optimum structure made by single thread bias and cross bias windings vs. shear modulus of the fabric is presented in Table 10. The critical loads for shells made by right angle winding also are presented.
The calculation results show that the optimum structure of a laminated cylindrical shell operating in axial compression is obtained by continuous cross bias winding of full strength fabrics at angle \( \phi = \pm 45^\circ \). It should be noted that the use of nonfull strength fabrics, for example, with 1:2 anisotropy, highly insignificantly reduces the critical load of the shell. With more clearly defined anisotropy, a more intensive decrease in critical load occurs.

66. Stability of Anisotropic Shells of Rotation as a Result of Uniform Pressure

Let a double curvature anisotropic shell be exposed to forces \( T_1^0 \), \( T_2^0 \), \( S^0 \). Then, by using the theory of tapered shells, the following system of differential equations can be obtained, which describe the local form of loss of stability

\[
I_1(w) = -k_1 \frac{\partial w}{\partial z} + k_1 \frac{\partial w}{\partial y} + T_1^0 \frac{\partial w}{\partial x} + T_2^0 \frac{\partial w}{\partial y} + 2S^0 \frac{\partial w}{\partial z} \frac{\partial w}{\partial y} ;
\]

\[
L_4(\Phi) = k_1 \frac{\partial w}{\partial z} + k_1 \frac{\partial w}{\partial y} ,
\]

where \( L_1() \), \( L_2() \) designate differential operators in up to fourth order partial derivatives

\[
L_1() = \frac{\partial^4}{\partial z^4} + \left[ c_{11} \frac{\partial^4}{\partial z^4} + 4c_{13} \frac{\partial^4}{\partial z^2 \partial y^2} + 2(c_{11} - 2c_{33}) \frac{\partial^4}{\partial z^2 \partial y^2} + 4c_{33} \frac{\partial^4}{\partial z \partial y^4} + c_{33} \frac{\partial^4}{\partial y^4} \right] ;
\]

\[
L_2() = \frac{1}{\rho} \left[ a_{11} \frac{\partial^4}{\partial z^4} - 2a_{13} \frac{\partial^4}{\partial z^2 \partial y^2} + (2a_{13} + a_{33}) \frac{\partial^4}{\partial z \partial y^4} + a_{11} \frac{\partial^4}{\partial y^4} \right] .
\]

System of Eq. (755) is equivalent to the following differential equation

\[
LL_4(w) + \left( k_1 \frac{\partial^2}{\partial z^2} + k_1 \frac{\partial^2}{\partial y^2} \right)^2 w = \left( T_1^0 \frac{\partial^2}{\partial z^2} + T_2^0 \frac{\partial^2}{\partial y^2} + 2S^0 \frac{\partial^2}{\partial z \partial y} \right) L_4(w).
\]

In the case of shells of rotation subjected to external uniform pressure,

\[
T_1^0 = -\frac{\rho R_1}{2} ; \quad T_2^0 = -\frac{\rho R_1}{2} \left( 2 - \frac{R_2}{R_1} \right),
\]

and, consequently, stability Eq. (757) takes the form
By substituting the solution in the form \( w = w_0 \cos (\lambda x + \eta y) \) in Eq. (759), we obtain

\[
\frac{6pR_e}{\delta^2} = \frac{1}{(2 - \frac{R_e}{R})^4} \left( c_{11} \lambda^2 + 4c_{12} \lambda^3 \eta + 2(c_{11} + 2c_{12}) \lambda^3 \eta^2 + 4c_{12} \lambda \eta \eta^2 + c_{22} \eta^4 + \frac{12(n_1 \lambda^2 + k_1 \eta^2)}{12(n_1 \lambda^2 + k_1 \eta^2)} \right). \tag{760}
\]

Instead of arbitrary parameters \( \lambda, \eta \), we introduce new parameters \( u, z \) according to the expressions

\[
u = \lambda^2 + \eta^2; \quad uz = k_2 \lambda^2 + k_1 \eta^2. \tag{761}
\]

It is easy to show that there are the following inequalities:

\[
k_1 < z < k_2, \quad \text{if} \quad k_1 < k_2,
\]

\[
k_2 < z < k_1, \quad \text{if} \quad k_1 > k_2.
\tag{762}
\]

By substitution of new parameters \( u, z \) for parameters \( \lambda, \eta \) in Eq. (760), the following formula can be obtained for determination of the critical external pressure

\[
\frac{6pR_e}{\delta^2} = \frac{1}{(2 - \frac{R_e}{R})^4} \left[ \frac{u}{(k_1 - k_2)^2} f_1(z) + \frac{z^2(k_1 - k_2)^2}{u f_1 (z)} \right], \tag{763}
\]

where homogeneous quadratic functions

\[
f_1(z) = c_{11} (z - k_1)^3 + 4c_{12} (z - k_1)^2 |z - k_1|^\frac{3}{2} (z - k_1) + \\
+ 2(c_{12} + 2c_{32}) |z - k_1|^2 |z - k_2| + 4c_{22} |z - k_2|^2 (z - k_2)( z - k_1)^{\frac{3}{2}} + \\
+ c_{22} (z - k_2)^2;
\]

\[
f_2(z) = \frac{12}{12} \left( a_{11} (z - k_1)^3 - 2a_{10} |z - k_1|^\frac{3}{2} (z - k_1) + \frac{1}{2} (z - k_1)(z - k_2)(z - k_3) - \\
+ (2a_{10} + a_{22}) |z - k_2| |z - k_3| + \\
- 2a_{10} |z - k_2| |z - k_3| \right). \tag{764}
\]
The right side of Eq. (763) has the least value for parameter $u$ at

$$u = \frac{(k_2 - k_1)^2 s}{V f'_(s) f''(s)}$$  \hspace{1cm} (765)$$

and it is determined by the expression

$$\frac{\partial P}{\partial z} = \frac{2s}{2 - R^2} \sqrt{\frac{f'_(s)}{f''(s)}}.$$  \hspace{1cm} (766)$$

At the poles of a shell of rotation, when they are spherical points, i.e., when $R_1 = R_2 = R$, the critical pressure is determined by the expression

$$\frac{p R^2 V^3}{28} = \sqrt{\frac{c_{11} + 4c_{12} \mu + 2(c_{11} - 2c_{20}) \mu^2 + 4c_{13} \mu^3 + c_{16} \mu^4}{a_{11} - 2a_{12} \mu + (2a_{13} + a_{22}) \mu^2 - 2a_{12} \mu^3 + a_{11} \mu^4}},$$  \hspace{1cm} (767)$$

consequently, the critical external pressure on a spherical shell is determined exactly by the same expressions as the critical axial load of a cylindrical shell.

In the general case, the critical external pressure is found by Eq. (766), by minimization of the right side relative to parameter $z$, which inequalities (762) satisfy.
CHAPTER 14. THREE PLY ORTHOTROPIC PLATES AND SHELLS WITH LIGHT ELASTIC FILLERS

67. Hypotheses and Basic Relationships Used

Despite the fact that three ply structures with elastic fillers have a whole set of valuable qualities which are necessary to products in the most diverse fields of the national economy, their use until recently was extremely limited, because of difficulties of high quality manufacture which ensured complete reliability of such structures in operation. With the appearance and extensive use of synthetic materials and reinforced plastics and resins, the possibilities of the development of reliable effective three ply structures with light elastic fillers increased sharply. Synthetic cements produced on phenol-formaldehyde and epoxy resin bases permit reliable joining of bearing layers with honeycomb or foam plastic fillers. More than that, in the manufacture of three ply structures, including metal structures, the cement compounds are generally speaking the only possible ones.

In many engineering applications, namely, when great flexural rigidity is required, the structure of three ply shells permits elimination of the basic structural defect of plastics, comparatively large yielding to deformation. Therefore, highly improved nonmetal three ply structures can be produced, especially with the use of fiberglass reinforced plastics.

The basic types of three ply structures are presented in Fig. 91.

Polyvinyl chloride, polystyrene or polyurethane foam plastics, glass honeycomb plastics, foam plastics, cork and balsa, corrugated or hollow thin walled elements and other light elastic materials can be used as fillers of three ply plates and shells.

Three ply plates and shells with fiberglass reinforced plastic bearing layers are anisotropic. Anisotropy of the elastic properties also arises in the use of asymmetrical honeycomb plastics, corrugated fillers or other anisotropic materials.

It should be noted that a rigorous solution of problems connected with the calculation of three ply plates and shells is an extremely complex problem. Therefore, to obtain visible calculation formulas and calculation methods accessible to engineering analysis, various simplifying assumptions and hypotheses must be introduced. Reference to modern computers does not change the situation, since the entire spectrum of problems encountered in engineering cannot be specified and
programed beforehand. However, of course, this does not exclude and does not reduce the great value of exact methods of calculation.

The total number of works which deal with the calculation of three ply plates and shells is extremely large. However, orthotopic three ply shells have been investigated less thoroughly.

We consider thin three ply plates and shells with light elastic fillers which are symmetrical relative to the mean surface of the structure (Fig. 92). It is assumed that the materials of which the shells are made are orthotropic and that their principal directions of anisotropy coincide with the coordinate directions on the surface of the shell.

In the discussion, we limit ourselves to a class of shells in which the loads acting on the mean surface are entirely absorbed by the bearing layers \(E_1 \delta_2 \ll E_1 \delta_1\).

We use the hypothesis of a rectilinear element, i.e., we will assume that rectilinear elements which are normal before deformation of the mean surface of the shell remain rectilinear after deformation but, generally, not the normal deformed mean surface.

According to this hypothesis, the stresses in the bearing layers and the filler layer are determined by the following expressions:

in bearing layers \(\frac{\delta_2}{2} \leq z \leq \frac{\delta}{2}\)

\[
\begin{align*}
\sigma_1^{(1)} &= E_1 \left[ \varepsilon_1 + \nu_2 \varepsilon_2 + z (\kappa_1 + \nu_2 \kappa_2) \right]; \\
\sigma_2^{(1)} &= E_2 \left[ \varepsilon_2 + \nu_1 \varepsilon_1 + z (\kappa_2 + \nu_1 \kappa_1) \right]; \\
\tau^{(1)} &= G \left( \omega + 2z \kappa_3 \right); \\
\end{align*}
\]

in the filler layer, \(\frac{\delta_2}{2} \leq z \leq \frac{\delta}{2}\)

\[
\begin{align*}
\sigma_1^{(2)} &= E_1^{(2)} \left[ \varepsilon_1 + \nu_2^{(2)} \varepsilon_2 + z (\kappa_1 + \nu_2^{(2)} \kappa_2) \right]; \\
\sigma_2^{(2)} &= E_2^{(2)} \left[ \varepsilon_2 + \nu_1^{(2)} \varepsilon_1 + z (\kappa_2 + \nu_1^{(2)} \kappa_1) \right]; \\
\tau^{(2)} &= G^{(2)} \left( \omega + 2z \kappa_3 \right); \\
\end{align*}
\]

the subscript "2" means that the corresponding value refers to the filler layer.

If the system of stresses which develops in normal sections of the shell is replaced by the statically equivalent system of elastic forces and moments and the effect of transverse compression of the fill-
layer is disregarded, the following basic elasticity relationships can be obtained which connect the stressed and deformed states of three-ply plates and shells:

\[
\begin{align*}
T_1 &= B_1 (\varepsilon_1 + v_1 \varepsilon_2); \\
T_2 &= B_2 (\varepsilon_2 + v_2 \varepsilon_3); \\
S &= B_3 \omega; \\
G_1 &= -D_1 (\kappa_1 + v_1 \kappa_3); \\
G_2 &= -D_2 (\kappa_2 + v_2 \kappa_3); \\
H &= -D_3 \kappa_3; \\
Q_1 &= -K_1 \varepsilon_1; \\
Q_3 &= -K_3 \varepsilon_3.
\end{align*}
\]

(770)

where the rigidity parameters of the shell are determined by the expression

\[
\begin{align*}
B_1 &= \overline{E} \delta_1; \\
B_2 &= \overline{E}_2 \delta_2; \\
B_3 &= G \delta_3; \\
D_1 &= \frac{\overline{E}_1 (\delta^1 - \delta^2)}{4}; \\
D_2 &= \frac{\overline{E}_2 (\delta^2 - \delta^3)}{4}; \\
D_3 &= \frac{G (\delta^2 - \delta^3)}{2}; \\
K_1 &= \frac{2G_{13} (\delta^1 - \delta^3)}{3\delta_1 (\delta^1 + \delta_3)}; \\
K_3 &= \frac{2G_{13} (\delta^2 - \delta^3)}{3\delta_1 (\delta^1 + \delta_3)}.
\end{align*}
\]

(771)

here, \( G_{13}, G_{23} \) are the transverse shear moduli of the filler.

For separate fillers, \( G_{13}, G_{23} \) should be understood to be the reduced shear moduli which are determined either from some theoretical considerations or experimentally.

Components of the deformed state of a three-ply shell \( c_1, \kappa_1, \gamma_1 \) are determined from Eq. (10)-(12).

Since the system of stresses was reduced to the statically equivalent system of elastic forces and moments in the mean surface of the shell, the equilibrium equations are written in the form of (17), with boundary conditions of the canonical type of (23).

In this manner, all results obtained in the preceding sections for the forces and moments of deformation and movements of orthotropic laminated plates and shells are completely valid for three-ply plates and shells with light elastic fillers, if the rigidity parameters are in accordance with Eq. (771).

The difference in calculations of three-ply shells appears only in calculation of the stresses, which are determined by Eq. (768) and (769).

It is easy to see that the results will be valid for three-ply plates and shells, the bearing layers of which are made of various ma-
terials and have varied thickness, if \( \delta_2 > \delta_1 \) and the conditions of elastic symmetry through the shell are satisfied, i.e., if there are the relationships

\[
E_1^\nu \delta_1^\nu - E_1^\nu \delta_1^\nu; \quad E_1^\nu \delta_1^\nu - E_1^\nu \delta_1^\nu; \quad G_1^\nu \delta_1^\nu - G_1^\nu \delta_1^\nu;
\]

\( \nu_1 = \nu_1; \quad \nu_2 = \nu_2. \quad (772) \)

Because the filler may have extremely low transverse strength, the normal transverse stresses must also be determined. The following can be obtained for them:

\[
\sigma_2 = \frac{3(\delta_2^\nu - \delta_1^\nu)}{2(\delta_2^\nu - \delta_1^\nu)} \left( \frac{C_1 + C_2}{N_1 + N_2} \right). \quad (773)
\]

68. Boundary Conditions and Estimate of Error of the Theory As Applied to Three Ply Plates and Shells

The random constants which occur in integration of the differential equations are determined from the boundary conditions. As has been noted, five independent parameters which define the deformed state of laminated shells, even in the case of homogeneous canonical boundary conditions, increase the diversity of types of support fastenings to a considerable extent.

For plates, the system of differential equations breaks down into two, one of which describes the planar stressed state of the plate and the other describes the bending of the mean surface. The boundary conditions are broken down correspondingly. In the case of bending of three ply plates, boundary conditions (22) can give the following graphic static geometric interpretation:

**DIAGRAMS OF SUPPORT FASTENINGS OF THREE PLY PLATES**

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Support Fastening diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G = H = Q = 0 )</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>( w = \phi = \psi = 0 )</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>( w = G = H = 0 )</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
<tr>
<td>( w = G = \psi = 0 )</td>
<td><img src="image" alt="Diagram" /></td>
</tr>
</tbody>
</table>
The first type of boundary condition corresponds to an unsupported end, the second corresponds to rigid fastening, and the remaining types of boundary conditions correspond to various cases of hinge support.

We now estimate the errors allowed by the initial hypotheses and assumptions. It is evident that the hypothesis of the rectilinear element does not take into account the bending of the bearing layers relative to the mean surface itself which occurs in transverse shearing in the filler layer.

A change of curvature and torsion of the mean surface of the bearing layers due to shearing of the filler layer are determined by the following expressions

\[
\begin{align*}
\kappa^i &= -\frac{1}{\kappa_i} \frac{\partial Q_i}{\partial x}; \quad \kappa^j &= -\frac{1}{\kappa_j} \frac{\partial Q_j}{\partial y}; \\
\kappa^c &= -\frac{1}{2} \left( \frac{1}{\kappa_i} \frac{\partial Q_i}{\partial y} + \frac{1}{\kappa_j} \frac{\partial Q_j}{\partial x} \right).
\end{align*}
\]

Consequently, the rectilinear element hypotheses adopted for the entire set of shells are equivalent to disregard of the following quantities in the equilibrium equations, compared with cutting forces \( Q_1, Q_2 \):

\[
\begin{align*}
\frac{D_1 \frac{\partial^2 Q_i}{\partial x^2} + D_2 \frac{\partial^2 Q_j}{\partial y^2} + C \frac{\partial^2 Q_i}{\partial x \partial y}}{K_1}, \\
\frac{D_1 \frac{\partial^2 Q_j}{\partial y^2} + D_2 \frac{\partial^2 Q_i}{\partial x^2} + C \frac{\partial^2 Q_j}{\partial x \partial y}}{K_2}.
\end{align*}
\]

i.e., in determination of the deformed state of the shell, error \( \xi_1 \)

\[1\]In the end effect zones, the errors permitted by the initial assumptions will be larger.
compared with unity is committed:

\[ \zeta_1 = \max \left( \frac{E_1 \delta_1}{G \delta_2 \beta}, \frac{F_1 \delta_1}{G \delta_2 \beta} \right), \]  \hspace{1cm} (776) 

where \( \alpha \) is the characteristic plan dimension of the shell.

In determination of the stressed state, a somewhat larger error is committed, namely, \( \zeta_2 \) compared with unity:

\[ \zeta_2 = \max \left( \frac{E_1 \delta_1 \beta}{G \delta_2 \beta}, \frac{F_1 \delta_1 \beta}{G \delta_2 \beta} \right). \]  \hspace{1cm} (777) 

Since usually \( \delta_2/\delta_1 \approx 0.1-0.3 \) in three ply shells, according to the rectilinear element hypothesis, the deformed state is defined with sufficient accuracy. The error in determination of the stressed state is somewhat higher. However, as needed, when error (777) is indeterminate, the stress can be refined if the accuracy in determination of the deformed state is sufficient.

Refinement of the stresses is carried out by Eq. (774), by which the bending of the bearing layers due to shear in the filler layer is determined. If a coordinate system in the mean surface of the bearing layers is selected (Fig. 93) and \( z \) designates the distance of the fibers from the mean surface of the bearing layer, the components of the deformation which originate in the bearing layers due to transverse shear in the filler layer are determined by the expressions

\[
\begin{align*}
\epsilon_i &= \bar{z} (x_i + v_i x_i); \\
\epsilon_j &= \bar{z} (x_j + v_j x_j); \\
\epsilon_n &= 2 \bar{z} x_i. 
\end{align*}
\]  \hspace{1cm} (778) 

Consequently, the maximum additional stresses which were disregarded are determined by the following expressions

\[
\begin{align*}
\sigma_{1 \text{ max}} &= E_1 \frac{\delta_1}{4} (x_1^2 + v_1 x_1^2); \\
\sigma_{2 \text{ max}} &= E_2 \frac{\delta_1}{4} (x_2^2 + v_2 x_2^2); \\
\tau_{\text{ max}} &= G \frac{\delta_1}{2} x_i. 
\end{align*}
\]  \hspace{1cm} (779) 

Finally, if the shell is such that errors in determination of the deformed state are inadmissible, theories must be used which are based on less rigid hypotheses. However, the complexity of solution of engineering problems increases significantly here.
It was assumed in the initial hypotheses that the effect of transverse deformations of the filler could be disregarded. It is easy to show that the error of such an assumption has the value $\zeta_3$ compared with unity:

$$\zeta_3 = \max \left( \frac{E_1 \delta_1}{E_f R^4}; \frac{E_2 \delta_2}{E_f R^4} \right).$$

(780)

where $E_z$ is the transverse modulus of elasticity of the filler.

In accordance with Eq. (780), in the case of very soft fillers, i.e., filler which satisfy the condition $E_z/E_1 \delta_1 \delta_2/R^2$, the effect of transverse deformations must be taken into account.

69. Differential Equation of Symmetrical Form of Loss of Stability

In Chapter 5, solutions of some problems of stability of laminated plates were obtained which, with the abovementioned stipulations, are valid for three ply plates with an elastic filler. The forms of loss of stability which were considered there are characterized by distortion of the mean plane of the plate, i.e., they are asymmetric relative to the mean plane.

As the calculation formulas obtained show, the critical loads in the asymmetric form of loss of stability increase with increase in thickness of the filler layer. However, this relationship occurs up to a certain thickness, beginning at which a further increase in thickness of the filler layer in order to increase the critical load is useless, since the possibility of loss of stability in a fundamentally different form appears (symmetrical wrinkling of the bearing layers occurs relative to the mean plane). The critical load of the symmetrical form of loss of stability depends little on filler layer thickness, and such a form of instability is characteristic only of three ply plates and shells with elastic fillers, although it is found in laminated structures in the form of flaking.

The forms of loss of stability which can occur in three ply plates and shells with light elastic fillers are shown in Fig. 94.

We obtain the differential equation of the symmetrical form of loss of stability by using the energy method. The total potential energy of the plate is made up of the potential energy of bending of the bearing layers, the potential energy of the filler and the work of external forces, and it is determined by the following expression [13]:

$$U = \int \int \Phi(w_{xx}, w_{yy}, w_{xy}, w_x, w_y, w) \, dx \, dy,$$

(787)
where

\[
\Phi(u_{xx}, u_{yy}, u_{xy}, u_x, u_y, w) = \begin{align*}
D_1^{(1)} u_{xx}^2 + 2D_2^{(1)} u_{xx} u_{xy} + & D_3^{(1)} u_{xy}^2 + 4D_3^{(1)} u_{yy}^2 + \\
& + \frac{1}{2} \left(T_0^x u_x^2 + T_0^y u_y^2 + 2S_0 u_{xx} u_{yy} \right) + \\
& + \frac{\pi h^2 - \pi^2}{2\pi h^2 \xi} \left(K_1 u_x^2 + K_2 u_y^2 \right) + \frac{\pi h^2 - \pi^2}{2\pi h^2 \xi} \mu^2.
\end{align*}
\]

(782)

here \(D_1^{(1)}, D_2^{(1)}, D_3^{(1)}, \ldots \) are the rigidity parameters of the bearing layers of the plate.

\[
D_1^{(1)} = \frac{E_1 \delta^2}{9(1-\nu_1\nu_2)}; \\
D_2^{(1)} = \frac{G \delta^2}{9(1-\nu_1\nu_2)}; \\
D_3^{(1)} = \frac{G \delta^2}{9}\; ; \\
D_4^{(1)} = \nu_1 D_4^{(1)} + \nu_2 D_4^{(1)}; \\
K_1 = G_{13} \delta_2; \quad K_2 = G_{23} \delta_2.
\]

(783)

\(E_z\) is the reduced modulus of normal transverse elasticity of the filler;

\[
E_z = \frac{\pi^4}{\sum_{i=1}^n \sum_{j=1}^n a_{ij}^4}.
\]

(784)

\(a_{ij}\) are the elasticity constants of an orthotropic filler as a three dimensional body; \(\xi\) is a random parameter proportional to filler layer thickness \(\delta_2\), determinable from the condition of the minimum critical load.

Because the potential energy of the plate has a minimum in the equilibrium position, deflection of the bearing layers in symmetrical wrinkling \(w\) should satisfy the following differential equation:

\[
\frac{\partial^4}{\partial z^4} \Phi_{w_z} + \frac{\partial^2}{\partial x \partial y} \Phi_{w_{xy}} + \frac{\partial^2}{\partial y^2} \Phi_{w_{yy}} - \frac{\partial}{\partial x} \Phi_{w_x} - \frac{\partial}{\partial y} \Phi_{w_y} + \Phi_w = 0.
\]

(785)

By expanding Eq. (785) according to Eq. (782), the following differential equation can be obtained, which describes the symmetrical form of loss of stability of three ply plates, i.e., the stability loss /242 phenomenon associated with wrinkling of the bearing layers:
\[ D_1^{(i)} \frac{\partial^2 w}{\partial x^2} + 2(D_1^{(i)} + 2D_2^{(i)}) \frac{\partial^2 w}{\partial x \partial y} + \]
\[ + D_2^{(i)} \frac{\partial^2 w}{\partial y^2} - \frac{q_1(\xi)}{4} \left( K_1 \frac{\partial^2 w}{\partial x^2} + K_2 \frac{\partial^2 w}{\partial y^2} \right) + E_r \frac{\partial w}{\partial x} \]
\[ = T_1^{(i)} \frac{\partial^2 w}{\partial x^2} + 2S_0 \frac{\partial^2 w}{\partial x \partial y} + T_2^{(i)} \frac{\partial^2 w}{\partial y^2}. \]  

(786)

where, for convenience in practical use, functions \( \phi_1, \phi_2 \) of random parameter \( \xi \) are introduced:

\[ q_1(\xi) = \frac{\xi (\sin 2\xi + 2\xi)}{\sin^2 \xi}; \quad q_2(\xi) = \frac{\sin 2\xi}{\xi \sin^2 \xi}. \]  

(787)

**TABLE 11. VALUES OF FUNCTIONS \( \phi_1, \phi_2, \phi_3 \)**

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<tr>
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<th>( \phi_1(t) )</th>
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The values of functions \( \phi_1, \phi_2 \) are presented in Table 11, where the value of function \( \phi_3(\xi) \) also is given, which can be determined by the expression

\[ q_3(\xi) = \frac{1}{4} \frac{q_1(\xi)}{\xi}. \]  

(788)
As an illustration, we consider some simplest examples of determination of critical loads in which symmetrical wrinkling of the bearing layers occurs relative to the mean plane.

Cylindrical wrinkling of bearing layers of three ply strip. Let a rectangular three ply plate with two opposite unsupported ends be compressed uniformly by distributed forces $T_i$ in the direction of the unsupported ends. It is evident that wrinkling of the bearing layers will occur in only one direction (Fig. 95) and, consequently, differential equation of stability (786) takes the form

$$D_1^{(1)} \frac{\partial^4 w}{\partial x^4} - \frac{1}{4} q_1(\xi) K_1 \frac{\partial^2 w}{\partial t^2} + E \frac{\partial^2 w}{\partial \xi^2} q_1(\xi) w = -T_i \frac{\partial^2 w}{\partial \xi^2}. \quad (790)$$

With hinge support of ends $x=0$, $x=\ell$, the solution can be sought in the form $w=A \sin \frac{m \pi x}{\ell}$. By substituting this solution in Eq. (790), the following expression can be obtained for determination of the critical load:

$$T_i = \left( \frac{m \pi}{\ell} \right)^4 D_1^{(1)} + \frac{E q_1(\xi)}{E_1} K_1 q_1(\xi). \quad (791)$$

Fig. 95. Stability of three ply panels in longitudinal compression. The right side of Eq. (791) will have the smallest value at

$$\frac{m \pi}{\ell} = \sqrt[4]{\frac{E_1 q_1(\xi)}{E q_1(\xi)}}. \quad (792)$$

Consequently, the critical force is determined by the expression

$$T_{cr} = 2 \sqrt{D_1^{(1)} E_1 q_1(\xi) + \frac{K_1}{4} q_1(\xi)}. \quad (793)$$

Parameter $\xi$ is selected from the condition of the minimum of the right side of Eq. (793).
In the case of soft fillers, when the number of half waves in the longitudinal direction is small, it should be kept in mind that parameter \( \xi \) satisfies condition (792).

If it turns out that the minimum of the right side of Eq. (793) occurs with a value of parameter \( \xi \) which corresponds to \( m = l \), this means that only one half wave forms in buckling in the direction of compression, i.e., the critical load should be found by the formula

$$ T_{cr} - \frac{\pi^2 D^{(1)}}{l^2} + \frac{E_1 \varphi_1(\xi)}{\pi b_1} + \frac{K_1}{4} \varphi_1(\xi), $$

(794)

where parameter \( \xi \) again is determined from the condition of the minimum of the right side.

It also is of interest to obtain calculation formulas for determination of the critical force of a three ply strip with a comparatively thick filler layer.

In this case, it should be assumed that \( \xi \to \infty \) and, consequently, according to Eq. (793),

$$ T_{cr} - \frac{\Omega_{13}}{1 - \nu^{(1)}_1 \nu^{(1)}_3} \frac{E^{(1)}_1 E_{13}}{E_1 G_{13}}. $$

(795)

Correspondingly, in the case of soft thick fillers, when one half wave develops along its length in loss of stability in the direction of compression, according to Eq. (794), the critical load is determined by the expression

$$ T_{cr} - \frac{\pi^2 D^{(1)}}{l^2} + \frac{2l}{\pi} E_1 G_{13}. $$

(796)

Stability of hinge supported plate during compression in one direction. In compression of a rectangular plate in one principal direction of anisotropy, differential equation of stability (786) takes the form

$$ D_1^{(1)} \frac{\partial^4 w}{\partial x^4} + 2(D_2^{(1)} + 2D_3^{(1)}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2^{(1)} \frac{\partial^4 w}{\partial y^4} - \frac{4}{4} (K_1 \frac{\partial^2 w}{\partial x^2} + K_3 \frac{\partial^2 w}{\partial y^2}) + E_1 \varphi_1(\xi) w = -T \frac{\partial^2 w}{\partial x^2}. $$

(797)

By substituting the solution in Eq. (797) in the form \( w = A \sin \lambda x \sin \eta y \), where

$$ \lambda = \frac{m \pi}{a} ; \quad \eta = \frac{n \pi}{b}, $$

(798)
for determination of the critical load, the following expression can be obtained:

\[
T_{cr} = D_1^{(1)} \frac{\lambda^2}{4} + \frac{D_2^{(1)} \eta^4 + \frac{K_1}{4} \varphi_4(\xi) + \frac{E_z}{\delta_1} \varphi_1(\xi)}{\xi^2} +
\]

\[
+ 2 (D_3^{(1)} + 2D_4^{(1)}) \eta^2 + \frac{K_1}{4} \varphi_2(\xi).
\]  

(799)

It is easy to see that the smallest value of the right side of Eq. (799) occurs with the smallest value of parameter \( \eta \), i.e., when, during buckling, one transverse half wave forms \( n=1, \eta=\pi/b \).

For determination of the critical load, the following formula can be obtained

\[
T_{cr} = \sqrt{D_1^{(1)} \left[ D_2^{(1)} \left( \frac{\pi}{b} \right)^4 + \frac{K_1}{4} \left( \frac{\pi}{b} \right)^2 \varphi_4(\xi) + \frac{E_z}{\delta_1} \varphi_1(\xi) \right] +
\]

\[
+ 2 (D_3^{(1)} + 2D_4^{(1)}) \left( \frac{\pi}{b} \right)^2 + \frac{K_1}{4} \varphi_2(\xi)}
\]  

(800)

with

\[
\frac{m \pi}{a} - \sqrt{\left[ \frac{K_1^{(1)}}{E_1^{(1)}} \left( \frac{\pi}{b} \right)^4 + \frac{K_1}{4u^{(1)}} \varphi_4(\xi) \right] + \frac{E_z}{\delta_1} \varphi_1(\xi)}.
\]  

(801)

Parameter \( \xi \) again is selected from the condition of the minimum of the right side of Eq. (800).

Condition (801) must be kept in mind in the case of soft fillers. If it turns out that \( m<1 \), the critical load must be determined by Eq. (800) with \( \lambda=\pi/a \).

**Axisymmetric wrinkling of bearing layers of three ply cylindrical shell in uniform axial compression.** We now consider axisymmetric buckling of the bearing layers of an orthotropic cylindrical shell in uniform axial compression (Fig. 96).

Because of symmetry, a unit strip of unit width can be considered. Because of the curvature and elasticity of the filler, the bearing layers are kept from buckling by a double elastic base as it were and, consequently, according to Eq. (790), the differential equation of stability can be written in the following form:

\[
D_1^{(1)} \frac{d^4 w}{dx^4} - \frac{K_1}{4} \varphi_1(\xi) \frac{d^2 w}{dx^2} + \left[ \frac{E_z}{\delta_1} \varphi_1(\xi) + \frac{E_z^{(1)}}{2H^2} \right] w = -T \frac{d^2 w}{dx^2}.
\]  

(802)

By substituting the solution in Eq. (802) in the form \( w=A \sin \) mmx/\( l \), the following expression can be obtained for determination of the critical load:
Fig. 96. Stability of three ply cylindrical shell in uniform axial compression.

\[ T_{cr} = D_1^{(1)} \left( \frac{m_1}{l} \right)^2 + \left[ \frac{E_1}{\delta_1} q_1(\xi) + \frac{E_2^{(1)} \delta_1}{2R^2} \right] + \frac{K_1}{4} q_4(\xi). \]  

(803)

The smallest value of the right side of Eq. (803) occurs with

\[ \frac{m_1}{l} = 4 \left( \frac{E_i}{\delta_1} q_1(\xi) + \frac{48E_1^{(1)}(1-v_1v_2)}{E_1^{(1)} R^2 \delta_1} \right) \]  

(804)

and it is determined by the expression

\[ T_{cr} = 2 \left[ D_1^{(1)} \left( \frac{E_1}{\delta_1} q_1(\xi) + \frac{E_2^{(1)} \delta_1}{2R^2} \right) + \frac{K_1}{4} q_4(\xi) \right]. \]  

(805)

Parameter \( \xi \) in Eq. (805) is determined from the condition of the minimum critical force.

In the case of a cylindrical shell with a quite thick filler layer, the critical load is determined by the formula

\[ T_{cr} = 2 \sqrt{D_1^{(1)}(E_1u + \frac{E_2^{(1)} \delta_1}{2R^2}) + \frac{G_{13}}{u}}, \]  

(806)

where parameter \( u \) is selected from the condition of the minimum of the right side or as the positive root of the equation

\[ u^4 - \frac{G_{13}^2}{D_1^{(1)} E_1} u - \frac{G_{13}^2 E_2^{(1)} \delta_1}{2D_1^{(1)} E_2 R^2} = 0. \]  

(807)
### Appendix 1

**VLASOV FUNCTIONS $\phi_1 - ch \gamma \cdot cos ky$**

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#### VLASOV FUNCTIONS $\phi_2 = y \cdot \sin ky$

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Appendix 2 continued

The table above contains data for various values of r and p, with columns for different values of the parameter. The values range from 0.05 to 1.00, with increments of 0.05. Each column represents a different value of the parameter, with the values increasing from left to right. The data is presented in a tabular format, with each row corresponding to a specific value of r - p and each column representing a different value of k = 1.00. The table is used to provide a quick reference for calculations involving the parameter values.
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### Appendix 3

**VLASOV FUNCTIONS $\phi_3 = \text{sh} y \cdot \cos ky**

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This table provides values for $\phi_3$ as a function of $y+p$, with $y$ and $p$ being the variables of interest. The values are calculated using the Vlasov functions, which are particularly useful in plasma physics and quantum mechanics.
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## Appendix 4

### Vlasov Functions \( \phi_y = \text{ch} (y \cdot \sin ky) \)

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