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LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS

V.I. Korolev

Translation of "Sloistyye anizotropnyye plastinki i obolochki iz armirovannykh plastmass," Mashinostroyeniye Press, Moscow, 1965, pp. 1-272

(NASA-TM-76585) LAMINATED ANISOTROPIC
REINFORCED PLASTIC PLATES AND SHELLS
(National Aeronautics and Space
Administration) 241 p HC A11/MF A01

N82-12474

Unclass
08336

CSCL 20K G3/39



STANDARD TITLE PAGE

1. Report No. NASA TM-76585	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS		5. Report Date October 1981	6. Performing Organization Code
7. Author(s) V.I. Korolev		8. Performing Organization Report No.	10. Work Unit No.
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		11. Contract or Grant No. NASW-3541	13. Type of Report and Period Covered Translation
12. Sponsoring Agency Name and Address National Aeronautics and Space Adminis- tration, Washington, D.C. 20546		14. Sponsoring Agency Code	
15. Supplementary Notes Translation of "Sloistyie anizotropnyye plastinki i obolochki iz armirovannykh plastmass," Mashinostroyeniye Press, Moscow, 1965, pp. 1-272			
16. Abstract Basic technical theories and engineering calculation equations for anisotropic plates and shells made of rigid reinforced plastics, mainly laminated fiberglass, are presented and discussed. Solutions are given for many problems of design of structural plates and shells, including curved sections and tanks, as well as two chapters on selection of the optimum materials, are given. Accounting for interlayer shearing and transverse separation, which are new engineering properties, are discussed. Application of the results obtained to thin three ply plates and shells with a light elastic filler is presented and discussed.			
17. Key Words (Selected by Author(s))		18. Distribution Statement Unlimited-Unclassified	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages	22.

ANNOTATION

The basic technical theories of anisotropic plates and shells made of rigid reinforced plastics are reported in the book. Solutions of numerous technical problems most often encountered in engineering practice are obtained, with recommendations on efficient design of elastic reinforced plastic parts. Some sections are entirely devoted to questions of selection of the optimum structure of the material.

The results obtained are valid for thin three ply plates and shells, if appropriate substitution is made of the rigidity parameters, which are among the most efficient stress schemes for reinforced plastic structures.

The book is intended for engineering and technical workers who are engaged in the development of thin walled reinforced plastic structures.

FOREWORD

Shells of laminated plastics made on a base of continuous fillers and various synthetic binders are multiply anisotropic, heterogeneous elastic systems. Some simplifying hypotheses permit study of deformed, stressed states of a shell to be reduced to study of bending and deformation of a surface of reduction. In this case, the system of stresses which develop in normal sections of the shell are replaced by a statically equivalent system of elastic forces and moments.

It follows from such a reduction that, to within the assumptions made, a structurally nonuniform laminated shell can be considered a uniform shell, i.e., laminated plastics can be considered uniform materials, which have some reduced properties which are determined by the properties of the initial components and the mutual location and orientation of the reinforcing filler.

Skillful use of the abovementioned properties permits the development of extremely efficient laminated plastic structures which, in many cases, are not inferior in efficiency and technical characteristics to excellent structures made of traditional materials. The defects inherent in welded metal structures, which significantly reduce the critical loads or result in premature destruction, are absent in thin walled laminated structures.

With the appearance of new structural materials, fiberglass plastics, the field of use of laminated plastics in engineering broadened substantially, and the technical and economic advantages of their use increased sharply. The development of methods of calculation of thin walled laminated plastic structures becomes of great practical importance in this connection. The attention of investigators was first drawn to these questions by the founder of the Soviet School of Anisotropic Plates S.G. Lekhnitskiy, the results of many years of study of which are reported in his monographs [16, 17].

The laminated plastics used in engineering have symmetrical elastic properties in the majority of cases, i.e., they are orthotropic materials. However, their principal directions of anisotropy may not coincide with the directions of the coordinate axes and, consequently, it becomes necessary to consider the elasticity relationships which correspond to the general case of anisotropy. For orthotropic materials, there are reliable methods of determination of the necessary mechanical characteristics in two principal directions of anisotropy. Moreover, fundamentally new characteristics of the laminated orthotropic material must be known, which usually do not have to be dealt with in isotropic uniform shells, namely: the shearing strength by layer and the transverse separation strength. These new characteristics of laminated plastics are associated with their structural inhomogeneity and the significant difference of the elastic and strength properties under various types of loads.

This book reports an approximate method of accounting for the effect of interlayer shear on the stressed, deformed state of laminated anisotropic plates and shells. In selection of simplifying hypotheses

for study of thin laminated shells, it was considered that the elastic properties of existing cements and binders are appreciably less than the corresponding elastic characteristics of the reinforcing fillers and, consequently, the interlayer shears which develop in the bending of laminated shells can significantly distort the pattern of the deformed state described by the hypotheses of nondeformable normals extensively used in the theory of shells, especially when the shell operates under heating conditions.

The results of thorough studies of thin laminated anisotropic shells, with interlayer shear and transverse deformation taken into account, are reported in the monograph of S.A. Ambartsumyan [1]. Since the corresponding rigidity parameters of a laminated sheet differ significantly, allowance for transverse deformation gives an extremely insignificant correction, and we will disregard its effect.

The proposed approximate method of calculation of laminated shells was used in study of three ply plates and shells, and it showed satisfactory correspondence with experimental results. Besides the usual elastic characteristics of a laminated shell, two new ones appear K_1 and K_2 , which define the connection of the cross forces to interlayer shears of the mean surface and characterize the resistance of the laminated shell to such shears in two mutually orthogonal directions. Laminated shell rigidity parameters K_1 and K_2 are determined experimentally in transverse bending tests of laminated strips and, consequently, they somewhat compensate the errors which are tolerated by the initial hypotheses adopted.

The results obtained in the work are valid for three ply plates and shells with a light elastic filler, if appropriate rigidity parameters are used. This question is presented in detail in the last chapter, where some stability problems characteristic only of three ply plates and shells with elastic fillers also are discussed.

In distinction from the traditional courses on the theory of shells, the author attempted to discuss problems connected with efficient design of plates and shells made of reinforced plastics, subjected to the effects of the loads most frequently encountered in engineering practice. Chapters 8 and 13 are completely devoted to questions of selection of the optimum structure of the material of cylindrical shells operating under axisymmetric loads.

The present work does not cover many questions raised by modern engineering practice and the needs of machine building. Problems connected with large displacements of the mean surface of a shell, including problems of stability are not touched on. Nonlinear elastic and inelastic deformations of laminated shells are not discussed, and questions of nonlinear oscillations are not covered. There is no doubt that they will be treated in the works of other investigators in the near future.

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LAMINATED ANISOTROPIC REINFORCED PLASTIC PLATES AND SHELLS

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CHAPTER 1. BASIC EQUATIONS OF TECHNICAL THEORY OF ANISOTROPIC PLATES AND SHELLS

1. Initial Hypotheses and Basic Relationships

We will consider laminated shells produced on a base of continuous fillers and synthetic binders as uniform anisotropic systems, the elastic constants of which are determined by conventional methods. /7*

We use the rectilinear element hypothesis [14] for description of the deformed state of the shell. This is satisfactorily consistent with experimental results for three ply plates with a light elastic filler.

Based on this hypothesis and the assumption that the normal stresses are independent of interlayer shear for the stresses which arise in normal sections of a shell, we have the following expressions

$$\left. \begin{aligned} \sigma_1 &= c_{11}(e_1 + z\kappa_1^e) + c_{12}(e_2 + z\kappa_2^e) + c_{13}(\omega + 2z\kappa_3^e); \\ \sigma_2 &= c_{12}(e_1 + z\kappa_1^e) + c_{22}(e_2 + z\kappa_2^e) + c_{23}(\omega + 2z\kappa_3^e); \\ \tau &= c_{13}(e_1 + z\kappa_1^e) + c_{23}(e_2 + z\kappa_2^e) + c_{33}(\omega + 2z\kappa_3^e), \end{aligned} \right\} \quad (1)$$

where c_{ij} ($i, j=1, 2, 3$) are the elastic constants of the material; e_1, e_2, ω are the relative elongation and shear of the mean surface of the shell; $\kappa_1^e, \kappa_2^e, \kappa_3^e$ are the effective changes in curvature and torsion of the mean surface of the shell.

As in [14], for shearing stresses which act between the layers of the shell, the following expressions can be obtained

$$\tau_1 = \frac{3}{2} \frac{(\delta^2 - 4z^2)}{\delta^3} Q_1; \quad \tau_2 = \frac{3}{2} \frac{(\delta^2 - 4z^2)}{\delta^3} Q_2. \quad (2)$$

In distinction from thin three ply shells with an elastic filler, shearing through the layers in a laminated shell change by a parabolic relationship and disappear at the bounding surfaces $z = \pm \delta/2$.

We will account for the effect of surface load X, Y on the shearing stresses in the membrane solution, on the assumption, as in [1], that the shearing stresses change linearly through the thickness of the /8

*Numbers in the margin indicate pagination in the foreign text.

shell.

We will characterize the shearing stresses between the layers which correspond to shearing stresses (2) by the maximum shears which arise in the mean surface of the shell. Like the stresses, shears through the thickness of the shell change parabolically.

By reducing stress system (1)-(2) to the statically equivalent system of elastic forces T_1 , T_2 , S and moments G_1 , G_2 , H , the following elasticity relationships can be obtained for a laminated anisotropic shell

$$\left. \begin{aligned} T_1 &= B_{11} \epsilon_1 + B_{12} \epsilon_2 + B_{13} \omega; \\ T_2 &= B_{12} \epsilon_1 + B_{22} \epsilon_2 + B_{23} \omega; \\ S &= B_{13} \epsilon_1 + B_{23} \epsilon_2 + B_{33} \omega; \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} G_1 &= -D_{11} \kappa_1' - D_{12} \kappa_2' - 2D_{13} \kappa_3'; \\ G_2 &= -D_{12} \kappa_1' - D_{22} \kappa_2' - 2D_{23} \kappa_3'; \\ H &= -D_{13} \kappa_1' - D_{23} \kappa_2' - 2D_{33} \kappa_3'; \end{aligned} \right\} \quad (4)$$

$$Q_1 = -K_1 \gamma_1; \quad Q_2 = -K_2 \gamma_2. \quad (5)$$

where shell rigidity parameters D_{ij} , B_{ij} , K_1 , K_2 ($i, j=1, 2, 3$), in the case of a sufficiently large number of layers, are determined through reduced elastic constants of the material c_{ij} ($i, j=1, 2, 3$), G_{13} , G_{23} and shell thickness δ by the following expressions

$$B_{ij} = c_{ij} \delta; \quad D_{ij} = \frac{1}{12} c_{ij} \delta^3 \quad (i, j = 1, 2, 3); \quad (6)$$

$$K_1 = \frac{5}{6} G_{13} \delta; \quad K_2 = \frac{5}{6} G_{23} \delta. \quad (7)$$

where G_{13} , G_{23} are the moduli of elasticity in interlayer shearing.

The rigidity parameters of a laminated shell also can be determined from the simplest experiments, which are described schematically in Section 8.

By solving expressions (3) for the components of deformation, the known relationships, which will be needed subsequently, can be obtained

$$\left. \begin{aligned} \epsilon_1 &= \frac{1}{\delta} (a_{11} T_1 + a_{12} T_2 + a_{13} S); \\ \epsilon_2 &= \frac{1}{\delta} (a_{12} T_1 + a_{22} T_2 + a_{23} S); \\ \omega &= \frac{1}{\delta} (a_{13} T_1 + a_{23} T_2 + a_{33} S). \end{aligned} \right\} \quad (8)$$

In the general case of anisotropy, elastic constants a_{ij} ($i, j=1, 2, 3$) are independent, and they are expressed through elastic constants c_{ij} by the expressions

$$\left. \begin{aligned} a_{11} &= \frac{c_{22}c_{33} - c_{23}^2}{\Delta}; & a_{12} &= \frac{c_{12}c_{23} - c_{13}c_{22}}{\Delta}; \\ a_{22} &= \frac{c_{22}c_{11} - c_{12}^2}{\Delta}; & a_{13} &= \frac{c_{12}c_{23} - c_{13}c_{22}}{\Delta}; \\ a_{33} &= \frac{c_{11}c_{22} - c_{12}^2}{\Delta}; & a_{23} &= \frac{c_{13}c_{12} - c_{11}c_{23}}{\Delta}; \\ \Delta &= c_{11}c_{22}c_{33} + 2c_{12}c_{13}c_{23} - c_{11}c_{23}^2 - \\ &\quad - c_{22}c_{12}^2 - c_{33}c_{13}^2. \end{aligned} \right\} \quad (9)$$

The deformed state of a laminated shell is defined by five random functions: three components of displacement u, v, w in the directions of coordinate axes α_1, α_2 , which coincide with the lines of curvature of the mean surface of the shell, and by the external normals to the mean surface and by two functions ϕ, ψ which characterize the bending of the shell without allowance for the effect of interlayer shearing

$$\left. \begin{aligned} e_1 &= \frac{1}{A_1} \frac{\partial u}{\partial \alpha_1} + \frac{v}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2} + \frac{w}{R_1}; \\ e_2 &= \frac{1}{A_2} \frac{\partial v}{\partial \alpha_2} + \frac{u}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} + \frac{w}{R_2}; \\ \omega &= \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{v}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{u}{A_1} \right); \end{aligned} \right\} \quad (10)$$

$$\left. \begin{aligned} \kappa_1^e &= \frac{1}{A_1} \frac{\partial \phi}{\partial \alpha_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2}; \\ \kappa_2^e &= \frac{1}{A_2} \frac{\partial \psi}{\partial \alpha_2} + \frac{\phi}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1}; \\ 2\kappa_3^e &= \frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{\psi}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{\phi}{A_1} \right); \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} \gamma_1 &= \phi + \frac{1}{A_1} \frac{\partial w}{\partial \alpha_1} - \frac{u}{R_1}; \\ \gamma_2 &= \psi + \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}. \end{aligned} \right\} \quad (12)$$

As curvilinear coordinates α_1, α_2 on the mean surface of the shell, it is advisable to consider only coordinates which form a rectilinear regular grid of coordinate lines. Geometrically, this condition is reduced to the requirement that vectors $\bar{r}_{\alpha_1}, \bar{r}_{\alpha_2}$, which are tangent to coordinate lines α_1, α_2 , not be collinear. For this, it is necessary analytically that, of the three jacobians

$$\frac{\partial(x, y)}{\partial(a_1, a_2)}, \frac{\partial(y, z)}{\partial(a_1, a_2)}, \frac{\partial(z, x)}{\partial(a_1, a_2)} \quad (13)$$

(where x, y, z are coordinates in the vector parametric representation of the surface) at least one be different from zero.

If the coordinate lines coincide with the lines of curvature of the mean surface, the curvilinear coordinate is orthogonal, and we subsequently will use primarily such a coordinate system.

In such a coordinate system, Lamé's constants A_1, A_2 are determined by the expressions

$$\begin{aligned} A_1 &= \sqrt{\left(\frac{\partial x}{\partial a_1}\right)^2 + \left(\frac{\partial y}{\partial a_1}\right)^2 + \left(\frac{\partial z}{\partial a_1}\right)^2}; \\ A_2 &= \sqrt{\left(\frac{\partial x}{\partial a_2}\right)^2 + \left(\frac{\partial y}{\partial a_2}\right)^2 + \left(\frac{\partial z}{\partial a_2}\right)^2}. \end{aligned} \quad (14)$$

The equations of continuity of deformation for laminated anisotropic shells are written as in the case of isotropic shells, and they have the form

$$\left. \begin{aligned} &\frac{\partial A_2 \kappa_2}{\partial a_1} - \frac{1}{A_1} \frac{\partial A_1^2 \kappa_2}{\partial a_2} - \frac{\partial A_2}{\partial a_1} \kappa_1 - \\ &- \frac{1}{H_1} \left(\frac{\partial A_2 \varepsilon_2}{\partial a_1} - \frac{1}{A_1} \frac{\partial A_1^2 \omega}{\partial a_1} - \frac{\partial A_2}{\partial a_1} \varepsilon_1 \right) + \\ &+ \frac{\partial}{\partial a_1} \left(\frac{A_2 \omega}{H_2} \right) + \frac{1}{H_2} \frac{\partial A_2}{\partial a_1} \frac{\omega}{2} = 0; \\ &\frac{\partial A_1 \kappa_1}{\partial a_2} - \frac{1}{A_2} \frac{\partial A_2^2 \kappa_1}{\partial a_1} - \frac{\partial A_1}{\partial a_2} \kappa_2 - \\ &- \frac{1}{H_2} \left(\frac{\partial A_1 \varepsilon_1}{\partial a_2} - \frac{1}{A_2} \frac{\partial A_2^2 \omega}{\partial a_1} - \frac{\partial A_1}{\partial a_2} \varepsilon_2 \right) + \\ &+ \frac{\partial}{\partial a_2} \left(\frac{A_1 \omega}{H_1} \right) + \frac{1}{H_1} \frac{\partial A_1}{\partial a_2} \frac{\omega}{2} = 0; \\ &\frac{\kappa_1}{H_1} + \frac{\kappa_2}{H_2} + \frac{1}{A_1 A_2} \left\{ \frac{\partial}{\partial a_1} \frac{1}{A_2} \left(\frac{\partial A_2 \varepsilon_2}{\partial a_2} - \frac{1}{A_1} \frac{\partial A_1^2 \omega}{\partial a_2} - \frac{\partial A_2}{\partial a_1} \varepsilon_1 \right) + \right. \\ &\left. + \frac{\partial}{\partial a_2} \frac{1}{A_1} \left(\frac{\partial A_1 \varepsilon_1}{\partial a_1} - \frac{1}{A_2} \frac{\partial A_2^2 \omega}{\partial a_1} - \frac{\partial A_1}{\partial a_2} \varepsilon_2 \right) \right\} = 0; \end{aligned} \right\} \quad (15)$$

here $\varepsilon_1, \varepsilon_2, \omega$ are the components of deformation and shear of the mean /11 surface of the shell determined by expressions (10), and $\kappa_1, \kappa_2, \kappa_3$ are

the total changes of curvature and torsion of the mean surface, which can be determined by the following expressions

$$\left. \begin{aligned} \kappa_1 &= \kappa_1^e - \frac{1}{A_1} \frac{\partial \gamma_1}{\partial \alpha_1} - \frac{\gamma_2}{A_1 A_2} \frac{\partial A_1}{\partial \alpha_2}; \\ \kappa_2 &= \kappa_2^e - \frac{1}{A_2} \frac{\partial \gamma_2}{\partial \alpha_2} - \frac{\gamma_1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1}; \\ 2\kappa_3 &= 2\kappa_3^e - \left[\frac{A_2}{A_1} \frac{\partial}{\partial \alpha_1} \left(\frac{\gamma_2}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial \alpha_2} \left(\frac{\gamma_1}{A_1} \right) \right]. \end{aligned} \right\} \quad (16)$$

2. Equations of Equilibrium of Laminated Shell and Boundary Conditions

If the positive directions of the elastic forces and moments are selected as indicated in Fig. 1, the equations of equilibrium in the system of curvilinear orthogonal coordinates selected can be written in conventional form

$$\left. \begin{aligned} \frac{\partial (A_2 T_1)}{\partial \alpha_1} + \frac{\partial (A_1 S)}{\partial \alpha_2} - T_2 \frac{\partial A_2}{\partial \alpha_1} + S \frac{\partial A_1}{\partial \alpha_2} - A_1 A_2 \frac{Q_1}{R_1} + \\ + A_1 A_2 X = 0; \\ \frac{\partial (A_1 T_2)}{\partial \alpha_2} + \frac{\partial (A_2 S)}{\partial \alpha_1} - T_1 \frac{\partial A_1}{\partial \alpha_2} + S \frac{\partial A_2}{\partial \alpha_1} - A_1 A_2 \frac{Q_2}{R_2} + A_1 A_2 Y = 0; \\ \frac{\partial (A_2 Q_1)}{\partial \alpha_1} + \frac{\partial (A_1 Q_2)}{\partial \alpha_2} + A_1 A_2 \left(\frac{T_1}{R_1} + \frac{T_2}{R_2} \right) - A_1 A_2 Z = 0; \\ \frac{\partial (A_2 G_1)}{\partial \alpha_1} + \frac{\partial (A_1 H)}{\partial \alpha_2} - G_2 \frac{\partial A_2}{\partial \alpha_1} + H \frac{\partial A_1}{\partial \alpha_2} - A_1 A_2 Q_1 = 0; \\ \frac{\partial (A_1 G_2)}{\partial \alpha_2} + \frac{\partial (A_2 H)}{\partial \alpha_1} - G_1 \frac{\partial A_1}{\partial \alpha_2} + H \frac{\partial A_2}{\partial \alpha_1} - A_1 A_2 Q_2 = 0; \end{aligned} \right\} \quad (17)$$

in addition, for determination of the normal transverse stress σ_z , the following expression can be obtained, which characterizes equilibrium of the internal and external forces in the normal direction

$$\begin{aligned} \sigma_z = \frac{1}{2\delta^3} [Z^+ (\delta^3 + 3\delta^2 z - 4z^3) - Z^- (\delta^3 - 3\delta^2 z + 4z^3)] + \\ + \frac{(\delta^2 - 4z^2)}{2\delta^3} \left[3 \left(\frac{C_1}{R_1} + \frac{G_2}{R_2} \right) - z \left(\frac{T_1}{R_1} + \frac{T_2}{R_2} \right) \right], \end{aligned} \quad (18)$$

where Z^+ , Z^- are components of the normal surface load applied to the upper ($z=\delta/2$) and lower ($z=-\delta/2$) bounding surfaces of the shell, respectively.

The boundary conditions for an anisotropic laminated shell, the deformed state of which is described by the rectilinear element hypotheses, differ significantly from the boundary conditions for shells which are deformed by the direct normals principle. We subsequently limit ourselves to cases when the edge of the shell coincides with the coordinate lines. In the case of closed shells or shells closed in one direction,

the boundary conditions are replaced by the corresponding periodicity conditions, which should ensure well defined movement at any point of the closed coordinate line.

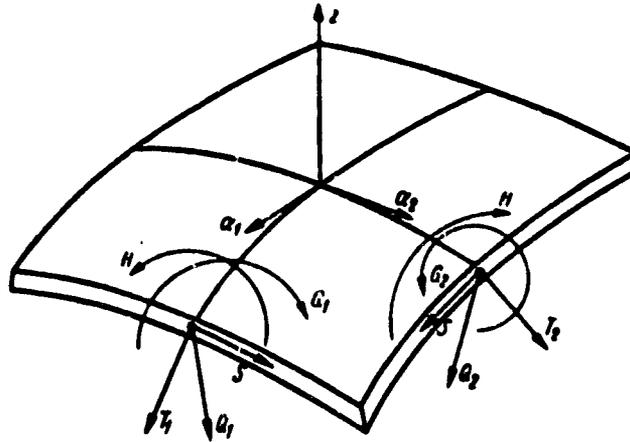


Fig. 1. Adopted positive directions of elastic forces and moments.

Variation of the potential energy of a laminated shell is determined by the following expression

$$\delta U = \iint (T_1 \delta \epsilon_1 + T_2 \delta \epsilon_2 + S \delta \omega - G_1 \delta \kappa_1^e - G_2 \delta \kappa_2^e - 2H \delta \kappa_3^e - Q_1 \delta \gamma_1 - Q_2 \delta \gamma_2 + X \delta u + Y \delta v + Z \delta w) A_1 A_2 da_1 da_2. \quad (19)$$

By substituting relationships (10)-(12) in Eq. (19), we obtain

$$\begin{aligned} \delta U = & \iint \left\{ T_1 \delta \left(\frac{1}{A_1} \frac{\partial u}{\partial a_1} + \frac{v}{A_1 A_2} \frac{\partial A_1}{\partial a_2} + \frac{w}{R_1} \right) + \right. \\ & + T_2 \delta \left(\frac{1}{A_2} \frac{\partial v}{\partial a_2} + \frac{u}{A_1 A_2} \frac{\partial A_2}{\partial a_1} + \frac{w}{R_2} \right) + \\ & + S \delta \left[\frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left(\frac{v}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial a_2} \left(\frac{u}{A_1} \right) \right] - \\ & - G_1 \delta \left(\frac{1}{A_1} \frac{\partial \varphi}{\partial a_1} + \frac{\psi}{A_1 A_2} \frac{\partial A_1}{\partial a_2} \right) - G_2 \delta \left(\frac{1}{A_2} \frac{\partial \psi}{\partial a_2} + \frac{\varphi}{A_1 A_2} \frac{\partial A_2}{\partial a_1} \right) - \\ & - H \delta \left[\frac{A_2}{A_1} \frac{\partial}{\partial a_1} \left(\frac{\psi}{A_2} \right) + \frac{A_1}{A_2} \frac{\partial}{\partial a_2} \left(\frac{\varphi}{A_1} \right) \right] - \\ & - Q_1 \delta \left(\varphi + \frac{1}{A_1} \frac{\partial w}{\partial a_1} - \frac{u}{R_1} \right) - Q_2 \delta \left(\psi + \frac{1}{A_2} \frac{\partial w}{\partial a_2} - \frac{v}{R_2} \right) + X \delta u + \\ & \left. + Y \delta v - p \delta w \right\} A_1 A_2 da_1 da_2. \quad (20) \end{aligned}$$

By partial integration, we obtain

$$\begin{aligned}
\delta U = & \int [T_1 A_2 \delta u + S A_2 \delta v - G_1 A_2 \delta \phi - H A_2 \delta \psi - Q_1 A_2 \delta w]_{\alpha_1}^{\alpha_2} da_2 + \\
& + \int [T_2 A_1 \delta v + S A_1 \delta u - G_2 A_1 \delta \psi - H A_1 \delta \phi - Q_2 A_1 \delta w]_{\beta_1}^{\beta_2} da_1 + \\
& + \iint \left\{ \left[\frac{\partial A_1 T_1}{\partial a_1} + \frac{\partial A_1 S}{\partial a_2} - T_2 \frac{\partial A_2}{\partial a_1} + S \frac{\partial A_1}{\partial a_2} - A_1 A_2 \frac{Q_1}{R_1} + A_1 A_2 X \right] \delta u + \right. \\
& + \left[\frac{\partial A_1 T_2}{\partial a_2} + \frac{\partial A_2 S}{\partial a_1} - T_1 \frac{\partial A_1}{\partial a_2} + S \frac{\partial A_2}{\partial a_1} - A_1 A_2 \frac{Q_2}{R_2} + A_1 A_2 Y \right] \delta v - \\
& - \left[\frac{\partial A_2 Q_1}{\partial a_1} + \frac{\partial A_1 Q_2}{\partial a_2} + A_1 A_2 \left(\frac{T_1}{R_1} + \frac{T_2}{R_2} \right) - A_1 A_2 P \right] \delta w - \\
& - \left[\frac{\partial A_2 G_1}{\partial a_1} + \frac{\partial A_1 H}{\partial a_2} - G_2 \frac{\partial A_2}{\partial a_1} + H \frac{\partial A_1}{\partial a_2} - A_1 A_2 Q_1 \right] \delta \phi - \\
& \left. - \left[\frac{\partial A_1 G_2}{\partial a_2} + \frac{\partial A_2 H}{\partial a_1} - G_1 \frac{\partial A_1}{\partial a_2} + H \frac{\partial A_2}{\partial a_1} - A_1 A_2 Q_2 \right] \delta \psi \right\} da_1 da_2.
\end{aligned} \tag{21}$$

On the strength of the independence of variations of δu , δv , δw , $\delta \phi$ and $\delta \psi$, the boundary conditions and equilibrium Eq. (17) follow as a result of this.

We present the boundary conditions for an edge which is bounded by the coordinate line $\alpha_1 = \text{const}$. The boundary conditions are decomposed into boundary conditions for tangential forces and movements and boundary conditions for normal forces, moments, deflection and displacement functions ϕ , ψ .

The following uniform boundary conditions can occur for the tangential forces and movements

$$\left. \begin{aligned}
T_1 = 0, \quad S = 0; \quad u = 0, \quad v = 0; \\
T_2 = 0, \quad v = 0; \quad u = 0, \quad v = 0.
\end{aligned} \right\} \tag{22}$$

Correspondingly, for the normal forces, moments, deflection and displacement functions ϕ , ψ , the following uniform boundary conditions (canonical form) can occur

$$\left. \begin{aligned}
G_1 = H = Q_1 = 0; \quad \varphi = H = Q_1 = 0; \\
G_1 = H = w = 0; \quad \varphi = H = w = 0; \\
G_1 = \psi = Q_1 = 0; \quad \varphi = \psi = Q_1 = 0; \\
G_1 = \psi = w = 0; \quad \varphi = \psi = w = 0.
\end{aligned} \right\} \tag{23}$$

In the case of flat plates, conditions (22) and (23) determine the boundary conditions for the two dimensional problem and for bending, respectively. For shells, boundary conditions (22) and (23) are set, i.e., when the deformed state of a laminated shell is described by the rectilinear element hypotheses, an extremely great diversity of boundary conditions can occur, namely, 32 different cases of canonical edge supporting anchors. /14

Thus, for example:

a. unsupported edge

$$T_1 = S = G_1 = H = Q_1 = 0;$$

b. rigidly fastened edge

$$u = v = w = \phi = \psi = 0;$$

c. edge unsupported in the tangential direction and rigidly fastened in the transverse direction

$$T_1 = S = w = \phi = \psi = 0;$$

d. an edge rigidly fastened in the tangential direction and unsupported in the transverse direction

$$u = v = G_1 = H = Q_1 = 0$$

etc.

There is a diversity of boundary conditions because, compared with the undeformable normal hypotheses, the rectilinear element hypotheses take into account two additional degrees of freedom of movement, which are characterized by the magnitudes of the interlayer shears in two directions.

3. Laminated Shells of Varied Orthotropic Structure

As has been noted, laminated shells produced on a continuous filler base can have diverse structures, depending on the mutual location and orientation of the filler.

We will assume each unit layer to be uniform and orthotropic, with elastic constants $E_1, E_2, G, \nu_1, \nu_2$; the well known relationship $E_1 \nu_2 = E_2 \nu_1$ occurs here.

1. If the principal axes of anisotropy coincide with the coordinate axis in production of the shell, the material of the laminated shell will be orthotropic, its elastic constants will coincide with the elastic constants of the unit layer and basic relationships (3)-(5), which connect the deformed and stressed states of the shell, are simplified and take the form

$$\left. \begin{aligned} T_1 &= B_1(\epsilon_1 + \nu_2 \epsilon_2); \\ T_2 &= B_2(\epsilon_2 + \nu_1 \epsilon_1); \\ S &= B_3 \omega; \end{aligned} \right\} \quad (24)$$

$$\left. \begin{aligned} G_1 &= -D_1(\kappa_1^e + \nu_2 \kappa_2^e); \\ G_2 &= -D_2(\kappa_2^e + \nu_1 \kappa_1^e); \\ H &= -2D_3 \kappa_3^e; \end{aligned} \right\} \quad (25)$$

$$Q_1 = -K_1 \gamma_1; \quad Q_2 = -K_2 \gamma_2. \quad (26)$$

where the rigidity parameters of the shell are determined by the expressions

$$\left. \begin{aligned} B_1 &= \frac{E_1 \delta}{1 - \nu_1 \nu_2}; \quad B_2 = \frac{E_2 \delta}{1 - \nu_1 \nu_2}; \quad B_3 = G \delta; \\ D_1 &= \frac{E_1 \delta^3}{12(1 - \nu_1 \nu_2)}; \quad D_2 = \frac{E_2 \delta^3}{12(1 - \nu_1 \nu_2)}; \\ D_3 &= \frac{G \delta^3}{12}; \\ K_1 &= \frac{5}{6} G_{13} \delta; \quad K_2 = \frac{5}{6} G_{23} \delta. \end{aligned} \right\} \quad (27)$$

By solving Eq. (24) for $\epsilon_1, \epsilon_2, \omega$, the following relationships can be obtained for an orthotropic shell

$$\left. \begin{aligned} \epsilon_1 &= \frac{T_1 - \nu_1 T_2}{B_1(1 - \nu_1 \nu_2)}; \\ \epsilon_2 &= \frac{T_2 - \nu_2 T_1}{B_2(1 - \nu_1 \nu_2)}; \\ \omega &= \frac{S}{B_3}. \end{aligned} \right\} \quad (28)$$

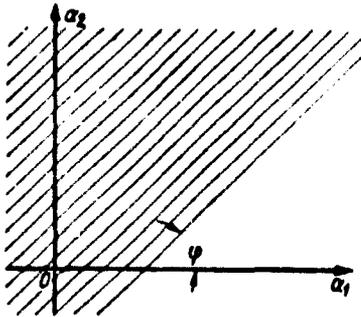


Fig. 2. Diagram of reinforcing of orthotropic plate reinforced in directions not coincident with the coordinate axes.

where

$$\left. \begin{aligned} c_{11} &= \bar{E}_1 \cos^4 \varphi + 2\bar{E}_3 \sin^2 \varphi \cos^2 \varphi + \bar{E}_2 \sin^4 \varphi; \\ c_{12} &= \bar{E}_1 \nu_2 + (\bar{E}_1 + \bar{E}_2 - 2\bar{E}_3) \sin^2 \varphi \cos^2 \varphi; \\ c_{22} &= \bar{E}_1 \sin^4 \varphi + 2\bar{E}_3 \sin^2 \varphi \cos^2 \varphi + \bar{E}_2 \cos^4 \varphi; \\ c_{33} &= G + (\bar{E}_1 + \bar{E}_2 - 2\bar{E}_3) \sin^2 \varphi \cos^2 \varphi; \\ c_{13} &= \frac{1}{2} (\bar{E}_2 \sin^2 \varphi - \bar{E}_1 \cos^2 \varphi + \bar{E}_3 \cos 2\varphi) \sin 2\varphi; \\ c_{23} &= \frac{1}{2} (\bar{E}_2 \cos^2 \varphi - \bar{E}_1 \sin^2 \varphi - \bar{E}_3 \cos 2\varphi) \sin 2\varphi, \end{aligned} \right\} \quad (29)$$

$$\bar{E}_1 = \frac{E_1}{1 - \nu_1 \nu_2}; \quad \bar{E}_2 = \frac{E_2}{1 - \nu_1 \nu_2}; \quad \bar{E}_3 = 2G + \bar{E}_2 \nu_1. \quad (30)$$

Elastic constants a_{ij} ($i, j=1, 2, 3$) in Eq. (8) also are not independent, and they are determined by the formulas

$$\left. \begin{aligned} a_{11} &= \frac{\cos^4 \varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1} \right) \sin^2 \varphi \cos^2 \varphi + \frac{\sin^4 \varphi}{E_2}; \\ a_{22} &= \frac{\sin^4 \varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1} \right) \sin^2 \varphi \cos^2 \varphi + \frac{\cos^4 \varphi}{E_2}; \\ a_{33} &= \frac{\cos^2 2\varphi}{G} + \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} \right) \sin^2 2\varphi; \\ a_{12} &= - \left[\frac{\nu_1}{E_1} - \frac{1}{4} \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G} \right) \sin^2 2\varphi \right]; \\ a_{13} &= \left[\frac{\sin^2 \varphi}{E_2} - \frac{\cos^2 \varphi}{E_1} + \frac{1}{2} \left(\frac{1}{G} - \frac{2\nu_1}{E_1} \right) \cos 2\varphi \right] \sin 2\varphi; \\ a_{23} &= \left[\frac{\cos^2 \varphi}{E_2} - \frac{\sin^2 \varphi}{E_1} - \frac{1}{2} \left(\frac{1}{G} - \frac{2\nu_1}{E_1} \right) \cos 2\varphi \right] \sin 2\varphi. \end{aligned} \right\} \quad (31)$$

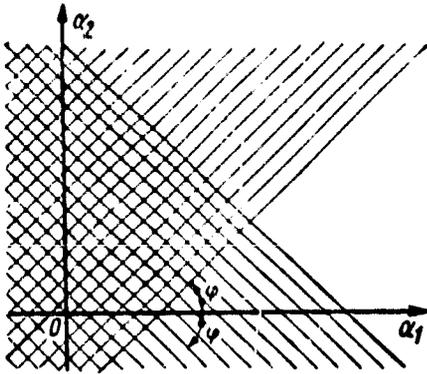


Fig. 3. Diagram of reinforcing of orthotropic plastic with crossed bias reinforcing.

3. Still another case of practical importance can be presented, when the structure of a laminated plastic ensures that it has orthotropic elastic properties [14]. With a large number of unit layers which are cross laid at angles $\pm\phi$ (Fig. 3), the laminated plastic can be considered orthotropic.

In this case, the basic relationships which connect the stressed and deformed states of the shell are represented by expressions (24)-(26). The rigidity parameters of the shell are determined by Eq. (27), where E_1 , E_2 , G , ν_1 are the elastic constants of the laminated material.

If the elastic constants of a unit orthotropic layer of filler E_1^0 , F_2^0 , G^0 , ν_1^0 are known, the following should be assumed in Eq. (27)

$$\left. \begin{aligned} E_1 &= c_{11}; & E_2 &= c_{22}; & G &= c_{33}; \\ \nu_2 &= \frac{c_{12}}{c_{11}}; & \nu_1 &= \frac{c_{12}}{c_{22}}, \end{aligned} \right\} \quad (32)$$

where c_{11} , c_{22} , c_{33} , c_{12} are determined by relationships (29), i.e., they depend on the cross laying angle of the filler.

CHAPTER 2. CYLINDRICAL BENDING OF RECTANGULAR PLATES

4. General Expressions for Calculation of Bending of Laminated Beams

Some simplest cases of determination of the deformed and stressed states in cylindrical bending of rectangular plates with bending rigidity D and rigidity K with respect to interlayer shear are discussed in this chapter. For extremely narrow plates, the Poisson coefficients in the expressions for D should be considered to equal zero. All the results obtained below are presented for a strip of unit width. /18

The coordinate system and symbols adopted in this chapter are indicated in Fig. 4. It is assumed that one of the principal directions of elasticity coincides with the x axis. The basic elasticity relationships in accordance with Eq. (24)-(26) have the form

$$G = -D\varphi'; \quad Q = -K(\varphi + w'). \tag{33}$$

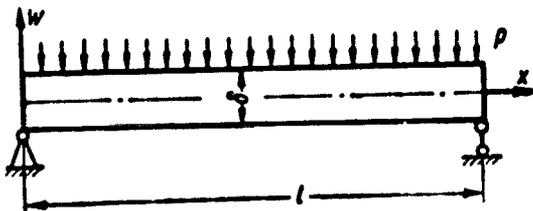


Fig. 4. Coordinate system and basic symbols.

From Eq. (33) and the equilibrium equations

$$Q' = -p; \quad G' = Q, \tag{34}$$

general expressions can be obtained for determination of the stressed and deformed states of rectangular strips in cylindrical bending. Since system of Eq. (33), (34) is equivalent to one fourth order differential equation four

random integration constants appear in the general solution, which are determined from the boundary conditions

$$\left. \begin{aligned} Q &= -px + C_1; \quad G = -\frac{px^2}{2} + C_1x + C_2; \\ \varphi &= \frac{px^3}{6D} - \frac{C_1x^2}{2D} - \frac{C_2x}{D} + C_3; \\ w &= -\frac{px^4}{24D} + \frac{C_1x^3}{6D} + \left(\frac{p}{K} + \frac{C_2}{D}\right)\frac{x^2}{2} - \\ &\quad - \left(\frac{C_1}{K} + C_3\right)x + C_4. \end{aligned} \right\} \tag{35}$$

General Eq. (35) are valid along the entire length of the strip, and only constants $C_i (i=1, 2, 3, 4)$ differ in sections which differ by the nature of the loading or by rigidities D or K . To determine the new random constants with each such section, the conditions of smoothness and continuity of conjugation /19

$$w_h = w_{h+1}; \quad \varphi_h = \varphi_{h+1}; \quad G_h = G_{h+1}; \quad Q_h = Q_{h+1}. \tag{36}$$

are added to the boundary conditions.

5. Bending of Cantilever Strip with Concentrated and Uniformly Distributed Loads

a. Bending of cantilever by concentrated force applied on unsupported end (Fig. 5). After determination of constants C_j ($j=1, 2, 3, 4$) from the boundary conditions

$$w=0; \quad \phi=0 \text{ at } x=0$$

$$G=0; \quad Q=p \text{ at } x=l;$$

from Eq. (35), with $p=0$, we obtain

$$\left. \begin{aligned} G &= P(x-l); \quad Q = P; \\ w &= \frac{Px}{6D} \left(x^2 - 3lx - 6\frac{D}{K} \right); \\ \varphi &= \frac{Px}{2D} (2l - x). \end{aligned} \right\} \quad (37)$$

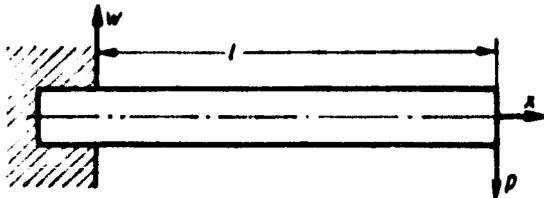


Fig. 5. Diagram of bending of cantilever by force applied to unsupported end.

The maximum deflection of the strip at the unsupported end /20

$$|w|_{\max} = \frac{Pl^2}{3D} (1 + 3\gamma) \quad (38)$$

Here and subsequently, the symbol for the relative give of the strip by interlayer shearing is introduced

$$\gamma = \frac{D}{Kl^2}. \quad (39)$$

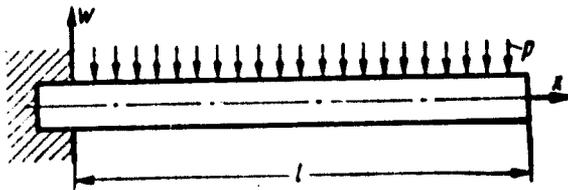


Fig. 6. Diagram of bending of cantilever by uniformly distributed load.

Correspondingly, the greatest values of the normal and shearing stresses are

$$\sigma_{\max} = \frac{6G}{\delta^3} = \frac{6Pl}{\delta^3}; \quad \tau_{\max} = \frac{3Q}{2\delta} = \frac{3P}{2\delta}. \quad (40)$$

b. Bending of cantilever by uniformly distributed load (Fig. 6). By determining the random constants in a manner similar to the preceding from the boundary conditions

$$w=0, \quad \phi=0 \text{ at } x=0;$$

$$G=0, \quad Q=0 \text{ at } x=l,$$

we obtain

$$\left. \begin{aligned} Q &= p(l-x); & G &= -\frac{p}{2}(x-l)^2; \\ \varphi &= \frac{px}{6D}(x^2 - 3lx + 3l^2); \\ w &= -\frac{px^2}{24D}[x^2 - 4lx^2 + 6l^2x - 12\gamma(x-2l)l^2]. \end{aligned} \right\} \quad (41)$$

The maximum deflection of the unsupported end of the beam is

$$|w|_{\max} = \frac{pl^4}{8D}(1+4\gamma). \quad (42)$$

6. Bending of Hinge Supported Strip

We consider several cases of loading of a strip which are most often encountered in various engineering applications.

a. Pure bending of strip (Fig. 7). Integration constants C_1 ($i=1, 2, 3, 4$) can be determined, for example, from these boundary conditions

$$\left. \begin{aligned} w &= 0, & Q &= 0, & G &= M_0 & \text{at } x=0; \\ \varphi &= 0 & \text{at } x = \frac{l}{2}; \\ Q &= 0, & G &= M_0; \\ \varphi &= \frac{M_0}{2D}(l-2x); \\ w &= \frac{M_0x}{2D}(x-l). \end{aligned} \right\} \quad (43)$$

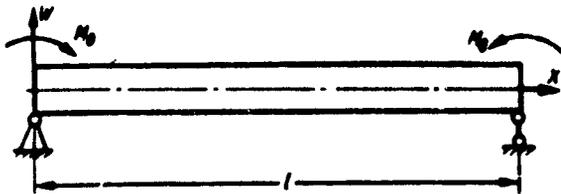


Fig. 7. Diagram of pure bending of strip.

The maximum deflection of the center section of the beam

$$|w|_{\max} = \frac{M_0 l^3}{8D}. \quad (44)$$

In pure bending, interlayer shear is absent, and the greatest normal stresses are constant over the length of the strip.



Fig. 8. Diagram of bending of hinge supported strip by concentrated force applied in center.

b. Bending of strip by concentrated force applied in middle section (Fig. 8). Because of symmetry, the left half of the strip alone can be considered $0 \leq x \leq l/2$.

Boundary conditions:

$$\begin{aligned} \text{at } x=0 & \quad w=0, \quad G=0, \quad Q=\frac{P}{2}; \\ \text{at } x=\frac{l}{2} & \quad \varphi=0. \end{aligned}$$

After determination of random constants C_j ($j=1, 2, 3, 4$), we obtain

$$\left. \begin{aligned} Q &= \frac{P}{2}; \quad G = \frac{Px}{2}; \\ \varphi &= \frac{P}{16D} (l^3 - 4x^3); \\ w &= \frac{Px}{48D} (4x^2 - 3l^2 - 24\gamma l^3). \end{aligned} \right\} \quad (45)$$

The maximum deflection in the center of the strip

22

$$|w|_{\max} = \frac{Pl^3}{48D} (1 + 12\gamma). \quad (46)$$

The greatest shearing stresses are constant in each half of the beam, and the absolute value is

$$\tau_{\max} = \frac{3}{4} \frac{P}{\delta}. \quad (47)$$

The normal stress in the critical section of the beam

$$\sigma_{\max} = \frac{3Pl}{2\delta^3}. \quad (48)$$

c. Bending of strip under uniformly distributed load (Fig. 9). After determination of the random constants from the boundary conditions

$$\text{at } x=0 \quad w=0, \quad G=0;$$

$$\text{at } x=\frac{l}{2} \quad \varphi=0, \quad Q=0,$$

we obtain

$$\left. \begin{aligned} Q &= \frac{P}{2} (l - 2x); \quad G = \frac{Px}{2} (l - x); \\ \varphi &= \frac{P}{24D} (4x^3 - 6lx^2 + l^3); \\ w &= -\frac{Px}{24D} [x^3 - 2lx^2 + l^3 + 12\gamma(l-x)l^2]. \end{aligned} \right\} \quad (49)$$

The maximum deflection of the strip in the center

23

$$|w|_{\max} = \frac{5Pl^4}{384D} (1 + 48\gamma). \quad (50)$$

The greatest shearing stress between the layers occurs at the ends of the strip

$$\tau_{\max} = \frac{3Pl}{4\delta}. \quad (51)$$



Fig. 9. Diagram of bending of hinge supported strip by uniformly distributed load.

The normal stresses in the critical section

$$\sigma_{\max} = \frac{3}{4} \frac{Pl^3}{\delta^3}. \quad (52)$$

7. Bending of Rigidly Fastened Strip

In the case of rigid fastening of the ends in the simplest cases of loading, it is especially easy to obtain expressions for the elastic forces and deflection.

a. Bending of strip by concentrated force applied in center (Fig. 10).

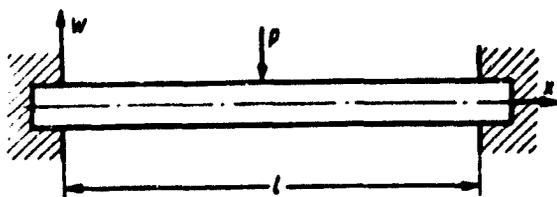


Fig. 10. Diagram of bending of strip with rigidly fastened ends by concentrated force.

The boundary conditions

$$\text{at } x=0 \quad w=0 \quad \phi=0, \quad Q=\frac{P}{2};$$

$$\text{at } x=\frac{l}{2} \quad \phi=0.$$

After determination of the random constants, we have

$$\left. \begin{aligned} Q &= \frac{P}{2}; \quad G = \frac{P}{8} (4x - l); \\ \varphi &= -\frac{Px}{8D} (2x - l); \\ w &= \frac{Px}{48D} (4x^2 - 3lx - 24\gamma l^2). \end{aligned} \right\} \quad (53)$$

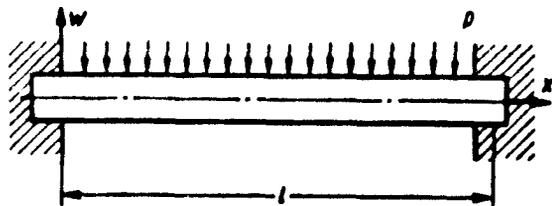


Fig. 11. Diagram of bending of strip with rigidly fastened ends by uniform loading.

The maximum deflection in the center of the strip

$$|w|_{\max} = \frac{Pl^3}{492D} (1 + 48\gamma). \quad (54)$$

The shearing stresses are constant $\frac{P}{24}$ over the length

$$\tau_{\max} = \frac{3}{4} \frac{P}{\delta}. \quad (55)$$

The normal stresses in the critical section

$$\sigma_{\max} = \frac{3Pl}{4\delta^3}. \quad (56)$$

b. Bending of strip by uniformly distributed load (Fig. 11).

The boundary conditions

at $x=0$ $w=0$, $\phi=0$;

at $x=\frac{l}{2}$ $\phi=0$, $Q=0$.

Accordingly, for the elastic forces and deformations of a sheet, the following expressions can be obtained

$$\left. \begin{aligned} Q &= \frac{p}{2}(l-2x); \quad G = -\frac{p}{12}(6x^2 - 6lx + l^2); \\ \varphi &= \frac{px}{12D}(2x^2 - 3lx + l^2); \\ w &= -\frac{p}{24D}[x^3(x-l)^2 + 12\gamma x(l-x)l^2]. \end{aligned} \right\} \quad (57)$$

The maximum deflection in the center

$$|w|_{\max} = \frac{pl^4}{384D}(1 + 48\gamma). \quad (58)$$

The greatest shearing stresses arise at the ends of the strip

$$\tau_{\max} = \frac{3pl}{4\delta}. \quad (59)$$

The normal stresses in the critical center section of the strip

$$\sigma_{\max} = \frac{pl^3}{8\delta^3}. \quad (60)$$

8. Experimental Determination of Elastic and Rigidity Parameters of Orthotropic Laminated Shells

For determination of the complete set of rigidity characteristics /25 of a laminated orthotropic material, tensile, torsion, clean and transverse bending tests of rectangular strips cut in the principal directions of anisotropy are required.

With standard specimens under tension, tensile rigidities B_1 , B_2 , moduli of elasticity E_1 , E_2 and Poisson coefficients ν_1 , ν_2 are obtained, which should satisfy the condition $E_1\nu_2 = E_2\nu_1$.

Flexural rigidities of the laminated shell D_1 , D_2 are determined by Eq. (27), if the reduced flexural and tensile and flexural moduli of elasticity are the same. Otherwise, flexural rigidities must be determined in clean bending tests of rectangular specimens according to the symmetrical two cantilever beam system (Fig. 12).

It evidently is advisable to provide for clean bending tests in the principal directions of anisotropy in all cases, if only as control tests, the more so that they are the simplest.

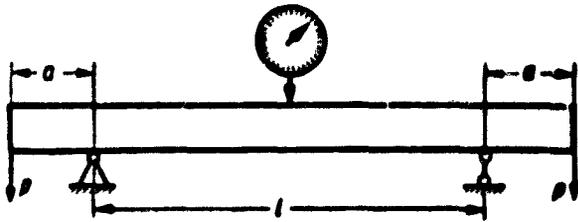


Fig. 12. Diagram of clean bending test.

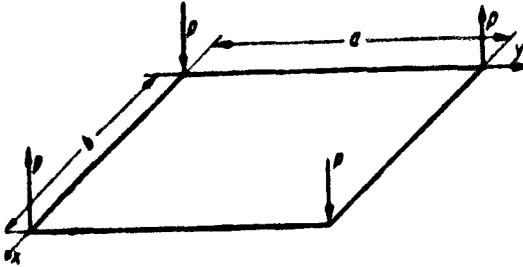


Fig. 13. Diagram of torsion tests.

If the deflection of a strip in the center section measured under load is w_0 and the width of the strip is b , the flexural rigidity is determined by the formula

$$D = \frac{Pl^3}{8bw_0}. \quad (61)$$

The torsional rigidity of a plate D_3 , shear modulus G and, consequently, shear rigidity B_3 are determined by torsion tests or by transverse bending tests of rectangular or square plates loaded with four equal balanced concentrated forces applied to the corners of the plate (Fig. 13).

Torsion tests of specimens cut in the other principal direction of anisotropy are control tests.

As the results obtained above show, the effect of interlayer shearing of laminated plates in the deformed and stressed states depends on the relative thickness of the plate, the boundary conditions and the nature of the load. It is evident that, for determination of the rigidity parameters of a plate, it is more advisable to use transverse bending tests by systems of hinge supported or rigidly fastened beams. /26

If, for example, the deflection of a hinge supported strip of width b , measured in transverse bending tests with loading force P applied in the center of the span is w_0 , the rigidity of the laminated plate with respect to shearing between the layers is determined by the formula

$$K = \frac{12DPl}{48Dbw_0 - Pl^3}. \quad (62)$$

By conducting such tests of strips cut in both principal directions of anisotropy, we obtain rigidities K_1 , K_2 .

In this manner, the simplest mechanical tests of rectangular strips cut from a laminated plate completely solve the problems associated with determination of the elastic and rigidity characteristics of laminated shells.

If a laminated shell is bent in one or both directions, the fabrication of flat control samples should be provided for, which are cut under the same technological conditions and go through the same heat treatment as the shell itself.

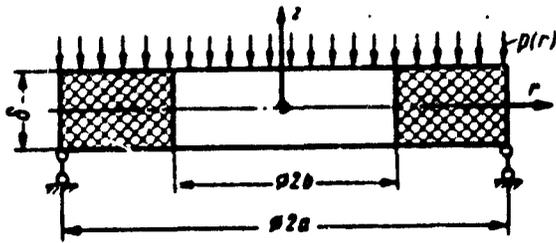
To obtain control samples, it is advisable to provide for technological margins in the fabrication of an actual structure, which are then cut into samples for mechanical testing.

CHAPTER 3. AXISYMMETRIC BENDING OF CIRCULAR PLATES WITH CYLINDRICAL ANISOTROPY

9. Equations of Bending of Circular Plate and General Solution under Axisymmetric Loading

We consider a circular plate of constant thickness made of a cylindrically orthotropic material (Fig. 14).

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It is assumed that the axis of anisotropy passes through the center of the plate perpendicular to the mean surface, and that the principal axes of elasticity coincide with the axis of anisotropy, with radial and circular directions. A load distributed symmetrically about the z axis acts on the plate normal to the mean surface.

Fig. 14. Sketch of circular plate and basic symbols.

In conformance with Eq. (10)-(12), (24), (26), in axisymmetric deformation of the plate, the basic relationships which connect the stressed and deformed states of the plate have the form

$$\left. \begin{aligned} T_1 &= B_1 \left(u' + \nu_2 \frac{u}{r} \right); \\ T_2 &= B_2 \left(\frac{u}{r} + \nu_1 u' \right). \end{aligned} \right\} \quad (63)$$

$$\left. \begin{aligned} G_1 &= -D_1 \left(\phi' + \nu_2 \frac{\phi}{r} \right); \\ G_2 &= -D_2 \left(\frac{\phi}{r} + \nu_1 \phi' \right). \end{aligned} \right\} \quad (64)$$

$$Q_1 = -K_1 (w' + \phi). \quad (65)$$

For determination of radial displacement u , plate deflection w and deformation function ϕ , we have three equations of equilibrium

$$\left. \begin{aligned} (rT_1)' - T_2 &= 0; \\ (rG_1)' - G_2 &= rQ_1; \\ (rQ_1)' + pr &= 0. \end{aligned} \right\} \quad (66)$$

System of differential equations (66) is decomposed into two systems relative to u and w , and ϕ

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$$u'' + \frac{u'}{r} - \frac{\lambda^2}{r^2} u = 0; \quad (67)$$

$$\left. \begin{aligned} \phi'' + \frac{\phi'}{r} - \frac{\lambda^2}{r^2} \phi &= \frac{pr}{2D_1} - \frac{C_1}{D_1} \frac{1}{r}; \\ w' + \phi &= \frac{pr}{2K_1} - \frac{C_1}{K_1} \frac{1}{r}. \end{aligned} \right\} \quad (68)$$

where C_1 is the random integration constant and

$$\lambda^2 = \frac{E_2}{E_1}. \quad (69)$$

General solutions of Eq. (68) have the form

$$\left. \begin{aligned} Q_1 &= -\frac{pr}{2} + \frac{C_1}{r}; \\ \varphi &= \frac{pr^3}{2(9-\lambda^2)D_1} + \frac{C_1 r}{(\lambda^2-1)D_1} + C_2 r^\lambda + C_3 r^{-\lambda}; \\ w &= \frac{pr^3}{4K_1} - \frac{pr^4}{8(9-\lambda^2)D_1} - \frac{C_1 r^3}{2(\lambda^2-1)D_1} - \\ &- \frac{C_1}{K_1} \ln r - \frac{C_2}{\lambda+1} r^{\lambda+1} + \frac{C_3}{\lambda-1} r^{1-\lambda} + C_4; \\ G_1 &= -D_1 \left[\frac{p(3+\nu_2)r^2}{2(9-\lambda^2)D_1} + \frac{(1+\nu_2)C_1}{(\lambda^2-1)D_1} + \right. \\ &+ C_2(\lambda+\nu_2)r^{\lambda-1} - C_3(\lambda-\nu_2)r^{-(\lambda+1)} \left. \right]; \\ G_2 &= -D_2 \left[\frac{(1+3\nu_1)pr^3}{2(9-\lambda^2)D_1} + \frac{(1+\nu_1)C_1}{(\lambda^2-1)D_1} + \right. \\ &+ C_2(1+\lambda\nu_1)r^{\lambda-1} + C_3(1-\lambda\nu_1)r^{-(\lambda+1)} \left. \right]. \end{aligned} \right\} \quad (70)$$

Integration constants C_1, C_2, C_3, C_4 are determined from the boundary conditions at the edges of the plate. In the case of a solid plate, the boundary condition on the inner profile is replaced by the condition in the center of the plate with $r=0$, which is reduced to the requirement of limitation of deflection of the plate or the finite nature of the bending moments, or the cutting force as a function of the type of load.

General solutions (70) permit various cases of symmetrical loading of the plate to be considered with diverse boundary conditions.

10. Bending of Solid Circular Plate by Uniform Load

Let a solid circular plate be bent by normal pressure uniformly distributed over the upper boundary of the plate (see Fig. 14).

In this case, by virtue of the finite nature of the deflection in the center of the plate and the absence of rotation of the normal, $C_1=C_3=0$ and, consequently,

$$\left. \begin{aligned} G_1 &= -D_1 \left[\frac{(3+\nu_2)pr^2}{2(9-\lambda^2)D_1} + C_2(\lambda+\nu_2)r^{\lambda-1} \right]; \\ G_2 &= -D_2 \left[\frac{(1+3\nu_1)pr^3}{2(9-\lambda^2)D_1} + C_2(1+\lambda\nu_1)r^{\lambda-1} \right]; \\ w &= \frac{pr^3}{4K_1} - \frac{pr^4}{8(9-\lambda^2)D_1} - \frac{C_2}{\lambda+1} r^{\lambda+1} + C_4; \\ \varphi &= \frac{pr^3}{2(9-\lambda^2)D_1} + C_2 r^\lambda. \end{aligned} \right\} \quad (71)$$

a. Edge of Plate Hinge Supported

By determining constants C_2 , C_4 from the boundary conditions with $r=a$, $w=0$, $G_1=0$, we obtain

$$\left. \begin{aligned} w &= \frac{pa^4}{8(9-\lambda^2)D_1} \left[\frac{(\lambda-3)(\lambda+4+\nu_2)}{(\lambda+\nu_2)(\lambda+1)} - \left(\frac{r}{a}\right)^4 + \right. \\ &+ \left. \frac{4(3+\nu_2)}{(\lambda+1)(\lambda+\nu_2)} \left(\frac{r}{a}\right)^{\lambda+1} + \frac{2(9-\lambda^2)D_1}{\Lambda_1 a^3} \left(\frac{r^3}{a^3} - 1\right) \right]; \\ \varphi &= \frac{pa^3}{2(9-\lambda^2)D_1} \left[\left(\frac{r}{a}\right)^3 - \frac{3+\nu_2}{\lambda+\nu_2} \left(\frac{r}{a}\right)^\lambda \right]; \\ G_1 &= -\frac{(3+\nu_2)pa^2}{2(9-\lambda^2)} \left[\left(\frac{r}{a}\right)^3 - \left(\frac{r}{a}\right)^{\lambda-1} \right]; \\ G_2 &= -\frac{(1+3\nu_1)\lambda^2 pa^2}{2(9-\lambda^2)} \left[\left(\frac{r}{a}\right)^3 - \frac{(3+\nu_2)(1+\lambda\nu_1)}{(\lambda+\nu_2)(1+3\nu_1)} \left(\frac{r}{a}\right)^{\lambda-1} \right]. \end{aligned} \right\} \quad (72)$$

The maximum deflection in the center of the plate is determined by the expression

$$|w|_{\max} = \frac{pa^4(\lambda+4+\nu_2)}{8(\lambda+3)(\lambda+\nu_2)(\lambda+1)D_1} + \frac{pa^3}{4K_1}; \quad (73)$$

and, at the edge of the plate ($r=a$)

$$G_1 = 0; \quad G_2 = \frac{pa^2\lambda^2(1-\nu_1\nu_2)}{2(\lambda+\nu_2)(\lambda+3)}. \quad (74)$$

In the center of the plate, bending moments $G_1=G_2=0$ if $\lambda>1$, or they increase indefinitely if $\lambda<1$, i.e., the stressed state depends essentially on the nature of the anisotropy. /30

b. Edge of Plate Rigidly Fastened

In this case, we obtain

$$\left. \begin{aligned} w &= \frac{pa^4}{8(9-\lambda^2)(\lambda+1)D_1} \left[\lambda - 3 + 4\left(\frac{r}{a}\right)^{\lambda+1} - \right. \\ &- \left. (\lambda+1)\left(\frac{r}{a}\right)^4 \right] + \frac{p(r^3-a^3)}{4K_1}; \\ \varphi &= \frac{pa^3}{2(9-\lambda^2)D_1} \left[\left(\frac{r}{a}\right)^3 - \left(\frac{r}{a}\right)^\lambda \right]; \\ G_1 &= -\frac{(3+\nu_2)pa^2}{2(9-\lambda^2)} \left[\left(\frac{r}{a}\right)^3 - \frac{\lambda+\nu_2}{3+\nu_2} \left(\frac{r}{a}\right)^{\lambda-1} \right]; \\ G_2 &= -\frac{(1+3\nu_1)\lambda^2 pa^2}{2(9-\lambda^2)} \left[\left(\frac{r}{a}\right)^3 - \frac{1+\lambda\nu_1}{1+3\nu_1} \left(\frac{r}{a}\right)^{\lambda-1} \right]. \end{aligned} \right\} \quad (75)$$

The greatest deflection in the center of the plate

$$|w|_{\max} = \frac{pa^4}{8(\lambda+3)(\lambda+1)D_1} + \frac{pa^3}{4K_1}. \quad (76)$$

The bending moments at the edge of the plate

$$G_1 = -\frac{pa^2}{2(3+\lambda)}; \quad G_2 = \nu_2 G_1. \quad (77)$$

In this case, the stressed state of the plate depends essentially on the nature of the anisotropy. If $\lambda^2 > 1$, the maximum stresses develop at the edge of the plate. If $\lambda^2 < 1$, the stresses in the center increase indefinitely.

Thus, if a laminated, cylindrically orthotropic, circular plate operates under uniform normal pressure, the modulus of elasticity of the material in the circular direction should be greater than the modulus of elasticity of the material in the radial direction. It is advisable to use laminated plates which do not satisfy this condition for the manufacture of circular parts. If, for some reason, a plate should be made of a material, for which $\lambda^2 = E_2/E_1 < 1$, the center of the plate must be reinforced with an absolutely rigid disk, i.e., a disk the flexural rigidity of which is considerably greater than the flexural rigidity of the plate.

11. Bending of Solid Circular Plate by Concentrated Force

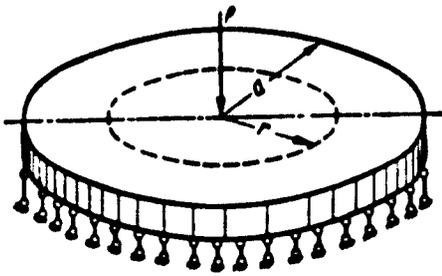
Let a solid circular plate made of a cylindrically orthotropic laminated plastic be bent by normal concentrated force P applied to the center of the plate (Fig. 15). In this case, $p=0$, $C_1 = -P/2\pi$, $C_3 = 0$ must be set in Eq. (70). We then obtain /31

$$\left. \begin{aligned} G_1 &= D_1 \left[\frac{P(1+\nu_2)}{2\pi(\lambda^2-1)D_1} - C_2(\lambda+\nu_2)r^{\lambda-1} \right]; \\ G_2 &= D_2 \left[\frac{P(1+\nu_1)}{2\pi(\lambda^2-1)D_1} - C_2(1+\lambda\nu_1)r^{\lambda-1} \right]; \\ w &= \frac{Pr^2}{4\pi(\lambda^2-1)D_1} + \frac{P \ln r}{2\pi K_1} - \frac{C_2}{\lambda+1} r^{\lambda+1} + C_4; \\ \varphi &= -\frac{Pr}{2\pi(\lambda^2-1)D_1} + C_2 r^\lambda. \end{aligned} \right\} \quad (78)$$

a. Edge of Plate $r=a$ Hinge Supported

The stressed, deformed state of the plate is determined by the expressions

$$\left. \begin{aligned} G_1 &= \frac{P(1+\nu_2)}{2\pi(\lambda^2-1)} \left[1 - \left(\frac{r}{a} \right)^{\lambda-1} \right]; \\ G_2 &= \frac{P(1+\nu_1)\lambda^2}{2\pi(\lambda^2-1)} \left[1 - \frac{(1+\nu_2)(1+\lambda\nu_1)}{(\lambda+\nu_2)(1+\nu_1)} \left(\frac{r}{a} \right)^{\lambda-1} \right]; \\ \varphi &= -\frac{Pa}{2\pi(\lambda^2-1)D_1} \left[\left(\frac{r}{a} \right) - \frac{1+\nu_2}{\lambda+\nu_2} \left(\frac{r}{a} \right)^\lambda \right]; \\ w &= \frac{Pa^2}{4\pi(\lambda^2-1)D_1} \left[\left(\frac{r}{a} \right)^2 - \frac{2(1+\nu_2)}{(\lambda+\nu_2)(\lambda+1)} \left(\frac{r}{a} \right)^{\lambda+1} - \right. \\ &\quad \left. - \frac{(\lambda-1)(\lambda+2+\nu_2)}{(\lambda+\nu_2)(\lambda+1)} \right] + \frac{P \ln \frac{r}{a}}{2\pi K_1}. \end{aligned} \right\} \quad (79)$$



The deflection in the center of the plate reverts to infinity. Consequently, in transmission of the force, distribution of the load over a definite area must be ensured, or a rigid disk must be incorporated in the center of the plate.

b. Edge of Plate Rigidly Fastened

In this case

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Fig. 15. Diagram of bending of circular plate by force applied to center.

$$\left. \begin{aligned} G_1 &= \frac{P(1+\nu_2)}{2\pi(\lambda^2-1)} \left[1 - \frac{(\lambda+\nu_2)}{1+\nu_2} \left(\frac{r}{a}\right)^{\lambda-1} \right]; \\ G_2 &= \frac{P(1+\nu_1)\lambda^2}{2\pi(\lambda^2-1)} \left[1 - \frac{1+\lambda\nu_1}{1+\nu_1} \left(\frac{r}{a}\right)^{\lambda-1} \right]; \\ \Psi &= -\frac{Pa}{2\pi(\lambda^2-1)D_1} \left[\left(\frac{r}{a}\right) - \left(\frac{r}{a}\right)^\lambda \right]; \\ w &= \frac{Pa^2}{4\pi(\lambda^2-1)(\lambda+1)D_1} \left[(\lambda+1)\left(\frac{r}{a}\right)^2 - \right. \\ &\quad \left. - 2\left(\frac{r}{a}\right)^{\lambda+1} - (\lambda-1) \right] + \frac{P \ln \frac{r}{a}}{2\pi K_1}. \end{aligned} \right\} \quad (80)$$

In this case, the load also should be transmitted through a rigid disk in the center or it should be distributed over a certain area in the center of the plate.

Relative to stress distribution, the same conclusions are valid as those made in the preceding section. For the fabrication of circular plates operating under locally distributed load applied in the center of the plate, it is advisable to use plates the modulus of elasticity in the radial direction of which is greater than the modulus of elasticity in the annular direction, i.e., $\lambda < 1$.

It is of interest to note that precisely such anisotropy of elastic properties develops in circular disks strengthened with radial stiffening ribs. However, radial ribs which converge in the center of the disk form a rigid hub. For more favorable stress distribution in the reinforced disks, annular strengthening ribs should be provided.

12. Bending of Circular Plate with Rigid Disk in Center by Uniform Pressure

We now consider a circular plate made of cylindrically orthotropic laminated plastic, subjected to uniformly distributed normal pressure. The inner profile of the plate is rigidly fastened to a massive disk located in the center (Fig. 16).

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In this case, we have $C_1=0$, $Q=-pr/2$ and, consequently,

$$\begin{aligned}
 w &= \frac{pa^3 \left(\frac{r^2}{a^2} - 1 \right)}{4K_1} - \frac{pa^4 \left(\frac{r^4}{a^4} - 1 \right)}{8(9-\lambda^2)D_1} - \\
 &- \frac{C_2 a^{\lambda+1} \left[\left(\frac{r}{a} \right)^{\lambda+1} - 1 \right]}{\lambda+1} + \frac{C_3 a^{1-\lambda} \left[\left(\frac{r}{a} \right)^{1-\lambda} - 1 \right]}{\lambda-1}; \\
 \varphi &= \frac{pr^2}{2(9-\lambda^2)D_1} + C_2 r^\lambda + C_3 r^{-\lambda}; \\
 G_1 &= -D_1 \left[\frac{(3+\nu_2)pr^2}{2(9-\lambda^2)D_1} + C_2(\lambda+\nu_2)r^{\lambda-1} - \right. \\
 &\quad \left. - C_3(\lambda-\nu_2)r^{-(\lambda+1)} \right]; \\
 G_2 &= -D_2 \left[\frac{(1+3\nu_1)pr^2}{2(9-\lambda^2)D_1} + C_2(1+\lambda\nu_1)r^{\lambda-1} + \right. \\
 &\quad \left. + C_3(1-\lambda\nu_1)r^{-(\lambda+1)} \right].
 \end{aligned} \tag{81}$$

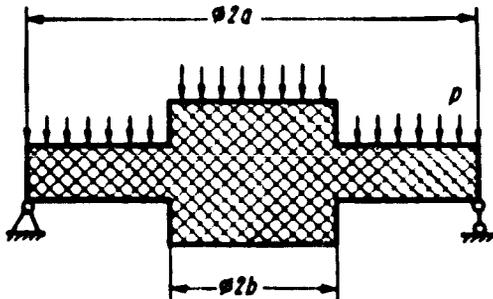


Fig. 16. Diagram of bending of hinge supported circular plate with rigid disk in center by uniformly distributed pressure.

a. Edge of plate $r=a$ hinge supported (see Fig. 16). By satisfying boundary conditions $w=0$, $G_1=0$, we obtain

$$\left. \begin{aligned}
 C_2 &= -\frac{pa^{3-\lambda}}{2(9-\lambda^2)D_1} \frac{(\lambda-\nu_2)q^{\lambda+3} + 3 + \nu_2}{(\lambda-\nu_2)q^{2\lambda} + \lambda + \nu_2}; \\
 C_3 &= \frac{pb^{\lambda+3}}{2(9-\lambda^2)D_1} \frac{(3+\nu_2)q^{\lambda-3} - (\lambda+\nu_2)}{(\lambda-\nu_2)q^{2\lambda} + \lambda + \nu_2},
 \end{aligned} \right\} \tag{82}$$

where $(q = \frac{b}{a})$.

At edge $r=a$ of the plate, the bending moments

$$G_1 = 0, G_2 = -\frac{pa^2 \lambda^2 (1-\nu_1 \nu_2)}{2(9-\lambda^2)} \frac{(\lambda+3)q^{2\lambda} - 2\lambda q^{\lambda+3} + \lambda - 3}{(\lambda-\nu_2)q^{2\lambda} + \lambda + \nu_2}.$$

At the edge of the plate around the the inner disk

$$\left. \begin{aligned}
 G_1 &= -\frac{pb^2}{2(9-\lambda^2)} \times \\
 &\times \frac{(\lambda-\nu_2)(3-\lambda)q^{2\lambda} - 2\lambda(3+\nu_2)q^{\lambda-3} + (\lambda+3)(\lambda+\nu_2)}{(\lambda-\nu_2)q^{2\lambda} + \lambda + \nu_2}; \\
 G_2 &= \nu_2 G_1.
 \end{aligned} \right\} \tag{84}$$

b. Edge of plate $r=0$ rigidly fastened (Fig. 17).

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$$\left. \begin{aligned}
 C_2 &= -\frac{pa^{3-\lambda}}{2(9-\lambda^2)D_1} \frac{1-q^{\lambda+3}}{1-q^{2\lambda}}; \\
 C_3 &= -\frac{pb^{\lambda+3}}{2(9-\lambda^2)D_1} \frac{1-q^{\lambda-3}}{1-q^{2\lambda}}.
 \end{aligned} \right\} \tag{85}$$

where $\rho = b/a$.

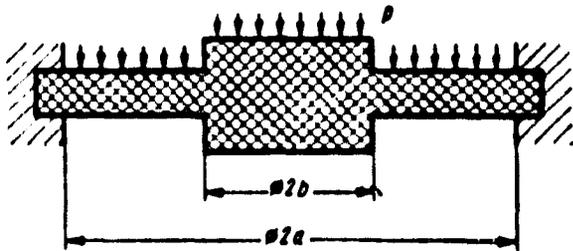


Fig. 17. Diagram of bending of circular plate with rigidly fastened rim and disk in center by uniformly distributed pressure.

At the edge of plate $r=a$, bending moments

$$\left. \begin{aligned} G_1 &= \frac{pa^2}{2(\lambda^2-1)} \frac{(\lambda-3)-2\lambda q^{\lambda+3}+(\lambda+3)q^{2\lambda}}{1-q^{2\lambda}}; \\ G_2 &= \nu_2 G_1. \end{aligned} \right\} \quad (86)$$

Correspondingly at the edge $r=b$, we obtain

$$\left. \begin{aligned} G_1 &= -\frac{pb^2}{2(\lambda^2-1)} \frac{\lambda+3-2\lambda q^{\lambda-3}+(\lambda-3)q^{2\lambda}}{1-q^{2\lambda}}; \\ G_2 &= \nu_2 G_1. \end{aligned} \right\} \quad (87)$$

In both cases considered, the greatest tangential stresses arise on the outer profile of the plate

$$\tau_{\max} = \frac{3pa}{\delta}. \quad (88)$$

13. Bending of Circular Plate by Forces Applied to Rigid Disk in Center

Let a circular plate fastened in an absolutely rigid disk be loaded by an axisymmetric system of normal forces applied to the disk. If the resultant equals P , $C_1 = -P/2\pi$ and, consequently, according to Eq.

(70), with $\mu=0$, the bending moments and deformation are determined by /35
the following expressions

$$\left. \begin{aligned} w &= \frac{P(r^2-a^2)}{4\pi(\lambda^2-1)D_1} + \frac{P}{2\pi K_1} \ln \frac{r}{a} - \frac{C_2(r^{\lambda+1}-a^{\lambda+1})}{\lambda+1} + \\ &\quad + \frac{C_3(r^{1-\lambda}-a^{1-\lambda})}{\lambda-1}; \\ \varphi &= -\frac{Pr}{2\pi(\lambda^2-1)D_1} + C_2 r^\lambda + C_3 r^{-\lambda}; \\ G_1 &= D_1 \left[\frac{P(1+\nu_2)}{2\pi(\lambda^2-1)D_1} - C_2(\lambda+\nu_2)r^{\lambda-1} + \right. \\ &\quad \left. + C_3(\lambda-\nu_2)r^{-(\lambda+1)} \right]; \\ G_2 &= D_2 \left[\frac{P(1+\nu_1)}{2\pi(\lambda^2-1)D_1} - C_2(1+\lambda\nu_1)r^{\lambda-1} - \right. \\ &\quad \left. - C_3(1-\lambda\nu_1)r^{-(\lambda+1)} \right], \\ Q_1 &= -\frac{P}{2\pi r}. \end{aligned} \right\} \quad (89)$$

a. Rim $r=0$ hinge supported (Fig. 18).

$$\left. \begin{aligned} C_2 &= \frac{Pa^{-(\lambda-1)}}{2\pi(\lambda^2-1)D_1} \frac{1+\nu_2+(\lambda-\nu_2)q^{\lambda+1}}{(\lambda-\nu_2)q^{2\lambda}+\lambda+\nu_2}; \\ C_3 &= \frac{Pb^{\lambda+1}}{2\pi(\lambda^2-1)D_1} \frac{\lambda+\nu_2-(1+\nu_2)q^{\lambda-1}}{(\lambda-\nu_2)q^{2\lambda}+\lambda+\nu_2}. \end{aligned} \right\} \quad (90)$$

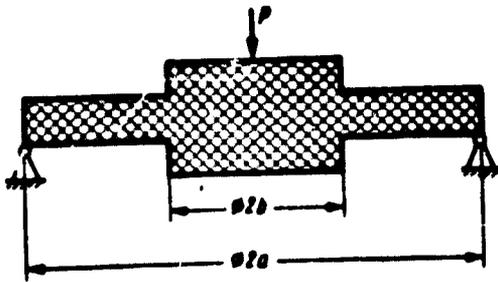


Fig. 18. Diagram of bending of hinge supported circular plate with rigid disk in center by forces applied to disk.

On the outer profile of the plate, bending moments

$$\left. \begin{aligned} G_1 &= 0; \\ G_2 &= \frac{P \lambda^2 (1 - \nu_1 \nu_2)}{2\pi (\lambda^2 - 1)} \frac{[(\lambda + 1) q^{2\lambda} - 2\lambda q^{\lambda+1} + \lambda - 1]}{(\lambda - \nu_2) q^{2\lambda} + \lambda + \nu_2}. \end{aligned} \right\} \quad (91)$$

Correspondingly, on the profile of the rigid disk, we obtain /36

$$\left. \begin{aligned} G_1 &= -\frac{P}{2\pi (\lambda^2 - 1)} \times \\ &\times \frac{(\lambda - 1) (\lambda - \nu_2) q^{2\lambda} + 2\lambda (1 + \nu_2) q^{(\lambda - 1)} - (\lambda + 1) (\lambda + \nu_2)}{(\lambda - \nu_2) q^{2\lambda} + \lambda + \nu_2}; \\ G_2 &= \nu_2 G_1. \end{aligned} \right\} \quad (92)$$

b. Rim of plate r=a rigidly fastened (Fig. 19). In this case, the following can be obtained

$$\left. \begin{aligned} C_2 &= \frac{P a^{-(\lambda - 1)}}{2\pi (\lambda^2 - 1) D_1} \frac{1 - q^{\lambda + 1}}{1 - q^{2\lambda}}; \\ C_3 &= \frac{P b^{\lambda + 1}}{2\pi (\lambda^2 - 1) D_1} \frac{1 - q^{\lambda - 1}}{1 - q^{2\lambda}}. \end{aligned} \right\} \quad (93)$$

The respective bending moments on the outer and inner profiles of the plate

$$G_1 = -\frac{P}{2\pi (\lambda^2 - 1)} \frac{(\lambda + 1) q^{2\lambda} - 2\lambda q^{\lambda + 1} + \lambda - 1}{1 - q^{2\lambda}}; \quad G_2 = \nu_2 G_1 \quad (94)$$

$$G_1 = \frac{P}{2\pi (\lambda^2 - 1)} \frac{(\lambda - 1) q^{2\lambda} - 2\lambda q^{\lambda - 1} + \lambda + 1}{1 - q^{2\lambda}}; \quad G_2 = \nu_2 G_1. \quad (95)$$

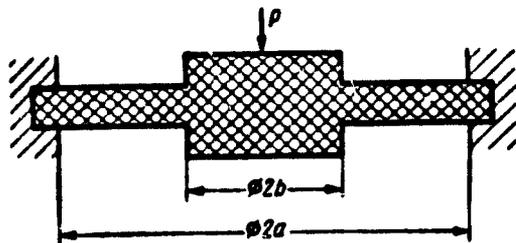


Fig. 19. Diagram of bending of circular plate with rigidly fastened rim and rigid disk in center by forces applied to disk.

In both of the boundary condition cases considered, the greatest shearing stresses arise on the inner profile of the plate

$$\tau_{\max} = \frac{3P}{4\pi b \delta}. \quad (96)$$

14. Bending of Annular Plate by Load Uniformly Distributed Over Inner Profile

Let an annular plate made of cylindrically orthotropic laminated plastic be loaded with load $P=2\pi b q$ uniformly distributed over the inner profile (Fig. 20). With different fastenings of the outer profile of the plate, the /37

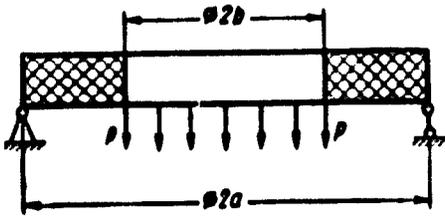


Fig. 20. Diagram of bending of annular plate with hinge supported outer rim by forces uniformly distributed over inner profile.

elastic forces and deformations are determined by general expressions (89).

a. Rim of plate $r=a$ hinge supported. In this case, it is easy to obtain

$$\left. \begin{aligned} C_1 &= \frac{P(1+\nu_2)a^{1-\lambda}}{2\pi(\lambda^2-1)D_1(\lambda+\nu_2)} \frac{(1-q^{\lambda+1})}{(1-q^{2\lambda})}; \\ C_2 &= -\frac{P(1+\nu_2)b^{\lambda+1}}{2\pi(\lambda^2-1)D_1(\lambda-\nu_2)} \frac{(1-q^{\lambda-1})}{(1-q^{2\lambda})}. \end{aligned} \right\} \quad (97)$$

The bending moments on the inner and outer profiles of the plate are determined by the following respective expressions

$$\left. \begin{aligned} G_1 &= 0; \\ G_2 &= -\frac{P\lambda^2(1-\nu_1\nu_2)}{2\pi(\lambda^2-1)(\lambda^2-\nu_2^2)} \times \\ &\times \frac{(\lambda+\nu_2)(\lambda+1)q^{2\lambda}-2\lambda(1+\nu_2)q^{\lambda+1}-(\lambda-1)(\lambda-\nu_2)}{1-q^{2\lambda}}; \end{aligned} \right\} \quad (98)$$

$$\left. \begin{aligned} G_1 &= 0; \\ G_2 &= -\frac{P\lambda^2(1-\nu_1\nu_2)}{2\pi(\lambda^2-1)(\lambda^2-\nu_2^2)} \times \\ &\times \frac{(\lambda-\nu_2)(\lambda-1)q^{2\lambda}+2\lambda(1+\nu_2)q^{\lambda-1}-(\lambda+1)(\lambda+\nu_2)}{1-q^{2\lambda}}. \end{aligned} \right\} \quad (99)$$

b. Rim of plate $r=a$ rigidly fastened (Fig. 21). In this case, the following can be obtained

$$\left. \begin{aligned} C_1 &= \frac{Pa^{1-\lambda}}{2\pi(\lambda^2-1)D_1} \frac{\lambda-\nu_2+(1+\nu_2)q^{\lambda+1}}{(\lambda-\nu_2)+(\lambda+\nu_2)q^{2\lambda}}; \\ C_2 &= -\frac{Pb^{\lambda+1}}{2\pi(\lambda^2-1)D_1} \frac{1+\nu_2-(\lambda+\nu_2)q^{\lambda-1}}{\lambda-\nu_2+(\lambda+\nu_2)q^{2\lambda}}. \end{aligned} \right\} \quad (100)$$

On the inner profile of plate $r=a$, the bending moments

$$\left. \begin{aligned} G_1 &= \frac{P}{2\pi(\lambda^2-1)} \times \\ &\times \frac{(\lambda+1)(\lambda+\nu_2)q^{2\lambda}-2\lambda(1+\nu_2)q^{\lambda+1}-(\lambda-\nu_2)(\lambda-1)}{\lambda-\nu_2+(\lambda+\nu_2)q^{2\lambda}}; \\ G_2 &= \nu_2 G_1. \end{aligned} \right\} \quad (101)$$

Correspondingly, on the inner profile, we obtain

$$G_1 = 0; \left. \begin{aligned} G_2 = \frac{P \lambda^2 (1 - \nu_1 \nu_2)}{2\pi (\lambda^2 - 1)} \frac{(\lambda - 1) \rho^{2\lambda} - 2\lambda \rho^{\lambda-1} + \lambda + 1}{\lambda - \nu_1 + (\lambda + \nu_2) \rho^{2\lambda}} \end{aligned} \right\} (102)$$

In both cases considered, the greatest shearing stresses arise on the inner profile of the plate

$$\tau_{\max} = \frac{3P}{4\pi b \delta} \quad (103)$$

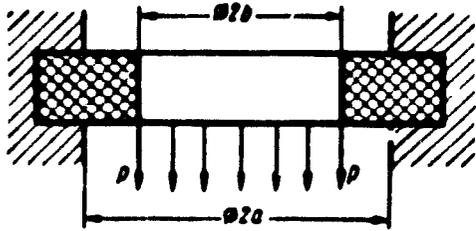


Fig. 21. Diagram of bending of annular plate with rigidly fastened outer rim by forces uniformly applied over inner profile.

It is easy to determine that it is advisable to make annular plates with a small opening in the center of laminated plastics, the modulus of elasticity in the radial direction of which is greater than the modulus of elasticity in the annular direction.

15. Bending of Circular Plates with Annular Fiber Reinforcing

A characteristic example of the practical use of circular cylindrically orthotropic laminated plates is circular plates made of synthetic polymers and reinforced in the annular direction with a fiber filler, fiberglass, for example. As has been noted, reinforcing of the plate only in the annular direction permits more efficient anisotropy of properties and, consequently, a more favorable distribution of stresses to be produced.

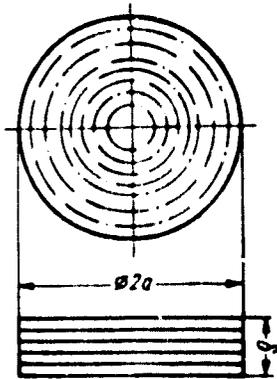


Fig. 22. Circular plate with annular fiber reinforcing.

We consider a circular plate reinforced in the annular direction with uniformly placed fibers of circular cross section (Fig. 22). /39

If the reinforcing fibers are located at uniform distance l in each layer, the basic relationships which connect the stressed and deformed states of the plate can be presented in the form

$$\left. \begin{aligned} G_1 &= -D_1 \left(\varphi' + \nu_2 \frac{\varphi}{r} \right); \\ G_2 &= -D_2 \left(\frac{\varphi}{r} + \nu_1 \varphi' \right); \\ Q_1 &= -K_1 (w' + \varphi). \end{aligned} \right\} (104)$$

Flexural rigidities of the plate in the radial and annular directions D_1 , D_2 and Poisson coefficients ν_1 , ν_2 are determined by the following expressions

$$\left. \begin{aligned} D_1 &= D = \frac{E_c \delta^3}{12(1-\nu^2)}; \\ D_2 &= D \left[1 + \frac{\pi^2 (1-\nu^2) E_H}{2l^2 E_c} \right] = D(1+k); \\ \nu_2 &= \nu; \quad \nu_1 = \nu \frac{D}{D_2}. \end{aligned} \right\} \quad (105)$$

where E_H , E_C are the moduli of elasticity of the fiber reinforcing and binder; ν is the Poisson coefficient of the binder; ζ is the cross section radius of the reinforcing fiber; l is the distance between the reinforcing rings.

Such plates have a favorable elastic property anisotropy, since the following relationship occurs

$$\lambda = \sqrt{\frac{D_2}{D_1}} = \sqrt{1+k} \geq 1. \quad (106)$$

Thus, for the calculation of laminated circular plates with uniform annular reinforcing, all the formulas obtained in the preceding sections are applicable. In the loading of such plates, both continuous and with an opening in the center, concentration of stresses and impermissible increase in deflection do not occur.

Simultaneous reinforcing of a plate in the annular and radial directions obviously is inadvisable with respect to stress distribution and complexity of production.

The basic difference of laminated circular plates reinforced only radially is that the anisotropy of their elastic properties changes radially. /40

16. Bending of Circular Plates with Radial Fiber Reinforcing

We consider a circular annular laminated plate obtained by bonding layers reinforced radially with a fiber filler (Fig. 23).

Evidently, the packing density of the fiber filler satisfies the relationship

$$0 \leq n \leq \frac{\pi b}{\zeta},$$

where ζ is the cross section radius of the fiber filler.

The aperture angle of the fiber reinforcing $\alpha = 2\pi\zeta/b$. The basic relationships which connect the stressed and deformed state of such a cylindrically orthotropic plate can be presented in the form

$$\left. \begin{aligned} G_1 &= -D_1(r)\varphi' - D\nu\frac{\varphi}{r}; \\ G_2 &= -D\frac{\varphi}{r} - D\nu\varphi'; \\ Q_1 &= -K_1(w' + \varphi). \end{aligned} \right\} \quad (107)$$

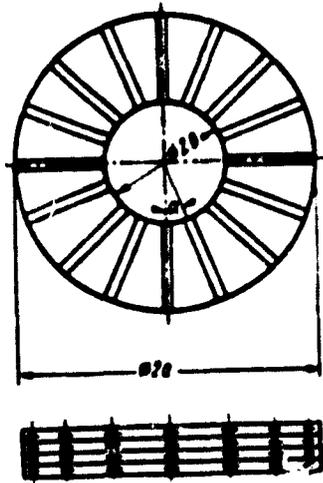


Fig. 23. Circular plate with radial fiber reinforcing.

where $D = \frac{E_c \delta^3}{12(1-\nu^2)}$ is the flexural rigidity of the unreinforced plate; E_c , ν are the modulus of elasticity and Poisson coefficient of the binder; K_1 is the rigidity of the laminated plate with respect to interlayer shearing; E_H is the modulus of elasticity of the fiber filler; D_1 is the radial flexural rigidity of the plate.

If the E_c/E_H ratio is disregarded compared with unity, the following expression for rigidity can be obtained

$$D_1 = D \left(1 + \frac{k}{r} \right), \quad (108)$$

where

$$k = \frac{\pi^2 (1-\nu^2)}{2a} \frac{E_H}{E_c}. \quad (109)$$

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By substituting Eq. (107) in plate equilibrium Eq. (66), we obtain the following system of differential equations, which describe the bending of the circular plate with radial reinforcing

$$\left. \begin{aligned} (r+k)\varphi'' + \varphi' - \frac{\varphi}{k} &= -\frac{Q_1 r}{D}; \\ w' &= -\frac{Q_1}{K_1} - \varphi; \\ Q_1 &= -\frac{pr}{2} + \frac{C_1}{r}. \end{aligned} \right\} \quad (110)$$

The following expressions also can be found for the bending moments and deformations

$$\left. \begin{aligned} G_1 &= -D_1 \left\{ \frac{C}{k^2} \left[A \left(1 + \nu + \frac{k}{r} \right) - \frac{k(1+\nu)}{r} + \right. \right. \\ &\quad \left. \left. + \left(1 + \nu + \frac{k}{r} \right) \ln \frac{r+k}{r} \right] + \frac{pk^2}{8D} \times \right. \\ &\times \left[\sum_{n=2}^{\infty} (-1)^n \frac{n+\nu}{n-1} \left(\frac{r}{k} \right)^{n-1} + \sum_{n=4}^{\infty} (-1)^n \frac{n}{n-1} \left(\frac{r}{k} \right)^{n-2} \right] - \\ &\quad \left. - \frac{C_1}{2D} \left[\sum_{n=2}^{\infty} (-1)^n \frac{n+\nu}{n-1} \left(\frac{r}{k} \right)^{n-1} + \right. \right. \\ &\quad \left. \left. + \sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1} \left(\frac{r}{k} \right)^{n-2} \right] \right\}; \end{aligned} \right\} \quad (111)$$

$$\begin{aligned}
G_0 = & -D \left\{ \frac{C}{k^2} \left[A(1+\nu) + (1+\nu) \ln \frac{r+k}{r} - \right. \right. \\
& \left. \left. - \frac{k(1+\nu)r+k}{r(r+k)} \right] + \frac{pk^2}{8D} \sum_{n=2}^{\infty} (-1)^n \frac{n\nu+1}{n-1} \left(\frac{r}{k}\right)^{n-1} - \right. \\
& \left. - \frac{C_1}{2D} \sum_{n=2}^{\infty} (-1)^n \frac{n\nu+1}{n-1} \left(\frac{r}{k}\right)^{n-1} \right\}; \tag{111} \\
\varphi = & \frac{C}{k^2} \left[Ar - k + r \ln \frac{r+k}{r} \right] + \frac{pk^2}{8D} \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \left(\frac{r}{k}\right)^n - \\
& - \frac{C_1 k}{2D} \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} \left(\frac{r}{k}\right)^n; \\
u = & \frac{pr^2}{4K_1} - \frac{C_1}{K_1} \ln r - \\
& - \frac{C}{2k^2} [Ar^2 - kr + (r^2 - k^2) \ln(r+k) - r^2 \ln r] - \\
& - \frac{pk^2}{8D} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \left(\frac{r}{k}\right)^{n+1} + \frac{C_1 k^2}{2D} \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} \left(\frac{r}{k}\right)^{n+1} + B.
\end{aligned}$$

Integration constants A, B, C, C_1 are determined from the boundary 42 conditions on the inner and outer profiles of the plate.

17. General Relationships and Differential Equations of Asymmetric Bending of Circular Anisotropic Plates

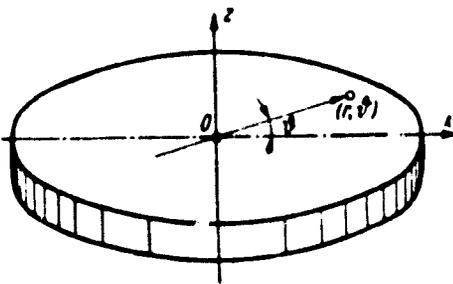


Fig. 24. Accepted coordinate system for circular plate.

in the trigonometric series

Let a circular plate made of laminated cylindrically orthotropic material be bent by a transverse load, which is distributed symmetrically about the polar axis of the plate x . This is practically the most frequently encountered case of loading.

We place the origin of the r, θ, z cylindrical coordinate system at the pole of anisotropy, and we direct the z axis along the axis of symmetry of elastic properties (Fig. 24).

A transverse load distributed symmetrically about the polar axis can be expanded

$$P = \sum_{n=0}^{\infty} P_n(r) \cos n \theta. \tag{112}$$

The general elasticity relationships are written in the form

$$\left. \begin{aligned} G_1 &= -D_1 \left[\frac{\partial \varphi}{\partial r} + \nu_0 \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\varphi}{r} \right) \right]; \\ G_2 &= -D_2 \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{\varphi}{r} + \nu_1 \frac{\partial \varphi}{\partial r} \right); \\ H &= -\frac{D_2}{2} \left(\frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \varphi}{\partial \theta} - \frac{\psi}{r} \right); \end{aligned} \right\} \quad (113)$$

$$\left. \begin{aligned} Q_1 &= -K_1 \left(\varphi + \frac{\partial w}{\partial r} \right); \\ Q_2 &= -K_2 \left(\psi + \frac{1}{r} \frac{\partial w}{\partial \theta} \right). \end{aligned} \right\} \quad (114)$$

In accordance with Eq. (17), with $A_1=1$, $A_2=r$, the plate equilibrium equations are written in the form /43

$$\left. \begin{aligned} \frac{\partial (rQ_1)}{\partial r} + \frac{\partial Q_2}{\partial \theta} &= pr; \\ \frac{\partial (rG_1)}{\partial r} + \frac{\partial H}{\partial \theta} - G_2 &= Q_1 r; \\ \frac{\partial G_2}{\partial \theta} + \frac{\partial (rH)}{\partial r} + H &= Q_2 r. \end{aligned} \right\} \quad (115)$$

Since bending of the plate will be symmetrical about the polar axis, the elastic forces and displacement can be sought in the trigonometric series

$$\left. \begin{aligned} \varphi &= \sum_{n=0}^{\infty} \varphi_n(r) \cos n \theta; \\ \psi &= \sum_{n=0}^{\infty} \psi_n(r) \sin n \theta; \\ w &= \sum_{n=0}^{\infty} w_n(r) \cos n \theta; \\ G_1 &= \sum_{n=0}^{\infty} G_1^n(r) \cos n \theta; \\ G_2 &= \sum_{n=0}^{\infty} G_2^n(r) \cos n \theta; \\ H &= \sum_{n=0}^{\infty} H^n(r) \sin n \theta; \\ Q_1 &= \sum_{n=0}^{\infty} Q_1^n(r) \cos n \theta; \\ Q_2 &= \sum_{n=0}^{\infty} Q_2^n(r) \sin n \theta. \end{aligned} \right\} \quad (117)$$

In accordance with Eq. (113), (114), the coefficients of expansion of the forces are connected to the coefficients of expansion of displacements φ_n , ψ_n , w_n by the following relationships

$$\left. \begin{aligned}
 G_1^n(r) &= -D_1 \left[\varphi_n' + \frac{\nu_2}{r} (n\psi_n + \varphi_n) \right]; \\
 G_2^n(r) &= -D_2 \left[\frac{n}{r} \psi_n + \frac{\varphi_n}{r} + \nu_1 \varphi_n' \right]; \\
 H^n(r) &= -\frac{D_2}{2} \left[\psi_n' - \frac{\psi_n}{r} - \frac{n}{r} \varphi_n \right]; \\
 Q_1^n(r) &= -K_1 (\varphi_n + w_n'); \\
 Q_2^n(r) &= -K_2 \left(\psi_n - \frac{n}{r} w_n \right).
 \end{aligned} \right\} \quad (118)$$

By substituting Eq. (117) in the equilibrium equations with Eq. (118) taken into account, we obtain differential equations of asymmetric bending of circular plates

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$$\left. \begin{aligned}
 \varphi_n'' + \frac{\varphi_n'}{r} - \left(\frac{n^2 \omega_1 + \lambda^2}{r^2} + k_1 \right) \varphi_n + \\
 + \frac{n(\omega_1 + \nu_2)}{r} \psi_n' - \frac{n(\omega_1 + \lambda^2)}{r^2} \psi_n &= k_1 w_n'; \\
 \omega_2 \psi_n'' + \frac{\omega_2}{r} \psi_n' - \left(\frac{n^2 + \omega_2}{r^2} + k_2 \right) \psi_n - \\
 - \frac{n(\omega_2 + \nu_1)}{r} \varphi_n' - \frac{n(\omega_2 + 1)}{r^2} \varphi_n &= -nk_2 \frac{w_n}{r}; \\
 r \varphi_n' + \varphi_n + g^2 n \psi_n &= -\frac{Pn r}{K_1} - r w_n' - \\
 - w_n' + \frac{n^2 g^2}{r} w_n.
 \end{aligned} \right\} \quad (119)$$

where

$$\left. \begin{aligned}
 \frac{D_2}{D_1} = \lambda^2; \quad \frac{D}{D_1} = \frac{G(1-\nu_1\nu_2)}{E_1} = \omega_1; \\
 \frac{D}{D_2} = \frac{G(1-\nu_1\nu_2)}{E_2} = \omega_2; \\
 \frac{K_1}{D_1} = k_1; \quad \frac{K_2}{D_2} = k_2; \quad \frac{K_2}{K_1} = \frac{G_{22}}{G_{12}} = g^2.
 \end{aligned} \right\} \quad (120)$$

If the load is distributed skew symmetrically about the polar axis, all the relationships are obtained by substitution of $\cos n\theta$ by $\sin n\theta$ and vice versa, with corresponding changes of signs of n .

In the most general case of asymmetric loading, the solution is found by summing the asymmetric, symmetric and skew symmetric solutions. In conclusion, we note some identity relationships between the coefficients introduced above

$$\left. \begin{aligned}
 g^2 \frac{k_1}{k_2} = \lambda^2; \quad \lambda^2 \omega_2 = \omega_1; \\
 \lambda(1 + \lambda\nu_1) = \lambda + \nu_2.
 \end{aligned} \right\} \quad (121)$$

18. Differential Equation of Bending of Anisotropic Rectangular Plates

We consider a rectangular plate made of laminated anisotropic material and loaded by forces which are normal to the mean surface of the plate before deformation. /45

We select the x, y, z rectangular coordinate system as indicated in Fig. 25.

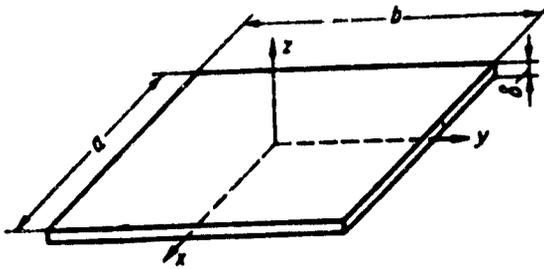


Fig. 25. Coordinate system and basic symbols.

In accordance with Eq. (4), (5) and (11), (12),

$$\left. \begin{aligned} G_1 &= -D_{11} \frac{\partial \varphi}{\partial x} - D_{13} \frac{\partial \psi}{\partial y} - D_{13} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \\ G_2 &= -D_{13} \frac{\partial \varphi}{\partial x} - D_{23} \frac{\partial \psi}{\partial y} - D_{23} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \\ H &= -D_{13} \frac{\partial \varphi}{\partial x} - D_{23} \frac{\partial \psi}{\partial y} - D_{23} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \end{aligned} \right\} \quad (122)$$

$$\left. \begin{aligned} Q_1 &= -K_1 \left(\varphi + \frac{\partial w}{\partial x} \right); \\ Q_2 &= -K_2 \left(\psi + \frac{\partial w}{\partial y} \right). \end{aligned} \right\} \quad (123)$$

By substituting Eq. (122), (123) in the equilibrium equations

$$\left. \begin{aligned} \frac{\partial G_1}{\partial x} + \frac{\partial H}{\partial y} &= Q_1; \\ \frac{\partial G_2}{\partial y} + \frac{\partial H}{\partial x} &= Q_2; \\ \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} &= -p, \end{aligned} \right\} \quad (124)$$

we obtain a system of differential equations of bending of rectangular anisotropic plates /46

$$\begin{aligned} K_1 \left(\varphi + \frac{\partial w}{\partial x} \right) &= D_{11} \frac{\partial^2 \varphi}{\partial x^2} + 2D_{13} \frac{\partial^2 \varphi}{\partial x \partial y} + D_{33} \frac{\partial^2 \varphi}{\partial y^2} + \\ &+ D_{13} \frac{\partial^2 \psi}{\partial x^2} + C \frac{\partial^2 \psi}{\partial x \partial y} + D_{23} \frac{\partial^2 \psi}{\partial y^2}; \end{aligned} \quad (125)$$

$$\begin{aligned} K_2 \left(\psi + \frac{\partial w}{\partial y} \right) &= D_{13} \frac{\partial^2 \varphi}{\partial x^2} + C \frac{\partial^2 \varphi}{\partial x \partial y} + D_{23} \frac{\partial^2 \varphi}{\partial y^2} + \\ &+ D_{23} \frac{\partial^2 \psi}{\partial x^2} + 2D_{23} \frac{\partial^2 \psi}{\partial x \partial y} + D_{33} \frac{\partial^2 \psi}{\partial y^2}; \end{aligned} \quad (126)$$

$$K_1 \left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + K_2 \left(\frac{\partial \psi}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) = p. \quad (127)$$

Eq. (125) and (126) can be reduced to the following symmetrical form

$$L_4(\varphi) = D_{22}K_1 \frac{\partial^2 w}{\partial x^2} + (2D_{23}K_1 - D_{13}K_2) \frac{\partial^2 w}{\partial x^2 \partial y} + \\ + (D_{33}K_1 - CK_2) \frac{\partial^2 w}{\partial x \partial y^2} - D_{22}K_2 \frac{\partial^2 w}{\partial y^2} - K_1K_2 \frac{\partial w}{\partial x}; \quad (128)$$

$$L_4(\psi) = D_{33}K_2 \frac{\partial^2 w}{\partial y^2} + (2D_{13}K_2 - D_{23}K_1) \frac{\partial^2 w}{\partial x \partial y} + \\ + (D_{11}K_2 - CK_1) \frac{\partial^2 w}{\partial x^2 \partial y} - D_{13}K_1 \frac{\partial^2 w}{\partial x^2} - K_1K_2 \frac{\partial w}{\partial y}, \quad (129)$$

where

$$C = D_{12} + D_{23}, \quad (130)$$

$$L_4(\cdot) = (D_{11}D_{33} - D_{13}^2) \frac{\partial^4}{\partial x^4} + 2(D_{13}D_{23} + D_{23}D_{11} - CD_{13}) \frac{\partial^4}{\partial x^2 \partial y} + \\ + (D_{33}^2 + 2D_{13}D_{23} + D_{11}D_{22} - C^2) \frac{\partial^4}{\partial x^2 \partial y^2} + \\ + 2(D_{23}D_{33} + D_{13}D_{22} - CD_{23}) \frac{\partial^4}{\partial x \partial y^3} + \\ + (D_{22}D_{33} - D_{23}^2) \frac{\partial^4}{\partial y^4} - (D_{11}K_2 + D_{23}K_1) \frac{\partial^3}{\partial x^2} - \\ - 2(D_{13}K_2 + D_{23}K_1) \frac{\partial^3}{\partial x^2 \partial y} - (D_{33}K_2 + D_{23}K_1) \frac{\partial^3}{\partial y^3} + K_1K_2, \quad (131)$$

By multiplying Eq. (127) by operator $L_4(\cdot)$ and with Eq. (128), (129) taken into account, a differential equation of bending of a laminar anisotropic rectangular plate can be obtained with interlayer shearing taken into account /47

$$(D_{11}D_{33} - D_{13}^2) K_1 \frac{\partial^6 w}{\partial x^4} + 2(D_{11}D_{23} - D_{13}D_{13}) K_1 \frac{\partial^6 w}{\partial x^2 \partial y} + \\ + [K_1(D_{33}^2 + 2D_{13}D_{23} + D_{11}D_{22} - C^2) + K_2(D_{11}D_{33} - D_{13}^2)] \times \\ \times \frac{\partial^6 w}{\partial x^2 \partial y^2} + 2[K_1(D_{22}D_{13} - D_{13}D_{23}) + \\ + K_2(D_{11}D_{23} - D_{13}D_{13})] \frac{\partial^6 w}{\partial x^2 \partial y^3} + \\ + [K_2(D_{33}^2 + 2D_{13}D_{23} + D_{11}D_{22} - C^2) + K_1(D_{22}D_{33} - D_{23}^2)] \frac{\partial^6 w}{\partial x^2 \partial y^4} + \\ + 2[D_{23}D_{13} - D_{13}D_{23}] K_2 \frac{\partial^6 w}{\partial x \partial y^5} + \\ + (D_{22}D_{33} - D_{23}^2) K_2 \frac{\partial^6 w}{\partial y^6} - K_1K_2 \times \\ \times \left[D_{11} \frac{\partial^4 w}{\partial x^4} + 4D_{13} \frac{\partial^4 w}{\partial x^2 \partial y} + 2(C + D_{33}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \right. \\ \left. + 4D_{23} \frac{\partial^4 w}{\partial x \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \right] = L_4(p). \quad (132)$$

For brevity, we introduce the following designations

$$\left. \begin{aligned} a_{40} &= \frac{D_{11}D_{33} - D_{13}^2}{a^4}; & a_{31} &= \frac{2(D_{11}D_{23} - D_{13}D_{13})}{a^2b}; \\ a_{22} &= \frac{D_{33}^2 + 2D_{13}D_{23} + D_{11}D_{22} - C^2}{a^2b^2}; \\ a_{13} &= \frac{2(D_{22}D_{13} - D_{13}D_{23})}{ab^2}; & a_{11} &= \frac{2(D_{13}K_2 + D_{23}K_1)}{ab}; \end{aligned} \right\} \quad (133)$$

$$\begin{aligned}
a_{04} &= \frac{D_{22}D_{33} - D_{23}^2}{b^4}; & a_{03} &= \frac{D_{22}K_1 + D_{23}K_2}{b^3}; \\
a_{00} &= \frac{D_{11}K_1 + D_{22}K_2}{a^3}; & a_{00} &= K_1K_2; \\
a_{00} &= \frac{K_1(D_{11}D_{22} - D_{12}^2)}{a^3}; & a_{00} &= \frac{K_2(D_{22}D_{33} - D_{23}^2)}{b^3}; \\
a_{01} &= \frac{2K_1(D_{11}D_{22} - D_{12}^2)}{a^3b}; & a_{10} &= \frac{2K_2(D_{22}D_{33} - D_{23}^2)}{ab^3}; \\
a_{02} &= \left\{ K_1(D_{22}^2 + 2D_{12}D_{23} + D_{11}D_{33} - C^2) + \right. \\
&\quad \left. + K_2(D_{11}D_{22} - D_{12}^2) \right\} \frac{1}{a^2b^2}; \\
a_{22} &= 2 \left\{ K_1(D_{22}D_{12} - D_{12}D_{23}) + K_2(D_{11}D_{23} - D_{12}D_{33}) \right\} \frac{1}{a^2b^2}; \\
a_{11} &= \left\{ K_1(D_{22}^2 + 2D_{12}D_{23} + D_{11}D_{33} - C^2) + \right. \\
&\quad \left. + K_2(D_{11}D_{22} - D_{12}^2) \right\} \frac{1}{a^2b^2};
\end{aligned} \tag{133}$$

$$\begin{aligned}
\alpha_1 &= \frac{D_{33}K_1}{a^3}; & \beta_1 &= \frac{D_{33}K_2}{b^3}; & \gamma_1 &= \frac{D_{11}}{a^4}; \\
\alpha_2 &= \frac{2K_1D_{23} - D_{12}K_2}{a^2b}; & \beta_2 &= \frac{2K_2D_{12} - D_{23}K_1}{ab^2}; & \gamma_2 &= \frac{4D_{12}}{a^2b}; \\
\alpha_3 &= \frac{D_{22}K_1 - CK_2}{ab^2}; & \beta_3 &= \frac{D_{11}K_2 - CK_1}{a^2b}; & \gamma_3 &= \frac{2(C + D_{22})}{a^2b^2}; \\
\alpha_4 &= \frac{D_{22}K_2}{b^3}; & \beta_4 &= \frac{D_{12}K_1}{a^3}; & \gamma_4 &= \frac{4D_{23}}{ab^2}; \\
\alpha_5 &= \frac{K_1K_2}{a}; & \beta_5 &= \frac{K_1K_2}{b}; & \gamma_5 &= \frac{D_{22}}{b^4}.
\end{aligned} \tag{134}$$

Then, in dimensionless coordinates $\xi=x/a$; $\eta=y/b$, system of differential equations of bending of an anisotropic plate (132), (128), (129) is written in the form

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$$L_6(w) = L_4(p); \tag{135}$$

$$\begin{aligned}
L_4(\varphi) &= \alpha_1 \frac{\partial^2 w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} + \alpha_3 \frac{\partial^2 w}{\partial \xi \partial \eta^2} - \alpha_4 \frac{\partial^2 w}{\partial \eta^2} - \alpha_5 \frac{\partial w}{\partial \xi}; \\
L_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_2 \frac{\partial^2 w}{\partial \xi \partial \eta} + \beta_3 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} - \beta_4 \frac{\partial^2 w}{\partial \xi^2} - \beta_5 \frac{\partial w}{\partial \eta};
\end{aligned} \tag{136}$$

where the following differential operators are written through $L_6()$, $L_4()$

$$\begin{aligned}
L_6() &= a_{00} \frac{\partial^4}{\partial \xi^4} + a_{01} \frac{\partial^4}{\partial \xi^3 \partial \eta} + a_{02} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \\
&\quad + a_{03} \frac{\partial^4}{\partial \xi \partial \eta^3} + a_{04} \frac{\partial^4}{\partial \eta^4} + a_{10} \frac{\partial^4}{\partial \xi^4 \partial \eta} + a_{11} \frac{\partial^4}{\partial \xi^3 \partial \eta^2} + a_{12} \frac{\partial^4}{\partial \xi^2 \partial \eta^3} + a_{13} \frac{\partial^4}{\partial \xi \partial \eta^4} - \\
&\quad - a_{00} \left(\gamma_1 \frac{\partial^4}{\partial \xi^4} + \gamma_2 \frac{\partial^4}{\partial \xi^3 \partial \eta} + \gamma_3 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \gamma_4 \frac{\partial^4}{\partial \xi \partial \eta^3} + \gamma_5 \frac{\partial^4}{\partial \eta^4} \right);
\end{aligned} \tag{137}$$

$$L_4(\) = a_{40} \frac{\partial^4}{\partial \xi^4} + a_{21} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + a_{12} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + a_{13} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \\ + a_{04} \frac{\partial^4}{\partial \eta^4} - a_{20} \frac{\partial^2}{\partial \xi^2} - a_{11} \frac{\partial^2}{\partial \xi \partial \eta} - a_{03} \frac{\partial^2}{\partial \eta^2} + a_{00} \quad (138)$$

19. General Equations of Bending of Orthotropic Rectangular Plates

If a laminated plate is made of an orthotropic material, the resulting general relationships and differential equations are simplified when the principal axes of anisotropy coincide with the coordinate axes, since $D_{13} = D_{23} = 0$.

By replacing D_{11} by D_1 , D_{22} by D_2 , D_{33} by D_3 and D_{12} by $D_1 \nu_2$ or $D_2 \nu_1$, for an orthotropic rectangular plate, we obtain /49

$$\left. \begin{aligned} G_1 &= -D_1 \left(\frac{\partial \varphi}{\partial x} + \nu_2 \frac{\partial \psi}{\partial y} \right); \\ C_2 &= -D_2 \left(\frac{\partial \psi}{\partial y} + \nu_1 \frac{\partial \varphi}{\partial x} \right); \\ H &= -D_3 \left(\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right); \end{aligned} \right\} \quad (139)$$

$$\left. \begin{aligned} Q_1 &= -K_1 \left(\varphi + \frac{\partial w}{\partial x} \right); \\ Q_2 &= -K_2 \left(\psi + \frac{\partial w}{\partial y} \right). \end{aligned} \right\} \quad (140)$$

System of differential Eq. (127)-(129) takes the form

$$\left. \begin{aligned} L_4(\varphi) &= D_2 K_1 \frac{\partial^2 w}{\partial x^2} + (D_2 K_1 - C K_2) \frac{\partial^2 w}{\partial x \partial y^2} - K_1 K_2 \frac{\partial w}{\partial x}; \\ L_4(\psi) &= D_1 K_2 \frac{\partial^2 w}{\partial y^2} + (D_1 K_2 - C K_1) \frac{\partial^2 w}{\partial x^2 \partial y} - K_1 K_2 \frac{\partial w}{\partial y}; \end{aligned} \right\} \quad (141)$$

$$K_1 \left(\frac{\partial \varphi}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + K_2 \left(\frac{\partial \psi}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) = p, \quad (142)$$

where $L_4(\)$ is a differential operator in up to fourth order partial derivatives

$$L_4(\) = D_1 D_2 \frac{\partial^4}{\partial x^4} + (D_1 D_2 + D_2^2 - C^2) \frac{\partial^4}{\partial x^2 \partial y^2} + \\ + D_2 D_2 \frac{\partial^4}{\partial y^4} - (D_1 K_2 + D_2 K_1) \frac{\partial^2}{\partial x^2} - \\ - (D_2 K_1 + D_1 K_2) \frac{\partial^2}{\partial y^2} + K_1 K_2. \quad (143)$$

Differential equation of bending of an orthotropic plate (132) takes the form

$$\begin{aligned}
 & D_1 D_3 K_1 \frac{\partial^4 w}{\partial x^4} + [D_1 (D_3 K_1 + D_2 K_2) + K_1 (D_3^2 - C^2)] \frac{\partial^4 w}{\partial x^2 \partial y^2} + \\
 & + [D_2 (D_1 K_2 + D_3 K_1) + K_2 (D_1^2 - C^2)] \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 D_3 K_2 \frac{\partial^4 w}{\partial y^4} - \\
 & - K_1 K_2 \left[D_1 \frac{\partial^4 w}{\partial x^4} + 2(C + D_3) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} \right] = F_6(p).
 \end{aligned} \tag{144}$$

System of differential equations of bending of an orthotropic plate (135), (136) is simplified, since $D_{13}=D_{23}=0$ and, consequently,

$$\begin{aligned}
 a_{30} = a_{31} = a_{13} = a_{21} = a_{31} = a_{13} = a_3 = a_4 = \\
 = \beta_3 = \beta_4 = \gamma_3 = \gamma_4 = 0.
 \end{aligned} \tag{145}$$

In dimensionless coordinates $\xi = x/a$, $\eta = y/b$, the system of equations has the form

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$$\left. \begin{aligned}
 F_4(\varphi) &= a_1 \frac{\partial^3 w}{\partial \xi^3} + a_3 \frac{\partial^3 w}{\partial \xi^2 \partial \eta^2} - a_5 \frac{\partial w}{\partial \xi}; \\
 F_4(\psi) &= \beta_1 \frac{\partial^3 w}{\partial \eta^3} + \beta_3 \frac{\partial^3 w}{\partial \xi^2 \partial \eta} - \beta_5 \frac{\partial w}{\partial \eta};
 \end{aligned} \right\} \tag{146}$$

$$F_6(w) = F_4(p), \tag{147}$$

where $F_6()$, $F_4()$ are the following differential operators in partial derivatives

$$\begin{aligned}
 F_6() &= a_{00} \frac{\partial^4}{\partial \xi^4} + a_{42} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + a_{24} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \\
 &+ a_{06} \frac{\partial^4}{\partial \eta^4} - a_{00} \left(\gamma_1 \frac{\partial^4}{\partial \xi^4} + \gamma_3 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + \gamma_5 \frac{\partial^4}{\partial \eta^4} \right);
 \end{aligned} \tag{148}$$

$$\begin{aligned}
 F_4() &= a_{40} \frac{\partial^4}{\partial \xi^4} + a_{22} \frac{\partial^4}{\partial \xi^2 \partial \eta^2} + a_{04} \frac{\partial^4}{\partial \eta^4} - \\
 &- a_{20} \frac{\partial^2}{\partial \xi^2} - a_{02} \frac{\partial^2}{\partial \eta^2} + a_{00}.
 \end{aligned} \tag{149}$$

Coefficients α_i , β_i , γ_i ($i=1, 3, 5$) a_{ij} are determined by Eq. (133), (134); with $D_{13}=D_{23}=0$, $D_{11}=D_1$, $D_{22}=D_2$, $D_{33}=D_3$, we have

$$\left. \begin{aligned}
 a_{40} &= \frac{D_1 D_3}{a^4}; & a_{04} &= \frac{D_2 D_3}{b^4}; \\
 a_{22} &= \frac{D_1 D_3 + D_3^2 - C^2}{a^2 b^2}; & a_{20} &= \frac{K_2 D_1 + K_1 D_3}{a^2};
 \end{aligned} \right\} \tag{150}$$

$$\left. \begin{aligned} a_{02} &= \frac{K_1 D_1 + K_2 D_2}{b^3}; & a_{00} &= K_1 K_2; \\ a_{20} &= \frac{D_1 D_2 K_1}{a^3}; & a_{08} &= \frac{D_2 D_3 K_2}{b^3}; \\ a_{22} &= [K_1 (D_1 D_2 + D_2^2 - C^2) + D_1 D_2 K_2] \frac{1}{a^2 b^3}; \\ a_{24} &= [K_2 (D_1 D_2 + D_2^2 - C^2) + D_1 D_2 K_1] \frac{1}{a^2 b^3}; \end{aligned} \right\} \quad (150)$$

$$\left. \begin{aligned} \alpha_1 &= \frac{D_2 K_1}{a^3}; & \beta_1 &= \frac{D_3 K_2}{b^3}; & \gamma_1 &= \frac{D_1}{a^2}; \\ \alpha_2 &= \frac{K_1 D_2 - C K_2}{a b^3}; & \beta_2 &= \frac{K_2 D_1 - C K_1}{a^2 b}; & \gamma_2 &= \frac{2(C + D_2)}{a^2 b^2}; \\ \alpha_3 &= \frac{K_1 K_2}{a}; & \beta_3 &= \frac{K_1 K_2}{b}; & \gamma_3 &= \frac{D_2}{b^2}. \end{aligned} \right\} \quad (151)$$

20. Energy of Deformation of Anisotropic Rectangular Plate

In the majority of cases of solution of specific engineering problems, an exact solution cannot successfully be obtained. Therefore, various approximate methods of analysis must be used. In the theory of shells, variation methods based on the principle of the deformation energy minimum are most widespread. If an anisotropic plate is bent by normal load p , the potential energy of bending is determined by the well known expression

$$U = -\frac{1}{2} \iint (G_1 \kappa_1^2 + G_2 \kappa_2^2 + 2H \kappa_3^2 + Q_1 \gamma_1 + Q_2 \gamma_2) dx dy. \quad (152)$$

By using the Hooke's law relationships, we obtain

$$\begin{aligned} U &= \frac{1}{2} \iint \{ D_{11} \kappa_1^2 + 2D_{12} \kappa_1 \kappa_2 + D_{22} \kappa_2^2 + 3D_{13} \kappa_1 \kappa_3 + \\ &+ 3D_{23} \kappa_2 \kappa_3 + 4D_{33} \kappa_3^2 + K_1 \gamma_1^2 + K_2 \gamma_2^2 \} dx dy. \end{aligned} \quad (153)$$

By substituting κ_1^e , κ_2^e , κ_3^e , γ_1 , γ_2 from Eq. (11), (12) in Eq. (153), we obtain

$$\begin{aligned} U &= \frac{1}{2} \iint \left[D_{11} \left(\frac{\partial \Phi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \Phi}{\partial x} \frac{\partial \Psi}{\partial y} + D_{22} \left(\frac{\partial \Psi}{\partial y} \right)^2 + \right. \\ &+ D_{33} \left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x} \right)^2 + \frac{3}{2} D_{13} \frac{\partial \Phi}{\partial x} \left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x} \right) + \\ &+ \frac{3}{2} D_{23} \frac{\partial \Psi}{\partial y} \left(\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x} \right) + K_1 \left(\Phi + \frac{\partial w}{\partial x} \right)^2 + \\ &\left. + K_2 \left(\Psi + \frac{\partial w}{\partial y} \right)^2 \right] dx dy. \end{aligned} \quad (154)$$

In this manner, while a possible deformed state of the plate is determined by functions ϕ , ψ , w , the actual deformed state differs from all the kinematically possible states, i.e., those which satisfy the boundary conditions given, insofar as, for the actually deformed state, the functional

$$\begin{aligned}
V = \frac{1}{2} \iint & \left[D_{11} \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_{22} \left(\frac{\partial \psi}{\partial y} \right)^2 + \right. \\
& + D_{33} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + \frac{3}{2} D_{13} \frac{\partial \varphi}{\partial x} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + \\
& + \frac{3}{2} D_{23} \frac{\partial \psi}{\partial y} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + \\
& \left. + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 - 2pw \right] dx dy
\end{aligned} \tag{155}$$

has the minimum value.

The integral is taken over the entire surface of the plate. If /52
the plate is orthotropic and the directions of coordinate axes x, y coincide with the principal directions of anisotropy, Eq. (155) is significantly simplified and takes the form

$$\begin{aligned}
V = \frac{1}{2} \iint & \left[D_1 \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_1 \nu_2 \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_2 \left(\frac{\partial \psi}{\partial y} \right)^2 + \right. \\
& + D_3 \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + \\
& \left. + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 - 2pw \right] dx dy.
\end{aligned} \tag{156}$$

The simplest alternate version of use of the principle of possible displacements, which is called the Ritz method, is as follows. Desired functions ϕ, ψ, w are assigned which satisfy the assigned boundary conditions at any values of random parameters $C_j (j=1, 2, \dots, n)$ and correspond as well as possible to the physical essence of the problem

$$\left. \begin{aligned}
\varphi &= \varphi(x, y; C_1, C_2, C_3, \dots, C_n); \\
\psi &= \psi(x, y; C_1, C_2, C_3, \dots, C_n); \\
w &= w(x, y; C_1, C_2, C_3, \dots, C_n).
\end{aligned} \right\} \tag{157}$$

By substituting these values in Eq. (155) for an anisotropic plate or in Eq. (156) for an orthotropic plate, after integration over x and y within limits which correspond to the entire surface of the plate, we obtain

$$V = V(C_1, C_2, C_3, \dots, C_n). \tag{158}$$

We select constants C_1, C_2, \dots, C_n in such a way that the energy of the system has the least value, i.e.,

$$\frac{\partial V}{\partial C_1} = \frac{\partial V}{\partial C_2} = \dots = \frac{\partial V}{\partial C_n} = 0. \tag{159}$$

Values of random constants C_1, C_2, \dots, C_n result from solution of the system of n equations, which determine the desired solution of the problem with the required degree of approximation. In the limit, as $n \rightarrow \infty$, an exact solution can be obtained. The accuracy of the solution depends on how successfully functions ϕ, ψ, w are selected.

21. Bending of Orthotropic Plate by Concentrated Force

Let a concentrated force applied at the point with coordinates x_0, y_0 act on an orthotropic rectangular plate, the principal axes of anisotropy of which coincide with the x, y coordinate axes (Fig. 26).

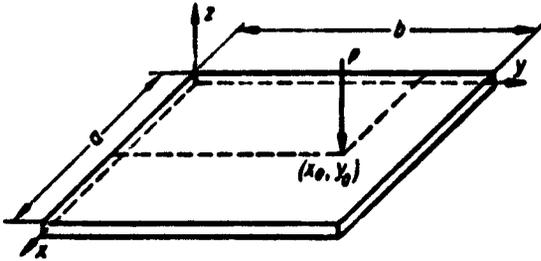


Fig. 26. Diagram of bending of hinge supported rectangular plate by concentrated force.

We will assume the sides of the plate to be supported and satisfy the following boundary conditions

$$\begin{aligned} w = G_1 = \psi = 0 & \quad \text{at} \quad x=0, x=a; \\ w = G_2 = \phi = 0 & \quad \text{at} \quad y=0, y=b. \end{aligned}$$

Such boundary conditions satisfy a possible deformed state, which is determined by the expressions

$$\left. \begin{aligned} \varphi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \\ \psi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}; \\ w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \end{aligned} \right\} \quad (160)$$

In accordance with Eq. (156), the functional of the elastic energy of the system

$$\begin{aligned} V &= \frac{1}{2} \int_0^a \int_0^b \left\{ D_1 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{m\pi}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 + \right. \\ &+ 2D_1 \nu_s \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \frac{m\pi}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right) \times \\ &\times \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right) + \\ &+ D_2 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 + \\ &+ D_3 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} \frac{n\pi}{b} + B_{mn} \frac{m\pi}{a} \right) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right]^2 \left. \right\} + \end{aligned} \quad (161)$$

$$\begin{aligned}
& + K_1 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(A_{mn} + C_{mn} \frac{m\pi}{a} \right) \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 + \\
& + K_2 \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(B_{mn} + C_{mn} \frac{n\pi}{b} \right) \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right]^2 \Big] dx dy - \\
& - P \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}.
\end{aligned} \tag{161}$$

Because the integrals of the product of the trigonometric functions discussed differ from zero only in the quadratic terms, we have

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$$\begin{aligned}
V = & \frac{ab}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[D_1 \left(\frac{m\pi}{a} \right)^2 A_{mn}^2 + 2D_1 \nu_2 \frac{\pi^2 mn}{ab} A_{mn} B_{mn} + \right. \\
& + D_2 \left(\frac{n\pi}{b} \right)^2 B_{mn}^2 + D_3 \left(A_{mn} \frac{n\pi}{b} + B_{mn} \frac{m\pi}{a} \right)^2 + \\
& + K_1 \left(A_{mn} + C_{mn} \frac{m\pi}{a} \right)^2 + K_2 \left(B_{mn} + C_{mn} \frac{n\pi}{b} \right)^2 \Big] - \\
& - P \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}.
\end{aligned} \tag{162}$$

The minimum of functional (162) is realized under the conditions

$$\left. \begin{aligned}
& A_{mn} \left[D_1 \left(\frac{m\pi}{a} \right)^2 + D_3 \left(\frac{n\pi}{b} \right)^2 + K_1 \right] + \\
& + B_{mn} (D_1 \nu_2 + D_3) \frac{\pi^2 mn}{ab} + C_{mn} K_1 \frac{m\pi}{a} = 0; \\
& A_{mn} (D_1 \nu_2 + D_3) \frac{\pi^2 mn}{ab} + B_{mn} \left[D_2 \left(\frac{n\pi}{b} \right)^2 + \right. \\
& \left. + D_3 \left(\frac{m\pi}{a} \right)^2 + K_2 \right] + C_{mn} K_2 \frac{n\pi}{b} = 0; \\
& A_{mn} K_1 \frac{m\pi}{a} + B_{mn} K_2 \frac{n\pi}{b} + C_{mn} \left[K_1 \left(\frac{m\pi}{a} \right)^2 + \right. \\
& \left. + K_2 \left(\frac{n\pi}{b} \right)^2 \right] = \frac{4P}{ab} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}
\end{aligned} \right\} \tag{163}$$

for $m, n=1, 2, 3, \dots$

By solving system of Eq. (162), we find

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$$\left. \begin{aligned}
A_{mn} &= -\frac{4P}{ab} \frac{\Delta_{1m}}{\Delta_m} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}; \\
B_{mn} &= -\frac{4P}{ab} \frac{\Delta_{2m}}{\Delta_m} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b}; \\
C_{mn} &= \frac{4P}{ab} \frac{\Delta_{3m}}{\Delta_m} \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b},
\end{aligned} \right\} \tag{164}$$

$(m, n = 1, 2, 3, \dots)$

where

$$\begin{aligned}
\Delta_{1m} &= K_1 D_2 \left(\frac{m\pi}{a} \right)^2 + (K_1 D_2 - K_2 C) \frac{m\pi}{a} \left(\frac{n\pi}{b} \right)^2 + \\
&\quad + K_1 K_2 \frac{m\pi}{a}; \\
\Delta_{2m} &= K_2 D_1 \left(\frac{n\pi}{b} \right)^2 + (K_2 D_1 - K_1 C) \left(\frac{m\pi}{a} \right)^2 \frac{n\pi}{b} + \\
&\quad + K_1 K_2 \frac{n\pi}{b}; \\
\Delta_{2m} &= D_1 D_2 \left(\frac{m\pi}{a} \right)^4 + (D_1 D_2 + D_2^2 - C^2) \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 + \\
&\quad + D_2 D_1 \left(\frac{n\pi}{b} \right)^4 + (D_1 K_2 + D_2 K_1) \left(\frac{m\pi}{a} \right)^2 + \\
&\quad + (D_2 K_1 + D_1 K_2) \left(\frac{n\pi}{b} \right)^2 + K_1 K_2; \\
\Delta_m &= D_1 D_2 K_1 \left(\frac{m\pi}{a} \right)^6 + [D_1 (D_2 K_1 + D_2 K_2) + \\
&\quad + K_1 (D_2^2 - C^2)] \left(\frac{m\pi}{a} \right)^4 \left(\frac{n\pi}{b} \right)^2 + [D_2 (D_1 K_2 + D_2 K_1) + \\
&\quad + K_2 (D_1^2 - C^2)] \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^4 + D_1 D_2 K_2 \left(\frac{n\pi}{b} \right)^6 + \\
&\quad + K_1 K_2 \left[D_1 \left(\frac{m\pi}{a} \right)^4 + 2(C + D_2) \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2 + \right. \\
&\quad \left. + D_2 \left(\frac{n\pi}{b} \right)^4 \right]; \\
C &= D_1 \nu_2 + D_2 = D_2 \nu_1 + D_1.
\end{aligned} \tag{165}$$

for $m, n=1, 2, 3, \dots$

The bending moments and cutting forces are determined by the ex- /56

$$\begin{aligned}
G_1 &= \frac{4PD_1\pi}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\Delta_{1m} \frac{m}{a} + \Delta_{2m} \frac{n}{b} \nu_2 \right)}{\Delta_m} \times \\
&\quad \times \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \\
G_2 &= \frac{4P\pi D_2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Delta_{2m} \frac{n}{b} + \nu_1 \Delta_{1m} \frac{m}{a}}{\Delta_m} \times \\
&\quad \times \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}; \\
H &= \frac{4\pi PD_2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\left(\Delta_{2m} \frac{m}{a} + \Delta_{1m} \frac{n}{b} \right)}{\Delta_m} \times
\end{aligned} \tag{166}$$

$$\begin{aligned}
& \times \sin \frac{m \pi x_0}{a} \sin \frac{n \pi y_0}{b} \cos \frac{m \pi x}{a} \cos \frac{n \pi y}{b} : \\
Q_1 &= \frac{4PK_1}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\Delta_{1m} - \Delta_{2m} \frac{m \pi}{a})}{\Delta_m} \times \\
& \times \sin \frac{m \pi x_0}{a} \sin \frac{n \pi y_0}{b} \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} ; \\
Q_2 &= \frac{4PK_2}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\Delta_{2m} - \Delta_{1m} \frac{n \pi}{b})}{\Delta_m} \times \\
& \times \sin \frac{m \pi x_0}{a} \sin \frac{n \pi y_0}{b} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} .
\end{aligned} \tag{166}$$

22. Bending of Orthotropic Plate by Uniformly Distributed Load

Let an orthotropic hinge supported rectangular plate with sides a , b be bent by a uniformly distributed load of intensity p . We select the coordinate system as indicated in Fig. 26.

We will seek a solution of Eq. (146), (147) in the binary trigonometric series /57

$$\left. \begin{aligned}
\varphi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos m \pi \xi \sin n \pi \eta ; \\
\psi &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin m \pi \xi \cos n \pi \eta ; \\
w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin m \pi \xi \sin n \pi \eta .
\end{aligned} \right\} \tag{167}$$

We represent load p in the form of the binary trigonometric series

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} \sin m \pi \xi \sin n \pi \eta , \tag{168}$$

where

$$P_{mn} = \frac{16p}{\pi^2 mn} (m, n = 1, 3, 5, \dots) . \tag{169}$$

By substituting Eq. (167), (168) in bending Eq. (146), (147), we obtain

$$\begin{aligned}
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} [\pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \\
& + \pi^2 (a_{20} m^2 + a_{02} n^2) + a_{00}] \cos m \pi \xi \sin n \pi \eta = \\
& = - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} [\pi^2 (a_1 m^2 + a_2 m n^2) + \pi a_3 m] \times \\
& \times \cos m \pi \xi \sin n \pi \eta ;
\end{aligned} \tag{170}$$

$$\begin{aligned}
& \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} [\pi^4 (a_{10} m^4 + a_{20} m^2 n^2 + a_{04} n^4) + \\
& + \pi^2 (a_{30} m^2 + a_{03} n^2) + a_{00}] \sin m \pi \xi \cos n \pi \eta = \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} [\pi^2 (\beta_1 n^2 + \beta_2 m^2 n) + \pi \beta_3 n] \times \\
& \quad \times \sin m \pi \xi \cos n \pi \eta; \\
& - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} [\pi^4 (a_{10} m^4 + a_{20} m^2 n^2 + a_{04} m^2 n^2 + a_{00} n^4) + \\
& + \pi^4 a_{00} (\gamma_1 m^4 + \gamma_2 m^2 n^2 + \gamma_3 n^4)] \sin m \pi \xi \sin n \pi \eta = \\
& = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P_{mn} [\pi^4 (a_{10} m^4 + a_{20} m^2 n^2 + a_{04} n^4) + \\
& + \pi^2 (a_{30} m^2 + a_{03} n^2) + a_{00}] \sin m \pi \xi \sin n \pi \eta.
\end{aligned} \tag{170}$$

According to Eq. (169), coefficients A_{mn} , B_{mn} , C_{mn} differ from zero only at odd values of indices $m, n=1, 3, 5, \dots$ therefore, we will not subsequently stipulate this, and we will understand that summing is carried out only over the odd indices.

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We introduce the following designations

$$\begin{aligned}
\zeta_0^{m,n} &= \pi^4 (a_{10} m^4 + a_{20} m^2 n^2 + a_{04} m^2 n^2 + a_{00} n^4) + \\
& + \pi^4 a_{00} (\gamma_1 m^4 + \gamma_2 m^2 n^2 + \gamma_3 n^4); \\
\zeta_1^{m,n} &= \pi^4 (a_{10} m^4 + a_{20} m^2 n^2 + a_{04} n^4) + \\
& + \pi^2 (a_{30} m^2 + a_{03} n^2) + a_{00}; \\
\zeta_2^{m,n} &= \pi^2 (\alpha_1 m^2 + \alpha_2 m n^2) + \pi a_3 m; \\
\zeta_3^{m,n} &= \pi^2 (\beta_1 n^2 + \beta_2 m^2 n) + \pi \beta_3 n
\end{aligned} \tag{171}$$

for $m, n=1, 3, 5, \dots$

In accordance with Eq. (170), we obtain

$$\left. \begin{aligned}
C_{mn} &= -P_{mn} \frac{\zeta_1^{m,n}}{\zeta_0^{m,n}}; \quad A_{mn} = P_{mn} \frac{\zeta_2^{m,n}}{\zeta_0^{m,n}}; \\
B_{mn} &= P_{mn} \frac{\zeta_3^{m,n}}{\zeta_0^{m,n}};
\end{aligned} \right\} \tag{172}$$

and, consequently, the solution of Eq. (146), (147) has the form

$$\left. \begin{aligned} \varphi &= \frac{16\rho}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_2^{m,n}}{mn \zeta_0^{m,n}} \cos m \pi \xi \sin n \pi \eta; \\ \psi &= \frac{16\rho}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_2^{m,n}}{mn \zeta_0^{m,n}} \sin m \pi \xi \cos n \pi \eta; \\ w &= -\frac{16\rho}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_1^{m,n}}{mn \zeta_0^{m,n}} \sin m \pi \xi \sin n \pi \eta. \end{aligned} \right\} \quad (173)$$

The maximum deflection in the center of the plate

$$|w|_{\max} = \frac{16\rho}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{m+n-2}{2}} \zeta_1^{m,n}}{mn \zeta_0^{m,n}}. \quad (174)$$

The bending moments, torques and cutting forces are determined by the following expressions

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$$\left. \begin{aligned} G_1 &= \frac{16\rho D_1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_1^{m,n} m + \zeta_2^{m,n} n}{mn \zeta_0^{m,n}} \sin m \pi \xi \sin n \pi \eta; \\ G_2 &= \frac{16\rho D_1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_3^{m,n} n + \zeta_1^{m,n} m}{mn \zeta_0^{m,n}} \sin m \pi \xi \sin n \pi \eta; \\ H &= -\frac{16\rho D_2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\zeta_2^{m,n} m + \zeta_3^{m,n} n}{mn \zeta_0^{m,n}} \cos m \pi \xi \cos n \pi \eta; \\ Q_1 &= \frac{16\rho K_1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \zeta_1^{m,n} m - \zeta_2^{m,n}}{mn \zeta_0^{m,n}} \cos m \pi \xi \sin n \pi \eta; \\ Q_2 &= \frac{16\rho K_2}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \zeta_1^{m,n} n - \zeta_3^{m,n}}{mn \zeta_0^{m,n}} \sin m \pi \xi \cos n \pi \eta. \end{aligned} \right\} \quad (175)$$

The maximum shearing stresses on the edges of the plate ($x=0$, $x=a$ and $y=0$, $y=b$) arise in the middles of the sides, and they are determined by the expressions

$$\left. \begin{aligned} \tau_{1 \max} &= \frac{24\rho K_1}{\pi^2 b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \zeta_1^{m,n} n - \zeta_2^{m,n}}{mn \zeta_0^{m,n}}; \\ \tau_{2 \max} &= \frac{24\rho K_2}{\pi^2 b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\pi \zeta_1^{m,n} m - \zeta_3^{m,n}}{mn \zeta_0^{m,n}}. \end{aligned} \right\} \quad (176)$$

where $\zeta_0^{m,n}$, $\zeta_1^{m,n}$, $\zeta_2^{m,n}$, $\zeta_3^{m,n}$ are determined by Eq. (133), (134) and (171).

A solution in the form of binary trigonometric series is inconven-

ient for practical use. Therefore, it is advisable to consider a solution presented in single series.

23. Bending of Orthotropic Rectangular Plate with Two Supported Edges by Uniformly Distributed Load

Let the principal axes of anisotropy of the plate be parallel to the sides. We select the coordinate system as shown in Fig. 27.

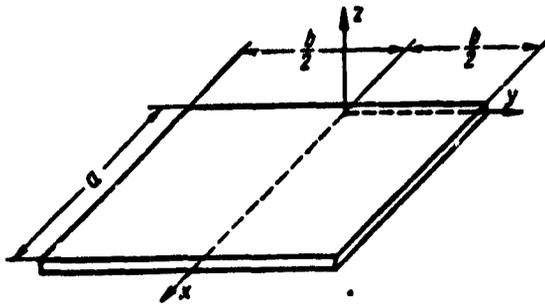


Fig. 27. Coordinate system selected.

We will assume the edges of the plate $x=0, x=a$ to be supported and to satisfy the following boundary conditions

$$w=G_1=\psi=0.$$

In this case, solution of plate bending system of differential Eq. (146), (147) can be sought in the form of single trigonometric series of the following type /60

$$\left. \begin{aligned} \varphi &= \sum_{n=1}^{\infty} \varphi_n(\eta) \cos \lambda_n \xi; \\ \psi &= \sum_{n=1}^{\infty} \psi_n(\eta) \sin \lambda_n \xi; \\ w &= \sum_{n=1}^{\infty} w_n(\eta) \sin \lambda_n \xi, \end{aligned} \right\} \quad (177)$$

where

$$(\lambda_n = n \pi).$$

We expand the uniformly distributed load in the trigonometric series

$$p = \sum_{n=1}^{\infty} A_n \sin \lambda_n \xi, \quad (178)$$

where

$$A_n = \frac{4p}{\lambda_n} \text{ for } n = 1, 3, 5, \dots \quad (179)$$

By substituting Eq. (177), (178) in bending Eq. (146), (147), we obtain

$$\left. \begin{aligned} \sum_{n=1,3}^{\infty} [a_{04} \varphi_n^{IV} - (\lambda_n^2 a_{22} + a_{02}) \varphi_n'' + (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}) \varphi_n] \cos \lambda_n \xi &= \sum_{n=1,3}^{\infty} [\lambda_n a_3 w_n' - \lambda_n (\lambda_n^2 a_1 + a_5) w_n] \cos \lambda_n \xi; \\ \sum_{n=1,3}^{\infty} [a_{04} \psi_n^{IV} - (\lambda_n^2 a_{22} + a_{02}) \psi_n'' + (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}) \psi_n] \sin \lambda_n \xi &= \sum_{n=1,3}^{\infty} [\beta_1 w_n''' - (\lambda_n^2 \beta_2 + \end{aligned} \right\} \quad (180)$$

$$\begin{aligned}
& + \beta_3) w_n] \sin \lambda_n \xi; \\
\sum_{n=1,3}^{\infty} [a_{04} w_n^{IV} - (\lambda_n^2 a_{22} + \gamma_3 a_{00}) w_n^{IV} + \lambda_n^2 (\lambda_n^2 a_{42} + \\
& + \gamma_3 a_{00}) w_n'' - \lambda_n^4 (\lambda_n^2 a_{60} + \gamma_1 a_{00}) w_n] \sin \lambda_n \xi = \\
& = \sum_{n=1,3}^{\infty} A_n (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}) \sin \lambda_n \xi.
\end{aligned} \tag{180}$$

Here and subsequently, summing is carried out only by odd indices /61
($n=1, 3, 5, \dots$).

System of Eq. (180) is satisfied if, for each $n=1, 3, 5, \dots$, there is a solution of the following system of conventional differential equations

$$\begin{aligned}
a_{04} \varphi_n^{IV} - (\lambda_n^2 a_{22} + a_{02}) \varphi_n'' + (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}) \varphi_n & = \\
& = \lambda_n a_2 w_n'' - \lambda_n (\lambda_n^2 a_1 + a_3) w_n; \\
a_{04} \psi_n^{IV} - (\lambda_n^2 a_{22} + a_{02}) \psi_n'' + (\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}) \psi_n & = \\
& = \beta_1 w_n''' - (\lambda_n^2 \beta_3 + \beta_5) w_n'; \\
a_{06} w^{VI} - (\lambda_n^2 a_{24} + \gamma_3 a_{00}) w_n^{IV} + \lambda_n^2 (\lambda_n^2 a_{42} + \gamma_3 a_{00}) w_n'' - \\
& - \lambda_n^4 (\lambda_n^2 a_{60} + \gamma_1 a_{00}) w_n = \frac{4\rho}{\lambda_n} (\lambda_n^2 a_{40} + \lambda_n^2 a_{20} + a_{00}).
\end{aligned} \tag{181}$$

We present system of Eq. (181) in canonical form, doing them correspondingly by coefficients a_{04} , a_{06} which, according to Eq. (150), are different from zero

$$\left. \begin{aligned}
\varphi_n^{IV} - e_n^{(1)} \varphi_n'' + e_n^{(2)} \varphi_n & = f_n^{(1)} w_n'' - f_n^{(2)} w_n; \\
\psi_n^{IV} - e_n^{(1)} \psi_n'' + e_n^{(2)} \psi_n & = f_n^{(3)} w_n''' - f_n^{(4)} w_n';
\end{aligned} \right\} \tag{182}$$

$$w_n^{VI} - \omega_n^{(1)} w_n^{IV} + \omega_n^{(2)} w_n'' - \omega_n^{(3)} w_n = \Omega_n \tag{183}$$

for $n=1, 3, 5, 7, \dots$, where

$$\left. \begin{aligned}
e_n^{(1)} & = \frac{\lambda_n^2 a_{22} + a_{02}}{a_{04}} = \frac{b^2}{D_2 D_3} \left[\frac{\lambda_n^2}{a^2} (D_1 D_2 + D_3^2 - C^2) + \right. \\
& \quad \left. + K_1 D_2 + K_2 D_3 \right]; \\
e_n^{(2)} & = \frac{\lambda_n^4 a_{40} + \lambda_n^2 a_{20} + a_{00}}{a_{04}} = \frac{b^4}{D_2 D_3} \left[\frac{\lambda_n^4}{a^4} D_1 D_3 + \frac{\lambda_n^2}{a^2} \times \right. \\
& \quad \left. \times (K_2 D_1 + K_1 D_3) + K_1 K_2 \right];
\end{aligned} \right\} \tag{184}$$

$$\begin{aligned}
f_n^{(1)} &= \frac{\lambda_n a_3}{a_{04}} = \frac{b^3}{D_1 D_3} \frac{\lambda_n}{a} (K_1 D_3 - 2CK_2); \\
f_n^{(2)} &= \frac{\lambda_n (\lambda_n^2 a_1 + a_2)}{a_{04}} = \frac{b^4}{D_1 D_3} \frac{\lambda_n}{a} \left(\frac{\lambda_n^2}{a^2} D_3 + K_2 \right) K_1; \\
f_n^{(3)} &= \frac{\beta_1}{a_{04}} = \frac{K_2 b}{D_1}; \\
f_n^{(4)} &= \frac{\lambda_n^2 \beta_3 + \beta_4}{a_{04}} = \frac{b^3}{D_1 D_3} \left[\frac{\lambda_n^2}{a^2} (K_2 D_1 - 2CK_2) + K_1 K_2 \right]; \\
\omega_n^{(1)} &= \frac{\lambda_n^2 a_{24} + \gamma_2 a_{00}}{a_{04}} = \\
&= \frac{b^2}{D_1 D_3 K_2} \left\{ \frac{\lambda_n^2}{a^2} [D_1 D_3 K_1 + K_2 (D_1 D_3 + D_3^2 - C^2)] + \right. \\
&\quad \left. + D_3 K_1 K_2 \right\}; \\
\omega_n^{(2)} &= \frac{\lambda_n^2 (\lambda_n^2 a_{42} + \gamma_3 a_{00})}{a_{04}} = \frac{b^4}{D_1 D_3 K_2} \times \\
&\times \frac{\lambda_n^2}{a^2} \left\{ \frac{\lambda_n^2}{a^2} [K_2 D_1 D_3 + K_1 (D_1 D_3 + D_3^2 - C^2)] + \right. \\
&\quad \left. + 2(C + D_3) K_1 K_2 \right\}; \\
\omega_n^{(3)} &= \frac{\lambda_n^4 (\lambda_n^2 a_{60} + \gamma_1 a_{00})}{a_{04}} = \\
&= \frac{b^4 D_1 K_1}{D_1 D_3 K_2} \frac{\lambda_n^4}{a^4} \left(\frac{\lambda_n^2}{a^2} D_3 + K_2 \right); \\
\Omega_n &= \frac{4p}{\lambda_n} \frac{b^4}{D_1 D_3 K_2} \left[\frac{\lambda_n^4}{a^4} D_1 D_3 + \right. \\
&\quad \left. + \frac{\lambda_n^2}{a^2} (K_2 D_1 + K_1 D_3) + K_1 K_2 \right]
\end{aligned} \tag{184}$$

for $n=1, 3, 5, \dots$

After finding the general solution for plate deflection w_n , for 63 deformation functions ϕ_n, ψ_n , partial solutions of Eq. (182) are taken. In this manner, the solution of system of differential Eq. (182), (183) is determined by the roots of the characteristic equation

$$k_0^4 - \omega_n^{(1)} k_0^4 + \omega_n^{(2)} k_0^3 - \omega_n^{(3)} = 0. \tag{185}$$

Eq. (185) always has two real roots. The remaining roots are determined by coefficients $\omega_n^{(j)}$ ($j=1, 2, 3$).

We consider the most general case of complex roots, i.e., we will assume that the roots of characteristic Eq. (185) are

$$\pm k_n; \pm (s_n \pm r_{n1}). \tag{186}$$

The general solution of differential Eq. (183) can then be presented in the following form (for brevity, we will omit index n of coefficients $C_1, A_1, B_1, k_1, s_1, r_1$)

$$w_n = -\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \operatorname{ch} k \eta + C_2 \Phi_1(\eta) + C_3 \Phi_2(\eta) + C_4 \operatorname{sh} k \eta + C_5 \Phi_3(\eta) + C_6 \Phi_4(\eta), \quad (187)$$

where the well known functions of V.Z. Vlasov are designated by $\Phi_i(\eta)$ ($i=1, 2, 3, 4$)

$$\left. \begin{aligned} \Phi_1(\eta) &= \operatorname{ch} s \eta \cos r \eta; & \Phi_3(\eta) &= \operatorname{sh} s \eta \cos r \eta; \\ \Phi_2(\eta) &= \operatorname{sh} s \eta \sin r \eta; & \Phi_4(\eta) &= \operatorname{ch} s \eta \sin r \eta. \end{aligned} \right\} \quad (188)$$

There are the following relationships for these functions

$$\left. \begin{aligned} \Phi_1' &= s \Phi_3 - r \Phi_4; & \Phi_3' &= s \Phi_1 - r \Phi_2; \\ \Phi_2' &= s \Phi_4 + r \Phi_3; & \Phi_4' &= s \Phi_2 + r \Phi_1; \\ \Phi_1'' &= (s^2 - r^2) \Phi_1 - 2rs \Phi_2; \\ \Phi_2'' &= (s^2 - r^2) \Phi_2 + 2rs \Phi_1; \\ \Phi_3'' &= (s^2 - r^2) \Phi_3 - 2rs \Phi_4; \\ \Phi_4'' &= (s^2 - r^2) \Phi_4 + 2rs \Phi_3; \\ \Phi_1''' &= s(s^2 - 3r^2) \Phi_3 + r(r^2 - 3s^2) \Phi_4; \\ \Phi_2''' &= s(s^2 - 3r^2) \Phi_4 - r(r^2 - 3s^2) \Phi_3; \\ \Phi_3''' &= s(s^2 - 3r^2) \Phi_1 + r(r^2 - 3s^2) \Phi_2; \\ \Phi_4''' &= s(s^2 - 3r^2) \Phi_2 - r(r^2 - 3s^2) \Phi_1; \\ \Phi_1^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_1 - 4rs(s^2 - r^2) \Phi_2; \\ \Phi_2^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_2 + 4rs(s^2 - r^2) \Phi_1; \\ \Phi_3^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_3 - 4rs(s^2 - r^2) \Phi_4; \\ \Phi_4^{IV} &= (s^4 - 6r^2s^2 + r^4) \Phi_4 + 4rs(s^2 - r^2) \Phi_3. \end{aligned} \right\} \quad (189)$$

A table of functions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ [15] is presented in the appendices. /64

In the coordinate system selected with a uniformly distributed load, solution of system of Eq. (182), (183) should be even relative to ϕ_n, w_n and odd relative to ψ_n , i.e., the solution must be sought in the form

$$\left. \begin{aligned} \varphi_n(\eta) &= \varphi_0 + A_1 \operatorname{ch} k \eta + A_2 \Phi_1(\eta) + A_3 \Phi_2(\eta); \\ \psi_n(\eta) &= B_1 \operatorname{sh} k \eta + B_2 \Phi_3(\eta) + B_3 \Phi_4(\eta); \\ w_n(\eta) &= -\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \operatorname{ch} k \eta + C_2 \Phi_1(\eta) + C_3 \Phi_2(\eta). \end{aligned} \right\} \quad (190)$$

By substituting Eq. (190) in differential Eq. (182), we obtain

$$\begin{aligned}
 & A_1 k^4 \operatorname{ch} k \eta + A_2 [(s^4 - 6r^2 s^2 + r^4) \Phi_1(\eta) - \\
 & - 4rs(s^2 - r^2) \Phi_2(\eta)] + A_3 [(s^4 - 6r^2 s^2 + r^4) \Phi_2(\eta) + \\
 & + 4rs(s^2 - r^2) \Phi_1(\eta)] - e_n^{(1)} \{ A_1 k^2 \operatorname{ch} k \eta + \\
 & + A_2 [(s^2 - r^2) \Phi_1(\eta) - 2rs \Phi_2(\eta)] + \\
 & + A_3 [(s^2 - r^2) \Phi_2(\eta) + 2rs \Phi_1(\eta)] \} + \\
 & + e_n^{(2)} [\varphi_0 + A_1 \operatorname{ch} k \eta + A_2 \Phi_1(\eta) + A_3 \Phi_2(\eta)] = \\
 & = f_n^{(1)} [C_1 k^2 \operatorname{ch} k \eta + C_2 [(s^2 - r^2) \Phi_1(\eta) - \\
 & - 2rs \Phi_2(\eta)] + C_3 [(s^2 - r^2) \Phi_2(\eta) + 2rs \Phi_1(\eta)] - \\
 & - f_n^{(2)} [w_0 + C_1 \operatorname{ch} k \eta + C_2 \Phi_1(\eta) + C_3 \Phi_2(\eta)]; \\
 & B_1 k^4 \operatorname{sh} k \eta + B_2 [(s^4 - 6r^2 s^2 + r^4) \Phi_3(\eta) - \\
 & - 4rs(s^2 - r^2) \Phi_4(\eta)] + B_3 [(s^4 - 6r^2 s^2 + r^4) \Phi_4(\eta) + \\
 & + 4rs(s^2 - r^2) \Phi_3(\eta)] - e_n^{(1)} \{ B_1 k^2 \operatorname{sh} k \eta + \\
 & + B_2 [(s^2 - r^2) \Phi_3(\eta) - 2rs \Phi_4(\eta)] + \\
 & + B_3 [(s^2 - r^2) \Phi_4(\eta) + 2rs \Phi_3(\eta)] \} + \\
 & + e_n^{(2)} [B_1 \operatorname{sh} k \eta + B_2 \Phi_3(\eta) + B_3 \Phi_4(\eta)] = \\
 & = f_n^{(3)} [C_1 k^2 \operatorname{sh} k \eta + C_2 [s(s^2 - 3r^2) \Phi_3(\eta) + \\
 & + r(r^2 - 3s^2) \Phi_4(\eta)] + C_3 [s(s^2 - 3r^2) \Phi_4(\eta) - \\
 & - r(r^2 - 3s^2) \Phi_3(\eta)] - f_n^{(4)} [C_1 k \operatorname{sh} k \eta + \\
 & + C_2 [s \Phi_3(\eta) - r \Phi_4(\eta)] + C_3 [s \Phi_4(\eta) + r \Phi_3(\eta)].
 \end{aligned} \tag{191}$$

Since equality (191) occurs with any values of n , on the assumption $\frac{1}{6}$ that $k^4 - e_n^{(1)} k^2 + e_n^{(2)} \neq 0$, the following expressions can be obtained for coefficients A_i, B_i ($i=1, 2, 3$)

$$\begin{aligned}
 \varphi_n &= \frac{f_n^{(2)}}{e_n^{(2)}} \frac{\Omega_n}{\omega_n^{(3)}}; \\
 A_1 &= C_1 \frac{f_n^{(1)} k^2 - f_n^{(2)}}{k^4 - e_n^{(1)} k^2 + e_n^{(2)}}; & B_1 &= C_1 \frac{k (f_n^{(3)} k^2 - f_n^{(4)})}{k^4 - e_n^{(1)} k^2 + e_n^{(2)}}; \\
 A_2 &= \Delta_n^{(1)} C_2 + \Delta_n^{(2)} C_3; & B_2 &= \Delta_n^{(3)} C_2 + \Delta_n^{(4)} C_3; \\
 A_3 &= -\Delta_n^{(3)} C_2 + \Delta_n^{(1)} C_3; & B_3 &= -\Delta_n^{(4)} C_2 + \Delta_n^{(3)} C_3.
 \end{aligned} \tag{192}$$

where

$$\begin{aligned}
 \Delta_n^{(1)} &= \frac{[f_n^{(1)}(s^2 - r^2) - f_n^{(2)}] [s^4 - 6r^2 s^2 + r^4 - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]}{[(s^4 - 6r^2 s^2 + r^4) - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]^2 + 4r^2 s^2 [e_n^{(1)} - 2(s^2 - r^2)]}; \\
 \Delta_n^{(2)} &= 2rs \frac{f_n^{(1)} [s^4 - 6r^2 s^2 + r^4 - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}] + [e_n^{(1)} - 2(s^2 - r^2)] [f_n^{(1)}(s^2 - r^2) - f_n^{(2)}]}{[(s^4 - 6r^2 s^2 + r^4) - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]^2 + 4r^2 s^2 [e_n^{(1)} - 2(s^2 - r^2)]}; \\
 \Delta_n^{(3)} &= s \frac{[f_n^{(3)}(s^2 - 3r^2) - f_n^{(4)}] [s^4 - 6r^2 s^2 + r^4 - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]}{[(s^4 - 6r^2 s^2 + r^4) - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]^2 + 4r^2 s^2 [e_n^{(1)} - 2(s^2 - r^2)]}.
 \end{aligned} \tag{193}$$

$$\left. \begin{aligned} & \frac{+e_n^{(2)} + 2r^3 [e_n^{(1)} - 2(s^2 - r^2)] [f_n^{(3)}(r^3 - 3s^2) + f_n^{(4)}]}{+4r^3 s^2 [e_n^{(1)} - 2(s^2 - r^2)]^2}; \\ \Delta_n^{(4)} = r & \frac{2s^3 [e_n^{(1)} - 2(s^2 - r^2)] [f_n^{(3)}(s^2 - 3r^2) - f_n^{(4)}] -}{[s^4 - 6r^3 s^2 + r^6 - e_n^{(1)}(s^2 - r^2) +} \\ & - \frac{[f_n^{(3)}(r^3 - 3s^2) + f_n^{(4)}] [s^4 - 6r^3 s^2 + r^6 - e_n^{(1)}(s^2 - r^2) + e_n^{(2)}]}{+e_n^{(2)} + 4r^3 s^2 [e_n^{(1)} - r(s^2 - r^2)]^2}. \end{aligned} \right\} \quad (193)$$

Thus,

$$\left. \begin{aligned} \varphi &= \sum_{n=1,3}^{\infty} [\varphi_0 + A_{1n} \operatorname{ch} k_n \eta + A_{2n} \Phi_{1n}(\eta) + A_{3n} \Phi_{2n}(\eta)] \cos \lambda_n \xi; \\ \psi &= \sum_{n=1,3}^{\infty} [B_{1n} \operatorname{sh} k_n \eta + B_{2n} \Phi_{3n}(\eta) + B_{3n} \Phi_{4n}(\eta)] \sin \lambda_n \xi; \\ w &= \sum_{n=1,3}^{\infty} \left[-\frac{\Omega_n}{\omega_n^{(3)}} + C_{1n} \operatorname{ch} k_n \eta + C_{2n} \Phi_{1n}(\eta) + \right. \\ & \left. + C_{3n} \Phi_{2n}(\eta) \right] \sin \lambda_n \xi. \end{aligned} \right\} \quad (194)$$

where coefficients A_{jn} , B_{jn} ($j=1, 2, 3$) are expressed through random constants C_{jn} ($j=1, 2, 3$) by Eq. (192), and k_n is a real root of characteristic Eq. (185). /66

Correspondingly, the following expressions can be obtained for the bending moments and forces

$$\left. \begin{aligned} G_1 &= D_1 \sum_{n=1}^{\infty} \left\{ \frac{\lambda_n}{a} [\varphi_0 + A_{1n} \operatorname{ch} k_n \eta + A_{2n} \Phi_{1n}(\eta) + \right. \\ & \left. + A_{3n} \Phi_{2n}(\eta)] - \frac{\nu_2}{b} [k_n B_{1n} \operatorname{ch} k_n \eta + \right. \\ & \left. + B_{2n} \Phi'_{3n}(\eta) + B_{3n} \Phi'_{4n}(\eta)] \right\} \sin \lambda_n \xi; \\ G_2 &= -D_2 \sum_{n=1}^{\infty} \left\{ \frac{1}{b} [k_n B_{1n} \operatorname{ch} k_n \eta + B_{2n} \Phi'_{3n}(\eta) + \right. \\ & \left. + B_{3n} \Phi'_{4n}(\eta)] - \frac{\lambda_n \nu_1}{a} [\varphi_0 + A_{1n} \operatorname{ch} k_n \eta + \right. \\ & \left. + A_{2n} \Phi_{1n}(\eta) + A_{3n} \Phi_{2n}(\eta)] \right\} \sin \lambda_n \xi; \\ H &= -D_3 \sum_{n=1}^{\infty} \left\{ \frac{1}{b} [A_{1n} k_n \operatorname{sh} k_n \eta + A_{2n} \Phi'_{1n}(\eta) + \right. \\ & \left. + A_{3n} \Phi'_{2n}(\eta)] + \frac{\lambda_n}{a} [B_{1n} \operatorname{sh} k_n \eta + B_{2n} \Phi_{3n}(\eta) + \right. \\ & \left. + B_{3n} \Phi_{4n}(\eta)] \right\} \cos \lambda_n \xi; \end{aligned} \right\} \quad (195)$$

$$\begin{aligned}
Q_1 &= -K_1 \sum_{n=1}^{\infty} \left\{ [\varphi_0 + A_{1n} \operatorname{ch} k_n \eta + A_{2n} \Phi_{1n}(\eta) + \right. \\
&\quad \left. + A_{3n} \Phi_{2n}(\eta)] + \frac{\lambda_n}{a} \left[-\frac{\Omega_n}{\omega_n^{(3)}} + C_{1n} \operatorname{ch} k_n \eta + \right. \right. \\
&\quad \left. \left. + C_{2n} \Phi_{1n}(\eta) + C_{3n} \Phi_{2n}(\eta) \right] \cos \lambda_n \xi; \\
Q_2 &= -K_2 \sum_{n=1}^{\infty} \left\{ [B_{1n} \operatorname{sh} k_n \eta + B_{2n} \Phi_{3n}(\eta) + \right. \\
&\quad \left. + B_{3n} \Phi_{4n}(\eta)] + \frac{1}{b} [C_{1n} k_n \operatorname{sh} k_n \eta + C_{2n} \Phi'_{1n}(\eta) + \right. \\
&\quad \left. + C_{3n} \Phi'_{2n}(\eta)] \right\} \sin \lambda_n \xi.
\end{aligned}$$

For each number $n=1, 3, 5$, random constants C_{1n}, C_{2n} are determined /67 from the boundary conditions at the edges of the plate $\eta=\pm 1/2$.

In conformance with conditions (22), the boundary conditions for the edge $\beta=\text{const}$ have the form

unsupported edges

$$G_2 = H = Q_2 = 0;$$

rigidly fastened

$$\psi = \phi = w = 0; \quad (196)$$

loosely supported

$$w = G_2 = H = 0; \quad \psi = H = Q_2 = 0;$$

$$w = \phi = G_2 = 0; \quad \psi = w = H = 0;$$

$$\psi = \phi = Q_2 = 0.$$

Besides these conditions, there can be different fastening of both edges, i.e., any pair combination of the boundary conditions written above. However, in these cases, the deformed and stressed states of the plate will not be symmetrical about the x axis and, consequently, all six random constant remain necessary in solution of (187).

24. Case of Different Real Roots of Characteristic Equation

Bicubic Eq. (185), by the known substitution

$$k = k_0^2 - \frac{\omega_n^{(1)}}{3},$$

can be reduced to canonical form

$$k^3 + 3pk + 2q = 0, \quad (197)$$

where

$$q = \frac{\omega_n^{(1)}\omega_n^{(2)}}{6} - \frac{\omega_n^{(1)3}}{27} - \frac{\omega_n^{(3)}}{2};$$

$$p = \frac{3\omega_n^{(3)} - \omega_n^{(1)3}}{9}. \quad (198)$$

The number of real roots of Eq. (197) depends on the sign of discriminant $D=q^2+p^2$

$$D = \frac{\omega_n^{(2)3}}{27} + \frac{\omega_n^{(3)3}}{4} + \frac{\omega_n^{(3)}\omega_n^{(1)3}}{27} - \frac{\omega_n^{(1)3}\omega_n^{(2)3}}{108} - \frac{\omega_n^{(1)}\omega_n^{(2)}\omega_n^{(3)}}{6}. \quad (199)$$

Consequently, a case of three different real roots can be visualized. Let these roots be $\bar{k}_1, \bar{k}_2, \bar{k}_3$. The roots of bicubic Eq. (185) will then be in the form $\pm k_1, \pm k_2, \pm i k_3$. We will assume that $e^{\pm k_i \eta}$ ($i=1, 2, 3$) are not solutions of uniform Eq. (182), i.e.,

$$k_i^4 - k_i^2 e_n^{(1)} + e_n^{(2)} \neq 0$$

$$(i = 1, 2, 3).$$

The solution of system of differential Eq. (182), (183) can then be presented in the following form

$$\left. \begin{aligned} \varphi_n(\eta) &= \varphi_0 + A_1 \operatorname{ch} k_1 \eta + A_2 \operatorname{ch} k_2 \eta + \\ &+ A_3 \cos k_3 \eta + A_4 \operatorname{sh} k_1 \eta + A_5 \operatorname{sh} k_2 \eta + A_6 \sin k_3 \eta; \\ \psi_n(\eta) &= B_4 \operatorname{ch} k_1 \eta + B_5 \operatorname{ch} k_2 \eta + B_6 \cos k_3 \eta + \\ &+ B_1 \operatorname{sh} k_1 \eta + B_2 \operatorname{sh} k_2 \eta + B_3 \sin k_3 \eta; \\ w_n(\eta) &= -\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \operatorname{ch} k_1 \eta + C_2 \operatorname{ch} k_2 \eta + \\ &+ C_3 \cos k_3 \eta + C_4 \operatorname{sh} k_1 \eta + C_5 \operatorname{sh} k_2 \eta + C_6 \sin k_3 \eta. \end{aligned} \right\} \quad (200)$$

In this case, it is easy to obtain a solution for the general case of asymmetric boundary conditions.

By substituting Eq. (200) in system of Eq. (182), we obtain

$$\left. \begin{aligned} &A_1 k_1^4 \operatorname{ch} k_1 \eta + A_2 k_2^4 \operatorname{ch} k_2 \eta + A_3 k_3^4 \cos k_3 \eta + \\ &+ A_4 k_1^4 \operatorname{sh} k_1 \eta + A_5 k_2^4 \operatorname{sh} k_2 \eta + A_6 k_3^4 \sin k_3 \eta - \\ &- e_n^{(1)} [A_1 k_1^2 \operatorname{ch} k_1 \eta + A_2 k_2^2 \operatorname{ch} k_2 \eta - A_3 k_3^2 \cos k_3 \eta + \\ &+ A_4 k_1^2 \operatorname{sh} k_1 \eta + A_5 k_2^2 \operatorname{sh} k_2 \eta - A_6 k_3^2 \sin k_3 \eta] + \end{aligned} \right\} \quad (201)$$

$$\begin{aligned}
& + e_n^{(2)} [\varphi_0 + A_1 \operatorname{ch} k_1 \eta + A_2 \operatorname{ch} k_2 \eta + A_3 \cos k_3 \eta + \\
& \quad + A_4 \operatorname{sh} k_1 \eta + A_5 \operatorname{sh} k_2 \eta + A_6 \sin k_3 \eta] = \\
& = f_n^{(1)} [C_1 k_1^2 \operatorname{ch} k_1 \eta + C_2 k_2^2 \operatorname{ch} k_2 \eta - C_3 k_3^2 \cos k_3 \eta + \\
& \quad + C_4 k_1^2 \operatorname{sh} k_1 \eta + C_5 k_2^2 \operatorname{sh} k_2 \eta - C_6 k_3^2 \sin k_3 \eta] - \\
& - f_n^{(2)} \left[-\frac{\Omega_n}{\omega_n^{(3)}} + C_1 \operatorname{ch} k_1 \eta + C_2 \operatorname{ch} k_2 \eta + C_3 \cos k_3 \eta + \right. \\
& \quad \left. + C_4 \operatorname{sh} k_1 \eta + C_5 \operatorname{sh} k_2 \eta + C_6 \sin k_3 \eta \right];
\end{aligned}$$

$$\begin{aligned}
& B_1 k_1^4 \operatorname{sh} k_1 \eta + B_2 k_2^4 \operatorname{sh} k_2 \eta + B_3 k_3^4 \sin k_3 \eta + \\
& \quad + B_4 k_1^4 \operatorname{ch} k_1 \eta + B_5 k_2^4 \operatorname{ch} k_2 \eta + B_6 k_3^4 \cos k_3 \eta - \\
& - e_n^{(1)} [B_1 k_1^2 \operatorname{sh} k_1 \eta + B_2 k_2^2 \operatorname{sh} k_2 \eta - B_3 k_3^2 \sin k_3 \eta + \\
& \quad + B_4 k_1^2 \operatorname{ch} k_1 \eta + B_5 k_2^2 \operatorname{ch} k_2 \eta - B_6 k_3^2 \cos k_3 \eta] + \\
& \quad + e_n^{(2)} [B_1 \operatorname{sh} k_1 \eta + B_2 \operatorname{sh} k_2 \eta + B_3 \sin k_3 \eta + \\
& \quad + B_4 \operatorname{ch} k_1 \eta + B_5 \operatorname{ch} k_2 \eta + B_6 \cos k_3 \eta] = \\
& = f_n^{(3)} [C_1 k_1^3 \operatorname{sh} k_1 \eta + C_2 k_2^3 \operatorname{sh} k_2 \eta + C_3 k_3^3 \sin k_3 \eta + \\
& \quad + C_4 k_1^3 \operatorname{ch} k_1 \eta + C_5 k_2^3 \operatorname{ch} k_2 \eta - \\
& \quad - C_6 k_3^3 \cos k_3 \eta] - f_n^{(1)} [C_1 k_1 \operatorname{sh} k_1 \eta + \\
& \quad + C_2 k_2 \operatorname{sh} k_2 \eta - C_3 k_3 \sin k_3 \eta + C_4 k_1 \operatorname{ch} k_1 \eta + \\
& \quad + C_5 k_2 \operatorname{ch} k_2 \eta + C_6 k_3 \cos k_3 \eta]
\end{aligned}$$

(201)

for $n=1, 3, 5, \dots$

Since Eq. (201) should be satisfied with any values of η , from Eq. (201) we obtain

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$$\begin{aligned}
\varphi_{0n} &= \frac{f_n^{(2)} \Omega_n}{e_n^{(2)} \omega_n^{(3)}}; \\
A_j &= C_j \frac{f_n^{(1)} k_j^2 - f_n^{(2)}}{k_j^4 - e_n^{(1)} k_j^2 + e_n^{(2)}} \quad (j=1,2); \\
A_{j+3} &= C_{j+3} \frac{f_n^{(1)} k_j^2 - f_n^{(2)}}{k_j^4 - e_n^{(1)} k_j^2 + e_n^{(2)}} \quad (j=1,2); \\
A_{j+3} &= -C_{j+3} \frac{f_n^{(1)} k_3^2 + f_n^{(2)}}{k_3^4 + e_n^{(1)} k_3^2 + e_n^{(2)}} \quad (j=0,3); \\
B_j &= C_j \frac{k_j (f_n^{(3)} k_j^2 - f_n^{(4)})}{k_j^4 - e_n^{(1)} k_j^2 + e_n^{(2)}}; \\
B_{j+3} &= C_{j+3} \frac{k_j (f_n^{(3)} k_j^2 - f_n^{(4)})}{k_j^4 - e_n^{(1)} k_j^2 + e_n^{(2)}}; \\
B_3 &= C_3 \frac{k_3 (f_n^{(3)} k_3^2 + f_n^{(4)})}{k_3^4 + e_n^{(1)} k_3^2 + e_n^{(2)}}; \\
B_6 &= -C_6 \frac{k_3 (f_n^{(3)} k_3^2 + f_n^{(4)})}{k_3^4 + e_n^{(1)} k_3^2 + e_n^{(2)}}.
\end{aligned}$$

(202)

In this manner,

$$\left. \begin{aligned}
 \varphi &= \sum_{n=1, 3, 5}^{\infty} [\varphi_{0n} + A_{1n} \operatorname{ch} k_{1n} \eta + A_{2n} \operatorname{ch} k_{2n} \eta + \\
 &+ A_{3n} \cos k_{3n} \eta + A_{4n} \operatorname{sh} k_{1n} \eta + A_{5n} \operatorname{sh} k_{2n} \eta + \\
 &\quad + A_{6n} \sin k_{3n} \eta] \cos \lambda_n \xi; \\
 \psi &= \sum_{n=1}^{\infty} [B_{1n} \operatorname{sh} k_{1n} \eta + B_{2n} \operatorname{sh} k_{2n} \eta + \\
 &+ B_{3n} \sin k_{3n} \eta + B_{4n} \operatorname{ch} k_{1n} \eta + B_{5n} \operatorname{ch} k_{2n} \eta + \\
 &\quad + B_{6n} \cos k_{3n} \eta] \sin \lambda_n \xi; \\
 w &= \sum_{n=1}^{\infty} \left[-\frac{\Omega}{\omega_n^{(3)}} + C_{1n} \operatorname{ch} k_{1n} \eta + C_{2n} \operatorname{ch} k_{2n} \eta + \right. \\
 &\quad + C_{3n} \cos k_{3n} \eta + C_{4n} \operatorname{sh} k_{1n} \eta + \\
 &\quad \left. + C_{5n} \operatorname{sh} k_{2n} \eta + C_{6n} \sin k_{3n} \eta \right] \sin \lambda_n \xi.
 \end{aligned} \right\} \quad (203)$$

The following corresponding expressions can be obtained for the elastic forces and moments

$$\left. \begin{aligned}
 G_1 &= -D_1 \sum_{n=1}^{\infty} G_{1n}(\eta) \sin \lambda_n \xi; \\
 G_2 &= -D_2 \sum_{n=1}^{\infty} G_{2n}(\eta) \sin \lambda_n \xi; \\
 H &= -D_3 \sum_{n=1}^{\infty} H_n(\eta) \cos \lambda_n \xi; \\
 Q_1 &= -K_1 \sum_{n=1}^{\infty} Q_{1n}(\eta) \cos \lambda_n \xi; \\
 Q_2 &= -K_2 \sum_{n=1}^{\infty} Q_{2n}(\eta) \sin \lambda_n \xi;
 \end{aligned} \right\} \quad (204)$$

where

$$\left. \begin{aligned}
 G_{1n}(\eta) &= -\frac{\lambda_n}{a} \left[\varphi_{0n} + \sum_{j=1}^2 A_{jn} \operatorname{ch} k_{jn} \eta + \right. \\
 &+ \sum_{j=1}^2 A_{j+3, n} \operatorname{sh} k_{jn} \eta + A_{3n} \cos k_{3n} \eta + A_{6n} \sin k_{3n} \eta \left. \right] + \\
 &+ \frac{\nu_2}{b} \left[\sum_{j=1}^2 k_{jn} B_{jn} \operatorname{ch} k_{jn} \eta + \sum_{j=1}^2 k_{jn} B_{j+3, n} \operatorname{sh} k_{jn} \eta + \right. \\
 &\quad \left. + k_{3n} B_{3n} \cos k_{3n} \eta - k_{3n} B_{6n} \sin k_{3n} \eta \right]; \\
 G_{2n}(\eta) &= \frac{1}{b} \left[\sum_{j=1}^2 (k_{jn} B_{jn} \operatorname{ch} k_{jn} \eta + \right. \\
 &\quad + k_{j+3, n} B_{j+3, n} \operatorname{sh} k_{jn} \eta) + k_{3n} B_{3n} \cos k_{3n} \eta - \\
 &\quad \left. - k_{3n} B_{6n} \sin k_{3n} \eta \right] - \frac{\lambda_n \nu_1}{a} \left[\varphi_{0n} + \sum_{j=1}^2 (A_{jn} \operatorname{ch} k_{jn} \eta + \right.
 \end{aligned} \right\} \quad (205)$$

$$\begin{aligned}
& + A_{j+3, n} \operatorname{sh} k_{jn} \eta + A_{3n} \cos k_{3n} \eta + A_{6n} \sin k_{3n} \eta \Big]; \\
H_n(\eta) = & \frac{1}{b} \left[\sum_{j=1}^2 (A_{jn} k_{jn} \operatorname{sh} k_{jn} \eta + A_{j+3, n} k_{jn} \operatorname{ch} k_{jn} \eta) - \right. \\
& \left. - A_{3n} k_{3n} \sin k_{3n} \eta + A_{6n} k_{3n} \cos k_{3n} \eta \right] + \\
& + \frac{\lambda_n}{a} \left[\sum_{j=1}^2 (B_{jn} \operatorname{sh} k_{jn} \eta + B_{j+3, n} \operatorname{ch} k_{jn} \eta) - \right. \\
& \left. - B_{3n} \sin k_{3n} \eta + B_{6n} \cos k_{3n} \eta \right]; \\
Q_{1n}(\eta) = & \left[\varphi_{0n} + \sum_{j=1}^2 (A_{jn} \operatorname{ch} k_{jn} \eta + A_{j+3, n} \operatorname{sh} k_{jn} \eta + \right. \\
& \left. + A_{3n} \cos k_{3n} \eta + A_{6n} \sin k_{3n} \eta \right] + \frac{\lambda_n}{a} \left[-\frac{\Omega_n}{\omega_n^{(3)}} + \right. \\
& \left. + \sum_{j=1}^2 (C_{jn} \operatorname{ch} k_{jn} \eta + C_{j+3, n} \operatorname{sh} k_{jn} \eta) + C_{3n} \cos k_{3n} \eta + \right. \\
& \left. + C_{6n} \sin k_{3n} \eta \right]; \\
Q_{2n}(\eta) = & \sum_{j=1}^2 (B_{jn} \operatorname{sh} k_{jn} \eta + B_{j+3, n} \operatorname{ch} k_{jn} \eta) + \\
& + B_{3n} \sin k_{3n} \eta + B_{6n} \cos k_{3n} \eta + \\
& + \frac{1}{b} \left[\sum_{j=1}^2 (C_{jn} k_{jn} \operatorname{sh} k_{jn} \eta + C_{j+3, n} k_{jn} \operatorname{ch} k_{jn} \eta) - \right. \\
& \left. - C_{3n} k_{3n} \sin k_{3n} \eta + C_{6n} k_{3n} \cos k_{3n} \eta \right]
\end{aligned} \tag{205}$$

for $n=1, 3, 5, \dots$

Coefficients A_{jn} , B_{jn} , ($j=1, 2, 3, 4, 5, 6$) are determined through 172 random constants C_{jn} by expressions (202).

For determination of constants C_{jn} ($j=1, 2, 3, 4, 5, 6$), boundary conditions (196) are used. In this manner, there can be 36 different combinations of possible support fastenings of the edges of the plate $n=\pm 1/2$.

If it turns out that any of functions $e^{\pm k_{jn}\eta}$ is a solution of uniform Eq. (182), i.e., $k_{jn}^4 - e_n^{(1)} k_{jn}^2 + e_n^{(2)} = 0$, the solution corresponding to the solution $w_n = C_1 \operatorname{ch} k_{jn} \eta + C_2 \operatorname{sh} k_{jn} \eta$ should be sought in the form

$$\left. \begin{aligned} \varphi_n(\eta) &= A_1 k_{jn} \eta \operatorname{sh} k_{jn} \eta + A_2 k_{jn} \operatorname{ch} k_{jn} \eta; \\ \psi_n(\eta) &= B_1 k_{jn} \eta \operatorname{ch} k_{jn} \eta + B_2 k_{jn} \eta \operatorname{sh} k_{jn} \eta. \end{aligned} \right\} \quad (206)$$

where

$$\left. \begin{aligned} A_1 &= C_1 \frac{f_n^{(1)} k_{jn}^2 - f_n^{(2)}}{2k_{jn}^2 (2k_{jn}^2 - \epsilon_n^{(1)})}; \\ A_2 &= C_1 \frac{f_n^{(1)} k_{jn}^2 - f_n^{(2)}}{2k_{jn}^2 (2k_{jn}^2 - \epsilon_n^{(1)})}; \\ B_1 &= C_1 \frac{k_{jn} (f_n^{(3)} k_{jn}^2 - f_n^{(4)})}{2k_{jn}^2 (2k_{jn}^2 - \epsilon_n^{(1)})}; \\ B_2 &= C_1 \frac{k_{jn} (f_n^{(3)} k_{jn}^2 - f_n^{(4)})}{2k_{jn}^2 (2k_{jn}^2 - \epsilon_n^{(1)})}. \end{aligned} \right\} \quad (207)$$

CHAPTER 5. STABILITY OF ANISOTROPIC RECTANGULAR PLATES

25. Formulation of Problem of Stability of Plates and General Methods of Determination of Critical Loads

The advent of high strength materials led to the extensive use of thin walled structures containing thin plates and shells as the basic elements in industry. Experience in the use of such structures shows that they, as a rule, turn out to be unsuccessful, not because the stresses which develop in them exceed permissible limits, but because of disturbance of the equilibrium of individual thin walled components. /73

Questions of the instability of equilibrium arise everywhere where there are thin walled structures. For thin walled structures made of laminated plastics, because of the low rigidity of the latter, assurance of stability is a particularly important problem in designing them.

If a parameter which characterizes the thin walled nature of a structural element, for example, the ratio of wall thickness to the characteristic plan dimension, is designated ϵ , questions of stability assurance will be significant in the event the critical load is determined by the relationship $p = A\epsilon^q$, where $q > 1$ since, in this case, a reduction in wall thickness will significantly decrease the critical load, while stress will increase only in proportion to the decrease in thickness. The critical load in such thin walled structures proves to be one or two orders of magnitude less than the load at which failure of the material occurs. For plates, $q=2$ and, consequently, if the bulging of units which consist of rectangular plates is an undesirable structure according to the operating conditions, proper selection of dimensions which ensure structural stability is an extremely important problem. /74
Dimensions can be selected with the availability of calculation formulas or nomograms which define the critical load as a function of geometric dimensions and elastic constants.

The problem of stability of a flat plate subjected to forces applied in the plane of the plate can be formulated in the following manner. It is assumed that the magnitude and principle of distribution of extreme forces remain constant and that parameter γ characterizes the external load. The critical value of parameter γ is determined at the time of appearance of other forms of plate equilibrium accompanied by distortion of its mean plane.

The theory of elastic stability has been worked out extremely thoroughly, and a number of effective methods are available. One method of determination of the critical load is as follows. On the assumption that, at some value of load parameter γ , the development of a distorted form of plate equilibrium is possible, differential bending equations are compiled with external forces $T_1 = \gamma T_1^0$, $\gamma T_2 = T_2^0$, $S = \gamma S^0$, which are applied in the mean plane of the plate and give bending component p normal to the mean plane of the plate, taken into account. The solution of such an equation, which contains γ as a parameter and which satisfies all boundary conditions, exists only with certain specific values of

parameter γ , which are called the eigenvalues of the problem.

Each eigenvalue γ_k ($k= 1, 2, 3, \dots$) defines a critical load which corresponds to a specific form of loss of stability. It is evident that only a load determined by the smallest eigenvalue of parameter γ is of practical importance.

Engineering practice usually is limited to obtaining approximate values, for which variation methods, based on general theorems of the equilibrium of mechanical systems and according to which the potential energy of the system has a minimum value in the equilibrium position, are extensively used

If U_0 is the potential energy of a plate in planar equilibrium and U is the potential energy of the plate in the distorted state of equilibrium, the critical load is determined from the equation

$$(208)$$

i.e., for determination of the critical value of load parameter γ , the work performed by external forces T_1^0, T_2, S in minor bending of the plate must be made equal to the potential energy of bending of the plate.

The solution of specific engineering problems by energy methods looks approximately as follows. Expressions are assigned for functions ϕ, ψ and plate deflection w , which satisfy the boundary conditions of the problem

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$$\left. \begin{aligned} \phi &= \sum_m \sum_n A_{mn} \phi_{mn}(x, y); \\ \psi &= \sum_m \sum_n B_{mn} \psi_{mn}(x, y); \\ w &= \sum_m \sum_n C_{mn} w_{mn}(x, y). \end{aligned} \right\} (209)$$

By substituting these expressions in variation Eq. (208), we obtain an equation of the type

$$F(A, B, C, \gamma) = \sum_m \sum_n [U_m(A, B, C) - \gamma V_{mn}(C)] = 0. \quad (210)$$

If a finite number of terms is taken in Eq. (209), Eq. (210) is not exactly satisfied. In this case, it is evident that the best approximation is obtained upon satisfaction of the conditions

$$\frac{\partial F}{\partial A_i} = \frac{\partial F}{\partial B_i} = \frac{\partial F}{\partial C_i} = 0, \quad (211)$$

$i=1, 2, 3, \dots, n.$

By setting the determinant of uniform linear system (211) equal to zero, a characteristic equation can be obtained for determination of the critical value of load parameter γ .

Investigation of stability can be approached from more general standpoints of the stability of motion. Here, instability or stability of the planar shape of a plate exposed to forces applied in the mean plane of the plate should be indicated. Together with this unperturbed form of equilibrium of the plate, perturbations of the form of motion similar to it are considered. If the smallest perturbations desired cause finite deviations from unperturbed equilibrium over time, the latter are called unstable.

As applied to plates, this method is reduced to the following. A differential equation of transverse vibrations are compiled, with the longitudinal forces taken into account. Further, natural oscillation frequency ω_{mn} is determined. It depends on the plate dimensions, elastic constants of the material c_{ij} and load parameter γ . At some values 76 of parameter γ , the frequencies may turn out to be zero or imaginary, and their corresponding deflections will increase indefinitely. Such values of parameter γ determine the critical load.

26. Differential and Variation Equations of Stability of Rectangular Plates

We consider a rectangular anisotropic plate with sides a, b . We select a coordinate system such that the x, y axes are along the sides of the plate. Let the plate be loaded along the edges with forces T_1^0, T_2^0, S^0 in the mean plane of the plate (Fig. 28).

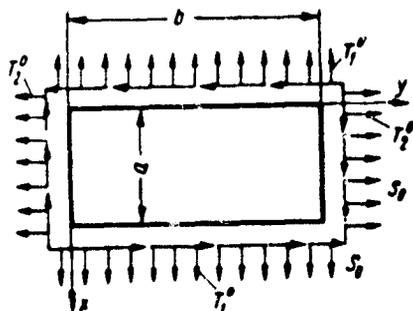


Fig. 28. Loading diagram and conventional symbols.

Let bulging of the plate occur at some combination of forces T_1^0, T_2^0, S^0 . It is evident that, with as small a distortion of the mean plane of the plate as desired, the equilibrium equations in addition to the internal forces in the plane of the plate which arise in bending do not depend on initial forces T_1^0, T_2^0, S^0 . More than that, these forces generally can be disregarded. The equations of equilibrium of the forces normal to the mean plane of the plate depend essentially on the initial forces, since the projections of these forces on the normal to the deformed mean plane are on the same order of smallness as the

cutting forces which arise upon bulging.

By projecting forces T_1^0 , T_2^0 , S^0 on the normal of the mean plane of the plate after bulging, we obtain

$$-p = T_1^0 \frac{\partial^2 w}{\partial x^2} + 2S^0 \frac{\partial^2 w}{\partial x \partial y} + T_2^0 \frac{\partial^2 w}{\partial y^2}, \quad (212)$$

i.e., the normal component of initial forces T_1^0 , T_2^0 , S^0 is equivalent to the distributed transverse load determined from Eq. (212). Since, because of the smallness of the bulge, all relationships obtained in study of the bending of a plate remain unchanged, the following system of differential equations can be obtained which describe the bulging of rectangular plates.

1. Differential equations of stability of anisotropic rectangular plate. For rectangular plates of anisotropic structure, the system of differential equations of stability has the following form

$$L_6(w) + \left(\frac{T_1^0}{a^2} \frac{\partial^2}{\partial \xi^2} + \frac{T_2^0}{b^2} \frac{\partial^2}{\partial \eta^2} + \frac{2S^0}{ab} \frac{\partial^2}{\partial \xi \partial \eta} \right) L_4(w) = 0; \quad (213)$$

$$\left. \begin{aligned} L_4(\varphi) &= \alpha_1 \frac{\partial^3 w}{\partial \xi^3} + \alpha_2 \frac{\partial^3 w}{\partial \xi^2 \partial \eta} + \alpha_3 \frac{\partial^3 w}{\partial \xi \partial \eta^2} - \alpha_4 \frac{\partial^3 w}{\partial \eta^3} - \alpha_5 \frac{\partial w}{\partial \xi}; \\ L_4(\psi) &= \beta_1 \frac{\partial^3 w}{\partial \eta^3} + \beta_2 \frac{\partial^3 w}{\partial \xi \partial \eta^2} + \beta_3 \frac{\partial^3 w}{\partial \xi^2 \partial \eta} - \\ &\quad - \beta_4 \frac{\partial^3 w}{\partial \xi^3} - \beta_5 \frac{\partial w}{\partial \eta}. \end{aligned} \right\} \quad (214)$$

where differential operators in partial derivatives $L_6()$, $L_4()$ are determined from Eq. (137), (138).

2. Differential equations of stability of orthotropic rectangular plates. In the case of orthotropic plates, the principal axes of anisotropy of which are parallel to the sides of the plate, differential equations of stability (213), (214) are simplified somewhat and they take the form

$$\left. \begin{aligned} F_6(w) + \left(\frac{T_1^0}{a^2} \frac{\partial^2}{\partial \xi^2} + \frac{T_2^0}{b^2} \frac{\partial^2}{\partial \eta^2} + \frac{2S^0}{ab} \frac{\partial^2}{\partial \xi \partial \eta} \right) F_4(w) &= 0; \\ F_4(\varphi) &= \alpha_1 \frac{\partial^3 w}{\partial \xi^3} + \alpha_3 \frac{\partial^3 w}{\partial \xi \partial \eta^2} - \alpha_5 \frac{\partial w}{\partial \xi}; \end{aligned} \right\} \quad (215)$$

$$F_4(\psi) = \beta_1 \frac{\partial^3 w}{\partial \eta^3} + \beta_3 \frac{\partial^3 w}{\partial \xi^2 \partial \eta} - \beta_5 \frac{\partial w}{\partial \eta}. \quad (216)$$

Differential operators in partial derivatives $F_6()$, $F_4()$ are determined by expressions (148), (149).

Eq. (213)-(216), together with boundary conditions (22), permit

determination of the critical values of forces T_1^0 , T_2^0 , S^0 , which are applied to the edges of the plate in the mean plane. It is particularly easy to obtain results in simple loading, i.e., when the external forces change in proportion to parameter γ alone.

It should be noted that, in the practical use of the method of direct integration of differential equations reported above, great difficulties arise in a number of cases, which are connected with satisfaction of the boundary conditions. Moreover, as a rule, the characteristic equations which can be obtained in determination of the critical load are transcendental, and they do not permit expression of the dependence of the critical load on the geometric dimensions of the plate in explicit form. Thus, it is highly advisable to have an approximate method for determination of the critical load. Approximate methods are based on consideration of the potential energy of bending of a plate upon bulging. /78

In derivation of the variation equation of stability, an expression must be obtained for the work of the external forces which is accomplished in bulging of the mean plane of the plate.

This work is determined by the expression [17]

$$\delta A = -\frac{1}{2} \int_0^a \int_0^b \left[T_1^0 \left(\frac{\partial w}{\partial x} \right)^2 + T_2^0 \left(\frac{\partial w}{\partial y} \right)^2 + 2S^0 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \quad (217)$$

Based on the general theorems of mechanics, the equilibrium is stable if the potential energy of the system is at a minimum. Consequently, the magnitude of the critical load is determined from the condition that the increment of potential energy of bending of the plate upon bulging equals the work of the external forces.

Since the potential energy of bending of the plate is determined by Eq. (155), (156), we have the following variation equations of stability of anisotropic rectangular plates.

1. Variation equation of stability of anisotropic plate. For a rectangular plate made of anisotropic laminated plastic, the variation equation of stability has the following form

$$\begin{aligned} & \int_0^a \int_0^b \left\{ D_{11} \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_{22} \left(\frac{\partial \psi}{\partial y} \right)^2 + D_{33} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + \right. \\ & + \frac{3}{2} D_{13} \frac{\partial \varphi}{\partial x} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + \frac{3}{2} D_{23} \frac{\partial \psi}{\partial y} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + \\ & \left. + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 \right\} dx dy = \int_0^a \int_0^b \left[T_1^0 \left(\frac{\partial w}{\partial x} \right)^2 + T_2^0 \left(\frac{\partial w}{\partial y} \right)^2 + \right. \\ & \left. + 2S^0 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \quad (218) \end{aligned}$$

2. Variation equation of stability of orthotropic rectangular

plate. For an orthotropic plate, when the principal axes of anisotropy are parallel to the sides of the plate, Eq. (218) is somewhat simplified, and it takes the form

$$\int_0^a \int_0^b \left[D_1 \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_2 \left(\frac{\partial \psi}{\partial y} \right)^2 + D_3 \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 \right] dx dy = - \int_0^a \int_0^b \left[T_1^0 \left(\frac{\partial w}{\partial x} \right)^2 + T_2^0 \left(\frac{\partial w}{\partial y} \right)^2 + 2S^0 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \quad (219)$$

27. Stability of Orthotropic Rectangular Plate Compressed in One Principal Direction of Anisotropy

Let a rectangular orthotropic plate be compressed in one principal /79 direction of anisotropy by forces T_1^0 uniformly distributed along the sides $x=0$, $x=a$ (Fig. 29).

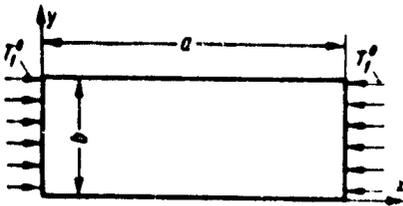


Fig. 29. Diagram of plate compressed in one direction.

We obtain a general expression for determination of the critical load by means of integration of the differential equation of stability. By increasing the intensity of load T_1^0 , such a state can be reached in which the planar form of equilibrium of the plate becomes unstable and bulging of the plate occurs.

The system of differential equations of stability has the form

$$F_6(w) = \frac{T_1^0}{a^2} \frac{\partial^2}{\partial \xi^2} F_4(w); \quad (220)$$

$$\left. \begin{aligned} F_4(\varphi) &= \alpha_1 \frac{\partial^2 w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 w}{\partial \xi^2 \partial \eta^2} - \alpha_3 \frac{\partial w}{\partial \xi}; \\ F_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_2 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} - \beta_3 \frac{\partial w}{\partial \eta}, \end{aligned} \right\} \quad (221)$$

where functionals $F_6()$ and $F_4()$ are determined by Eq. (148) and (149).

a. Stability of plate hinge supported on profile. The solution of Eq. (220) which satisfies the boundary conditions of hinge support along the contour

$$\text{at } x=0, x=a \quad w=G_1=\psi=0;$$

$$\text{at } y=0, y=b \quad w=G_2=\phi=0,$$

can be sought in the form

$$\left. \begin{aligned} \varphi &= A \cos m \pi \xi \sin n \pi \eta; \\ \psi &= B \sin m \pi \xi \cos n \pi \eta; \\ \omega &= C \sin m \pi \xi \sin n \pi \eta. \end{aligned} \right\} \quad (222)$$

By substituting Eq. (222) in system of Eq. (220), (221), we obtain

$$\begin{aligned} & \pi^4 (a_{40} m^4 + a_{42} m^2 n^2 + a_{24} m^2 n^4 + a_{04} n^4) + \\ & + a_{00} \pi^4 (\gamma_1 m^4 + \gamma_3 m^2 n^2 + \gamma_5 n^4) = \\ = & \frac{T_1^0}{a^4} m^2 \pi^2 \{ \pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{20} m^2 + a_{02} n^2) + a_{00} \}; \end{aligned} \quad (223)$$

$$\left. \begin{aligned} A \{ \pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{20} m^2 + a_{02} n^2) + \\ + a_{00} \} = -C \{ \alpha_1 m^3 \pi^3 + \alpha_3 m n^3 \pi^3 + \alpha_5 m \pi \}; \\ B \{ \pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{20} m^2 + a_{02} n^2) + \\ + a_{00} \} = -C \{ \beta_1 n^3 \pi^3 + \beta_3 m^2 n \pi^3 + \beta_5 n \pi \}. \end{aligned} \right\} \quad (224)$$

In conformance with Eq. (223), the critical load is determined by /80
the following expression as a function of two integer parameters m, n ,
which determine the mode of wave formation

$$\begin{aligned} T_1^0 &= \frac{\pi^2 a^2}{m^2} \times \\ & \times \frac{\pi^2 (a_{40} m^4 + a_{42} m^2 n^2 + a_{24} m^2 n^4 + a_{04} n^4) + a_{00} (\gamma_1 m^4 + \gamma_3 m^2 n^2 + \gamma_5 n^4)}{\pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{20} m^2 + a_{02} n^2) + a_{00}}. \end{aligned} \quad (225)$$

We present Eq. (225) in a more convenient form for practical use.
We introduce the following designations

$$\left. \begin{aligned} \zeta_1(q) &= \pi^2 (a_{40} q^4 + a_{42} q^2 + a_{24} q + a_{04}); \\ \zeta_2(q) &= a_{00} (\gamma_1 q^2 + \gamma_3 q + \gamma_5); \\ \zeta_3(q) &= \pi^4 (a_{40} q^4 + a_{22} q^2 + a_{04}); \\ \zeta_4(q) &= \pi^2 (a_{20} q + a_{02}), \end{aligned} \right\} \quad (226)$$

where

$$q = \frac{m^2}{n^2}; \quad n^2 = u. \quad (227)$$

The critical load is then determined by the simple expression

$$T_1^0 = \frac{\pi^2 a^2}{q} \frac{u^2 \zeta_1(q) + u \zeta_2(q)}{u^2 \zeta_3(q) + u \zeta_4(q) + a_{00}}. \quad (228)$$

From expression (228), we obtain

$$\frac{\partial T_1^0}{\partial u} = \frac{\pi^2 a^2}{q} \frac{(\zeta_1 \zeta_4 - \zeta_2 \zeta_3) u^2 + 2a_{00} \zeta_1 u + a_{00} \zeta_2}{(\zeta_3 u^2 + \zeta_4 u + a_{00})^2}. \quad (229)$$

It is easy to note that the sign of the right side of Eq. (229) is determined by the coefficient

$$\begin{aligned} \zeta_1 \zeta_4 - \zeta_2 \zeta_3 = \pi^4 & \{ (a_{00} a_{20} - a_{40} a_{00} \gamma_1) q^4 + a_{04} a_{02} - a_{04} a_{00} \gamma_2 + \\ & + (a_{42} a_{20} + a_{00} a_{02} - a_{40} a_{00} \gamma_2 - a_{22} a_{00} \gamma_1) q^2 + \\ & + (a_{24} a_{20} + a_{42} a_{02} - a_{40} a_{00} \gamma_2 - a_{22} a_{00} \gamma_2 - a_{04} a_{00} \gamma_1) q^2 + \\ & + (a_{04} a_{20} + a_{24} a_{02} - a_{22} a_{00} \gamma_2 - a_{04} a_{00} \gamma_2) q \}, \end{aligned} \quad (230)$$

or, according to Eq. (150)

$$\begin{aligned} \zeta_1 \zeta_4 - \zeta_2 \zeta_3 = \pi^4 & \left\{ \frac{D_1 D_2^2 K_1^2}{a^4} q^4 + \frac{D_2}{a^2 b^2} [D_1^2 K_2^2 + (2D_1 D_2 + D_2^2 - C^2) K_1^2 - \right. \\ & - 2CD_1 K_1 K_2] q^2 + \frac{1}{a^2 b^2} [(D_1 D_2 + 2D_2^2 - C^2) (D_1 K_2^2 + D_2 K_1^2) - \\ & - 2(D_1 D_2 + D_2^2 - C^2) CK_1 K_2] q^2 + \frac{D_2}{a^2 b^2} [D_1^2 K_1^2 + \\ & \left. + (2D_1 D_2 + D_2^2 - C^2) K_2^2 - 2CD_2 K_1 K_2] q + \frac{D_1 D_2^2 K_1^2}{b^4} \right\}. \end{aligned} \quad (231)$$

In this manner, over a wide range of change of plate rigidity parameters $\frac{\partial T^0}{\partial u} \gg 0$ and, consequently, in bulging of a hinge supported plate compressed in one principal direction of anisotropy, one half wave forms transverse to the compression, i.e., $n=1$.

The critical load is determined by the smallest value of the expression

$$(T^0)_{cr} = \frac{\pi^2 a^4}{q} \frac{\zeta_1(q) + \zeta_2(q)}{\zeta_3(q) + \zeta_4(q) + a_{00}}, \quad (232)$$

where $q=1^2, 2^2, \dots, m^2$.

It now is advisable to consider the case frequently encountered in practice of the cylindrical shape of loss of stability of a rectangular plate upon compression in the direction of the unsupported edges.

b. Stability of plate with two supported and two unsupported edges. In this case, the approximate solution of system of Eq. (220), (221) can be sought in the form

$$\varphi = A \cos m \pi \xi; \quad \psi = 0; \quad w = C \sin m \pi \xi. \quad (233)$$

By substituting Eq. (233) in the system of differential equations of stability, we obtain

$$\pi^4 m^4 a_{00} + \pi^4 m^4 a_{00} \gamma_1 = \frac{T^0}{a^2} m^2 \pi^2 (\pi^4 m^4 a_{40} + \pi^2 m^2 a_{20} + a_{00}); \quad (234)$$

$$A (m^4 \pi^4 a_{40} + m^2 n^2 a_{20} + a_{00}) = -B (m^3 \pi^3 a_1 + m \pi a_2), \quad (235)$$

whence, for determination of the critical load, the following expression can be obtained

$$T_1 = D_1 K_1 \left(\frac{m\pi}{a} \right)^2 \cdot \frac{D_2 \left(\frac{m\pi}{a} \right)^2 + K_2}{D_1 D_2 \left(\frac{m\pi}{a} \right)^2 + (D_1 K_2 + D_2 K_1) \left(\frac{m\pi}{a} \right)^2 + K_1 K_2}, \quad (236)$$

or

$$T_1^0 = \frac{D_1 K_1 \left(\frac{m\pi}{a} \right)^2}{D_1 \left(\frac{m\pi}{a} \right)^2 + K_1}. \quad (237)$$

In accordance with Eq. (237), in cylindrical bulging of a laminated strip in the direction of compression, one half wave forms ($m=1$) and, consequently

$$T_{cr} = \frac{\pi^2 D_1 K_1}{\pi^2 D_1 + K_1 a^2}. \quad (238)$$

The critical load for higher forms of loss of stability ($m>1$) does not tend toward infinity, as occurs in the case of uniform shells, but toward finite limit K_1 . The equivalent of this is that, with decrease in length of a rod, the critical load, upon increasing, asymptotically tends towards finite limit K_1 . This phenomenon should be taken into account in the use of reinforcing ribs made of laminated plastics to strengthen a cylindrical shell. The carrying capacity of such stiffening ribs can be limited by their rigidity in transverse shear.

28. Stability of Hinge Supported Rectangular Orthotropic Plate in Compression in Two Principal Directions of Anisotropy

We now consider the problem of the stability of a rectangular hinge supported plate with sides a , b , in which the principal axes of anisotropy are parallel to the sides and which is compressed by uniformly distributed forces T_1^0 , T_2^0 (Fig. 30).

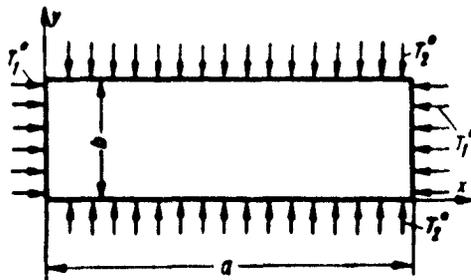


Fig. 30. Diagram of plate compressed in principal directions of anisotropy.

We obtain a solution by using variation equation of stability (219). We assign the deformed state of the plate after bulging in the form

$$\left. \begin{aligned} \varphi &= A \cos \lambda x \sin \eta y; \\ \psi &= B \sin \lambda x \cos \eta y; \\ w &= E \sin \lambda x \sin \eta y, \end{aligned} \right\} \quad (239)$$

where

$$\lambda = \frac{m\pi}{a}; \quad \eta = \frac{n\pi}{b}; \quad (240)$$

the following can be obtained

$$\left. \begin{aligned} \frac{\partial \varphi}{\partial x} &= -A \lambda \sin \lambda x \sin \eta y; & \frac{\partial \varphi}{\partial y} &= -B \eta \sin \lambda x \sin \eta y; \\ \frac{\partial \varphi}{\partial y} &= A \eta \cos \lambda x \cos \eta y; & \frac{\partial \psi}{\partial x} &= E \lambda \cos \lambda x \sin \eta y; \\ \frac{\partial \psi}{\partial x} &= B \lambda \cos \lambda x \cos \eta y; & \frac{\partial \psi}{\partial y} &= E \lambda \sin \lambda x \cos \eta y. \end{aligned} \right\} \quad (241)$$

By substituting Eq. (239), (241) in variation equation of stability (219), we obtain /83

$$\begin{aligned} U &= \int_0^a \int_0^b \{ D_1 \lambda^2 \sin^2 \lambda x \sin^2 \eta y + 2C \lambda \eta AB \sin^2 \lambda x \sin^2 \eta y + \\ &+ D_2 B^2 \eta^2 \sin^2 \lambda x \sin^2 \eta y + D_3 (\eta A + \lambda B)^2 \cos^2 \lambda x \cos^2 \eta y + \\ &+ K_1 (A + \lambda E)^2 \cos^2 \lambda x \sin^2 \eta y + K_2 (B + \eta E)^2 \sin^2 \lambda x \cos^2 \eta y - \\ &- T_1^0 \lambda^2 E^2 \cos^2 \lambda x \sin^2 \eta y - T_2^0 \eta^2 E^2 \sin^2 \lambda x \cos^2 \eta y \} dx dy = \\ &= \frac{ab}{4} \{ D_1 \lambda^2 A^2 + 2C \lambda \eta AB + D_2 \eta^2 B^2 + D_3 (\eta A + \lambda B)^2 + \\ &+ K_1 (A + \lambda E)^2 + K_2 (B + \eta E)^2 - (T_1^0 \lambda^2 + T_2^0 \eta^2) E^2 \}, \end{aligned} \quad (242)$$

where, as before, $C = D_{12} + D_3$.

The minimum potential energy condition has the form

$$\frac{\partial U}{\partial A} = \frac{\partial U}{\partial B} = \frac{\partial U}{\partial E} = 0, \quad (243)$$

or, after reduction,

$$\left. \begin{aligned} A (D_1 \lambda^2 + D_3 \eta^2 + K_1) + B (C + D_3) \lambda \eta + E \lambda K_1 &= 0; \\ A (C + D_3) \lambda \eta + B (D_2 \eta^2 + D_3 \lambda^2 + K_2) + E \eta K_2 &= 0; \\ A \lambda K_1 + B \eta K_2 + E (\lambda^2 K_1 + \eta^2 K_2 - T_1^0 \lambda^2 - T_2^0 \eta^2) &= 0. \end{aligned} \right\} \quad (244)$$

The condition of nontriviality of the solution of this system gives the necessary characteristic equation for determination of the critical load

$$\begin{vmatrix} D_1 \lambda^2 + D_2 \eta^2 + K_1 & C \lambda \eta & K_1 \lambda \\ C \lambda \eta & D_1 \eta^2 + D_2 \lambda^2 + K_2 & K_2 \eta \\ K_1 \lambda & K_2 \eta & K_1 \lambda^2 + K_2 \eta^2 - (T_1^0 \lambda^2 + T_2^0 \eta^2) \end{vmatrix} = 0 \quad (245)$$

or

$$\begin{aligned} & (T_1^0 \lambda^2 + T_2^0 \eta^2) [(D_1 \lambda^2 + D_2 \eta^2 + K_1)(D_2 \eta^2 + D_3 \lambda^2 + K_2 - C^2 \lambda^2 \eta^2)] = \\ & = (D_1 \lambda^2 + D_2 \eta^2 + K_1)(D_2 \eta^2 + D_3 \lambda^2 + K_2)(K_1 \lambda^2 + K_2 \eta^2) + \\ & \quad + 2CK_1 K_2 \lambda^2 \eta^2 - (D_2 \eta^2 + D_3 \lambda^2 + K_2) K_1^2 \lambda^2 - \\ & \quad - (D_1 \lambda^2 + D_2 \eta^2 + K_1) K_2^2 \eta^2 - C^2 (K_1 \lambda^2 + K_2 \eta^2) \lambda^2 \eta^2. \end{aligned} \quad (246)$$

As a result, for determination of the critical load of a hinge supported orthotropic plate compressed in the principal directions of anisotropy, the following expression can be obtained /84

$$T_1^0 \left(\frac{m}{a}\right)^2 + T_2^0 \left(\frac{n}{b}\right)^2 = \pi^2 \frac{\zeta_1(m, n) + \zeta_2(m, n)}{\zeta_3(m, n) + \zeta_4(m, n) + a_{00}}, \quad (247)$$

where, in accordance with Eq. (150), the following designations are introduced

$$\left. \begin{aligned} \zeta_1(m, n) &= \pi^2 (a_{40} m^6 + a_{42} m^4 n^2 + a_{24} m^2 n^4 + a_{04} n^6); \\ \zeta_2(m, n) &= a_{00} (\gamma_1 m^4 + \gamma_3 m^2 n^2 + \gamma_5 n^4); \\ \zeta_3(m, n) &= \pi^4 (a_{40} m^4 + a_{24} m^2 n^2 + a_{04} n^4); \\ \zeta_4(m, n) &= \pi^2 (a_{20} m^2 + a_{02} n^2). \end{aligned} \right\} \quad (248)$$

In simple loading, when the load along all edges increases in proportion to a single parameter, i.e., when $T_1^0 = T$, $T_2^0 = \omega T$, the critical value of parameter T is determined from the expression

$$T = \frac{\pi^2}{\left(\frac{m}{a}\right)^2 + \omega \left(\frac{n}{b}\right)^2} \frac{\zeta_1(m, n) + \zeta_2(m, n)}{\zeta_3(m, n) + \zeta_4(m, n) + a_{00}}, \quad (249)$$

i.e., the problem is reduced to finding the smallest value of the right side of Eq. (249) as a function of integer parameters m, n.

29. Stability of Infinitely Wide Orthotropic Plate in Compression Along Short Sides

Let an extremely wide orthotropic plate of length a be compressed by forces T_1^0 , which are uniformly distributed along the wide edges. We will assume that the principal directions of anisotropy coincide with the sides (Fig. 31). In this case, a cylindrical form of loss of stability can be considered, i.e., it can be assumed that all components

of displacements and forces do not depend on coordinate y . The system of differential equations of stability then takes the form

$$\left. \begin{aligned} a_{00}w^{VI} - a_{00}\gamma_1w^{IV} &= \frac{T^0}{a^2} (a_{10}w^{VI} - a_{20}w^{IV} + a_{00}w''); \\ a_{10}\varphi^{IV} - a_{20}\varphi'' + a_{00}\varphi &= a_1w''' - a_1w'. \end{aligned} \right\} \quad (250)$$

In accordance with Eq. (150), system of Eq. (250) can be presented in the following form /85

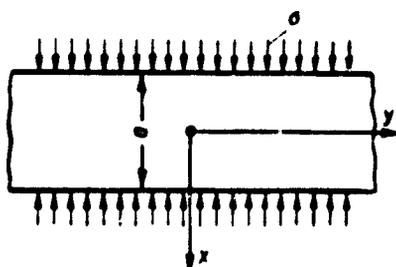


Fig. 31. Loading diagram of infinitely wide strip.

$$\left. \begin{aligned} (D_3 \frac{d^2}{dx^2} - K_2) D_1 K_1 \frac{d^4 w}{dx^4} &= \\ = T (D_3 \frac{d^2}{dx^2} - K_2) (D_1 \frac{d^2}{dx^2} - K_1) \frac{d^4 w}{dx^4}; \\ (D_3 \frac{d^2}{dx^2} - K_2) (D_1 \frac{d^2 \varphi}{dx^2} - K_1 \varphi) &= \\ = (D_3 \frac{d^2}{dx^2} - K_2) K_1 w'. \end{aligned} \right\} \quad (251)$$

System of Eq. (251) satisfies the solution of the system

$$\left. \begin{aligned} w^{IV} + k^2 w'' &= 0; \\ \varphi'' - p^2 \varphi &= p^2 w, \end{aligned} \right\} \quad (252)$$

where

$$k^2 = \frac{TK_1}{D_1(K_1 - T)}; \quad p^2 = \frac{K_1}{D_1}. \quad (253)$$

The general solution of system of differential Eq. (252) has the form

$$\left. \begin{aligned} w &= C_1 \cos kx + C_2 \sin kx + C_3 x + C_4; \\ \varphi &= \frac{p^2 k}{p^2 + k^2} (C_1 \sin kx - C_2 \cos kx) - C_3. \end{aligned} \right\} \quad (254)$$

It also is easy to find the bending moment and cutting force

$$\left. \begin{aligned} G_1 &= -\frac{D_1 p^2 k^2}{p^2 + k^2} (C_1 \cos kx + C_2 \sin kx); \\ Q_1 &= \frac{K_1 k^3}{p^2 + k^2} (C_1 \sin kx - C_2 \cos kx). \end{aligned} \right\} \quad (255)$$

Further, we consider some partial cases of fastening of the plate edges $x = \pm a/2$.

a. Edge of plate hinge supported. Because of symmetry, an even solution for x can be considered, i.e., it can be assumed that $C_2 = C_3 = 0$.

Consequently, the boundary conditions are satisfied if $\cos ka/2=0$ or $ka=\pi$, and the critical load is determined by the expression

$$T_{cr} = \frac{\pi^2 D_1 K_1}{\pi^2 D_1 + K_1 a^2}. \quad (256)$$

b. Edge of plate rigidly fastened. The boundary conditions have the form

$$\left. \begin{aligned} C_1 \cos \frac{ka}{2} + C_4 &= 0; \\ C_1 \frac{p^2 k}{p^2 + k^2} \sin \frac{ka}{2} &= 0; \end{aligned} \right\} \quad (257)$$

as a result, it follows that $ka=2\pi$ and, consequently,

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$$T_{cr} = \frac{4\pi^2 D_1 K_1}{4\pi^2 D_1 + K_1 a^2}; \quad (258)$$

c. Edge $x=-a/2$ rigidly fastened, edge $x=a/2$ unsupported. In this case, the boundary conditions have the form

$$\left. \begin{aligned} C_1 \cos \alpha - C_2 \sin \alpha - C_3 \frac{a}{2} + C_4 &= 0; \\ C_1 \sin \alpha + C_2 \cos \alpha + C_3 \frac{p^2 + k^2}{p^2 k} &= 0; \\ C_1 \cos \alpha + C_2 \sin \alpha &= 0; \\ C_3 T &= 0, \end{aligned} \right\} \quad (259)$$

where

$$\alpha = \frac{ka}{2} = \frac{a}{2} \sqrt{\frac{TK_1}{D_1(K_1 - T)}}. \quad (260)$$

The characteristic equation for determination of the critical load has the form

$$\begin{vmatrix} \sin \alpha & \cos \alpha \\ \cos \alpha & \sin \alpha \end{vmatrix} = 0, \quad (261)$$

from which we have $k^2 a^2 = \pi^2/4$ and, consequently,

$$T_{cr} = \frac{\pi^2 D_1 K_1}{\pi^2 D_1 + 4K_1 a^2}. \quad (262)$$

d. Edge $x=-a/2$ rigidly fastened, edge $x=a/2$ hinge supported. For such boundary conditions

$$\text{at } x = -\frac{a}{2} \quad w = \varphi = 0;$$

$$\text{at } x = \frac{a}{2} \quad w = G_1 = 0,$$

from which we obtain the following system of linear equations

$$\left. \begin{aligned} C_1 \cos \alpha - C_2 \sin \alpha - C_3 \frac{a}{2} + C_4 &= 0; \\ C_1 \sin \alpha + C_2 \cos \alpha + C_3 \frac{p^2 + k^2}{p^2 k} &= 0; \\ C_1 \cos \alpha + C_2 \sin \alpha + C_3 \frac{a}{2} + C_4 &= 0; \\ C_1 \cos \alpha + C_2 \sin \alpha &= 0. \end{aligned} \right\} \quad (263)$$

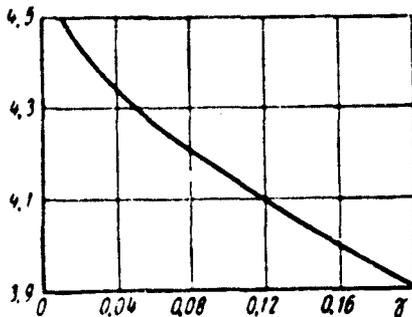
From Eq. (263), it is easy to obtain the following transcendental equation for determination of the critical load

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$$\operatorname{tg} ka = \frac{p^2 ka}{p^2 + k^2} = \frac{ka}{1 + \gamma k^2 a^2}, \quad (264)$$

where parameter γ characterizes the effect of interlayer shearing on the critical load of the plate and is determined by the expression

$$\gamma = \frac{D_1}{K_1 a^2}. \quad (265)$$



If the least root of Eq. (264) is designated by ω , the critical load is determined by the following formula

$$T_{cr} = \frac{\omega^2 D_1 K_1}{\omega^2 D_1 + K_1 a^2}. \quad (266)$$

The least root of Eq. (266) as a function of γ is determined by the graph presented in Fig. 32.

Fig. 32. Graph for determination of least root of equation $\tan x = \frac{x}{1 + \gamma x^2}$.

30. Formulation of Problem of Transverse Oscillations of Anisotropic Laminated Plates

We will consider small bending oscillations of uniform anisotropic /88 plates of constant thickness bounded by a simple profile. We will assume the bending deformations which arise in the oscillations to be small elastic oscillations which are governed by the generalized Hooke's law. Such oscillations are described by differential equations which are similar to the differential bending equations. Their fundamental difference is the dependence of the external load and, consequently, deformation functions ϕ , ψ and plate deflection w on time, as well as the presence of additional terms which define the inertial load.

Forced oscillations of the plate which arise as a result of variable transverse load $p(x, y; t)$ should be distinguished from the natural free oscillations. We will state that the plate accomplishes free transverse oscillations if any forces which impart deflections and velocities to the particles of the mean surface are instantaneously removed.

Thus, the system of differential equations of oscillations of an anisotropic plate can be written in the following form

$$L_6(w) = \rho \frac{\partial^2}{\partial t^2} L_4(w) + L_4(q); \quad (267)$$

$$\left. \begin{aligned} L_4(\phi) &= \alpha_1 \frac{\partial^2 w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} + \alpha_3 \frac{\partial^2 w}{\partial \xi \partial \eta^2} - \alpha_4 \frac{\partial^2 w}{\partial \eta^2} - \alpha_5 \frac{\partial w}{\partial \xi}; \\ L_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_2 \frac{\partial^2 w}{\partial \xi \partial \eta^2} + \beta_3 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} - \beta_4 \frac{\partial^2 w}{\partial \xi^2} - \beta_5 \frac{\partial w}{\partial \eta}; \end{aligned} \right\} \quad (268)$$

where ρ is the plate material density; q is the variable transverse load applied to the plate.

Differential operators $L_6()$, $L_4()$ and coefficients α_1 , β_1 are /89 determined from Eq. (134), (137) and (138).

In the case of a rectangular orthotropic plate, the principal axes of anisotropy of which are parallel to the sides, the system of differential equations is simplified, and it takes the form

$$F_6(w) = \rho \delta \frac{\partial^2}{\partial t^2} F_4(w) + F_4(p); \quad (269)$$

$$\left. \begin{aligned} F_4(\phi) &= \alpha_1 \frac{\partial^2 w}{\partial \xi^2} + \alpha_3 \frac{\partial^2 w}{\partial \xi^2 \partial \eta^2} - \alpha_5 \frac{\partial w}{\partial \xi}; \\ F_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_3 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} - \beta_5 \frac{\partial w}{\partial \eta}. \end{aligned} \right\} \quad (270)$$

Operators $F_6()$, $F_4()$ and coefficients α_1 , β_1 are determined by

Eq. (148)-(151). The free oscillations of the plate are determined by solution of uniform system of Eq. (269), (270).

Deformation functions ϕ and ψ , as well as plate deflection w should satisfy boundary conditions which depend on the fastening conditions of the bounded profile of the plate and the initial conditions which define the form and velocity of displacement of the particles of the mean surface at the initial moment of time, i.e., at $t=0$, the following conditions should be satisfied

$$\left. \begin{aligned} \varphi &= \varphi_0(\xi, \eta); \quad \psi = \psi_0(\xi, \eta); \quad w = w_0(\xi, \eta); \\ \dot{\varphi} = \dot{\psi} = 0; \quad \frac{\partial w}{\partial t} &= v_0(\xi, \eta). \end{aligned} \right\} \quad (271)$$

The solution of the problem of free oscillations of the plate is reduced to determination of the form of the oscillations, which is determined by the mode of functions ϕ , ψ , w and the natural oscillation frequency. It should be noted that, in oscillation theory, the eigenfrequencies of an elastic system are of extremely great importance.

Following S.G. Lekhnitskiy [17], we reproduce the trend of the solution of the problem of free bending oscillations of an anisotropic plate by the Fourier method.

We represent the solution of system of Eq. (267), (268) in the form of the product of the periodic time functions by the amplitude of the corresponding functions, i.e., we set

$$\left. \begin{aligned} \varphi &= (C_1 \cos pt + C_2 \sin pt) \Phi(\xi, \eta); \\ \psi &= (C_1 \cos pt + C_2 \sin pt) \Psi(\xi, \eta); \\ w &= (C_1 \cos pt + C_2 \sin pt) W(\xi, \eta). \end{aligned} \right\} \quad (272)$$

where p is the frequency of the natural oscillations of the plate.

By substituting Eq. (272) in Eq. (267), (268) for determination of Φ , Ψ , W , we obtain the system of differential equations

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$$\left. \begin{aligned} L_4(W) + p^2 \rho \delta L_4(W) &= 0; \\ L_4(\Phi) &= a_1 \frac{\partial^2 W}{\partial \xi^2} + a_2 \frac{\partial^2 W}{\partial \xi^2 \partial \eta} + \\ &+ a_3 \frac{\partial^2 W}{\partial \xi \partial \eta^2} - a_4 \frac{\partial^2 W}{\partial \eta^2} - a_5 \frac{\partial W}{\partial \xi}; \\ L_4(\Psi) &= \beta_1 \frac{\partial^2 W}{\partial \eta^2} + \beta_2 \frac{\partial^2 W}{\partial \xi \partial \eta^2} + \\ &+ \beta_3 \frac{\partial^2 W}{\partial \xi^2 \partial \eta} - \beta_4 \frac{\partial^2 W}{\partial \xi^2} - \beta_5 \frac{\partial W}{\partial \eta}. \end{aligned} \right\} \quad (273)$$

By satisfying the assigned boundary conditions of the problem, as is done in determination of the critical load, we obtain a characteristic equation which determines the presence of the nontrivial solution

$\Delta(p)=0$, which gives an infinite spectrum of the eigenfrequencies of oscillation of the plate.

The eigenfrequencies of oscillation of the plate depend on two integer parameters $m, n=1, 2, 3, \dots$. The lowest frequency is called the eigenfrequency of the primary tone, and the remaining frequencies are called frequencies of the second, third, etc. order. Each eigenfrequency p_{mn} corresponds to the form of the natural oscillations $\Phi_{mn}, \Psi_{mn}, W_{mn}$, which is determined to within an arbitrary factor.

Functions $\Phi_{mn}, \Psi_{mn}, W_{mn}$ frequently are called eigenfunctions. They are used in solution of problems of oscillations of a plate.

If the deformed state of the plate must be determined at any moment of time, the following procedure is used: initial functions ϕ_0, ψ_0, w_0 and initial velocity $v_0(\xi, \eta)$ are expanded in series by the eigenfunctions

$$\left. \begin{aligned} \phi_0(\xi, \eta) &= \sum_m \sum_n a_{mn} \Phi_{mn}(\xi, \eta); \\ \psi_0(\xi, \eta) &= \sum_m \sum_n \beta_{mn} \Psi_{mn}(\xi, \eta); \\ w_0(\xi, \eta) &= \sum_m \sum_n \gamma_{mn} W_{mn}(\xi, \eta); \\ v_0(\xi, \eta) &= \sum_m \sum_n \delta_{mn} W_{mn}(\xi, \eta); \end{aligned} \right\} \quad (274)$$

and the solution is found in the form of the analogous series

$$\left. \begin{aligned} \varphi(\xi, \eta; t) &= \sum_m \sum_n (C_{1mn} \cos pt + C_{2mn} \sin pt) \Phi_{mn}(\xi, \eta); \\ \psi(\xi, \eta; t) &= \sum_m \sum_n (C_{1mn} \cos pt + C_{2mn} \sin pt) \Psi_{mn}(\xi, \eta); \\ w(\xi, \eta; t) &= \sum_m \sum_n (C_{1mn} \cos pt + C_{2mn} \sin pt) W_{mn}(\xi, \eta). \end{aligned} \right\} \quad (275)$$

There is no difficulty in finding constants C_{1mn}, C_{2mn} and, consequently, the deformed state is determined by the sum of simple harmonic oscillations.

The possibility of expansion of the solution in series by eigenfunctions is based on the orthogonal nature of the latter. Actually, let the plate accomplish simple harmonic oscillations of frequency p , when the inertial load acting on the plate is $\rho \delta p^2 w(\chi, \eta)$. Since the

plate acquirer deflection w_1 as a result of characteristic load $\rho \delta p_1^2 w_1$, and deflection w_k as a result of characteristic load $\rho \delta p_k^2 w_k$, according to the reciprocity principle of the work of Beatty, we have

$$(276)$$

$$\rho \delta \iint p_i^2 w_i w_k d\xi d\eta = \rho \delta \iint p_k^2 w_k w_i d\xi d\eta,$$

from which it follows that

$$(p_i^2 - p_k^2) \iint w_i(\xi, \eta) w_k(\xi, \eta) d\xi d\eta = 0, \quad (277)$$

i.e., eigenfunctions w_1 are orthogonal. The orthogonal nature of eigenfunctions ϕ_1, ψ_1 ($i=1, 2, 3, \dots$) is proved similarly.

Thus, if deflection is presented in the form of an expansion by eigenfunctions

$$w(\xi, \eta) = \sum_i a_i w_i(\xi, \eta),$$

the coefficients of expansion are determined by the expressions

$$a_i = \frac{\iint w(\xi, \eta) w_i(\xi, \eta) d\xi d\eta}{\iint w_i^2(\xi, \eta) d\xi d\eta}. \quad (278)$$

If the eigenfunctions are normalized, i.e., if

$$\iint w_i^2(\xi, \eta) d\xi d\eta = 1, \quad (279)$$

the coefficients of expansion by eigenfunctions are determined by the expressions

$$a_i = \iint w(\xi, \eta) w_i(\xi, \eta) d\xi d\eta. \quad (280)$$

31. Variation Equation of Transverse Oscillations of Rectangular Plates

Exact determination of the form and frequency of oscillation of a /92
plate, with the exception of the simplest cases of a hinge supported rectangular plate, involves the solution of extremely complicated systems of differential Eq. (267), (268) for anisotropic plates or Eq. (269), (270) for orthotropic plates. In the solution of specific engineering problems, approximate methods based on some general principles of mechanics are extremely effective. In theories of core systems, such methods permit rapid determination of the frequency of oscillation of the primary tones, which are of the greatest practical interest, without integration of differential equations. These methods can be generalized for the case of transverse oscillations of plates.

We consider the action functional of Ostrogradskiy-Hamilton

$$S = \int_{t_A}^{t_B} (T - U) dt, \quad (281)$$

where T and U are, respectively, the kinetic and potential energies of a plate accomplishing transverse oscillations.

From the class of permissible functions which describe bending oscillations of a plate, we take the population of principal oscillations with frequency p.

By integrating over time for one period of oscillation $t_B - t_A = 2\pi/p$, we obtain the variation equation of transverse oscillations of the plate in the form

$$\delta(T_{\max} - U_{\max}) = 0; \quad (282)$$

the natural primary oscillations satisfy this equation.

We now write Eq. (282) in expanded form. For this, we determine the maximum values of the kinetic and potential energies of the plate.

If the plate accomplishes transverse oscillations $\phi_0(x, y; t)$, $\psi_0(x, y; t)$, $w_0(x, y; t)$ the corresponding potential energies for a plate with a general type of anisotropy and an orthotropic plate, the principal axes of anisotropy of which are parallel to the coordinate axes, are determined by the expressions

$$\begin{aligned} U_A = \frac{1}{2} \iint & \left[D_{11} \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi_0}{\partial x} \frac{\partial \psi_0}{\partial y} + D_{22} \left(\frac{\partial \psi_0}{\partial y} \right)^2 + \right. \\ & + D_{33} \left(\frac{\partial \varphi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right)^2 + \frac{3}{2} D_{13} \frac{\partial \varphi_0}{\partial x} \left(\frac{\partial \varphi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right) + \\ & + \frac{3}{2} D_{23} \frac{\partial \psi_0}{\partial y} \left(\frac{\partial \varphi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right) + K_1 \left(\varphi_0 + \frac{\partial w_0}{\partial x} \right)^2 + \\ & \left. + K_2 \left(\psi_0 + \frac{\partial w_0}{\partial y} \right)^2 \right] dx dy; \end{aligned} \quad (283)$$

$$\begin{aligned} U_0 = \frac{1}{2} \iint & \left[D_1 \left(\frac{\partial \varphi_0}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi_0}{\partial x} \frac{\partial \psi_0}{\partial y} + D_{22} \left(\frac{\partial \psi_0}{\partial y} \right)^2 + \right. \\ & \left. + D_3 \left(\frac{\partial \varphi_0}{\partial y} + \frac{\partial \psi_0}{\partial x} \right)^2 + K_1 \left(\varphi_0 + \frac{\partial w_0}{\partial x} \right)^2 + K_2 \left(\psi_0 + \frac{\partial w_0}{\partial y} \right)^2 \right] dx dy. \end{aligned} \quad (284)$$

The kinetic energy of a plate accomplishing transverse oscillations is determined by the known expression

$$T = \frac{1}{2} \rho \delta \iint \left(\frac{\partial w}{\partial t} \right)^2 dx dy. \quad (285)$$

When the plate accomplishes one of the primary oscillations, i.e., when

$$\left. \begin{aligned} \varphi_0(x, y; t) &= \varphi(x, y) \sin(pt + \alpha); \\ \psi_0(x, y; t) &= \psi(x, y) \sin(pt + \alpha); \\ w_0(x, y; t) &= w(x, y) \sin(pt + \alpha), \end{aligned} \right\} \quad (286)$$

the greatest potential energies are determined by Eq. (284), (285) where, instead of deformation components ϕ_0 , ψ_0 , w_0 , their amplitude values ϕ , ψ , w stand, and the greatest kinetic energy of the oscillating plate

$$T_{\max} = \frac{1}{2} \rho \delta p^2 \iint w^2(x, y) dx dy. \quad (287)$$

Consequently, the variation equations for the principal natural oscillations of the plate can be written in the following form:

a. variation equation of oscillation of anisotropic plate

$$\begin{aligned} \delta' L = \delta' \left\{ \iint \left[D_{11} \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_{22} \left(\frac{\partial \psi}{\partial y} \right)^2 + \right. \right. \\ \left. \left. + D_{33} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + \frac{3}{2} D_{13} \frac{\partial \varphi}{\partial x} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + \right. \right. \\ \left. \left. + \frac{3}{2} D_{23} \frac{\partial \psi}{\partial y} \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 \right] dx dy - \right. \\ \left. - \rho \delta p^2 \iint w^2(x, y) dx dy \right\} = 0; \quad (288) \end{aligned}$$

b. variation equation of oscillation of orthotropic plate, /94
the principal axes of anisotropy of which coincide with the coordinate axes

$$\begin{aligned} \delta' L = \delta' \left\{ \iint \left[D_1 \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2D_{12} \frac{\partial \varphi}{\partial x} \frac{\partial \psi}{\partial y} + D_2 \left(\frac{\partial \psi}{\partial y} \right)^2 + \right. \right. \\ \left. \left. + D_3 \left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} \right)^2 + K_1 \left(\varphi + \frac{\partial w}{\partial x} \right)^2 + \right. \right. \\ \left. \left. + K_2 \left(\psi + \frac{\partial w}{\partial y} \right)^2 - \rho \delta p^2 w^2 \right] dx dy \right\}. \quad (289) \end{aligned}$$

In Eq. (288), (289), δ' designates the variation of the functional.

The solution of the variation problem of transverse oscillations of a plate, as in the case of static bending and stability, can be obtained, for example, by the Ritz method, namely, deformation components ϕ , ψ , w are assigned in the form of an infinite sum with the indeterminate coefficients

$$\left. \begin{aligned} \varphi(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \varphi_{mn}(x, y); \\ \psi(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \psi_{mn}(x, y); \\ w(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} w_{mn}(x, y). \end{aligned} \right\} \quad (290)$$

where φ_{mn} , ψ_{mn} , w_{mn} are complete systems of continuous functions dependent on two parameters and satisfying the conditions of fastening of the edges of the plate. After formulation of Eq. (290) in variation equations and integration, the problem is reduced to finding the minimum of the quadratic functions of arguments A_{mn} , B_{mn} , C_{mn} , i.e., to solution of a system of uniform linear algebraic equations for A_{mn} , B_{mn} , C_{mn} .

The condition of nontriviality of the solution leads to the characteristic equation for determination of the oscillation frequency

$$|a_{ij}(p)| = 0. \quad (291)$$

The smallest value of the root gives an approximate value of the oscillation frequency of the primary tone. The remaining roots are the frequencies of the higher tones.

32. Determination of Frequencies of Natural Oscillations of Orthotropic Rectangular Plate

We consider bending oscillations of a rectangular plate made of an orthotropic material. We will assume the principal axes of anisotropy to be parallel to the sides of the plate (Fig. 33).

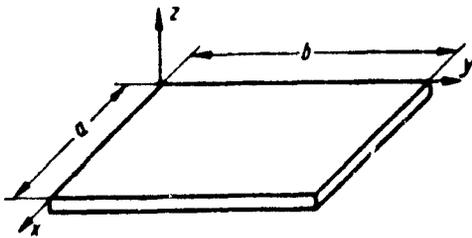


Fig. 33. Basic designations.

a. Free oscillations of hinge supported rectangular plate. It is particularly simple to obtain a solution of the problem for a hinge supported rectangular plate. In accordance with Eq. (273), the system of differential equations of the natural oscillations of an orthotropic plate has the form

$$F_6(w) + q\delta p^2 F_4(w) = 0; \quad (292)$$

$$\left. \begin{aligned} F_4(\varphi) &= \alpha_1 \frac{\partial^2 w}{\partial \xi^2} + \alpha_2 \frac{\partial^2 w}{\partial \xi \partial \eta^2} - \alpha_3 \frac{\partial w}{\partial \xi}; \\ F_4(\psi) &= \beta_1 \frac{\partial^2 w}{\partial \eta^2} + \beta_2 \frac{\partial^2 w}{\partial \xi^2 \partial \eta} - \beta_3 \frac{\partial w}{\partial \eta}. \end{aligned} \right\} \quad (293)$$

where operators $F_6(\cdot)$, $F_4(\cdot)$ and coefficients α_1 , β_1 are determined by

Eq. (148)-(151).

The solution of system of Eq. (292), (293) which satisfies the conditions of hinge support of the plate, i.e., the conditions

$$\text{at } x=0, \quad x=a \quad w=\psi=G_1=0;$$

$$\text{at } y=0, \quad y=b \quad w=\phi=G_2=0,$$

can be sought in the form

$$\left. \begin{aligned} \varphi &= A \cos m \pi \xi \sin n \pi \eta; \\ \psi &= B \sin m \pi \xi \cos n \pi \eta; \\ w &= C \sin m \pi \xi \sin n \pi \eta. \end{aligned} \right\} \quad (294)$$

where m, n are whole numbers.

The oscillation frequency is determined from Eq. (292). By substituting Eq. (294) in this equation, we obtain

$$\rho \delta p^2 = \frac{\pi^4 (a_{00} m^4 + a_{22} m^2 n^2 + a_{22} m^2 n^2 + a_{00} n^4) + \pi^4 a_{00} (\gamma_1 m^4 + \gamma_3 m^2 n^2 + \gamma_3 n^4)}{\pi^4 (a_{40} m^4 + a_{22} m^2 n^2 + a_{04} n^4) + \pi^2 (a_{20} m^2 + a_{02} n^2) + a_{00}} \quad (295)$$

where coefficients a_{ij}, γ_1 are determined by expressions (150), (151).

The oscillation frequency of the primary tone at $m=1, n=1$

$$p_{11} = \frac{\pi^4}{\rho \delta} \frac{\pi^2 (a_{00} + a_{22} + a_{22} + a_{00}) + a_{00} (\gamma_1 + \gamma_3 + \gamma_3)}{\pi^4 (a_{40} + a_{22} + a_{04}) + \pi^2 (a_{20} + a_{02}) + a_{00}} \quad (296)$$

b. Free oscillations of rigidly fastened rectangular plate. For a plate with rigidly fastened edges, it is extremely complicated to obtain a precise solution. Therefore, for determination of the frequency of the natural oscillations, we use variation Eq. (289), which we write in dimensionless coordinates $\xi=x/a; \eta=y/b;$

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$$\begin{aligned} \delta \int_0^1 \int_0^1 & \left[\frac{D_1}{a^2} \left(\frac{\partial \varphi}{\partial \xi} \right)^2 + \frac{2D_{12}}{ab} \frac{\partial \varphi}{\partial \xi} \frac{\partial \psi}{\partial \eta} + \frac{D_2}{b^2} \left(\frac{\partial \psi}{\partial \eta} \right)^2 + \right. \\ & + D_3 \left(\frac{1}{b} \frac{\partial \varphi}{\partial \eta} + \frac{1}{a} \frac{\partial \psi}{\partial \xi} \right)^2 + K_1 \left(\varphi + \frac{1}{a} \frac{\partial w}{\partial \xi} \right)^2 + \\ & \left. + K_2 \left(\psi + \frac{1}{b} \frac{\partial w}{\partial \eta} \right)^2 - \rho \delta p^2 w^2 \right] d\xi d\eta = 0. \end{aligned} \quad (297)$$

We will seek the form of oscillation of the primary tone in the form

$$\left. \begin{aligned} \varphi &= A \sin \frac{2\pi x}{a} \left(1 - \cos \frac{2\pi y}{b}\right); \\ \psi &= B \left(1 - \cos \frac{2\pi x}{a}\right) \sin \frac{2\pi y}{b}; \\ w &= F \left(1 - \cos \frac{2\pi x}{a}\right) \left(1 - \cos \frac{2\pi y}{b}\right). \end{aligned} \right\} \quad (298)$$

By substituting Eq. (298) in the variation equation, the following can be obtained

$$\begin{aligned} \delta' \left\{ 3 \left(\frac{D_1}{a^2} + \frac{D_2}{3b^2} + \bar{K}_1 \right) A^2 + 3 \left(\frac{D_2}{b^2} + \frac{D_3}{3a^2} + \bar{K}_2 \right) B^2 + \right. \\ \left. + \left(\frac{3K_1}{a^2} + \frac{3K_2}{b^2} - \frac{9Q\delta p^2}{4\pi^2} \right) F^2 + \frac{2(D_{12} + D_3)}{ab} AB + \right. \\ \left. + \frac{3K_1}{\pi a} AF + \frac{3K_2}{\pi b} BF \right\} = 0, \end{aligned} \quad (299)$$

where

$$\bar{K}_1 = \frac{K_1}{4\pi^2}; \quad \bar{K}_2 = \frac{K_2}{4\pi^2}. \quad (300)$$

The condition of nontriviality of the system of linear equations

$$\frac{\partial L}{\partial A} = \frac{\partial L}{\partial B} = \frac{\partial L}{\partial F} = 0 \quad (301)$$

has the form

$$\begin{vmatrix} 3 \left(\frac{D_1}{a^2} + \frac{D_2}{3b^2} + \bar{K}_1 \right); & \frac{C}{ab} & \frac{3K_1}{2\pi a} \\ \frac{C}{ab} & 3 \left(\frac{D_2}{b^2} + \frac{D_3}{3a^2} + \bar{K}_2 \right) & \frac{3K_2}{2\pi b} \\ \frac{3K_1}{2\pi a} & \frac{3K_2}{2\pi b} & \frac{3K_1}{a^2} + \frac{3K_2}{b^2} - \frac{9Q\delta p^2}{4\pi^2} \end{vmatrix} = 0; \quad (302)$$

from which we find

$$\begin{aligned} \frac{3Q\delta p^2}{4\pi^2} \left[(\lambda_1 + \bar{K}_1)(\lambda_2 + \bar{K}_2) - \frac{C^2}{9a^2b^2} \right] &= \frac{2C}{3a^2b^2} \bar{K}_1 K_2 + \\ + \left(\frac{K_1}{a^2} + \frac{K_2}{b^2} \right) \left[(\lambda_1 + \bar{K}_1)(\lambda_2 + \bar{K}_2) - \frac{C^2}{9a^2b^2} \right] &- \\ - \frac{K_2 \bar{K}_2}{b^2} (\lambda_1 + \bar{K}_1) - \frac{K_1 \bar{K}_1}{a^2} (\lambda_2 + \bar{K}_2), & \end{aligned} \quad (303)$$

where

$$\lambda_1 = \frac{D_1}{a^2} + \frac{D_2}{3b^2}; \quad \lambda_2 = \frac{D_2}{b^2} + \frac{D_3}{3a^2}. \quad (304)$$

Consequently, the frequency of natural oscillation of the primary tone

$$p_{11} = \frac{2\pi}{\sqrt{3Q\delta}} \times \sqrt{\frac{K_1 \bar{K}_1 \left[\frac{D_1}{a^4} + \frac{2(C+D_2)}{3a^2b^2} + \frac{D_2}{b^4} \right] + \left(\frac{K_1}{a^2} + \frac{K_2}{b^2} \right) \left(\lambda_1 \lambda_2 - \frac{C^2}{9a^2b^2} \right)}{(\lambda_1 + \bar{K}_1)(\lambda_2 + \bar{K}_2) - \frac{C^2}{9a^2b^2}}. \quad (305)$$

As $K_1 \rightarrow \infty$, $K_2 \rightarrow \infty$, i.e., without accounting for interlayer shearing, the known formula for the frequency of natural oscillations of an orthotropic rectangular plate with rigidly fastened edges follows from Eq. (305)

$$p_{11} = \frac{22.79}{\sqrt{Q\delta}} \sqrt{\frac{D_1}{a^4} + \frac{2(C+D_2)}{3a^2b^2} + \frac{D_2}{b^4}}. \quad (306)$$

c. Transverse oscillations of laminated strips. Free primary oscillations of laminated strips are described by the following system of differential equations

$$\left. \begin{aligned} w^{IV} + 2r^2 w'' - s^4 w &= 0; \\ \varphi'' - \omega^2 \varphi &= \omega^2 w', \end{aligned} \right\} \quad (307)$$

where

$$\omega^2 = \frac{K_1}{D_1}; \quad 2r^2 = \frac{Q\delta p^2}{K_1}; \quad s^4 = \frac{Q\delta p^4}{D_1}. \quad (308)$$

If

$$k_1^2 = \sqrt{r^4 + s^4} - r^2; \quad k_2^2 = \sqrt{r^4 + s^4} + r^2, \quad (309)$$

is designated, it is easy to obtain a solution of Eq. (307) in the form /98

$$\left. \begin{aligned} w &= C_1 \operatorname{ch} k_1 x + C_2 \operatorname{sh} k_1 x + C_3 \cos k_2 x + C_4 \sin k_2 x; \\ \varphi &= \frac{k_1 \omega^2}{k_1^2 - \omega^2} (C_1 \operatorname{sh} k_1 x + C_2 \operatorname{ch} k_1 x) + \\ &\quad + \frac{k_2 \omega^2}{k_2^2 + \omega^2} (C_3 \sin k_2 x - C_4 \cos k_2 x). \end{aligned} \right\} \quad (310)$$

There also can be found

$$\left. \begin{aligned} G_1 &= -\frac{D_1 k_1^2 \omega^2}{k_1^2 - \omega^2} (C_1 \operatorname{ch} k_1 x + C_2 \operatorname{sh} k_1 x) - \\ &\quad - \frac{D_1 k_2^2 \omega^2}{k_2^2 + \omega^2} (C_3 \cos k_2 x + C_4 \sin k_2 x); \end{aligned} \right\} \quad (311)$$

$$Q_1 = - \frac{K_1 k_1^3}{k_1^3 - \omega^2} (C_1 \operatorname{sh} k_1 x + C_2 \operatorname{ch} k_1 x) + \left. \begin{aligned} &+ \frac{K_1 k_2^3}{k_2^3 + \omega^2} (C_3 \sin k_2 x - C_4 \cos k_2 x). \end{aligned} \right\} \quad (311)$$

We select a coordinate system as indicated in Fig. 34, and we determine the free oscillation frequency with various fastenings of the ends of the strip $x = \pm a/2$.

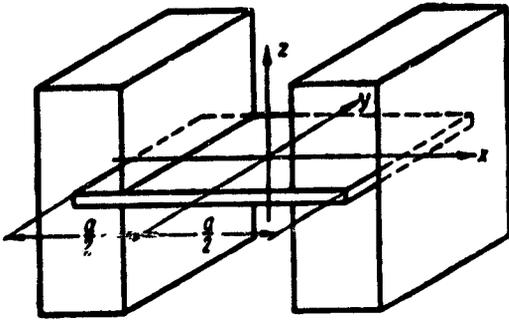


Fig. 34. Strip with rigidly fastened ends.

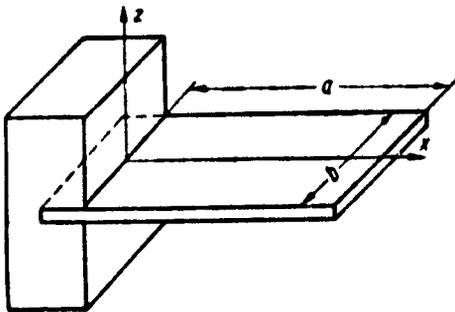


Fig. 35. Diagram of bracket and coordinate system.

a. Edge of strip hinge supported.
By satisfying the boundary conditions

$$w = G_1 = 0 \text{ at } x = \pm \frac{a}{2}$$

the following can be obtained

$$\left. \begin{aligned} C_1 \operatorname{ch} \alpha_1 + C_2 \cos \alpha_2 &= 0; \\ C_1 \frac{k_1^3}{k_1^3 - \omega^2} \operatorname{ch} \alpha_1 + C_3 \frac{k_2^3}{k_2^3 + \omega^2} \cos \alpha_2 &= 0; \\ C_2 \operatorname{sh} \alpha_1 + C_4 \sin \alpha_2 &= 0; \\ C_2 \frac{k_1^3}{k_1^3 - \omega^2} \operatorname{sh} \alpha_1 + C_4 \frac{k_2^3}{k_2^3 + \omega^2} \sin \alpha_2 &= 0, \end{aligned} \right\} \quad (312)$$

where

$$\left. \begin{aligned} \alpha_1 &= \frac{k_1 a}{2} = \frac{ar}{2} \sqrt{\sqrt{1 + 2 \frac{\omega^2}{r^2}} - 1}; \\ \alpha_2 &= \frac{k_2 a}{2} = \frac{ar}{2} \sqrt{\sqrt{1 + 2 \frac{\omega^2}{r^2}} + 1}. \end{aligned} \right\} \quad (313)$$

The characteristic equation can be written in the form

$$\cos \alpha_2 = 0; \quad \sin \alpha_2 = 0;$$

and, consequently, the frequency spectrum of free oscillations of a laminated strip is determined by the expression

$$p = \frac{\pi^2 m^2}{\sqrt{Q\delta}} \sqrt{\frac{K_1 D_1}{\pi^2 m^2 D_1 + K_1 a^2}}. \quad (314)$$

b. Edge of strip $x=0$ rigidly fastened, edge $x=a$ unsupported (Fig. 35). By satisfying the boundary conditions

$$w = \phi = 0 \quad \text{at} \quad x = 0,$$

$$G_1 = Q_1 = 0 \quad \text{at} \quad x = a,$$

we obtain the following system of linear equations for determination of the frequencies

$$\left. \begin{aligned}
& C_1 + C_3 = 0; \\
& \frac{k_1}{k_1^2 - \omega^2} C_2 - \frac{k_2}{k_2^2 + \omega^2} = 0; \\
& \frac{k_1}{k_1^2 - \omega^2} (C_1 \operatorname{ch} 2\alpha_1 + C_2 \operatorname{sh} 2\alpha_1) + \\
& + \frac{k_2}{k_2^2 + \omega^2} (C_3 \cos 2\alpha_2 + C_4 \sin 2\alpha_2) = 0; \\
& \frac{k_1}{k_1^2 - \omega^2} (C_1 \operatorname{sh} 2\alpha_1 + C_2 \operatorname{ch} 2\alpha_1) - \\
& - \frac{k_2}{k_2^2 + \omega^2} (C_3 \sin 2\alpha_2 - C_4 \cos 2\alpha_2) = 0.
\end{aligned} \right\} \quad (315)$$

Consequently, the frequency spectrum of the natural oscillations of a laminated bracket is determined by the equation /10

$$\left(\frac{\omega}{r} \right)^2 + \left(1 + \frac{\omega^2}{r^2} \right) \operatorname{ch} 2\alpha_1 \cos 2\alpha_2 - \frac{1}{\sqrt{2}} \frac{\omega}{r} \operatorname{sh} 2\alpha_1 \sin 2\alpha_2 = 0, \quad (316)$$

where α_1 and α_2 are determined by Eq. (313).

As $K_1 \rightarrow \infty$, there is the known equation for determination of the frequency of natural oscillations of the bracket

$$\operatorname{ch} k_1 a \cos k_1 a = 1. \quad (317)$$

33. Axisymmetric Transverse Oscillations of Circular Plate

If a circular plate made of a cylindrically orthotropic laminated plastic executes axisymmetric transverse oscillations, the system of differential equations has the form [14]

$$\left. \begin{aligned}
& \frac{\partial (rG_1)}{\partial r} - G_2 = Q_1 r; \\
& \frac{\partial (rQ_1)}{\partial r} = -\rho \delta r \frac{\partial^2 w_1}{\partial t^2} - q(r, t) r,
\end{aligned} \right\} \quad (318)$$

where ρ is the density of the laminated plastic; q is the external transverse load which changes over time.

By substituting basic relationships (64), (65) in Eq. (318), we obtain

$$\left. \begin{aligned}
& \frac{\partial^2 \Phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi_1}{\partial r} - \frac{\lambda^2}{r^2} \Phi_1 = \frac{K_1}{D_1} \left(\frac{\partial w_1}{\partial r} + \Phi_1 \right); \\
& \frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r} \frac{\partial w_1}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} (r \Phi_1) = \frac{\rho \delta}{K_1} \frac{\partial^2 w_1}{\partial t^2} + \frac{q(r, t)}{K_1}.
\end{aligned} \right\} \quad (319)$$

For the principal normal oscillations of the plate

$$\left. \begin{aligned} \varphi_1(r, t) &= \varphi(r) (C_1 \cos \omega t + C_2 \sin \omega t); \\ w_1(r, t) &= w(r) (C_1 \cos \omega t + C_2 \sin \omega t); \end{aligned} \right\} \quad (320)$$

and, consequently, the oscillation amplitudes comply with the following system of differential equations

$$\varphi'' + \frac{\varphi'}{r} - \left(\frac{\lambda^2}{r^2} + k^2 \right) \varphi = k^2 w'; \quad (321)$$

$$w'' + \frac{w'}{r} + p^2 w = - \left(\varphi' + \frac{\varphi}{r} \right); \quad (322)$$

where

$$k^2 = \frac{K_1}{D_1}; \quad \lambda^2 = \frac{E_2}{E_1}; \quad p^2 = \frac{\rho \delta \omega^2}{K_1}. \quad (323)$$

Eq. (321), (322) are equivalent to the following differential equations

$$\begin{aligned} &\varphi^{IV} + \frac{2}{r} \varphi''' - \left(\frac{\lambda^2 + 2}{r^2} - p^2 \right) \varphi'' + \\ &+ \left(\frac{3\lambda^2}{r^2} + \frac{p^2}{r} \right) \varphi' - \left(\frac{3\lambda^2}{r^4} + \frac{\lambda^2 p^2}{r^2} + p^2 k^2 \right) \varphi = 0; \end{aligned} \quad (324)$$

$$w = - \frac{1}{k^2 p^2} \left(\varphi''' + \frac{2}{r} \varphi'' - \frac{\lambda^2}{r^2} \varphi' + \frac{\lambda^2}{r^2} \varphi \right). \quad (325)$$

Eq. (324) is a fourth order Fuchs class differential equation. We will seek its solution in the form

$$\varphi = r^{q_0} \sum_{m=0}^{\infty} A'_m r^m. \quad (326)$$

By substituting Eq. (326) in Eq. (324), the following can be obtained

$$\begin{aligned} &\sum_0^{\infty} [a + bm + cm(m-1) + em(m-1)(m-2) + \\ &\quad + m(m-1)(m-2)(m-3)] A'_m r^m + \\ &+ \sum_2^{\infty} [d + f(m-2) + p^2(m-2)(m-3)] A'_{m-2} r^m - \\ &\quad - k^2 p^2 \sum_4^{\infty} A'_{m-4} r^m = 0, \end{aligned} \quad (327)$$

where

$$\left. \begin{aligned} a &= q_0^4 - 4q_0^3 + (3 - \lambda^2) q_0^2 + 4\lambda^2 q_0 - 3\lambda^2; \\ b &= 4q_0^3 - 6q_0^2 - 2(\lambda^2 + 1) q_0 + 3\lambda^2; \\ c &= 6q_0^2 - (\lambda^2 + 2); \\ d &= p^2 (q_0^3 - \lambda^2); \\ e &= 2(2q_0 + 1); \\ f &= p^2 (2q_0 + 1). \end{aligned} \right\} \quad (328)$$

The following recurrent formulas for coefficients A_m' can be obtained from Eq. (327)

$$\begin{aligned} m = 0, \quad m = 1 \\ [a + bm + cm(m-1) + em(m-1)(m-2) + \\ + m(m-1)(m-2)(m-3)] A_m' = 0; \end{aligned} \quad (329)$$

$$\begin{aligned} m = 2, \quad m = 3 \\ A_m' = \\ = \frac{-[d + f(m-2) + p^2(m-2)(m-3)]}{a + bm + cm(m-1) + em(m-1)(m-2) + m(m-1)(m-2)(m-3)} A_{m-2}' \end{aligned} \quad (330)$$

$$\begin{aligned} m \geq 4 \\ A_m' = \frac{k^2 p^2 A_{m-4}' - [d + f(m-2) + p^2(m-2)(m-3)] A_{m-3}'}{a + bm + cm(m-1) + em(m-1)(m-2) + m(m-1)(m-2)(m-3)}. \end{aligned} \quad (331)$$

In accordance with Eq. (329), the characteristic equation for determination of ρ_0 has the form

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$$\rho_0^4 - 4\rho_0^3 + (3 - \lambda^2)\rho_0^2 + 4\lambda^2\rho_0 - 3\lambda^2 = 0. \quad (332)$$

The roots of this characteristic equation are

$$\rho_0^I = 1; \quad \rho_0^{II} = 3; \quad \rho_0^{III} = \lambda; \quad \rho_0^{IV} = -\lambda. \quad (333)$$

It is easy to note that $A_{2m+1}' = 0$ ($m=0, 1, 2, 3, \dots$).

In the case of a continuous plate, the solution which corresponds to root $\rho_0 = -\lambda$ should be set equal to zero, and roots $\rho_0^I = 1$, $\rho_0^{II} = 3$ gives linearly dependent solutions. The missing solution should be sought in the form

$$\varphi = r^2 \left(A_0 \varphi_2 + \sum_{m=0}^{\infty} A_m'' r^m \right), \quad (334)$$

where φ_2 is the solution which corresponds to the root $\rho_0^{II} = 3$.

By substituting Eq. (334) in Eq. (324), we obtain

$$\begin{aligned} \sum_{m=0}^{\infty} [3(16 - \lambda^2)m + (52 - \lambda^2)m(m-1) + 14m(m-1)(m-2) + \\ + m(m-1)(m-2)(m-3)] A_m'' r^m + \sum_2^{\infty} [p^2(9 - \lambda^2) + 7p^2(m-2) + \\ + p^2(m-2)(m-3)] A_{m-2}'' r^m - k^2 p^2 \sum_4^{\infty} A_{m-4}'' r^m = \\ = A_0 [2(9 - \lambda^2) + 2(34 - \lambda^2)m + 36m(m-1) + \end{aligned} \quad (335)$$

$$\begin{aligned}
& + 4m(m-1)(m-2)](A'_m)_1 r^m + \\
& + \sum_2^{\infty} [6p^3 + 2p^3(m-2)](A'_{m-2})_1 r^m.
\end{aligned} \tag{335}$$

In this manner, for coefficients A_m'' , the following recurrent re- /103
relationships can be obtained

$$\begin{aligned}
& m = 2 \\
& A_2'' = - \frac{1}{8(25-\lambda^2)} \left\{ p^3(9-\lambda^2) A_0'' + \right. \\
& \left. + A_0 \left[6k^3(A'_0)_1 - \frac{2k^3(113-3\lambda^2)(9-\lambda^2)}{8(25-\lambda^2)} (A'_1)_1 \right] \right\},
\end{aligned} \tag{336}$$

where

$$(A'_0)_1 = - \frac{p^3(9-\lambda^2)}{3(25-\lambda^2)} (A'_1)_1; \tag{337}$$

for $m \geq 2n$ ($n=1, 2, 3, \dots$)

$$\begin{aligned}
& A_m'' = - \frac{1}{[3(16-\lambda^2)m + (52-\lambda^2)m(m-1) + 14m(m-1)(m-2)]} \times \\
& \times \left\{ [p^3(9-\lambda^2) + 7p^3(m-2) + p^3(m-2)(m-3)] A_{m-2}'' - \right. \\
& \left. - k^3 p^3 A_{m-4}' + A_0 [2(9-\lambda^2) + 2(34-\lambda^2)m + \right. \\
& \left. + 36m(m-1) + 4m(m-1)(m-2)] (A'_m)_1 + \right. \\
& \left. + A_0 [6k^3 + 2k^3(m-2)] (A'_{m-2})_1 \right\},
\end{aligned} \tag{338}$$

where coefficients $(A'_m)_1$ are determined by the expression

$$\begin{aligned}
& (A'_m)_1 = \\
& = \frac{k^3 p^3 (A'_{m-4})_1 - [p^3(9-\lambda^2) + 7p^3(m-2)(m-3)] (A'_{m-2})_1}{m[3(16-\lambda^2) + (52-\lambda^2)(m-1) + 14(m-1)(m-2) + (m-1)(m-2)(m-3)].}
\end{aligned} \tag{339}$$

The odd coefficients again equal zero.

CHAPTER 7. MEMBRANE THEORY OF ANISOTROPIC LAMINATED SHELLS

34. Region of Applicability of Membrane Theory of Shells and Boundary Conditions

The simplest alternate version of the general theory of shells is /104 the membrane theory, which is widely used for calculation of various engineering structures and buildings. The explanation of this is that the membrane theory quite satisfactorily describes the behavior of thin shells under various loads which have to be of concern in engineering. The simplicity and value of membrane theory is not only significant mathematical simplification of the basic differential equations of the theory of shells but also that, in many cases, the results of the basic stage of the theory, which consists of determination of the nature of transmission of forces from the equations of equilibrium, are valid for any thin shells regardless of their structure and nature of deformation. Structural inhomogeneity within the shell material appears in subsequent stages of solution of the problem, which are connected with determination of the deformed state and the nature of distribution of stresses through the shell.

As in the case of isotropic or anisotropic shells [1, 8], we will call membrane theory an approximate method of calculation, based on the assumption that bending stresses are small compared with the stresses uniformly distributed through the shell. This assumption is mathematically equivalent to the assumption that cutting forces Q_1 , Q_2 can be disregarded in the first three equilibrium Eq. (17). With the intention that only shells of rotation will be considered subsequently, we write the basic equations of membrane theory for this partial case. Membrane theory of anisotropic shells is discussed in greater detail in the monograph of S.A. Ambartsumyan [1].

As curvilinear Gaussian coordinates which define the positions of /105 points on the mean surface of a shell, we use arc length s , reckoned from the initial parallel (point M_0 in Fig. 36) and angle β between two planes passing through the axis of rotation. One such plane was selected as the initial plane. We introduce two more coordinates: shell cross section radius r and angle α between the normal to meridian \bar{n} and the axis of rotation.

Principal radii of curvature R_1 , R_2 are determined by the expressions [8]

$$R_1 = \frac{ds}{d\alpha}; \quad R_2 = \frac{r}{\sin \alpha}. \quad (340)$$

Two Gauss-Codacci relationships are satisfied identically, and the third has the form

$$\frac{dr}{d\alpha} = \frac{d(R_2 \sin \alpha)}{d\alpha} = R_1 \cos \alpha. \quad (341)$$

The last relationship can be obtained from geometric considera-

tions (Fig. 37)

$$\frac{\Delta r}{R_1 \Delta \alpha} = \cos \alpha.$$

If the components of the distributed surface load acting on the shell are X, Y, Z , equilibrium Eq. (17), with $A_1=1, A_2=r$, take the form

$$\left. \begin{aligned} r \frac{\partial T_1}{\partial s} - (T_2 - T_1) \cos \alpha + \frac{\partial S}{\partial \beta} &= -Xr; \\ \frac{\partial T_2}{\partial \beta} + r \frac{\partial S}{\partial s} + 2S \cos \alpha &= -Yr; \\ \frac{T_1}{R_1} + \frac{T_2}{R_2} &= Z. \end{aligned} \right\} \quad (342)$$

Fig. 36. Sketch of shell of rotation and conventional symbols.

In accordance with Eq. (10), the components of deformation of the mean surface are determined by the expressions

$$\left. \begin{aligned} e_1 &= \frac{\partial u}{\partial s} + \frac{w}{R_1}; \\ e_1 &= \frac{1}{r} \frac{\partial v}{\partial \beta} + \frac{u}{r} \cos \alpha + \frac{w}{R_2}; \\ \omega &= \frac{1}{r} \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial s} - \frac{v}{r} \cos \alpha. \end{aligned} \right\} \quad (343)$$

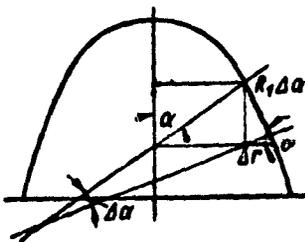


Fig. 37. Geometric interpretation of Gauss-Codacci relationship.

For shells with undetermined anisotropy of elastic properties and for shells made of orthotropic materials, the principal axes of anisotropy of which do not coincide with the coordinate axes, in accordance with Eq. (8), (343), Hooke's law has the form

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$$\left. \begin{aligned} \frac{\partial u}{\partial s} + \frac{w}{R_1} &= \frac{1}{\delta} (a_{11}T_1 + a_{12}T_2 + a_{13}S); \\ \frac{1}{r} \frac{\partial v}{\partial \beta} + \frac{u}{r} \cos \alpha + \frac{w}{R_2} &= \frac{1}{\delta} (a_{12}T_1 + a_{22}T_2 + a_{23}S); \\ \frac{1}{r} \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial s} - \frac{v}{r} \cos \alpha &= \frac{1}{\delta} (a_{13}T_1 + a_{23}T_2 + a_{33}S). \end{aligned} \right\} \quad (344)$$

Correspondingly, for orthotropic laminated shells, the principal axes of anisotropy of which coincide with the coordinate axes, in accordance with Eq. (28) and (343), the elasticity relationships have the form

$$\left. \begin{aligned} \frac{\partial u}{\partial s} + \frac{w}{R_1} &= \frac{T_1 - \nu_1 T_2}{B_1 (1 - \nu_1 \nu_2)}; \\ \frac{1}{r} \frac{\partial v}{\partial \beta} + \frac{u}{r} \cos \alpha + \frac{w}{R_2} &= \frac{T_2 - \nu_2 T_1}{B_2 (1 - \nu_1 \nu_2)}; \\ \frac{1}{r} \frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial s} - \frac{v}{r} \cos \alpha &= \frac{S}{B_3}. \end{aligned} \right\} \quad (345)$$

Thus, the systems of differential equations of membrane theory can be integrated in two successive stages:

stage 1. by solution of system of Eq. (342), elastic forces T_1, T_2, S of the shell are determined;

stage 2. from system of Eq. (344) or (345), displacements u, v, w are determined for anisotropic and orthotropic shells.

Since, by the definition of membrane theory, interlayer shears γ_1, γ_2 are absent or negligibly small under load, by Eq. (12) we obtain

$$\varphi = -\frac{\partial w}{\partial s} + \frac{u}{R_1}; \quad \psi = -\frac{1}{r} \frac{\partial w}{\partial \beta} + \frac{v}{R_2}. \quad (346)$$

The components of effective change of curvature and torsion of the mean surface, in accordance with Eq. (11) and (346), are determined by the following expressions

$$\left. \begin{aligned} \kappa_1^e &= -\frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right); \\ \kappa_2^e &= - \left[\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right]; \end{aligned} \right\} \\ \\ \left. \begin{aligned} 2\chi_3^e &= - \left[r \frac{\partial}{\partial s} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ &\left. + \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) - \frac{\cos \alpha}{r} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) \right]. \end{aligned} \right\} \quad (347)$$

Bending moments and torques G_1, G_2, H can now be determined by Eq. (4) or (25) /107

For shells of rotation with random anisotropy of properties and for orthotropic shells, the principal axes of anisotropy of which do not coincide with the coordinate axes, there are the following expressions for determination of the bending moments and torque

$$\left. \begin{aligned} G_1 &= D_{11} \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) + D_{12} \left[\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ &\quad \left. + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right] + D_{13} \left[r \frac{\partial}{\partial s} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ &\quad \left. + \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) - \frac{\cos \alpha}{r} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) \right]; \\ G_2 &= D_{12} \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) + D_{22} \left[\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ &\quad \left. + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right] + D_{23} \left[r \frac{\partial}{\partial s} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ &\quad \left. + \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) - \frac{\cos \alpha}{r} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) \right]; \\ H &= D_{13} \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) + D_{23} \left[\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \end{aligned} \right\} \quad (348)$$

$$\left. \begin{aligned} & + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right] + D_{22} \left[r \frac{\partial}{\partial s} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \\ & \left. + \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) - \frac{\cos \alpha}{r} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) \right]. \end{aligned} \right\} \quad (348)$$

Correspondingly, for orthotropic shells of rotation, the principal axes of anisotropy of which coincide with the coordinate axes,

$$\left. \begin{aligned} G_1 &= D_1 \left\{ \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) + \nu_2 \left[\frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \right. \right. \\ & \quad \left. \left. + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right] \right\}; \\ G_2 &= D_2 \left\{ \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \frac{\cos \alpha}{r} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) + \right. \\ & \quad \left. + \nu_1 \frac{\partial}{\partial s} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) \right\}; \\ H &= D_3 \left[r \frac{\partial}{\partial s} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) + \frac{1}{r} \frac{\partial}{\partial \beta} \left(\frac{\partial w}{\partial s} - \frac{u}{R_1} \right) - \right. \\ & \quad \left. - \frac{\cos \alpha}{r} \left(\frac{1}{r} \frac{\partial w}{\partial \beta} - \frac{v}{R_2} \right) \right]. \end{aligned} \right\} \quad (349)$$

From the last two of equilibrium Eq. (17), cutting forces $Q_1, Q_2, \underline{/108}$ which were eliminated in equilibrium Eq. (342), can be determined

$$\left. \begin{aligned} rQ_1 &= \frac{\partial (rG_1)}{\partial s} + \frac{\partial H}{\partial \beta} - G_2 \cos \alpha; \\ rQ_2 &= \frac{\partial G_2}{\partial \beta} + \frac{\partial (rH)}{\partial s} + H \cos \alpha. \end{aligned} \right\} \quad (350)$$

System of Eq. (348), (350) or (349), (350) are supplementary in membrane theory, and they are used only for checking the possibilities of its use, namely, if it turns out that the bending stresses actually are negligibly small compared with the membrane stresses, i.e., they are uniformly distributed in the thickness of the membrane, this is confirmation of the applicability of membrane theory.

In some cases, it can be foreseen that membrane theory cannot sufficiently well describe the axisymmetric stressed and deformed states of a shell of rotation. This will occur in those cases when there is a break in continuity of geometric dimensions δ, R_1, R_2 , rigidity characteristics c_{ij} , including rigid fastening or other kinematic connections or, finally, there are areas of discontinuity of external surface load X, Y, Z .

The condition of the presence of areas of slight disturbance of the geometric, elastic or strength parameters can be replaced by more general ones, namely, for inapplicability of membrane theory, it is sufficient that the abovementioned parameters have a large index of variability.

In areas of a shell where there are such features, additional stresses can develop, which cause local bending of the mean surface of the shell. Exact solutions show that areas of bending stresses are extremely small and, consequently, at some distance from such areas, shell calculations can be carried out according to membrane theory.

Thus, membrane theory is described by systems of Eq. (342), (344) for shells with random anisotropy and for orthotropic shells, the principal axes of anisotropy of which do not coincide with the coordinate axes, and by system of Eq. (342), (345) for orthotropic shells, the principal axes of anisotropy of which coincide with the coordinate axes.

In accordance with Eq. (22), the boundary conditions of membrane theory have the form:

$$1. \text{ unsupported edge } T=S=0; \quad (351)$$

$$2. \text{ rigidly fastened edge } u=v=0; \quad (352)$$

$$3. \text{ hinge supported edge } T_1=v=0 \text{ or } u=S=0. \quad (353)$$

It follows from boundary conditions (351)-(353) that membrane theory is applicable in the event the shell is not loaded by cutting forces and moments on the edges, since end effects, i.e., local bending of the shell, will develop on the edges. /109

35. Membrane Theory of Symmetrically Loaded Shells of Rotation

If a shell of rotation is loaded symmetrically about the axis of rotation, the surface loading components should be functions of arc s alone, i.e., they should not depend on angle β :

$$X=X(s); \quad Y=Y(s); \quad Z=Z(s). \quad (354)$$

Since all geometric parameters R_1, R_2, r of shells of rotation depend on arc s , the elastic forces and displacements also are functions of coordinate s alone and, consequently, the equations of equilibrium of a symmetrically loaded shell of rotation have the form

$$\left. \begin{aligned} r \frac{dT_1}{ds} - (T_2 - T_1) \cos \alpha &= -Xr; \\ r \frac{dS}{ds} + 2S \cos \alpha &= -rY; \\ \frac{T_1}{R_1} + \frac{T_2}{R_2} &= Z. \end{aligned} \right\} \quad (355)$$

If, following V.V. Novozhilov [21], stress functions are introduced,

$$\Phi(s) = T_1 r \sin \alpha; \quad \Psi(s) = Sr^2; \quad (356)$$

the following can be obtained from the first two of Eq. (355)

$$\left. \begin{aligned} \frac{d\Phi}{ds} &= r(Z \cos \alpha - X \sin \alpha); \\ \frac{d\Psi}{ds} &= -r^2 Y. \end{aligned} \right\} \quad (357)$$

Consequently, the stress functions are determined by the expressions

$$\Phi(s) = \int_0^s r(Z \cos \alpha - X \sin \alpha) ds + \Phi_0; \quad (358)$$

$$\Psi(s) = -\int_0^s r^2 Y ds + \Psi_0. \quad (359)$$

By determining force T_2 from the last equation of equilibrium, in accordance with Eq. (356), (358), (359), we obtain

$$T_1 = \frac{\Phi}{r \sin \alpha}; \quad T_2 = R_1 \left(Z - \frac{T_1}{R_1} \right); \quad S = -\frac{\Psi}{r^2}. \quad (360)$$

According to membrane theory, the normal, tangential and shearing stresses are determined by the simple expressions /110

$$\left. \begin{aligned} \sigma_1^0 &= \frac{T_1}{\delta}; \quad \sigma_2^0 = \frac{T_2}{\delta}; \quad \tau_0 = \frac{S}{\delta}; \\ \tau_1^0 &= \frac{z}{\delta} (X^+ + X^-) + \frac{X^+ - X^-}{2}; \\ \tau_2^0 &= \frac{z}{\delta} (Y^+ + Y^-) + \frac{Y^+ - Y^-}{2}, \end{aligned} \right\} \quad (361)$$

where X^+ , X^- , Y^+ , Y^- are components of the external surface load applied to the upper ($z=\delta/2$) and lower ($z=-\delta/2$) bounding surfaces of the shell, respectively.

As was noted in the preceding section, the expressions for the elastic forces of an anisotropic laminated shell coincide with the corresponding for an isotropic shell.

In accordance with Eq. (344), movements of a symmetrically loaded shell of rotation are determined by the following system of linear differential equations

$$\left. \begin{aligned} \frac{du}{ds} + \frac{w}{R_1} &= \frac{1}{\delta} (a_{11}T_1 + a_{12}T_2 + a_{13}S); \\ \frac{u}{r} \cos \alpha + \frac{w}{R_2} &= \frac{1}{\delta} (a_{12}T_1 + a_{22}T_2 + a_{23}S); \\ \frac{dv}{ds} - \frac{v}{r} \cos \alpha &= \frac{1}{\delta} (a_{13}T_1 + a_{23}T_2 + a_{33}S). \end{aligned} \right\} \quad (362)$$

System of Eq. (362) is equivalent to the following:

$$w = \frac{R_1}{\delta} [(a_{11} - a_{12} Q) T_1 + a_{13} S + a_{12} R_2 Z] - R_1 \frac{du}{ds}; \quad (363)$$

$$\left. \begin{aligned} \frac{du}{ds} - Q \frac{\cos \alpha}{r} u &= (a_{11} - 2a_{12} Q + a_{22} Q^2) \frac{T_1}{\delta} + \\ &+ (a_{13} - a_{23} Q) S + (a_{12} - a_{22} Q) \frac{R_2}{\delta} Z; \\ \frac{dv}{ds} - \frac{\cos \alpha}{r} v &= (a_{12} - a_{22} Q) \frac{T_1}{\delta} + a_{23} \frac{S}{\delta} + a_{22} R_2 \frac{Z}{\delta}, \end{aligned} \right\} \quad (364)$$

where

$$Q = \frac{R_2}{R_1}. \quad (365)$$

If, as in determination of stresses, the deformation functions

$$\xi = \frac{u}{\sin \alpha}; \quad \eta = \frac{v}{r}, \quad (366)$$

are introduced, from Eq. (364), it is easy to obtain

$$\left. \begin{aligned} \frac{d\xi}{ds} &= \frac{1}{\delta \sin \alpha} [(a_{11} - 2a_{12} Q + a_{22} Q^2) T_1 + \\ &+ (a_{13} - a_{23} Q) S + (a_{12} - a_{22} Q) R_2 Z]; \\ \frac{d\eta}{ds} &= \frac{1}{r \delta} [(a_{12} - a_{22} Q) T_1 + a_{23} S + a_{22} R_2 Z], \end{aligned} \right\} \quad (367)$$

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whence

$$\left. \begin{aligned} \xi &= \frac{1}{\delta} \int_{s_0}^s [(a_{11} - 2a_{12} Q + a_{22} Q^2) T_1 + (a_{13} - a_{23} Q) S + \\ &+ (a_{12} - a_{22} Q) R_2 Z] \frac{ds}{\sin \alpha} + \psi_0; \\ \eta &= \frac{1}{\delta} \int_{s_0}^s [(a_{12} - a_{22} Q) \frac{T_1}{r} + a_{23} \frac{S}{r} + a_{22} \frac{R_2}{r} Z] ds + \psi_0; \end{aligned} \right\} \quad (368)$$

where ϕ_0 , ψ_0 are the values of the deformation functions at the edge of the shell.

In this manner, the components of movement of an anisotropic symmetrically loaded shell of rotation are determined by the following expressions

$$\left. \begin{aligned} u &= \xi \sin \alpha; \quad v = r \eta; \\ w &= -\xi \cos \alpha + \frac{R_1}{\delta} [(a_{12} - a_{22} Q) T_1 + a_{23} S + a_{22} R_2 Z]. \end{aligned} \right\} \quad (369)$$

As should be expected, in distinction from isotropic shells, with any boundary conditions, each movement depends on all three components of external surface load X, Y, Z. The nature of the stressed state depends essentially on the boundary conditions, namely, only with static indeterminate boundary conditions will the forces depend on all three components of the external load.

Eq. (369) also determine movements of orthotropic shells of rotation, the principal directions of anisotropy of which do not coincide

with the coordinate axes.

In the case of orthotropic shells of rotation, the principal directions of anisotropy of which coincide with the coordinate axes, the expressions for the elastic forces remain as before, and the movement functions are determined by the formulas

$$\left. \begin{aligned} \xi &= \frac{1}{\delta} \int_{s_0}^s \left[\left(\frac{1}{E_1} + \frac{2\nu_1}{E_1} \nu + \frac{Q^2}{E_1} \right) T_1 - \right. \\ &\quad \left. - \left(\frac{\nu_1}{E_1} + \frac{Q}{E_2} \right) R_2 Z \right] \frac{ds}{\sin \alpha} + \xi_0; \\ \eta &= \frac{a_{20}}{\delta} \int_{s_0}^s \frac{S}{r} ds + \eta_0; \end{aligned} \right\} \quad (370)$$

$$\left. \begin{aligned} u &= \xi \sin \alpha; \quad v = \eta r; \\ w &= -\xi \cos \alpha + \frac{R_2}{\delta} \left[\frac{R_2 Z}{E_2} - \left(\frac{\nu_1}{E_1} + \frac{Q}{E_2} \right) T_1 \right]. \end{aligned} \right\} \quad (371)$$

In this case, movements u , w are caused by the radial and meridional components of the external load Z , X , and displacement v is caused by annular forces Y , i.e., the same as in isotropic shells of rotation. /112

In this manner, calculation of symmetrically loaded anisotropic and orthotropic shells of rotation is reduced to determination of four random integration constants ϕ_0 , ψ_0 , ϕ_1 , ψ_1 . Consequently, on each edge of the shell $s=s_0$; $s=s_1$, for an unambiguous solution, two boundary conditions each must be assigned. In this case, at least two of the boundary conditions should be kinematic. Otherwise, the existence of the stressed membrane state will be impossible, i.e., bending of the mean surface of the shell without stretching (compressing) or shearing will occur, or displacement of the shell as a solid will be possible.

We now consider some examples of calculation of symmetrically loaded shells of rotation according to membrane theory.

36. Calculation of Closed Containers Operating under Constant Internal Pressure

Shells of rotation in the form of cylindrical and conical shells closed by end plates of different geometric shape and of spherical and toroidal containers are exceptionally widely used in industry. Particularly in chemical equipment, these shells operate under uniform internal pressure. Such structures are calculated according to membrane theory, with the exception of small end effect areas, where more exact equations, which will be obtained later, must be used for the calculation. In such zones, special design measures must be used to moderate stress concentrations and more uniformly distribute the stress.

In shells of rotation subjected to uniform internal pressure, $X=Y=0$, $Z=p$, where $p=\text{const}$ is the intensity of the internal pressure.

In accordance with Eq. (358), the stress function

$$\Phi = p \int_{\alpha_0}^{\alpha} R_2 \sin \alpha d(R_2 \sin \alpha) + \Phi_0 = \frac{p R_2^2 \sin^2 \alpha}{2} + C. \quad (372)$$

Consequently, the elastic forces are determined by the expressions /113

$$\left. \begin{aligned} T_1 &= \frac{p R_2}{2} - \frac{C}{R_2 \sin^2 \alpha}; \\ T_2 &= \frac{p R_2}{2} \left(2 - \frac{R_2}{R_1} \right) - \frac{C}{R_1 \sin^2 \alpha}. \end{aligned} \right\} \quad (373)$$

It is evident that, for shells of rotation closed at the top, $C=0$ must be set and, consequently, for such shells

$$T_1 = \frac{p R_2}{2}; \quad T_2 = \frac{p R_2}{2} \left(2 - \frac{R_2}{R_1} \right). \quad (374)$$

The deformation functions for closed shells of rotation, according to Eq. (370),

$$\xi = \frac{p}{2E_1 \delta} \int_{\alpha_0}^{\alpha} \left[1 - 2\nu_1 - \frac{2\nu_1(1-\nu_2)}{\nu_2} \varrho + \frac{\nu_1}{\nu_2} \varrho^2 \right] \frac{R_1 R_2 d\alpha}{\sin \alpha} + \xi_0. \quad (375)$$

and the components of movement

$$\left. \begin{aligned} u &= \frac{p \sin \alpha}{2E_1 \delta} \int_{\alpha_0}^{\alpha} \left[1 - 2\nu_1 - \frac{2\nu_1(1-\nu_2)}{\nu_2} \varrho + \frac{\nu_1}{\nu_2} \varrho^2 \right] \times \\ &\quad \times \frac{R_1 R_2 d\alpha}{\sin \alpha} + \xi_0 \sin \alpha; \\ w &= -\frac{p \cos \alpha}{2E_1 \delta} \int_{\alpha_0}^{\alpha} \left[1 - 2\nu_1 - \frac{2\nu_1(1-\nu_2)}{\nu_2} \varrho + \frac{\nu_1}{\nu_2} \varrho^2 \right] \times \\ &\quad \times \frac{R_1 R_2 d\alpha}{\sin \alpha} - \xi_0 \cos \alpha + \frac{p R_2^2}{2E_2 \delta} (2 - \nu_2 - \varrho). \end{aligned} \right\} \quad (376)$$

Consequently, radial movement of the shell, i.e., movement perpendicular to the axis of rotation, is determined by the expression

$$\zeta = \Delta r = u \cos \alpha + w \sin \alpha = \frac{p R_2^2}{2E_2 \delta} (2 - \nu_2 - \varrho). \quad (377)$$

Correspondingly, the angle of rotation of a meridional element

$$\chi_0 = \frac{pR_2 \cos \alpha}{2E_s \delta \sin \alpha} \left[\varrho^2 + 2(1 + \nu_2)\varrho + 4 - \frac{\nu_2}{\nu_1} \right] - \frac{pR_2^2}{2E_s \delta} \frac{d\varrho}{ds}. \quad (378)$$

a. Spherical container or spherical bottom (Fig. 38). In the case of a spherical container

$$R_1 = R_2 = a$$

and, consequently, the forces generated in the shell

$$T_1 = T_2 = \frac{pa}{2}. \quad (379)$$

The radial deformation and angular displacement of an element of the meridian are determined by the respective expressions

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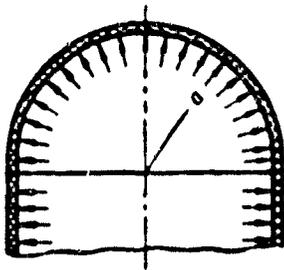


Fig. 38. Hemispherical bottom.

$$\zeta_0 = \frac{pa^2(1 - \nu_2)}{2E_s \delta}; \quad \chi_0 = \frac{pa \operatorname{ctg} \alpha}{2E_s \delta} \left(7 + 2\nu_2 - \frac{\nu_2}{\nu_1} \right). \quad (380)$$

b. Closed cylindrical container (Fig. 39). In this case, $R_1 \rightarrow \infty$; $\rho = 0$; $R_2 = r = a$ and, consequently, according to Eq. (374)

$$T_1 = \frac{pa}{2}; \quad T_2 = pa. \quad (381)$$

Correspondingly, the radial deformation and angular displacement

$$\zeta_0 = \frac{pa^2(2 - \nu_2)}{2E_s \delta}; \quad \chi_0 = 0. \quad (382)$$

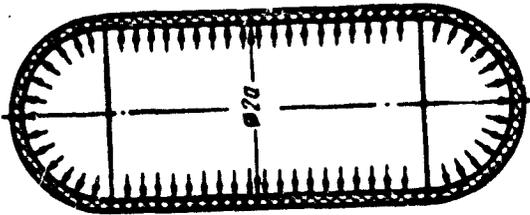


Fig. 39. Closed cylindrical container.

c. Ellipsoidal bottom (Fig. 40). In this case

$$e = \frac{a^2}{b^2} - 1.$$

The radii of curvature of an ellipsoidal bottom are determined by the known expressions

$$R_1 = \frac{a\sqrt{1+e}}{\sqrt{(1+e)\sin^2 \alpha}}; \quad R_2 = \frac{a\sqrt{1+e}}{\sqrt{1+e\sin^2 \alpha}}. \quad (383)$$

Fig. 40. Ellipsoidal bottom.

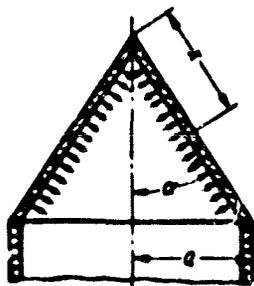
Consequently, the elastic forces and deformations

$$\left. \begin{aligned}
 T_1 &= \frac{pa}{2} \sqrt{\frac{1+e}{1+e \sin^2 \alpha}}; \\
 T_2 &= \frac{pa}{2} (1 - e \sin^2 \alpha) \sqrt{\frac{1+e}{1+e \sin^2 \alpha}}; \\
 e &= \frac{R_2}{R_1} = 1 + e \sin^2 \alpha; \\
 \zeta_0 &= \frac{pa^2 (1+e)}{2E_1 \delta (1+e \sin^2 \alpha)} (1 - \nu_2 - e \sin^2 \alpha); \\
 \chi_0 &= \frac{pa}{2E_1 \delta} \sqrt{\frac{1+e}{1+e \sin^2 \alpha}} \frac{\cos \alpha}{\sin \alpha} \times \\
 &\times \left[e^2 + 2(1 + \nu_2)e + 4 - \frac{\nu_2}{\nu_1} - 2qe \sin^2 \alpha \right].
 \end{aligned} \right\} (384)$$

d. Coneal bottom (Fig. 41). In this case

$$R_1 = \infty; \quad R_2 = x \operatorname{tg} \gamma; \quad \rho = 0.$$

If distance x is reckoned along the generatrix of the cone from the top, it is easy to obtain



$$\left. \begin{aligned}
 T_1 &= \frac{px \operatorname{tg} \gamma}{2}; \quad T_2 = px \operatorname{tg} \gamma; \\
 \zeta_0 &= \frac{px^2 \operatorname{tg}^2 \gamma}{2E_1 \delta} (2 - \nu_2); \\
 \chi_0 &= \frac{px \operatorname{tg}^2 \gamma}{2E_1 \delta} \left(4 - \frac{\nu_2}{\nu_1} \right).
 \end{aligned} \right\} (385)$$

Fig. 41. Conical bottom.

e. Toroidal container (Fig. 42). In this case

$$R_2 = \frac{R+r \sin \alpha}{\sin \alpha}; \quad R_1 = r.$$

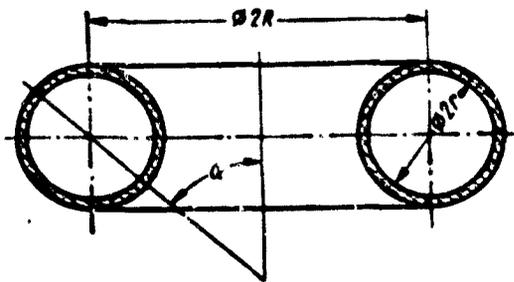


Fig. 42. Closed toroidal container.

A toroidal reservoir is not closed at the top. Consequently, the forces are determined by Eq. (373), and the value of constant C must be determined.

From the conditions of equilibrium of an element of a torus cut along a plane passing through the curvilinear axis of the torus, and of a cylindrical surface passing through the curvilinear axis, we find

$$(T_1)_{\alpha = \frac{\pi}{2}} = \frac{pr}{2} \frac{(2R+r)}{R+r}; \quad (386)$$

and, consequently, the desired constant is $C = -pR^2/2$. Thus,

$$T_1 = \frac{pr}{2} \frac{2R + r \sin \alpha}{R + r \sin \alpha}; \quad T_2 = \frac{pr}{2}. \quad (387)$$

The respective radial and angular displacements

$$\zeta_0 = \frac{pR^3}{2E_1 \delta} (2 - \nu_1 - \epsilon);$$

$$\chi_0 = \frac{pR_0 \epsilon \tan \alpha}{2E_1 \delta} \left[\epsilon^3 + 2(1 - \nu_1)\epsilon + 4 - \frac{\nu_1}{\nu_1} + \frac{R_0 \epsilon}{r \sin \alpha} \right]. \quad (388)$$

Further, we consider some problems in determination of efficient bottom parameters, which ensure strength with the least possible weight.

37. Some Problems Connected with Determination of Parameters of Least Weight Bottom

Let a cylindrical container of radius a be subjected to internal pressure of intensity p . Spherical and conical bottoms of the least possible weight must be selected.

Since the problem is solved from the point of view of membrane theory of shells, some simplifying assumptions must be made. Namely, we will assume that, in uneven joining of the bottom with the cylindrical part of the container, the resulting thrust is taken up by a reinforcing ring which is mounted in the butt section. We designate the permissible yield strength of the material by σ and the specific weight of the material by γ .

a. Spherical bottom (Fig. 43). In this case

$$R = \frac{a}{\sin \alpha}.$$

Since the stresses are uniform in a spherical shell, the bottom should be made of constant thickness and provide isotropic structure of the laminated plastic.

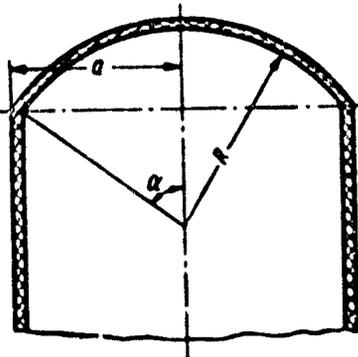


Fig. 43. Spherical bottom.

The weight of the bottom is determined by the formula

$$G_{sh} = 2\pi\gamma(1 - \cos \alpha) R^2 \delta.$$

The required bottom thickness

$$\delta = \frac{pR}{2\sigma}.$$

Consequently, with bottom aperture angle α , the weight of the bottom shell

$$G_{sh} = \frac{\pi p \gamma a^3}{\sigma} \frac{1 - \cos \alpha}{\sin^2 \alpha}. \quad (389)$$

The required weight of the reinforcing ring in the butt section

$$G_r = \frac{\pi p \gamma_{\text{лп}} a^3}{\sigma_{\text{лп}}} \frac{\cos \alpha}{\sin^2 \alpha} \quad (390)$$

The total weight of the spherical bottom shell with the reinforcing ring

$$G = \frac{\pi p a^3 \gamma}{\sigma} \cdot \frac{1 + (k-1) \cos \alpha}{\sin^2 \alpha} \quad (391)$$

where

$$k = \frac{\left(\frac{\sigma}{\gamma}\right)_{\text{ш}}}{\left(\frac{\sigma}{\gamma}\right)_r} \quad (392)$$

Fig. 44 presents the results of calculation of the weight of the bottom as a function of the relative specific strength of the shell and ring material k and of aperture angle α of the spherical bottom.

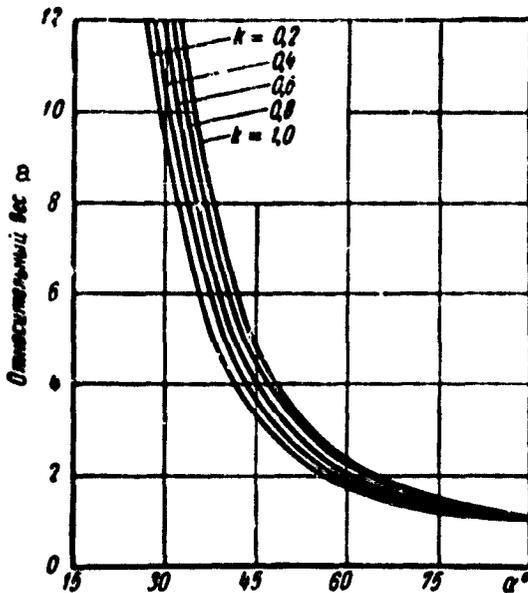


Fig. 44. Spherical bottom weight vs. aperture angle and relative specific strength of plastic.

Key: a. Relative weight

Eq. (391) and the calculation results show that it is advisable to reinforce the butt section of the bottom with a ring of a material, the specific strength of which is considerably greater than the specific strength of the laminated plastic. The maximum possible weight advantage over a hemispherical bottom reaches 28%. In reinforcing with the same material, because of the unidirectional nature of the laminated plastic, the relative specific strength $k=0.5$.

b. Constant thickness conical bottom (Fig. 45). The shell weight of a conical bottom of constant thickness /118

$$G_{\text{ш}} = \frac{\pi p \gamma a^3}{\sigma} \frac{2}{\sin 2\alpha}; \quad (393)$$

the required weight of the reinforcing ring

$$G_r = \frac{\pi p \gamma_r a^3}{\sigma_r} \operatorname{tg} \alpha. \quad (394)$$

Consequently, the total weight of the conical bottom shell and reinforcing ring

$$G = \frac{\pi p \sigma^2 \gamma}{\sigma} \left[\frac{2}{\sin \alpha} + k \operatorname{tg} \alpha \right]. \quad (395)$$

The results of calculation of the bottom weight by Eq. (395) are presented in Fig. 46 as a function of k and the half aperture angle of the cone.

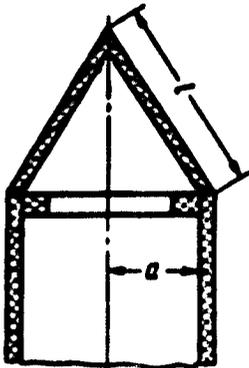


Fig. 45. Conical bottom with reinforcing ring.

The minimum weight advantage of the use of a conical bottom over a hemispherical bottom is 50%, if it is considered that the specific strength of a conical bottom made of uniform oriented laminated plastic is greater than the specific strength of a hemispherical bottom. Actually, if the yield strength of a unidirectional plastic is σ , the yield strength of a full strength plastic is $\sigma/2$, and the yield strength of the plastic of a conical bottom is $2/3\sigma$.

c. Variable thickness conical bottom. In the use of laminated plastics for manufacture of bottoms, available technological methods of continuous winding permit a variable thickness bottom to be obtained without difficulty.

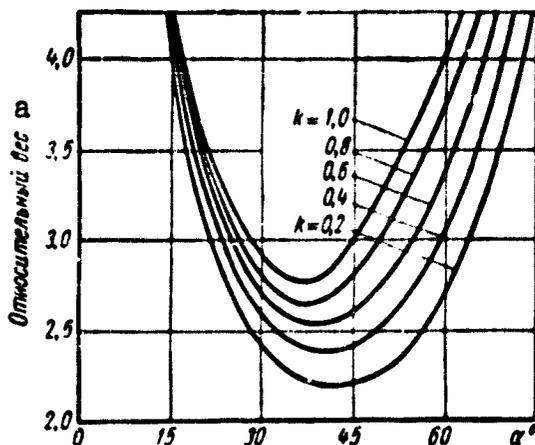


Fig. 46. Constant thickness conical bottom weight vs. half aperture angle and relative specific strength of plastic.

Key: a. Relative weight

Since the stressed state of a conical bottom is variable along the generatrix, it is more advisable to make a variable thickness conical bottom. Evidently, the relationship of change in thickness of the bottom is the following

$$\delta(x) = \frac{p x \operatorname{tg} \alpha}{\sigma}. \quad (396)$$

Consequently, the weight of the bottom shell

$$G_{sh} = \frac{\pi p \gamma a^2}{\sigma} \frac{4}{3 \sin 2\alpha}. \quad (397)$$

Since the weight of the reinforcing ring remains the same, the total weight of a variable thickness conical bottom

$$G = \frac{\pi p \gamma a^2}{\sigma} \left(\frac{4}{3 \sin 2\alpha} + k \operatorname{tg} \alpha \right). \quad (398)$$

The results of calculation of the bottom weight by Eq. (398) is presented in Fig. 47 as a function of cone half aperture angle α and k .

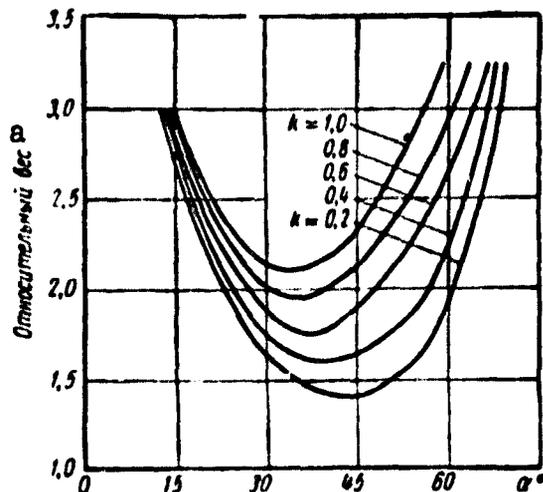


Fig. 47. Variable thickness conical bottom weight vs. half aperture angle and relative specific strength of plastic.

Key: a. Relative weight

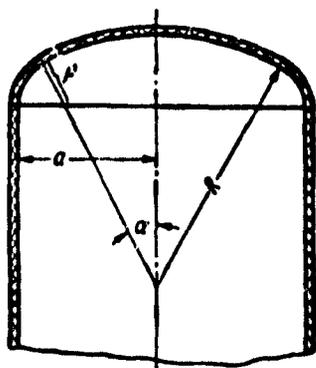


Fig. 48. Box bottom.

It is easy to note that the weight of the bottom decreases with decrease in ϵ and, consequently, the smallest joint radius based on design or other considerations must be used.

The maximum weight advantage will be at $\alpha=60^\circ$ and $\epsilon \rightarrow 0$, i.e., the toroidal part of the bottom, by ensuring even joining, replaces the ring, as it were. The weight advantage is 23% over a hemispherical bottom.

The results of calculation of the box bottom weight as a function of ϵ and α are presented in Fig. 49.

The minimum weight advantage of the use of a variable thickness conical bottom over a hemispherical bottom with $k=0$ is 33%. If it is considered that the specific strength of the conical bottom is greater than the specific strength of a hemispherical bottom, with $k=0$, the minimum possible weight of a variable thickness conical bottom equals the weight of the hemispherical bottom. This could have been expected beforehand, since both bottoms are full strength. /120

d. Box bottom (Fig. 48). We now consider a box bottom, obtained by even joining of the spherical part with the cylindrical part through a toroidal shell.

The weight of the spherical part of the bottom

$$G_{\text{sph}} = \frac{\pi \gamma a^3 \rho}{\sigma} \frac{(1 - \epsilon + \epsilon \sin \alpha)^2 (1 - \cos \alpha)}{\sin^3 \alpha}. \quad (399)$$

The weight of the toroidal part of the bottom

$$G_{\text{tor}} = \frac{\pi \rho \gamma a^3}{\sigma} \left\{ \epsilon (2 - \epsilon) \left[\left(\frac{\pi}{2} - \alpha \right) (1 - \epsilon) + \epsilon \cos \alpha \right] \right\}. \quad (400)$$

Thus, the total weight of the box bottom

$$G = \frac{\pi \rho a^3 \gamma}{\sigma} \left\{ \frac{(1 - \epsilon + \epsilon \sin \alpha)^2 (1 - \cos \alpha)}{\sin^3 \alpha} + \epsilon (2 - \epsilon) \left[\left(\frac{\pi}{2} - \alpha \right) (1 - \epsilon) + \epsilon \cos \alpha \right] \right\}. \quad (401)$$

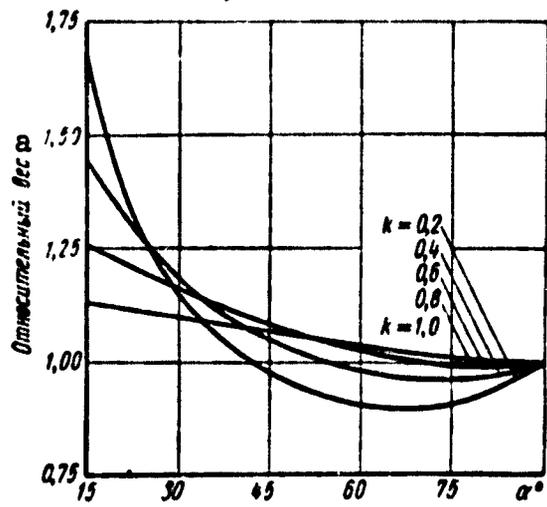


Fig. 49. Box bottom weight vs. geometric parameters.

Key: a. Relative weight

CHAPTER 8. OPTIMUM METHODS OF CONTINUOUS WINDING OF CYLINDRICAL FIBER-GLASS REINFORCED PLASTIC SHELLS

38. Basic Concepts and Initial Hypotheses

One of the most convenient and widespread methods of production of laminated shells is continuous winding. There are various methods of continuous winding, which differ in the method of placement on the mandrel and type of filler, as well as the nature of impregnation of the filler. Rotation of the mandrel combined with forward motion of the carriage with the bobbin along the mandrel permits the most diverse filler orientation to be achieved. After winding, the shell together with the mandrel go through heat treatment, as a result of which hardening of the binder occurs. After heat treatment, the shell is removed from the mandrel. To make removal of the shell from the mandrel easier the latter is covered with a film before winding, which prevents adhesion of the filler.

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For mass production of cylindrical shells and tapes, high capacity coil winders usually are used. One of them is shown in Fig. 50.

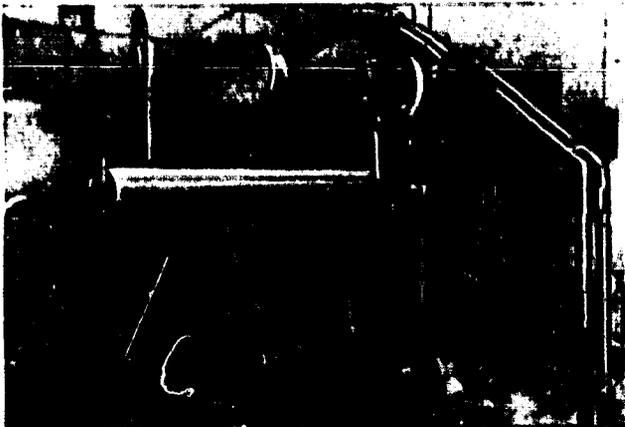


Fig. 50. Unit for continuous winding of cylindrical shells.

Fabric and nonfabric glass filler, in the form of threads, tapes, bands and fabrics are used for continuous winding of fiberglass reinforced plastic shells. Polyester, phenol, epoxy, organosilicon resins and various modifications of them are used as the binders.

in orientation and mutual location of the filler during production. It is expedient to call such anisotropy of the material controllable technological anisotropy, in distinction from structural anisotropy, which is produced by strengthening the shells with stiffening ribs.

Fiberglass reinforced plastic cylindrical shells obtained by continuous winding are anisotropic laminated materials. In distinction from natural anisotropic materials, the nature of the anisotropy of the fiberglass reinforced plastics and other reinforcing plastics can be regulated by change

The most efficient reinforced plastic structures are those in which anisotropy of the elastic properties most profitably corresponds to the stressed state of the shell or ensures its maximum rigidity with respect to a given load. Determination of the optimum structure of laminated plastics in various structures presents interesting new problems of the theory of elasticity and the theory of shells.

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This chapter discusses the problem of the selection of optimum structure of a fiberglass reinforced plastic in a cylindrical shell

which is exposed to axisymmetric loads, which produce a uniform state of stress in it, determined by the components of the normal forces T_1 , T_2 .

It is assumed that the binder has ideal elastic, strength and adhesive properties, which ensure compatibility of the deformation of individual glass filler elements all the way to failure. The glass filler is considered in the form of circular cross section fibers, which does not restrict the generality of the results obtained.

It is easy to show that, in elastic deformations, forces T_1 , T_2 will be taken up by the filler and binder in proportion to their moduli of elasticity E_H , E_c and their volumetric content in the material.

If the relative volumetric content of binder is designated ζ , the relative fraction of the forces which are taken up by the binder is determined by the expression

$$q = \frac{E_c \zeta}{E_H (1 - \zeta)}. \quad (402)$$

The moduli of elasticity of available resins change between $3 \cdot 10^2$ and $7 \cdot 10^2$ n/m², and the modulus of elasticity of glass $E_H = 7 \cdot 10^3$ n/m². The optimum content of binder in fiberglass reinforced plastic is approximately 30% and, consequently, by Eq. (402), $q = 2-4\%$.

Thus, the normal and shearing forces in fiberglass reinforced plastic shells are primarily taken up by the glass filler. This determines the carrying capacity of the structure. Based on this, we will assume that the effective load on the shell is taken up by the glass filler.

We will call continuous winding the optimum if it ensures equilibrium of the glass filler without the binder. It should be noted that, in nonlinear deformations in the binder and in plastic deformations of the material at the time of failure, the fraction of the load which is taken up by the binder decreases sharply. Therefore, selection of the optimum winding general speaking is of decisive importance for increasing the carrying capacity of a shell. We will call a shell composed of fibers alone the basic system.

Since actual resins which are used as binders in the manufacture of shells have various properties, they provide compatibility of deformation of the glass filler in different ways, and this explains the results of studies in which a significant effect of the binder on the elastic and strength properties of fiberglass reinforced plastics was found. Actually, with slight adhesion of the binder to the glass filler, the distribution of forces through the shell will be irregular. This leads to both premature destruction of the filler in the most stressed fibers and to overstress and failure of the binder, i.e., an increase in irregularity of distribution of the forces and subsequent reduction of carrying capacity of the shell.

If the axial force in a cylindrical shell is T_1 and the annular force is $(1+\alpha) T_1$, the normal forces which act on a surface which is located at angle θ to the generatrix of the shell is determined by the expression

$$T_\theta = T_1 (1 + \alpha \cos^2 \theta). \quad (403)$$

Let glass fibers be wound on a cylindrical mandrel at angle ϕ to the generatrix (Fig. 51).

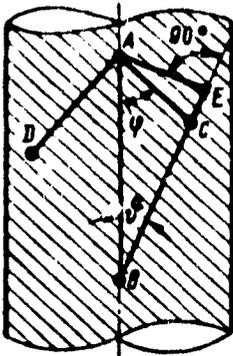


Fig. 51. Winding diagram and conventional symbols.

It is easy to determine that the same number of fibers passes through segments AB, AD and BC (Fig. 51). If the length of a segment of the generatrix $AB=a$, the length of a segment perpendicular to the fiber direction equals $a \sin \phi$. Consequently, $na \sin \phi$ fibers pass through the segments indicated above, where n is the glass fiber packing density, i.e., the number of fibers passing through a unit segment perpendicular to the fibers.

The angle between the normal to area BC dropped from point A ($AE \perp BE$), and the length of segment BC

$$\alpha = \frac{\pi}{2} - (\theta + \phi); \quad x = |BC| = \frac{a \sin \phi}{\sin (\theta + \phi)}. \quad (404)$$

Consequently, the normal force which arises in area BC in stretching of the fibers by force f is determined from the expression

$$T_\theta = \frac{fna \sin \phi \cos \theta}{x} = fn \sin^2 (\theta + \phi). \quad (405)$$

According to Eq. (405), the distribution of normal forces in a cylindrical shell depends essentially on the orientation of the glass filler during winding, i.e., on angle ϕ .

According to Eq. (405), winding of the fibers at one constant angle ϕ does not ensure equilibrium of the basic system. We will subsequently assume Eq. (405) to be written for the limiting state of the shell, i.e., we will assume fiber tension f to be equal to the breaking force of the fiber. For convenience in use, it is advisable to subsequently present Eq. (405) in the form

$$T_\theta = \frac{fn}{2} [1 + \cos 2\phi - 2 \cos 2\phi \cos^2 \theta + \sin 2\phi \sin 2\theta]. \quad (406)$$

39. Continuous Winding of Cylindrical Shells with Unidirectional Glass Fillers at Optimum Angles to Shell Generatrix

Since winding a cylindrical shell at one constant angle does not

ensure equilibrium of the basic system, we consider the case when the shell is wound with two glass fiber systems at angles ϕ_1 and ϕ_2 to the shell generatrix (Fig. 52).

We initially study the simplest case, when each layer of one system of fibers corresponds to one layer of the other system. According to Eq. (406), the force on the area θ is presented in the form

$$T_\theta = \frac{fn}{2} (2 + \cos 2\phi_1 + \cos 2\phi_2 - 2(\cos 2\phi_1 + \cos 2\phi_2) \cos^2 \theta + (\sin 2\phi_1 + \sin 2\phi_2) \sin 2\theta). \quad (407)$$

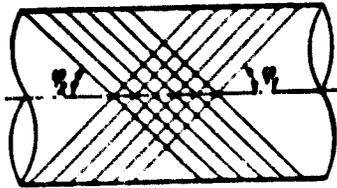


Fig. 52. Bias cross winding.

The equilibrium conditions of the glass fibers have the form

$$\left. \begin{aligned} \cos 2\phi_1 + \cos 2\phi_2 &= -\frac{2\alpha}{2+\alpha}; \\ \sin 2\phi_1 + \sin 2\phi_2 &= 0. \end{aligned} \right\} \quad (408)$$

From Eq. (408), it is easy to find

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$$\phi = \phi_1 = -\phi_2 + m\pi \quad (m = 0, 1, 2, \dots); \quad (409)$$

$$\cos 2\phi = -\frac{\alpha}{2+\alpha}, \quad (410)$$

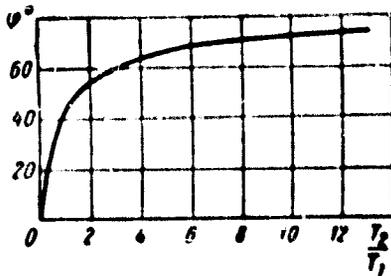


Fig. 53. Optimum winding angles vs. stress of cylindrical shell.

The results of calculation by Eq. (410) are presented in Table 1, and they are illustrated in Fig. 53.

The basic relationship which connects the carrying capacity of the shell with the strength of the glass fibers has the form

$$T_1 = \frac{2}{2+\alpha} fn. \quad (411)$$

We now consider a more general case of continuous winding, when N layers wound at angle ϕ_2 to the generatrix are applied to each layer wound at angle ϕ_1 to the generatrix.

The equilibrium conditions of the fibers have the form

$$\left. \begin{aligned} \cos 2\varphi_1 + N \cos 2\varphi_2 &= -\frac{(N+1)a}{2+a}; \\ \sin 2\varphi_1 + N \sin 2\varphi_2 &= 0. \end{aligned} \right\} \quad (412)$$

TABLE 1. OPTIMUM WINDING ANGLES OF CYLINDRICAL SHELL UNDER AXISYMMETRIC LOAD VS. T_2/T_1

а	Вид нагружения оболочки	T_2/T_1	φ°
б	Осевое растяжение	0	0°
		1	45°
с	Внутреннее давление с осевой силой	2	54° 44'
		3	60°
		4	63° 25'
		5	65° 55'
		9	71° 35'
		∞	90°

Key: a. Type loading of shell
 b. Axial tension
 c. Internal pressure with axial force

Consequently, the optimum winding angles are determined by the following expressions

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$$\left. \begin{aligned} \cos 2\varphi_1 &= \frac{2(a+1)N - (a^2 + 2a + 2)}{(2+a)a}; \\ \cos 2\varphi_2 &= \frac{(a^2 + 2a + 2)N - 2(a+1)}{(2+a)aN}. \end{aligned} \right\} \quad (413)$$

The carrying capacity of the shell

$$T_1 = \frac{(N-1)}{(2+a)} f_n. \quad (414)$$

Determination of the optimum winding angles by Eq. (413) is not difficult in any axisymmetric stress of the shell. Further, we study the cases of loading with uniform internal pressure most frequently encountered in practice.

If a shell operates under uniform internal pressure, the optimum winding angles are determined by the formulas

$$\cos 2\varphi_1 = \frac{4N-5}{3}; \quad \cos 2\varphi_2 = \frac{4-5N}{3N}. \quad (415)$$

It follows from Eq. (415) that

$$1 \leq N \leq 2.$$

(416)

The carrying capacity of the shell

$$pR = \frac{2}{3}(N + 1)/n.$$

(417)

TABLE 2. OPTIMUM WINDING ANGLES OF CYLINDRICAL SHELLS OPERATING UNDER INTERNAL PRESSURE VS. LAYER RATIO N

N	φ_1	φ_2
1	54° 44'	125° 16'
1.1	50° 46'	121° 29'
1.2	46° 54'	118° 08'
1.3	43° 06'	115° 04'
1.4	40° 14'	112° 12'
1.5	35° 16'	109° 28'
1.6	31° 06'	106° 46'
1.7	26° 34'	104° 02'
1.8	21° 28'	101° 06'
1.9	14° 58'	97° 36'
2	0°	90°

The results of calculation of the optimum winding angles by Eq. (415) are presented in Table 2.

N=1 corresponds to bias cross winding; N=2 corresponds to longitudinal-transverse winding, when two layers are wound in the annular direction ($\phi_2=90^\circ$) on one longitudinally laid layer.

The calculation results presented in Table 2 are illustrated in Fig. 54.

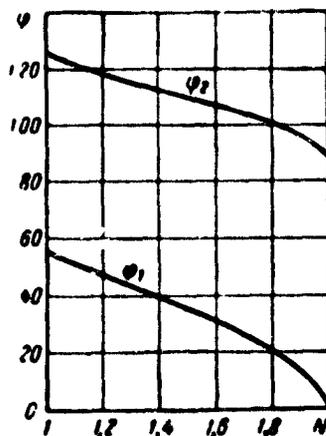


Fig. 54. Optimum winding angles of cylindrical shells operating under internal pressure.

We now compare the yield strengths of shell materials obtained by continuous winding with the same glass fibers and different anisotropy of the strength properties. According to basic relationship (414), which connects the carrying capacity of the shell with the strength of the fibers, the following results can be obtained (Table 3).

40. Optimum Continuous Winding of Cylindrical Shells with Fiberglass Fabrics

Fiberglass fabric is an aggregate of two mutually orthogonal glass fiber systems connected together with varied amounts of interweaving in textile processing. By type and amount of interweaving of the warp and woof fibers, card, satin and serge cloth are distinguished (Fig. 55).

TABLE 3. FIBERGLASS REINFORCED PLASTIC STRENGTH VS. NATURE OF ANISOTROPY

a Тип стеклопластика	b Характер анизотропии	c Пределы прочности в главных направлениях анизотропии	
		σ_{10}	σ_{20}
d Однонаправленный	1:0	f/n	0
e Равнопрочный	1:1	$\frac{1}{2} f/n$	$\frac{1}{2} f/n$
—	2:1	$\frac{2}{3} f/n$	$\frac{1}{3} f/n$
$m > n$	$m:n$	$\frac{m}{m+n} f/n$	$\frac{n}{m+n} f/n$

Key: a. Fiberglass plastic type
 b. Nature of anisotropy
 c. Yield strength in principal directions of anisotropy
 d. Unidirectional
 e. Full strength

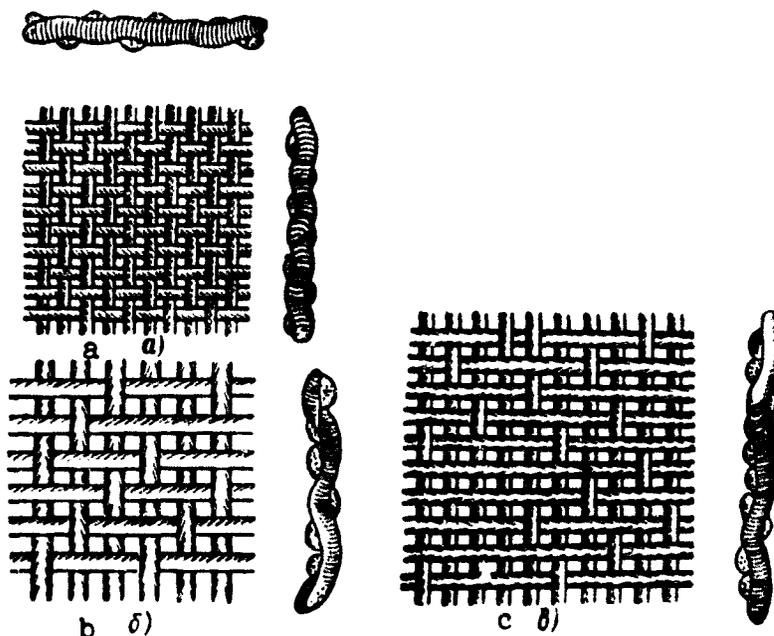


Fig. 55. Fiberglass fabric structure: a. card; b. satin; c. serge.

Satin fabrics have the greatest flexibility, they ensure the highest quality packing in continuous winding, and they have the best capacity for impregnation with binders. Fiberglass reinforced plastics produced from fabrics usually are called fiberglass laminates.

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For convenience, we introduce the following designations:

f_1, f_2 are the breaking forces of the warp and woof fibers;

n_1, n_2 are the packing density of the warp and woof fibers;

$k = f_2 n_2 / f_1 n_1$ is the relative strength of the fabric ($k \geq 1$).

As in winding with unidirectional glass fillers, winding of a cylindrical shell with fabrics at a constant angle does not ensure equilibrium of the basic system, except for straight winding of the warp in the annular or longitudinal directions. In this case, of course, the anisotropy of the fabric should be $k = T_2 / T_1$ or $k = T_1 / T_2$.

We consider the general case of cross winding of a cylindrical shell with two fabric systems, which are laid so that the direction of the warp fabric is at angles ϕ_1, ϕ_2 with the shell generatrix.

Both systems are composed of the same fabric of relative strength k in which, on each layer of the first system, there are N of the second system.

In accordance with Eq. (406), the normal force on area θ

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$$T_0 = \frac{f_1 n_1}{2} [(N+1)(k+1) - (k-1)(\cos 2\varphi_1 + N \cos 2\varphi_2) + \\ + 2(k-1)(\cos 2\varphi_1 + N \cos 2\varphi_2) \cos^2 \theta - \\ - (k-1)[\sin 2\varphi_1 + N \sin 2\varphi_2] \sin 2\theta]. \quad (418)$$

Consequently, the equilibrium conditions of the basic system are written in the form

$$\left. \begin{aligned} \cos 2\varphi_1 + N \cos 2\varphi_2 &= \frac{(N+1)(k+1)a}{(k-1)(2+a)}; \\ \sin 2\varphi_1 + N \sin 2\varphi_2 &= 0. \end{aligned} \right\} \quad (419)$$

The relationship which connects the carrying capacity of the reinforced shell with the strength of the fibers has the form

$$T_1 = \frac{f_1 n_1}{2+a} (N+1)(k+1). \quad (420)$$

The following expressions for determination of the optimum winding angles can be obtained from Eq. (419):

$$\left. \begin{aligned} \cos 2\varphi_1 &= \\ &= - \frac{(2+a)^2 (k-1)^2 N^2 - (k+1)^2 a^2 (N+1)^2 - (k-1)^2 (2+a)^2}{2(N+1)(k^2-1)a(2+a)}; \\ \cos 2\varphi_2 &= \\ &= \frac{(k-1)^2 (2+a)^2 N^2 + (k+1)^2 a^2 (N+1)^2 - (k-1)^2 (2+a)^2}{2N(N+1)(k^2-1)a(2+a)}. \end{aligned} \right\} \quad (421)$$

Since it is quite complex to study Eq. (421) in the general case of loading a cylindrical shell, we consider the case of loading a shell with uniform internal pressure ($\alpha=1$) in greater detail, for cross wind-

ing $N=1$.

In this partial case, $\phi_1 = -\phi_2 = \phi$

$$\cos 2\varphi = \frac{(k+1)}{3(k-1)}; \quad (422)$$

it follows from this formula that, for manufacture of fiberglass fabric cylindrical shells operating under uniform internal pressure, fiberglass fabrics must be used which satisfy the condition

$$0 < k < \frac{1}{2}, \quad (423)$$

i.e., the fabric strength along the woof should not be greater than half the fabric strength along the warp. Otherwise, the excess strength of the fabric along the woof cannot be used, even in bias winding. /130

The results of the calculation by Eq. (422) are presented in Table 4 and are depicted in Fig. 56.

TABLE 4. OPTIMUM CROSS WINDING ANGLES OF FIBERGLASS FABRICS

k	φ	k	φ
0	54° 44'	0.25	61° 48'
0.05	55° 48'	0.30	64° 07'
0.10	57° 07'	0.35	66° 50'
0.15	58° 24'	0.40	70° 32'
0.20	60°	0.5	90°

According to Eq. (420), the carrying capacity of a shell under internal pressure equals

$$pR = \frac{4}{3} f_1 n_1 (k + 1). \quad (424)$$

It can be concluded from this that fiberglass laminate shells are considerable inferior in strength to shells made by winding unidirectional glass fillers, the carrying capacity of which equals

$$pR = \frac{8}{3} fn. \quad (425)$$

Actually, the relationship $f_1 n_1 > fn$ always occurs, since the fiber strength in textile processing only decreases, i.e., $f_1 < f$. Besides, the fiber packing density in fabrics is less than the fiber packing density in winding: $n_1 < n$. Consequently,

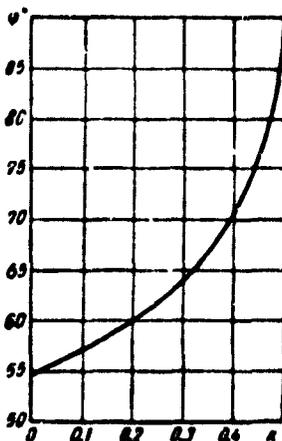


Fig. 56. Optimum winding angle of cylindrical shell vs. relative warp and woof strength of fabric.

even in the most favorable case, when $f_1 n_1 \sim \sin$, the carrying capacity of a fiberglass laminate shell is 25% less than the carrying capacity of a shell produced by winding unidirectional glass fillers.

It also follows from Eq. (424) that with a given warp strength of fiberglass fabric, the maximum carrying capacity of the shell is reached at $k=0.5$, i.e., in straight winding of fabrics, the warp strength of which is twice the woof strength.

It seems advisable to consider a still simpler method of manufacture of shells, when annular winding is accompanied by laying the fabric warp along the shell generatrix.

Let N layers be wound in the annular direction on each layer of fabric laid along the generatrix or vice versa. In this case, the normal force on area θ is represented by the expressions /131

or
$$T_\theta = f_1 n_1 (N_1 k + 1) \left[1 - \frac{(1-k)(1-N_1)}{N_1 k + 1} \cos^2 \theta \right] \quad (426)$$

$$T_\theta = f_1 n_1 (N_2 + k) \left[1 + \frac{(1-k)(1-N_2)}{N_2 + k} \cos^2 \theta \right]. \quad (427)$$

The relative fabric strength is not limited by bias winding conditions, and it changes in the $0 \leq k \leq 1$ interval.

Thus, the number of annular layers on one layer of fabric laid along the generatrix is determined by the formula

$$N_1 = \frac{1+a-k}{1-k-ak} \quad \text{or} \quad N_2 = \frac{1-k-ka}{k+a-1}. \quad (428)$$

Since $N > 0$, the fiberglass fabric should satisfy the conditions

$$\left. \begin{aligned} k &\leq \min \left(1+a; \frac{1}{1+a} \right) \leq 1; \\ 1 &\geq k \geq \max \left(1+a; \frac{1}{1+a} \right). \end{aligned} \right\} \quad (429)$$

We consider several possible cases.

1. Axial tension of shell $\alpha = -1$; the optimum fabrics which ensure the greatest strength of the shell should be unidirectional ($k=0$). According to Eq. (428) $N_1 = 0$.
2. Uniform tension of shell $\alpha = 0$; for the manufacture of a cylindrical shell, the annular stress of which equals the axial stress, it is advisable to use full strength fabrics ($k=1$). According to Eq. (428), $N_1 = 1$.
3. Shell under internal pressure $\alpha = 1$; in this case, the

optimum fabrics should have relative wool strength $k=1/2$ and be wound only annularly. According to Eq. (428), $N_2=0$.

41. Optimum Methods of Combined Winding of Cylindrical Shells

There is still another method of continuous winding of shells with fiberglass fabrics alternately with some unidirectional glass fillers. This combined winding method permits the improvement of fabrics which do not satisfy the conditions of optimality. Moreover, in laying the fabric warp in the direction of the generatrix, the strength of the shell is increased correspondingly compared with fiberglass laminate shells. /132

We initially consider the combined continuous winding method as applied to the manufacture of shells which operate under uniform internal pressure. Let the warp of the fabric form angle ϕ_1 with the generatrix of the shell and the unidirectional filler be wound at angle ϕ_2 to the generatrix.

If N layers of unidirectional glass filler are wound on each layer of fabric, the optimum winding angles are determined by the following expressions

$$\left. \begin{aligned} \cos 2\phi_1 &= \frac{4k_1^2 - 5k^2 - k_1k + 8k - k_1 - 5}{3(1-k)(k_1+k+1)}; \\ \cos 2\phi_2 &= -\frac{5k_1^2 - 4k^2 + k_1k + k_1 + 10k - 4}{3k_1(k_1+k+1)}. \end{aligned} \right\} \quad (430)$$

where

$$k_1 = \frac{fn}{f_1n_1} N,$$

f is the tensile strength of the fiber; n is the packing density of the unidirectional filler.

We also consider the case of winding of the greatest practical importance, when the fabric is laid with the warp in the axial direction ($\phi_1=0$), and the unidirectional glass filler is cross wound at angles $\pm\phi$.

In this case, the normal force on area θ is represented by the formula

$$T_\theta = f_1n_1 \left[1 + \frac{k_1(1 + \cos 2\phi)}{2} - (k_1 \cos 2\phi + 1 - k) \cos^2 \theta \right]. \quad (431)$$

The carrying capacity of the shell is determined by the expression

$$pR = \frac{2}{3} (f_1n_1 + f_2n_2 + fnN), \quad (432)$$

and the optimum cross winding angle of the unidirectional glass filler is found by the formula

$$\cos 2\varphi = \frac{2k + k_1 - 4}{3k_1} \quad (433)$$

where parameters k_1 , k should satisfy the condition

$$\xi = k_1 + k \geq 2. \quad (434)$$

In the individual case when $k_1 + k = 2$, $\varphi = 90^\circ$, i.e., the unidirectional glass filler should be wound in the annular direction.

TABLE 5. OPTIMUM COMBINED WINDING ANGLES

k	$\xi = k_1 + k$							
	2	3	4	5	6	7	8	9
0.1	90°	70° 11'	65° 35'	63° 08'	61° 37'	60° 34'	59° 48'	59° 12'
0.2	90°	69° 49'	65° 14'	62° 51'	61° 21'	60° 20'	59° 31'	59°
0.3	90°	69° 26'	64° 52'	62° 32'	61° 04'	60° 05'	59° 22'	58° 48'
0.4	90°	69° 02'	64° 31'	62° 12'	60° 48'	59° 50'	59° 08'	58° 36'
0.5	90°	68° 35'	64° 08'	61° 52'	60° 30'	59° 35'	58° 54'	58° 24'
0.6	90°	68° 07'	63° 43'	61° 32'	60° 12'	59° 19'	58° 40'	58° 12'
0.7	90°	67° 37'	63° 18'	61° 10'	59° 54'	59° 03'	58° 26'	57° 09'
0.8	90°	67° 06'	62° 51'	60° 48'	59° 34'	58° 46'	58° 12'	57° 46'
0.9	90°	66° 32'	62° 22'	60° 25'	59° 15'	58° 30'	57° 56'	57° 32'
1	90°	65° 50'	61° 50'	60°	58° 54'	58° 12'	57° 42'	57° 18'

The results of calculation by Eq. (433) are presented in Table 5 and Fig. 57.

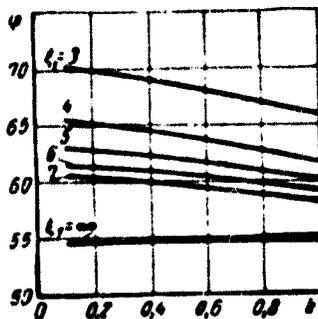


Fig. 57. Optimum combined winding angles.

We now study the general case when it is necessary to determine the optimum combined winding of a cylindrical shell operating under axisymmetric load $T_2/T_1 = 1 + \alpha$. We consider the simplest longitudinal-transverse winding, when the fabric warp is laid long the axis of the shell or is wound with the warp in the annular direction, depending on the nature of the stress of the shell. /133

Similarly to the preceding, for determination of the optimum winding angles of the unidirectional glass filler, the following calculation formulas can be obtained

$$\cos 2\varphi = \frac{2k - k_1 \alpha - 2(1 + \alpha)}{(2 + \alpha) k_1}; \quad (435)$$

$$\cos 2\varphi = \frac{2(1 - \alpha) - 2k(1 + \alpha) - k_1 \alpha}{k_1(2 + \alpha)}. \quad (436)$$

When the warp of the fabric is laid axially, according to Eq. (435), the filler parameters should satisfy the conditions

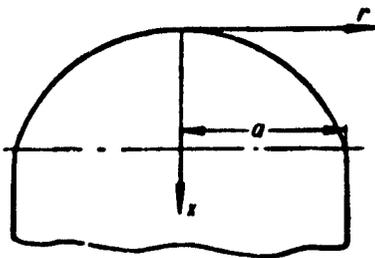
$$k_1 + k > 1 + a; \quad 1 + k_1 > \frac{k}{1+a}. \quad (437)$$

If the warp of the fabric is wound annularly, the filler parameters should satisfy the conditions

$$k_1 + k \geq \frac{1-a}{1+a}; \quad k - \frac{k_1}{1+a} \leq \frac{1-a}{1+a}. \quad (438)$$

42. Optimum Winding Angles of Bottoms of Varied Geometric Shape

We will consider a bottom manufactured by continuous winding, in the form of a shell of rotation and smoothly joined to a cylindrical body of radius a (Fig. 58). /134



If the (r, x) coordinate origin is placed in the pole of the bottom, the principal radii of curvature of the bottom are determined by the known expressions

$$R_1 = -\frac{\sqrt{(1+r'^2)^3}}{r}; \quad R_2 = r\sqrt{1+r'^2}. \quad (439)$$

Fig. 58. Coordinate system for shell of rotation.

The meridional and annular forces which are generated in the bottom as a result of uniform internal pressure are equal to

$$\left. \begin{aligned} T_1 &= \frac{pr}{r} \sqrt{1+r'^2}; \\ T_2 &= \frac{pr}{2} \sqrt{1+r'^2} \left(2 + \frac{rr''}{1+r'^2} \right). \end{aligned} \right\} \quad (440)$$

Since a shell of rotation with positive Gaussian curvature $1/R_1 R_2 > 0$ is a nondevelopable surface, the bottom can be made only by continuous winding of glass fibers.

Let nc fibers pass through small segment c perpendicular to the fiber direction. We consider an element of the bottom cut by two axial planes and two conical surfaces, so that the condition $ds_1 \sin \phi = c$, $ds_2 \cos \phi = c$, where ϕ is the winding angle, i.e., the angle between the fiber direction and the meridian of the surface, is satisfied (Fig. 59).

The same number of fibers, equal to the product of the packing density and the length of segment c (perpendicular to the fiber direction), i.e., equal to nc , passes through segments ds_2, c and ds_1 . In distinction from a cylindrical shell, the fiber packing density on a /135

double curvature shell is not constant, but it changes direction toward the pole of the shell, i.e., the fiber packing density in winding shells of rotation is a function of the cross section radius or axial coordinate x .

The normal forces which are generated in area ds_1 , ds_2 , with the fibers under tension of force f ,

$$T_1 = fn \cos^2 \phi; \quad T_2 = fn \sin^2 \phi. \quad (441)$$

According to Eq. (440), (441), a system of differential equations which determines the optimum continuous winding of shells of rotation has the form

$$\left. \begin{aligned} pr \sqrt{1+r'^2} &= 2fn \cos^2 \phi; \\ pr \sqrt{1+r'^2} \left(2 + \frac{rr''}{1+r'^2} \right) &= 2fn \sin^2 \phi. \end{aligned} \right\} \quad (442)$$

By dividing the second equation of (442) by the first, we obtain an expression for the square of the tangent of the optimum winding angle of the bottom as a function of the shape of the bottom

$$\tan^2 \phi = 2 + \frac{rr''}{1+r'^2}. \quad (443)$$

This basic relationship permits the pattern of change of winding angles ϕ in the manufacture of shells of rotation of arbitrary shape to be found:

Shape of shell of rotation

Optimum winding angle

Hemispherical

Ellipsoidal

Ellipsoidal $b=a/\sqrt{2}$

Conical

Ogival

Box

$$\begin{aligned} & \arctg \sqrt{2 - \frac{a^4}{a^4 - (a^2 - b^2)r^2}} \\ & \arctg \sqrt{\frac{2(1-\zeta^2)}{2-\zeta^4}} \\ & 54^\circ 44' \\ & \arctg \sqrt{2 - \frac{r}{k+r-a}} \\ & \arctg \sqrt{2 - \frac{r}{r-\frac{a}{2}}} \end{aligned}$$

However, in continuous winding of shells with a nonzero Gaussian curvature, it must be kept in mind that, besides satisfying the condition of equilibrium of the basic system, i.e., equilibrium of the glass filler without the binder, still another no less important condition must be met which specifies no slipping of the glass filler from the shell surface, and it is geometrically reduced to winding along the geodetic lines of the surface. According to the Kleró theorem for a surface of rotation, /136

this condition can be written in the form

$$r \sin \phi = h, \quad (444)$$

where h is the geodetic line parameter.

With an opening at the pole and continuous winding of the opening, the geodetic line parameter equals the cross section radius of the opening.

According to Eq. (443), (444), the differential equation which determines the shape of the bottom and the optimum winding can be written in the form

$$\frac{rr''}{1+r'^2} + \frac{2r^2-3h^2}{r^2-h^2} = 0. \quad (445)$$

By replacement of the variables

$$\frac{r}{h} = \zeta; \quad r' = U(\zeta), \quad (446)$$

differential Eq. (445) can be reduced to the form

$$\frac{U' dU}{1+U^2} = \frac{(3-2\zeta^2) d\zeta}{\zeta(\zeta^2-1)}, \quad (447)$$

from which, after integration, the desired shape of the bottom is determined in the form of a simple quadrature as a function of parameter h

$$x = h \int_{\zeta_0}^{\zeta} \frac{\zeta^2 d\zeta}{c^2(\zeta^2-1) - \zeta^4} + x_0. \quad (448)$$

where

$$c^2 = \frac{\left(\frac{a}{h}\right)^4}{\left(\frac{a}{h}\right)^2 - 1}. \quad (449)$$

According to Eq. (448), the optimum shape only exists in regions where the subintegral expression is positive.

Optimum shape of bottom with longitudinal-transverse winding.

There is interest in determination of the optimum shape of a bottom with longitudinal-transverse winding of the cylindrical shell.

In longitudinal-transverse winding, the glass fibers will go in the direction of the meridian of the shell. Consequently, to ensure equilibrium of the basic system in the annular direction, a bottom shape must be selected in which the annular stresses equal zero.

According to Eq. (440), the desired bottom shape is determined by the following differential equation

$$rr'' + 2r' \dot{\zeta} + 2 = 0. \quad (450)$$

The solution of Eq. (450) is represented by the elliptical integral /13

$$x = \int_0^r \frac{t^2 dt}{\sqrt{a^2 - t^2}}, \quad (451)$$

which can be expressed through elliptical functions.

In the dimensionless coordinates $\eta = x/a$, $\zeta = r/a$, the shape of the bottom is determined by the equation

$$\eta = \Phi(\zeta), \quad (452)$$

where function $\Phi(\zeta)$ and its first derivative are presented in Table 6.

TABLE 6. OPTIMUM SHAPE OF BOTTOM WITH LONGITUDINAL-TRANSVERSE WINDING

ζ	$\Phi(\zeta) \cdot 10^3$	$\Phi'(\zeta)$	ζ	$\Phi(\zeta) \cdot 10^3$	$\Phi'(\zeta)$
0	0	0	0.52	4.7629	0.2809
0.02	$0.3 \cdot 10^{-3}$	0.0004	0.54	5.3484	0.3049
0.04	$0.213 \cdot 10^{-3}$	0.0016	0.56	5.9834	0.3303
0.06	$0.720 \cdot 10^{-3}$	0.0036	0.58	6.6702	0.3572
0.08	0.0177	0.0064	0.60	7.4131	0.3859
0.10	0.0333	0.0100	0.62	8.2152	0.4164
0.12	0.0576	0.0144	0.64	9.0797	0.4490
0.14	0.0915	0.0196	0.66	10.013	0.4839
0.16	0.1365	0.0256	0.68	11.017	0.5215
0.18	0.1944	0.0324	0.70	12.098	0.5621
0.20	0.2667	0.0400	0.72	13.266	0.6062
0.22	0.3594	0.0485	0.74	14.521	0.6544
0.24	0.4613	0.0577	0.76	15.880	0.7076
0.26	0.5865	0.0678	0.78	17.344	0.7666
0.28	0.7327	0.0786	0.80	18.931	0.8329
0.30	0.9015	0.0904	0.82	20.531	1.1008
0.32	1.0947	0.1029	0.84	22.270	$1/\Phi'(\zeta)$
0.34	1.3139	0.1164	0.86	23.172	0.9371
0.36	1.5608	0.1307	0.88	26.301	0.8170
0.38	1.8372	0.1459	0.90	28.687	0.7240
0.40	2.1450	0.1621	0.92	31.435	0.6292
0.42	2.4861	0.1792	0.94	34.682	0.5294
0.44	2.8621	0.1973	0.96	38.702	0.4212
0.46	3.2764	0.2165	0.98	44.184	0.2901
0.48	3.7294	0.2368	1	58.115	0.0000
0.50	4.2242	0.2582			

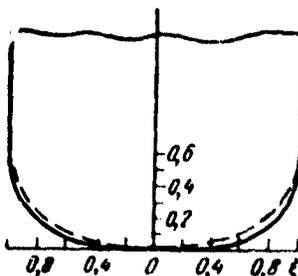


Fig. 60. Optimum shape of bottom with longitudinal-transverse winding.

The optimum bottom shape is presented in Fig. 60, where an ellipsoidal bottom with the same semi-axes as the optimum bottom is noted by the dashed line.

43. Differential Equation of Axisymmetrical Deformation of Cylindrical Shell

We consider an orthotropic cylindrical shell, the principal axes of anisotropy of which coincide with coordinate axes x, β , which characterize the distance along the shell generatrix and the circumferential angle (Fig. 61). /138

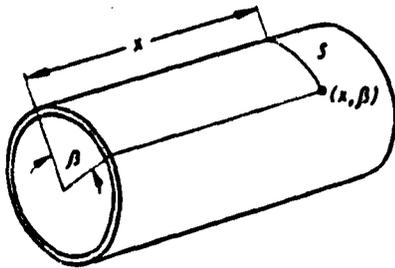


Fig. 61. Gaussian coordinates on surface of cylindrical shell.

According to Eq. (10)-(12) and (24)-(26), in axisymmetric loading of an orthotropic cylindrical shell, the basic elasticity relationships are presented in the form

$$\left. \begin{aligned} T_1 &= B_1 \left(u' + \nu_2 \frac{w}{R} \right), \\ T_2 &= B_2 \left(\frac{w}{R} + \nu_1 u' \right); \\ Q_1 &= -K_1 (\varphi + w'); \\ G_1 &= -D_1 \varphi'; \quad G_2 = \nu G_1. \end{aligned} \right\} \quad (453)$$

The equilibrium equations of a shell subjected to a surface load and end forces can be written in the form

$$\left. \begin{aligned} T_1' + X(x) &= 0; \\ Q_1' + \frac{T_2}{R} &= Z(x); \\ G_1' &= Q_1. \end{aligned} \right\} \quad (454)$$

The axial force generated in the shell is determined by the expression

$$T_1 = - \int_{x_0}^x X(\xi) d\xi + T_0, \quad (455)$$

where T_0 is the axial force applied to the end of shell $x=x_0$.

From the first elasticity relationship, with Eq. (455) taken into /139 account, the following can be obtained

$$u' = - \frac{1}{B_1} \int_{x_0}^x X(\xi) d\xi + \frac{T_0}{B_1} - \nu_2 \frac{w}{R}. \quad (456)$$

By substituting Eq. (456) in the second elasticity relationship, we obtain

$$T_2 = B_2(1 - \nu_1\nu_2) \frac{w}{R} + \nu_2 T_0 - \nu_2 \int_{x_0}^x X(\xi) d\xi. \quad (457)$$

If stress function $\Phi = -D_1 \phi$ is introduced, the elastic forces, moments and movements of the shell are determined by the following expressions

$$\left. \begin{aligned} G_1 &= \Phi'; & Q_1 &= \Phi''; \\ T_2 &= RZ(x) - R\Phi'''; \\ w &= \frac{R^3}{E_2 \delta} \left[Z(x) - \nu_2 \frac{T_2}{R} + \frac{\nu_2}{R} \int_{x_0}^x X(\xi) d\xi - \Phi''' \right]; \\ u &= -\frac{1}{B_1} \int_{x_0}^x dx \int_{x_0}^x X(\xi) d\xi + \frac{(x-x_0)T_0}{B_1} - \frac{\nu_2}{R} \int_{x_0}^x w dx + u_1, \end{aligned} \right\} \quad (458)$$

where u_1 is the axial movement of the end of shell $x=x_0$.

The differential equation which describes the axisymmetric deformation of an orthotropic cylindrical shell made of laminated plastics has the form

$$\Phi^{IV} - 2Q^2 \Phi'' + k^4 \Phi = -\nu_2 \frac{X}{R} + Z'(x), \quad (459)$$

where

$$2Q^2 = \frac{E_2 \delta}{K_1 R^3}; \quad k^4 = \frac{12(1 - \nu_1 \nu_2) E_2}{R^2 \delta^2 E_1}. \quad (460)$$

If the partial solution of Eq. (459), which is found by conventional methods, is designated $\phi_0(x)$, the forces, moments and movements which 140 correspond to this partial solution are written in the form

$$\left. \begin{aligned} G_0 &= \Phi_0'; & Q_0 &= \Phi_0''; & T_2^0 &= R(Z - \Phi_0'''); \\ \varphi_0 &= -\frac{\Phi_0}{D_1}, \\ w_0 &= \frac{\nu_2 R}{E_2 \delta} \int_{x_0}^x X(\xi) d\xi - \frac{\nu_2 R}{E_2 \delta} T_0 - \frac{R^2 \Phi_0'''}{E_2 \delta} + \frac{R^2 Z}{E_2 \delta}; \\ u_0 &= -\frac{1}{B_1} \int_{x_0}^x dx \int_{x_0}^x X(\xi) d\xi - \frac{(x-x_0)T_0}{B_1} - \frac{\nu_2}{R} \int_{x_0}^x w_0 dx + u_2. \end{aligned} \right\} \quad (461)$$

The solution of homogeneous differential Eq. (459), which determines the end effects generated near zones of abrupt change of geometric and

rigidity parameters or external load, has the form, for long cylindrical shells,

$$\Phi(x) = C_1 \Phi_1(x) + C_2 \Phi_2(x), \quad (462)$$

where

$$\Phi_1(x) = e^{-sx} \cos rx; \quad \Phi_2(x) = e^{-sx} \sin rx; \quad (463)$$

$$s = \sqrt{\frac{k^2 + q^2}{2}}; \quad r = \sqrt{\frac{k^2 - q^2}{2}}; \quad (464)$$

The following known relationships satisfy functions $\Phi_1(x)$, $\Phi_2(x)$

$$\left. \begin{aligned} \Phi_1' &= -(r\Phi_2 + s\Phi_1); & \Phi_2' &= r\Phi_1 - s\Phi_2; \\ \Phi_1'' &= (s^2 - r^2)\Phi_1 + 2rs\Phi_2; & \Phi_2'' &= (s^2 - r^2)\Phi_2 - \\ & & & - 2rs\Phi_1; \\ \Phi_1''' &= -s(s^2 - 3r^2)\Phi_1 + r(r^2 - 3s^2)\Phi_2; \\ \Phi_2''' &= -s(s^2 - 3r^2)\Phi_2 - r(r^2 - 3s^2)\Phi_1; \end{aligned} \right\} \quad (465)$$

for short cylindrical shells,

$$\Phi(x) = C_1 \phi_1 + C_2 \phi_2 + C_3 \phi_3 + C_4 \phi_4; \quad (466)$$

where $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, $\phi_4(x)$ are the functions of V.Z. Vlasov, defined by Eq. (188), satisfied by Eq. (189).

In short cylindrical shells, there is a general bent state of the shells, and, consequently, the "end effect" concept itself loses meaning.

It follows from Eq. (462)-(464) that, in laminated orthotropic shells, the end effect damping zone is determined not only by the geometric dimensions, but depends essentially on the nature of the anisotropy and rigidity with respect to interlayer shear. /141

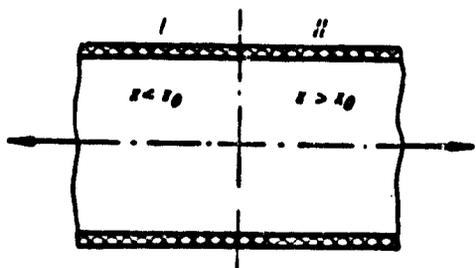
The end effect zone decreases with increase in parameters E_2/E_1 , E_2/G_{13} , i.e., the presence of interlayer shearing contributes to damping the stressed and deformed state of the shell. The concepts of "short" and "infinitely long" is not purely geometrical for laminated orthotropic shells, and it is determined by the value of parameter sl .

It is assumed that, in actual laminated shells, the relationship $k^2 > \rho^2$ or the relationship

$$\frac{G_{13}R}{E_1 \delta} > \frac{1}{2\sqrt{3(1-\nu_1\nu_2)}}, \quad (467)$$

where G_{13} is the interlayer shear modulus along the shell axis, is satisfied.

Long cylindrical shells. We select the distance from section $x = x_0$ as the axial coordinate in which the end effect is studied (Fig. 62).



With this coordinate system, in order to maintain the validity of the expressions obtained above for the elastic forces and movements, the change of sign in differentiation in region I, located to the left of the section, should be taken into account. In the expressions presented below, the upper sign of the double sign concerns region II, where $x \geq x_0$ and the lower sign, region I, where $x < x_0$:

Fig. 62. Hypothetical regions of propagation of end effects.

$$\left. \begin{aligned} Q_1 &= Q_0 + C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + \\ &\quad + C_2 [(s^2 - r^2) \Phi_2 - 2rs \Phi_1]; \\ G_1 &= G_0 \mp [C_1 (s \Phi_1 + r \Phi_2) - C_2 (r \Phi_1 - s \Phi_2)]; \\ u &= u_0 + \frac{v_1 R}{E_1 \delta} [\Phi'(x) - \Phi'(x_0)]; \\ \varphi &= \varphi_0 - \frac{1}{D_1} (C_1 \Phi_1 + C_2 \Phi_2); \\ w &= w_0 \pm \frac{R_2}{E_2 \delta} [C_1 [s (s^2 - 3r^2) \Phi_1 - r (r^2 - 3s^2) \Phi_2] + \\ &\quad + C_2 [r (r^2 - 3s^2) \Phi_1 + s (s^2 - 3r^2) \Phi_2]]. \end{aligned} \right\} (468)$$

The two random integration constants are determined from the boundary conditions in section $x=0$.

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Short cylindrical shells. In short cylindrical shells, for convenience in practical calculations, it is more convenient to express the solution of Eq. (466) in initial parameters ϕ_0, w_0, G_0, Q_0 , which designate the angle of rotation of the normal, deflection, bending moment and cutting force on the end of the shell ($x=0$). Since two constants will be assigned on one end, in specific calculations, the problem is reduced to determination of the two remaining constants from the boundary conditions on the other end of the shell.

According to Eq. (461), for short shells,

$$\varphi = -\frac{\Phi}{D_1}; \quad G = \Phi'; \quad Q = \Phi''; \quad w = -\frac{R^2 \Phi'''}{R_1 \delta}. \quad (469)$$

The following expressions for the basic components of the forces and deformations can be obtained in the initial parameters

$$\left. \begin{aligned} \varphi &= A_{\varphi\varphi}(x) \varphi_0 + A_{\varphi w}(x) w + A_{\varphi Q}(x) Q_0 + A_{\varphi G}(x) G_0; \\ w &= A_{w\varphi}(x) \varphi_0 + A_{ww}(x) w + A_{wQ}(x) Q_0 + A_{wG}(x) G_0; \\ Q &= A_{Q\varphi}(x) \varphi_0 + A_{Qw}(x) w + A_{QQ}(x) Q_0 + A_{QG}(x) G_0; \\ G &= A_{G\varphi}(x) \varphi_0 + A_{Gw}(x) w + A_{GQ}(x) Q_0 + A_{GG}(x) G_0; \end{aligned} \right\} (470)$$

Influence coefficients A_{ij} are determined through hyperbolotrigonometric Vlasov functions (188), by the expressions

$$\begin{aligned}
 A_{\varphi\varphi}(x) &= \Phi_1 - \frac{s^2 - r^2}{2rs} \Phi_2; \\
 A_{\varphi w}(x) &= \frac{E_2 \delta}{D_1 R^3} \frac{1}{2rs(s^2 + r^2)} (s \Phi_4 - r \Phi_3); \\
 A_{\varphi Q}(x) &= -\frac{\Phi_2}{2rs D_1}; \\
 A_{\varphi G}(x) &= \frac{1}{2rs(s^2 + r^2) D_1} [r(r^2 - 3s^2) \Phi_2 + \\
 &\quad + s(s^2 - 3r^2) \Phi_3]; \\
 A_{w\varphi}(x) &= -\frac{1}{2rs} (s \Phi_4 + r \Phi_3); \\
 A_{ww}(x) &= \Phi_1 + \frac{s^2 - r^2}{2rs} \Phi_2; \\
 A_{wQ}(x) &= -\frac{R^2}{E_2 \delta} \frac{1}{2rs} [s(s^2 - 3r^2) \Phi_4 - r(r^2 - 3s^2) \Phi_3]; \\
 A_{wG}(x) &= \frac{\Phi_2}{2rs D_1}; \\
 A_{Q\varphi}(x) &= \frac{D_1 (s^2 + r^2)}{2rs} \Phi_2; \\
 A_{Qw}(x) &= -\frac{E_2 \delta}{R^3} \frac{1}{2rs} (s \Phi_4 + r \Phi_3); \\
 A_{QQ}(x) &= \Phi_1 + \frac{s^2 - r^2}{2rs} \Phi_2; \\
 A_{QG}(x) &= -\frac{r^2 + s^2}{2rs} (s \Phi_4 - r \Phi_3); \\
 A_{G\varphi}(x) &= \frac{D_1 (s^2 + r^2)}{2rs} (s \Phi_4 - r \Phi_3); \\
 A_{Gw}(x) &= -\frac{E_2 \delta \Phi_2}{R^3 2rs}; \\
 A_{GQ}(x) &= \frac{1}{2rs} (s \Phi_4 + r \Phi_3); \\
 A_{GG}(x) &= \Phi_1 - \frac{s^2 - r^2}{2rs} \Phi_2.
 \end{aligned} \tag{471}$$

In finding the initial parameters from the boundary conditions to 143 solution (470), the corresponding solution obtained from membrane theory must be added.

44. Stressed and Deformed States of Cylindrical Shell Generated by Annular Concentrated Forces

We consider a long cylindrical shell compressed by a concentrated annular force of intensity q (Fig. 63).

In view of the symmetry, only one half of the shell can be considered, for example, that located to the left of the section where the pressure is applied.

Since there are no surface loads or axial forces on the end of

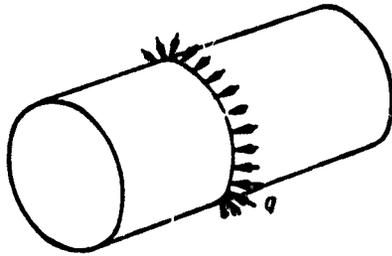


Fig. 63. Cylindrical shell subjected to concentrated annular pressure.

the shell ($X=Z=T_0=0$), the initial forces, moments and movements equal zero. Consequently,

$$\left. \begin{aligned} Q_1 &= C_1[(s^2 - r^2)\Phi_1 + 2rs\Phi_2] + C_2[(s^2 - r^2)\Phi_2 - \\ &\quad - 2rs\Phi_1]; \\ G_1 &= -[C_1(s\Phi_1 + r\Phi_2) - C_2(r\Phi_1 - s\Phi_2)]; \\ u &= \frac{\nu_2 R}{E_2 \delta} [Q_1(x) - Q_1(x_0)] + u_0; \\ \psi &= -\frac{1}{D_1} (C_1\Phi_1 + C_2\Phi_2); \\ w &= \frac{R^2}{E_2 \delta} [C_1[s(s^2 - 3r^2)\Phi_1 - r(r^2 - 3s^2)\Phi_2] + \\ &\quad + C_2[r(r^2 - 3s^2)\Phi_1 + s(s^2 - 3r^2)\Phi_2]]. \end{aligned} \right\} \quad (472)$$

Integration constants C_1, C_2 are determined from the boundary conditions

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$$\left. \begin{aligned} \text{at } x=0 \quad \varphi=0; \quad Q_1 &= -\frac{q}{2}; \\ C_1 &= 0; \quad C_2 = \frac{q}{4rs}. \end{aligned} \right\} \quad (473)$$

Consequently, the stressed and deformed states of a laminated cylindrical shell compressed by a concentrated annular force are determined by the following expressions

$$\left. \begin{aligned} Q_1 &= -\frac{q}{4rs} [2rs\Phi_1 - (s^2 - r^2)\Phi_2]; \\ G_1 &= \frac{q}{4rs} (r\Phi_1 - s\Phi_2); \\ u &= \frac{q\nu_2 R}{2E_2 \delta} \left[(1 - \Phi_1) + \frac{s^2 + r^2}{2rs} \Phi_2 \right]; \\ \psi &= -\frac{q}{4rsD_1} \Phi_2; \\ w &= \frac{qR^2}{4rsE_2 \delta} [r(r^2 - 3s^2)\Phi_1 + s(s^2 - 3r^2)\Phi_2]. \end{aligned} \right\} \quad (474)$$

The maximum deflection in the section where the annular force is applied

$$w_{\max} = -\frac{(3s^2 - r^2)qR^2}{4sE_2 \delta}. \quad (475)$$

The maximum flexural stress and the greatest shearing stress between the layers are generated in section $x=0$:

$$\sigma_{\max} = \frac{3q}{2sE_2}, \quad \tau_{\max} = \frac{3q}{4\delta}. \quad (476)$$

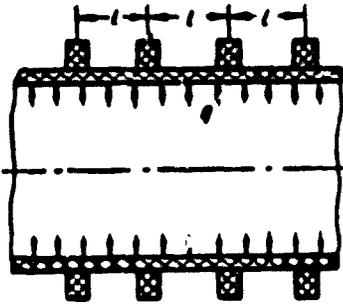


Fig. 64. Sketch of cylindrical shell with annular ribs.

We investigate the stressed and deformed states of a laminated cylindrical shell subjected to uniform internal pressure, which is strengthened by equally spaced reinforcing rings (Fig. 64).

We will assume that a segment of the shell of length l located between two rings can be considered infinitely long in the sense indicated above, i.e., the stressed and deformed shell states described by Eq. (474) are completely damped in distance $l/2$. In actual shells, these conditions usually are always satisfied, since the reinforcing rings must be larger to preserve the circular shape of the shell than to increase the strength.

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Let the rigidity of the reinforcing ring under tension equal EF . The end forces and surface load components

$$T_0 = \frac{pR}{2}; \quad Z=p; \quad X=0. \quad (477)$$

Consequently, the solution of the inhomogeneous equation has the form

$$\left. \begin{aligned} G_0 = Q_0 = \varphi_0 = 0; \\ w_0 = \frac{(2-\nu_2) p R^2}{2E_2 \delta}. \end{aligned} \right\} \quad (478)$$

The intensity of the force of interaction between the reinforcing ring and the shell q is determined from the condition of compatibility of deformation of the ring and shell

$$\frac{pR^2}{EF} - \frac{(2-\nu_2) p R^2}{2E_2 \delta} - \frac{(3s^2 - r^2) q R^2}{4sE_2 \delta}. \quad (479)$$

Consequently, the intensity of the force of interaction

$$q = \frac{(2-\nu_2) p}{2} \cdot \frac{1}{\left[\frac{E_2 \delta}{EF} + \frac{3s^2 - r^2}{4s} \right]}. \quad (480)$$

The normal stress generated in the reinforcing ring

$$\sigma_{\max} = \frac{(2-\nu_2) p R}{2\delta} \cdot \frac{1}{\left[\frac{E_2}{E} + \frac{(3s^2 - r^2) F}{4s\delta} \right]}. \quad (481)$$

The stressed and deformed states of the shell are the sum of the momentless state generated by the internal pressure and the stressed and deformed states generated by the resistance of the ring to deformation

of the shell. The latter is determined by Eq. (474), where q is found by Eq. (480).

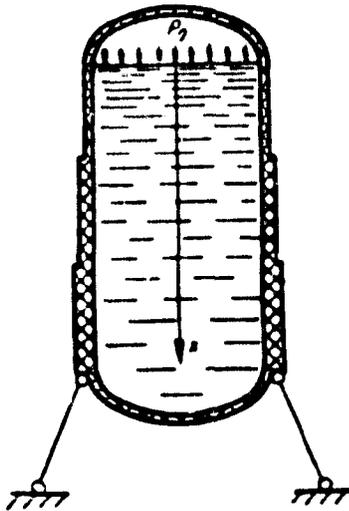
Thus, the greatest normal stresses and maximum shearing stresses between the layers develop in the reinforced sections of the shell, and they are determined by the expressions

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$$\left. \begin{aligned} \sigma_{1 \max} &= \frac{pR}{2\delta} + \frac{3(2-\nu_2)p}{4s\delta^2} \frac{1}{\left[\frac{E_2\delta}{EF} + \frac{3s^2-r^2}{4s} \right]} ; \\ \sigma_{2 \max} &= \frac{pR}{\delta} + \frac{3(2-\nu_2)\nu_2 p}{4s\delta^2} \frac{1}{\left[\frac{E_2\delta}{EF} + \frac{3s^2-r^2}{4s} \right]} ; \\ \tau_{\max} &= \frac{3(2-\nu_2)p}{8\delta} \cdot \frac{1}{\left[\frac{E_2\delta}{EF} + \frac{3s^2-r^2}{4s} \right]} . \end{aligned} \right\} \quad (482)$$

45. Calculation of Laminated Cylindrical Shell with Variable Wall Thickness Changing by Steps

We determine the stresses generated in a closed laminated cylindrical container, which is supported by the base on a circular support and is filled with liquid of specific weight γ and is under low pressurization p_0 , which is required to give the container the necessary rigidity (Fig. 65).



The lateral hydrostatic pressure of the liquid on the container wall is proportional to the height of the liquid column to the section under consideration and the specific weight of the liquid. If the coordinate origin is selected on the open surface of the liquid,

$$X=0; T_0 = \frac{p_0 R}{2}; Z(x) = (\gamma x + p_0). \quad (483)$$

In accordance with Eq. (461), the momentless state of the shell is determined by the expressions

$$\left. \begin{aligned} G_0 = Q_0 = 0; T_0^* &= (p_0 + \gamma x)R; \\ \varphi_0 &= -\frac{\gamma R^2}{E_2 \delta}; w_0 = \frac{(2-\nu_2)p_0 R^2}{2E_2 \delta(x)} + \frac{\gamma x R^2}{E_2 \delta(x)}. \end{aligned} \right\} \quad (484)$$

Fig. 65. Variable thickness chemical container.

Consequently, it is inadvisable to make the walls of such containers of constant thickness. Let the wall thickness be increased toward the base, changing by steps. We investigate the end effects which arise in the sections of the container where there is wall thickness discontinuity $\delta(x)$.

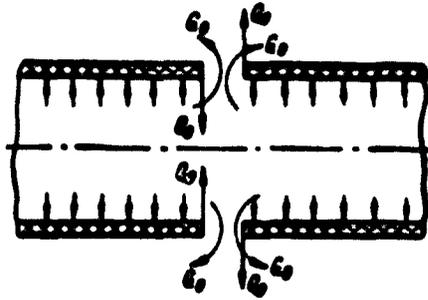


Fig. 66. Initial parameters G_0, Q_0 in cylindrical shell.

Let the thickness of adjacent sections of the shell be δ_1, δ_2 ($\delta_2 > \delta_1$). We consider two sections of the tank separately, on the assumption that they are sufficiently long: $l \gg \pi/s$ (Fig. 66).

For section I

$$\left. \begin{aligned} Q_1 &= C_1 [(s_1^2 - r_1^2) \Phi_1 + 2r_1 s_1 \Phi_2] + C_2 [(s_1^2 - r_1^2) \Phi_2 - 2r_1 s_1 \Phi_1]; \\ G_1 &= C_1 (s_1 \Phi_1 + r_1 \Phi_2) - C_2 (r_1 \Phi_1 - s_1 \Phi_2); \\ w_0 &= w_0^1 - \frac{R^2}{E_1 \delta_1} [C_1 [s_1 (s_1^2 - 3r_1^2) \Phi_1 - r_1 (r_1^2 - 3s_1^2) \Phi_2] + C_2 [r_1 (r_1^2 - 3s_1^2) \Phi_1 + s_1 (s_1^2 - 3r_1^2) \Phi_2]]; \\ \psi &= -\frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3} (C_1 \Phi_1 + C_2 \Phi_2). \end{aligned} \right\} (485)$$

By satisfying the boundary conditions, the following can be obtained

$$C_1 = \frac{2s_1 G_0 - Q_0}{s_1^2 + r_1^2}; \quad C_2 = \frac{(s_1^2 - r_1^2) G_0 - s_1 Q_0}{r_1 (s_1^2 + r_1^2)}. \quad (486)$$

Consequently, in the initial parameters, Eq. (485) take the form

$$\left. \begin{aligned} Q_1 &= G_0 \frac{s_1^2 + r_1^2}{r_1} \Phi_2 + Q_0 \left(\Phi_1 - \frac{s_1}{r_1} \Phi_2 \right); \\ G_1 &= G_0 \left(\Phi_1 + \frac{s_1}{r_1} \Phi_2 \right) - Q_0 \frac{\Phi_2}{r_1}; \\ w_0 &= w_0^1 + \frac{R^2}{E_1 \delta_1} \left[G_0 (r_1^2 + s_1^2) \left(\Phi_1 - \frac{s_1}{r_1} \Phi_2 \right) - Q_0 \left(2s_1 \Phi_1 - \frac{s_1^2 - r_1^2}{r_1} \Phi_2 \right) \right]; \\ \psi &= -\frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3 (s_1^2 + r_1^2)} \left[G_0 \left(2s_1 \Phi_1 + \frac{s_1^2 - r_1^2}{r_1} \Phi_2 \right) - Q_0 \left(\Phi_1 + \frac{s_1}{r_1} \Phi_2 \right) \right]. \end{aligned} \right\} (487)$$

The movement and angle of rotation of the end of the shell $x=0$, respectively, /14

$$\left. \begin{aligned} w_1 &= \frac{(2-\nu_2) \rho_0 R^2}{2E_2 \delta_2} + \frac{\gamma_0 R^2}{E_2 \delta_2} + \frac{R^2}{E_2 \delta_2} [G_0 (r_1^2 + s_1^2) - 2s_1 Q_0]; \\ \varphi_1 &= -\frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3 (s_1^2 + r_1^2)} (2s_1 G_0 - Q_0) - \frac{\gamma R^2}{E_2 \delta_2}. \end{aligned} \right\} (488)$$

For section II

$$\left. \begin{aligned}
 Q_1 &= C'_1 [(s_2^2 - r_2^2) \Phi_1 + 2s_2 \Phi_2] + C'_2 [(s_2^2 - r_2^2) \Phi_2 - \\
 &\quad - 2r_2 s_2 \Phi_1]; \\
 G_1 &= -C'_1 (s_2 \Phi_1 + r_2 \Phi_2) + C'_2 (r_2 \Phi_1 - s_2 \Phi_2); \\
 \varphi &= -\frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3} (C'_1 \Phi_1 + C'_2 \Phi_2); \\
 w &= w_0^{II} + \frac{R^2}{E_1 \delta_1} \{ C'_1 [s_2 (s_2^2 - 3r_2^2) \Phi_1 - \\
 &\quad - r_2 (r_2^2 - 3s_2^2) \Phi_2] + C'_2 [r_2 (r_2^2 - 3s_2^2) \Phi_1 + \\
 &\quad + s_2 (s_2^2 - 3r_2^2) \Phi_2] \}.
 \end{aligned} \right\} \quad (489)$$

We again find constants C_1' , C_2' from the boundary conditions

$$\left. \begin{aligned}
 C'_1 (s_2^2 - r_2^2) - C'_2 2r_2 s_2 &= Q_0; \\
 -C'_1 s_2 + C'_2 r_2 &= G_0.
 \end{aligned} \right\} \quad (490)$$

from which the following values of the constants can be obtained

$$C'_1 = -\frac{2s_2 G_0 + Q_0}{s_2^2 + r_2^2}; \quad C'_2 = -\frac{(s_2^2 - r_2^2) G_0 + s_2 Q_0}{r_2 (s_2^2 + r_2^2)}. \quad (491)$$

In initial parameters G_0 , Q_0 , Eq. (489) are presented in the form

$$\left. \begin{aligned}
 Q_1 &= -G_0 \frac{s_2^2 + r_2^2}{r_2} \Phi_2 + Q_0 \left(\Phi_1 + \frac{s_2}{r_2} \Phi_2 \right); \\
 G_1 &= G_0 \left(\Phi_1 + \frac{s_2}{r_2} \Phi_2 \right) + Q_0 \frac{\Phi_2}{r_2}; \\
 w &= w^{II} + \frac{R^2}{E_1 \delta_1} \left[G_0 (r_2^2 + s_2^2) \left(\Phi_1 - \frac{s_2}{r_2} \Phi_2 \right) + \right. \\
 &\quad \left. + Q_0 \left(2s_2 \Phi_1 - \frac{s_2^2 - r_2^2}{r_2} \Phi_2 \right) \right]; \\
 \varphi &= \frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3 (s_2^2 + r_2^2)} \left[G_0 \left(2s_2 \Phi_1 + \frac{s_2^2 - r_2^2}{r_2} \Phi_2 \right) + \right. \\
 &\quad \left. + Q_0 \left(\Phi_1 + \frac{s_2}{r_2} \Phi_2 \right) \right].
 \end{aligned} \right\} \quad (492)$$

The deflection and angle of rotation of the end of the shell $x=0$ /149

$$\left. \begin{aligned}
 w_{11} &= \frac{(2-\nu_2) r_0 R^2}{2E_2 \delta_2} + \frac{\gamma a R^2}{E_2 \delta_2} + \frac{R^2}{E_1 \delta_1} [G_0 (r_2^2 + s_2^2) + 2s_2 Q_0]; \\
 \varphi_{11} &= \frac{12(1-\nu_1\nu_2)}{E_1 \delta_1^3 (s_2^2 + r_2^2)} (2s_2 G_0 + Q_0) - \frac{\gamma R^2}{E_2 \delta_2}.
 \end{aligned} \right\} \quad (493)$$

The conditions of compatibility of the deformations in the butt sections of the shell can be reduced to the form

$$\left. \begin{aligned} G_0 [\delta_1 (s_2^2 + r_2^2) - \delta_2 (s_1^2 + r_1^2)] + 2Q_0 (s_2 \delta_1 + s_1 \delta_2) = \\ = \left(\frac{2-\nu_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1); \\ 2G_0 [s_2 (s_1^2 + r_1^2) \delta_1^2 + s_1 (s_2^2 + r_2^2) \delta_2^2] + Q_0 [(s_1^2 + r_1^2) \delta_1^2 - \\ - (s_2^2 + r_2^2) \delta_2^2] = \frac{\gamma R^2 (\delta_1 - \delta_2) \delta_1^2 \delta_2^2}{12(1-\nu_1 \nu_2) \lambda^3} (s_1^2 + r_1^2) (s_2^2 + r_2^2); \end{aligned} \right\} \quad (494)$$

From this, for determination of end forces G_0 , Q_0 , we obtain the following expressions

$$\left. \begin{aligned} G_0 = \left(\frac{2-\nu_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1) f_1 + \\ + \frac{\gamma R^2 (\delta_2 - \delta_1) \delta_1^2 \delta_2^2}{6\lambda^2 (1-\nu_1 \nu_2)} f_2; \\ Q_0 = -2 \left(\frac{2-\nu_2}{2} p_0 + \gamma a \right) (\delta_2 - \delta_1) f_3 - \\ - \frac{\gamma R^2 (\delta_2 - \delta_1) \delta_1^2 \delta_2^2}{6\lambda^2 (1-\nu_1 \nu_2)} f_4, \end{aligned} \right\} \quad (495)$$

$$\left. \begin{aligned} f_1 &= \frac{(s_1^2 + r_1^2) \delta_1^2 - (s_2^2 + r_2^2) \delta_2^2}{\Delta}; \\ f_2 &= \frac{(s_1^2 + r_1^2) (s_2^2 + r_2^2) (s_2^2 \delta_1 + s_1^2 \delta_2)}{\Delta}; \\ f_3 &= \frac{s_2 (s_1^2 + r_1^2) \delta_1^2 + s_1 (s_2^2 + r_2^2) \delta_2^2}{\Delta}; \\ f_4 &= \frac{(s_1^2 + r_1^2) (s_2^2 + r_2^2) [(s_2^2 + r_2^2) \delta_1 - (s_1^2 + r_1^2) \delta_2]}{\Delta}; \\ \Delta &= [(s_2^2 + r_2^2) \delta_1 - (s_1^2 + r_1^2) \delta_2] [(s_1^2 + r_1^2) \delta_1 - \\ - (s_2^2 + r_2^2) \delta_2] - 4(s_2 \delta_1 + s_1 \delta_2) [s_2 (s_1^2 + r_1^2) \delta_1^2 + \\ + s_1 (s_2^2 + r_2^2) \delta_2^2]. \end{aligned} \right\} \quad (496)$$

After determination of the end forces, the stressed state in sections in which the thickness changes by steps is determined by the expressions obtained above. /150

We note that the manufacture of cylindrical shells made of laminated plastics with variable wall thickness presents no technical difficulties.

46. Calculation of Laminated Orthotropic Cylindrical Shell Subjected to Axial Eccentrically Applied Forces

In actual shells, axial forces may be applied to the side surface only with some eccentricity with respect to the mean surface (Fig. 67).

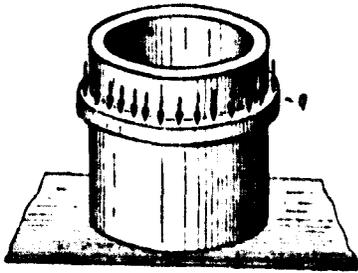


Fig. 67. Cylindrical shell subjected to eccentrically applied axial load.

We will assume that the contact of the shell with the reinforcing ring is linear. The axial eccentrically applied load can be reduced to axial forces applied to the mean surface of the shell and to reactive bending moments uniformly distributed around the circumference of the shell cross section. The strength of the reactive bending moments is determined from the conditions of compatibility of the deformations of the ring and shell.

By separating the reinforcing ring from the shell and replacing the force of interaction between them by reactive moments of unknown intensity m , we have the calculation scheme shown in Fig. 68. M is the external torque generated by the eccentricity of the axial load.

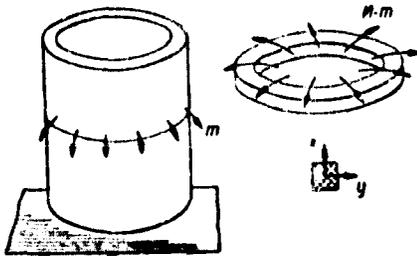


Fig. 68. Calculation scheme of cylindrical shell.

The calculation formulas for determination of the deformed and stressed states of an orthotropic laminated cylindrical shell, subjected to uniformly distributed bending moments m , are obtained by differentiation of the solution obtained for the case of annular pressure with simultaneous substitution of m for q [18]. However, in taking account of interlayer shearing, this method /151 can lead to errors, since the effect of shearing in the limiting transition is not taken into account exactly. Therefore, we use general solution (468).

For the upper and lower sections of the shell, respectively

$$\left. \begin{aligned}
 Q_1 &= C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + C_2 [(s^2 - r^2) \Phi_2 - 2rs \Phi_1]; \\
 G_1 &= \mp [C_1 (s \Phi_1 + r \Phi_2) - C_2 (r \Phi_1 - s \Phi_2)]; \\
 w &= \pm \frac{R^2}{E_1 \delta} [C_1 [s(s^2 - 3r^2) \Phi_1 - r(r^2 - 3s^2) \Phi_2] + \\
 &\quad + C_2 [r(r^2 - 3s^2) \Phi_1 + s(s^2 - 3r^2) \Phi_2]]; \\
 \varphi &= -\frac{1}{D_1} (C_1 \Phi_1 + C_2 \Phi_2); \\
 u &= u_0 + \frac{\nu_2 R}{E_2 \delta} [Q_1(x) - Q_1(x_0)].
 \end{aligned} \right\} (497)$$

In section $x=0$, the bending moment has a discontinuity of m . Consequently, integration constants C_1 , C_2 can be obtained from the following boundary conditions.

$$\text{at } x=0 \quad G_1 = \frac{m}{2}; \quad w=0. \quad (498)$$

By substituting Eq. (497) in Eq. (498), we obtain

$$\left. \begin{aligned} G_1 s (s^2 - 3r^2) - C_2 r (3s^2 - r^2) &= 0; \\ C_1 s - C_2 r &= -\frac{m}{2}. \end{aligned} \right\} \quad (499)$$

The following can be obtained from Eq. (499)

$$C_1 = \frac{m(3s^2 - r^2)}{4s(s^2 + r^2)}; \quad C_2 = -\frac{m(s^2 - 3r^2)}{4r(s^2 + r^2)}. \quad (500)$$

In this manner, there are the following calculation formulas for a shell subjected to bending moments of intensity m uniformly distributed around the perimeter of the section

$$\left. \begin{aligned} Q_1 &= -\frac{m(s^2 + r^2)}{4rs} (r\Phi_1 + s\Phi_2); \\ G_1 &= \mp \frac{m}{4rs} [2rs\Phi_1 + (s^2 - r^2)\Phi_2]; \\ \varphi &= \frac{m}{4rs(r^2 + s^2)D_1} [r(3s^2 - r^2)\Phi_1 + s(s^2 - 3r^2)\Phi_2]; \\ w &= \pm \frac{mR^2}{4E_s \delta} \frac{(s^2 + r^2)^2}{rs} \Phi_2; \\ u &= \frac{\nu_s R}{E_s \delta} [Q_1(x) - Q_1(x_0)]. \end{aligned} \right\} \quad (501)$$

In Eq. (501), the upper sign refers to upper region I of the shell and the lower sign, to lower region II of the shell (see Fig. 68). /152

For shells made of laminated plastics, reinforcing rings with a continuous rectangular cross section are most acceptable. The resistance of such a ring to axisymmetrical torsion is determined by the flexural rigidity of the ring EJ_y , i.e., the rigidity of the ring when it is bent out of the plane of curvature.

From the compatibility condition of deformations of the shell and reinforcing ring $\phi_r = \phi_s$, it can be found that

$$m = \frac{M}{1 + \zeta}, \quad (502)$$

where the geometric rigidity parameter is designated ζ

$$\zeta = \frac{EJ_y (3s^2 - r^2)}{4sD_1 R^2 (r^2 + s^2)}. \quad (503)$$

In accordance with Eq. (502), an external uniformly distributed bending moment which is transmitted to the shell through an elastic reinforcing ring is taken up by both the shell and the reinforcing ring. The fractions taken up by the shell and reinforcing ring are determined by parameter ζ . An increase of this parameter entails a decrease in load which is transmitted to the shell. Thus, the stress and deformation of the shell can be regulated by increasing the flexural rigidity of the ring. Since this rigidity is characterized by moment of inertia J_y , for transmission of eccentrically applied axial forces to the laminated cylindrical shell, it is advisable to use wide reinforcing rings.

It should be noted that, with large shell dimensions, the effect of the reinforcing ring decreases sharply in proportion to $R^{3/2}$, i.e., in thin cylindrical shells of large diameter, the external bending moment is almost entirely absorbed by the shell.

The stresses and deformations due to uniform compression should be added in the lower part of the shell to the stresses and deformations which are determined by Eq. (501).

47. Initial Relationships and Basic Differential Equations

We consider laminated shells of rotation loaded symmetrically /153
 relative to the axis, including the reactions of the support connections. We select circumferential angle β and angle α , formed by the normal to the mean surface with the axis of rotation as the curvilinear Gaussian coordinates on the mean surface of the shell (Fig. 69).

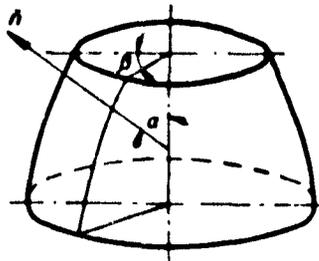


Fig. 69. Curvilinear coordinates on surface of shell of rotation.

We will assume that the principal directions of anisotropy of an orthotropic laminated plastic are coincident with the coordinate directions, i.e., with the lines of curvatures of the mean surface of the shell.

Under axisymmetric loading, a shell of rotation will be deformed symmetrically relative to the axis. Consequently, the forces, bending moments and movements of the mean surface of the shell will be functions of angular coordinate α alone.

From the elastic forces in the shell, only normal forces T_1, T_2 , transverse cutting force Q_1 and bending moments G_1, G_2 will result. Movements of points of the mean surface are completely determined by two components u, w , i.e., by movement along the meridian and deflection of the shell.

In accordance with basic Eq. (24)-(26) and Eq. (10)-(12), in axisymmetric deformation of a shell of rotation, /154

$$\left. \begin{aligned} T_1 &= B_1 \left(\frac{u' + w}{R_1} + \nu_2 \frac{u \operatorname{ctg} \alpha + w}{R_2} \right); \\ T_2 &= B_2 \left(\frac{u \operatorname{ctg} \alpha + w}{R_2} + \nu_1 \frac{u' + w}{R_1} \right); \end{aligned} \right\} \quad (504)$$

$$\left. \begin{aligned} Q_1 &= -K_1 \left(\varphi + \frac{w' - u}{R_1} \right); \\ G_1 &= -D_1 \left(\frac{\varphi'}{R_1} + \nu_2 \frac{\varphi \operatorname{ctg} \alpha}{R_2} \right); \\ G_2 &= -D_2 \left(\frac{\varphi \operatorname{ctg} \alpha}{R_2} + \nu_1 \frac{\varphi'}{R_1} \right). \end{aligned} \right\} \quad (505)$$

where R_1, R_2 are the principal radii of curvature of the mean surface, and the primes designated differentiation over angular coordinate α .

For symmetrically loaded shells of rotation, equilibrium Eq. (17) take the form

$$\left. \begin{aligned} \frac{d(T_1 R_2 \sin \alpha)}{d\alpha} - T_2 R_1 \cos \alpha - Q_1 R_2 \sin \alpha &= -X R_2 \sin \alpha; \\ \frac{d(Q_1 R_2 \sin \alpha)}{d\alpha} + R_1 R_2 \sin \alpha \left(\frac{T_1}{R_1} + \frac{T_2}{R_2} \right) &= Z R_2 \sin \alpha; \\ \frac{d(G_1 R_2 \sin \alpha)}{d\alpha} - G_2 R_1 \cos \alpha - Q_1 R_1 R_2 \sin \alpha &= 0. \end{aligned} \right\} \quad (506)$$

As the solution of nonhomogeneous system of Eq. (506), we will use the solution which corresponds to the membrane theory of shells. Further, we obtain the solution of homogeneous system of Eq. (506) on the assumption that $X=Y=0$.

Following Meissner, we introduce the stress function

$$V = R_2 Q_1. \quad (507)$$

Then, from the first two equilibrium Eq. (506), we obtain

$$T_1 = -\frac{V \operatorname{ctg} \alpha}{R_2}; \quad T_2 = -\frac{V'}{R_1}. \quad (508)$$

For axisymmetric loading of shells of rotation, relationships (8) take the form

$$\left. \begin{aligned} u' + w &= \frac{1}{E_1 \delta} \left(\nu_1 V' - \frac{R_1}{R_2} V \operatorname{ctg} \alpha \right); \\ u \operatorname{ctg} \alpha + w &= \frac{1}{E_2 \delta} \left(\nu_2 V \operatorname{ctg} \alpha - \frac{R_2}{R_1} V' \right). \end{aligned} \right\} \quad (509)$$

By substituting them in the obvious identity which expresses the condition of deformation compatibility

$$w' - u = (u \operatorname{ctg} \alpha + w - u' - w) \operatorname{ctg} \alpha + (u \operatorname{ctg} \alpha + w)', \quad (510)$$

we obtain one differential equation

$$\begin{aligned} V'' \frac{R_2}{R_1} + V' \left[\left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \operatorname{ctg} \alpha \right] - V \left[\lambda^2 \frac{R_1}{R_2} \operatorname{ctg}^2 \alpha + \right. \\ \left. + \frac{6E_2}{5G_{13}} \frac{R_1}{R_2} - \nu_2 \right] = E_2 \delta R_1 \varphi. \end{aligned} \quad (511)$$

By substituting Eq. (505) in the remaining unused equilibrium equation, a second equation can be obtained for stress and deformation functions V and ϕ

$$\varphi'' \frac{R_2}{R_1} + \varphi' \left[\left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \operatorname{ctg} \alpha \right] - \varphi \left[\nu_2 + \frac{R_1}{R_2} \lambda^2 \operatorname{ctg}^2 \alpha \right] = -\frac{R_1}{D_1} V. \quad (512)$$

Thus, the axisymmetric deformation of shells of rotation is described by system of Eq. (511), (512), which it is convenient to present in the following symmetrical form

$$L(V) + \left(\frac{v_2}{R_1} - \frac{k_1}{R_2} \right) V = E_2 \delta \varphi, \quad (513)$$

$$L(\varphi) - \frac{v_2}{R_1} \varphi = -\frac{V}{D_1}, \quad (514)$$

where $L()$ designates the differential operator in second order partial derivatives

$$L() = R_2 \frac{d^2}{d\zeta^2} + \left[\left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \operatorname{ctg} \alpha \right] \frac{d}{d\zeta} - \frac{\lambda^2}{R_2} \operatorname{ctg}^2 \alpha (). \quad (515)$$

$$k_1 = \frac{B_2(1-\nu_1\nu_2)}{K_1} = \frac{6E_2}{5G_{12}},$$

here $\zeta = R_1 \alpha$ is the length of the arc of the meridian of the mean surface of the shell.

Consideration of shells of rotation of variable thickness $\delta = \delta(\alpha)$ is of practical interest, since the manufacture of such shells from laminated plastics involves no fundamental difficulties compared with shells of constant thickness.

Without repeating the calculation, for axisymmetrically loaded shells of rotation of variable thickness the following system of differential equations can be obtained

$$\left. \begin{aligned} L(V) + \left(\frac{v_2}{R_1} - \frac{k_1}{R_2} \right) V - \frac{\delta}{\delta} (R_2 \dot{V} - v_2 V \operatorname{ctg} \alpha) &= E_2 \delta \varphi; \\ L(\varphi) - \frac{v_2}{R_1} \varphi + 3 \frac{\delta}{\delta} (R_2 \dot{\varphi} + v_2 \varphi \operatorname{ctg} \alpha) &= -\frac{V}{D_1}, \end{aligned} \right\} \quad (516)$$

where the points designate differentiation over the arc of the meridian ζ .

Eq. (513), (514) and (516) are valid for laminated anisotropic shells of rotation of constant and variable thickness, which are made of laminated plastics of unchanging elastic constants along the meridian. However, if a convex shell of rotation is made by continuous winding of fiber fillers, as was pointed out above, the elastic constants of the material will change along the meridian, since the fiber packing density increases toward the pole of the shell. /156

The structure of the equations and asymptotic analysis of the solutions show that system of Eq. (513), (514) or (516) describe the simple end effects phenomenon, i.e., for not very "short" shells, the solutions contain terms which are rapidly damped with increase in the argument of ζ [1]. Consequently, the equations obtained will be valid

for shells of rotation, where there are factors which cause local flexural deformations.

If the indices of variability of the external loads, elastic constants and geometric dimensions of the shells are not very large, Eq. (513), (514) for constant thickness shells of rotation and Eq. (516) for variable thickness shells of rotation remain valid. However, further simplifications can be made in these equations.

48. Differential Equations of Technical Theory of Axisymmetrically Loaded Shells of Rotation

Exact solutions of Eq. (513), (514) present great mathematical difficulties, and they are obtained only with some simplifying hypotheses for individual cases. At the same time, for engineering applications in many cases, it is quite sufficient to be limited to the first approximation of asymptotic integration [7]. The error of the simplified equations which correspond to the first approximation is $\lambda\sqrt{\delta/R}$ compared with unity, i.e., it is completely sufficient for "thin" shells.

The technical theory equations can be obtained, if it is assumed that the index of variability of displacement functions ϕ and stress functions V are considerably greater than the indices of variability of geometric dimensions R_1, R_2, δ , elastic constants c_{1j}, a_{1j} and the external load applied to the shell.

By multiplying Eq. (514) by random complex constant $z=x+iy$ and summing with Eq. (513), we obtain

$$z \left[L(\phi) - \frac{\nu_2}{R_1} \phi - \frac{R_2 \delta}{z} \phi \right] + \left[L(V) - \left(\frac{k_1}{R_2} - \frac{\nu_2}{R_1} \right) V + \frac{V}{D_1} z \right] = 0. \quad (517)$$

Following A.I. Lur'ye [18], we set

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$$\left. \begin{aligned} \frac{\nu_2}{R_1} - \frac{k_1}{R_2} + \frac{z}{D_1} &= a + bi; \\ \frac{\nu_2}{R_1} + \frac{E_2 \delta}{z} &= -(a + bi). \end{aligned} \right\} \quad (518)$$

By substituting $z=x+iy$ in Eq. (518) and separating the real and imaginary parts, we obtain

$$\left. \begin{aligned} \frac{\nu_2}{R_1} - \frac{k_1}{R_2} + \frac{x}{D_1} &= a; \\ y &= D_1 b; \\ \frac{\nu_2}{R_1} + \frac{E_2 \delta x}{x^2 + y^2} &= -a; \\ \frac{y E_2 \delta}{x^2 + y^2} &= b. \end{aligned} \right\} \quad (519)$$

By disregarding the small terms, the following values of the constants introduced can be obtained:

$$\left. \begin{aligned}
 x &\approx \frac{k_1 D_1}{2R_2} \approx \omega^2 D_1 R_2; \\
 y &= D_1 R_2 \sqrt{\frac{12\lambda^2(1-\nu_1\nu_2)}{R_2^2 \delta^2} - \frac{k_1^2}{4R_2^2}} \approx -D_1 D_2 \sqrt{k^4 - \omega^4}; \\
 a &= -\frac{k_1}{2R_2} = -\omega^2 R_2; \\
 b &= R_2 \sqrt{\frac{12\lambda^2(1-\nu_1\nu_2)}{R_2^2 \delta^2} - \frac{k_1^2}{4R_2^2}} = -R_2 \sqrt{k^4 - \omega^4}.
 \end{aligned} \right\} \quad (520)$$

where

$$\left. \begin{aligned}
 \omega^2 &= \frac{k_1}{2R_2} = \frac{\beta_2(1-\nu_1\nu_2)}{2K_1 R_2^2}; \\
 k^2 &= \frac{\lambda \sqrt{12(1-\nu_1\nu_2)}}{R_2 \delta}.
 \end{aligned} \right\} \quad (521)$$

Consequently, Eq. (517) takes the form

$$L(U) + (a+bi)U = 0, \quad (522)$$

where U is the complex function of stresses and deformations determined from the expression

$$U + V + (x+iy)\phi. \quad (523)$$

Because of the assumptions made as to the nature of variability of the quantities, operator $L()$, which can be determined by Eq. (515), can be simplified

$$L() = R_2 \frac{d^2}{d\zeta^2} + \left[\left(\frac{R_2}{R_1} \right)' + \frac{R_2}{R_1} \operatorname{ctg} \alpha \right] \frac{d}{d\zeta} - \frac{\lambda^2}{R_2} \operatorname{ctg}^2 \alpha. \quad (524)$$

Far from the pole of the shell ($\alpha=0$), the last terms in this operator can be disregarded and, consequently, the deformed and stress-/158
 ed states of the axisymmetrically loaded shell of rotation in zones quite far from the pole are described by the following second order differential equation

$$\frac{d^2 U}{d\zeta^2} - [\omega^2 - i] \sqrt{k^4 - \omega^4} U = 0. \quad (525)$$

The elastic forces, bending moments and radial movement of the shell far from the pole are determined by the formulas

$$\left. \begin{aligned}
 Q_1 &= \frac{V}{R_2}; \quad T_1 = -\frac{V \operatorname{ctg} \alpha}{R_2}; \quad T_2 = -\frac{dV}{d\zeta}; \\
 G_1 &= -D_1 \frac{d\varphi}{d\zeta}; \quad G_2 = \nu_2 G_1; \\
 \xi &= \Delta r = -\frac{R_2}{E_2 \delta} \frac{dV}{d\zeta} \sin \alpha.
 \end{aligned} \right\} \quad (526)$$

Correspondingly, the stressed and deformed states of the shell of rotation near the pole ($\alpha=0$) are described by the following differential equation

$$\frac{d^2U}{d\zeta^2} + \frac{1}{\zeta} \frac{dU}{d\zeta} - \left[\omega^2 + \frac{c^2}{\zeta^2} - i\sqrt{k^2 - \omega^2} \right] U = 0, \quad (527)$$

where

$$c^2 = \frac{\lambda^2 R_1^2}{R_2^2}. \quad (528)$$

The elastic forces, bending moments and radial movements of the shell in the area of the pole are determined through the stress and deformation function by the formulas

$$\left. \begin{aligned} Q_1 &= \frac{V}{R_1}; & T_1 &= -\frac{VR_1}{\zeta R_2}; & T_2 &= -\frac{dV}{d\zeta}; \\ G_1 &= -D_1 \left(\frac{d\varphi}{d\zeta} + \nu_2 \frac{\varphi R_1}{\zeta R_2} \right); \\ G_2 &= -D_2 \left(\frac{\varphi R_1}{\zeta R_2} + \nu_1 \frac{d\varphi}{d\zeta} \right); \\ \xi = \Delta r &= -\frac{R_2 \sin \alpha}{E_2 \delta} \left(\frac{dV}{d\zeta} - \nu_2 \frac{VR_1}{\zeta R_2} \right). \end{aligned} \right\} \quad (529)$$

In accordance with Eq. (523), if a solution of differential Eq. (525) or (527) is found, i.e., if the following complex function of stresses and deformations is found

$$U(\zeta) = \text{Re}U(\zeta) + i\text{Im}U(\zeta), \quad (530)$$

functions ϕ and V are determined by the following expressions

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$$\left. \begin{aligned} \varphi &= \frac{\text{Im}U(\zeta)}{y} = -\frac{\text{Im}U(\zeta)}{2r_2 D_1 R_2}; \\ V &= \text{Re}U(\zeta) + \frac{s^2 - r^2}{2r_2} \text{Im}U(\zeta), \end{aligned} \right\} \quad (531)$$

where

$$s = \sqrt{\frac{k^2 + \omega^2}{2}}; \quad r = \sqrt{\frac{k^2 - \omega^2}{2}}. \quad (532)$$

49. Calculation of Axisymmetrically Loaded Shells of Rotation in Their Coupling Zones

The examples of axisymmetrically loaded shells of rotation most often found in practice are the bottoms of cylindrical containers operating under internal pressure. In chemical containers, a bottom made up of smoothly joined spherical, conical and toroidal shells is used. At the joint sites in such shells, local bending stresses and deformations appear which are described by differential Eq. (525).

The solution of Eq. (525) which disappears at infinity can be written in the following form

$$U = (C'_1 + iC'_2) e^{-\xi_0} (\sin rs + i \cos rs). \quad (533)$$

Consequently, the real and imaginary parts

$$\left. \begin{aligned} \operatorname{Re} U(\zeta) &= C'_1 \Phi_2 - C'_2 \Phi_1; \\ \operatorname{Im} U(\zeta) &= C'_1 \Phi_1 + C'_2 \Phi_2. \end{aligned} \right\} \quad (534)$$

where $\Phi_1(\zeta)$, $\Phi_2(\zeta)$ are exponential trigonometric functions which disappear at infinity and are determined by Eq. (463). They are satisfied by Eq. (465), which are very convenient for practical strength calculations.

In accordance with Eq. (531), we obtain

$$\left. \begin{aligned} \varphi &= -\frac{1}{D_1} (C_1 \Phi_1 + C_2 \Phi_2); \\ \frac{v}{R_2} &= C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + C_2 [(s^2 - r^2) \Phi_2 - 2rs \Phi_1]. \end{aligned} \right\} \quad (535)$$

The elastic forces, bending moments and radial movements of an axisymmetrically loaded shell of rotation are determined by the following formulas /160

$$\left. \begin{aligned} Q_1 &= Q_0 + C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + \\ &\quad + C_2 [(s^2 - r^2) \Phi_2 - 2rs \Phi_1]; \\ G_1 &= G_0 \mp [C_1 (s \Phi_1 + r \Phi_2) - C_2 (r \Phi_1 - s \Phi_2)]; \\ T_1 &= -[C_1 [(s^2 - r^2) \Phi_1 + 2rs \Phi_2] + \\ &\quad + C_2 [(s^2 - r^2) \Phi_2 - 2rs \Phi_1]] \operatorname{ctg} \alpha + T_1^0; \\ T_2 &= \pm [C_1 [s(3r^2 - s^2) \Phi_1 - r(3s^2 - r^2) \Phi_2] + \\ &\quad + C_2 [r(3s^2 - r^2) \Phi_1 + s(3r^2 - s^2) \Phi_2]] R_2 + T_2^0; \\ \xi &= \pm \frac{R_2^2}{E_2 \delta} [C_1 [s(3r^2 - s^2) \Phi_1 - r(3s^2 - r^2) \Phi_2] + \\ &\quad + C_2 [r(3s^2 - r^2) \Phi_1 + s(3r^2 - s^2) \Phi_2]] \sin \alpha + \xi_0. \end{aligned} \right\} \quad (536)$$

where quantities obtained according to membrane theory are noted by the symbol "o." In Eq. (535), (536), angular coordinate β is reckoned from the section where the local stresses are investigated. The upper and lower signs concern the lower (I) and upper (II) regions of the shell, respectively (Fig. 70).

We consider a quite "long" shell of rotation, at one end of which $\beta=0$ bending moments G_0 and thrusts P_0 are applied (Fig. 71).

For anisotropic shells of rotation in study of the deformed and stressed states under axisymmetrical loading, the concepts "long" and

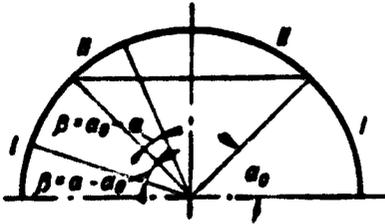


Fig. 70. Diagram of arbitrary regions of propagation of end effects in shell of rotation.

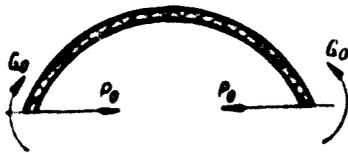


Fig. 71. End forces in shell of rotation.

"short," as was pointed out earlier, are not purely geometrical but, rather, they are determined by elastic constants of the shell λ^2 , k_1 or by parameter sl .

With given thrust P_0 , cutting force $Q_0 = P_0 \sin \alpha$ and, consequently, integration constants C_1 , C_2 are determined through initial parameters G_0 , Q_0 by the following expressions /161

$$C_1 = \frac{2sG_0 - Q_0}{s^2 + r^2}; \quad C_2 = \frac{(s^2 - r^2)G_0 - sQ_0}{r(s^2 + r^2)}. \quad (537)$$

Thus, in expressions of the elastic forces, bending moments and deformations through the initial parameters, there are the following formulas

$$\left. \begin{aligned} Q_1 &= G_0 \frac{s^2 + r^2}{r} \Phi_2 + Q_0 \left(\Phi_1 - \frac{s}{r} \Phi_2 \right) + Q_0; \\ G_1 &= G_0 \left(\Phi_1 + \frac{s}{r} \Phi_2 \right) - Q_0 \frac{\Phi_2}{r} + G_0; \\ \xi &= \xi_0 + \frac{r^2}{E_2 b} \left[G_0 (r^2 + s^2) \left(\Phi_1 - \frac{s}{r} \Phi_2 \right) - \right. \\ &\quad \left. - Q_0 \left(2s \Phi_1 - \frac{s^2 - r^2}{r} \Phi_2 \right) \right] \sin \alpha; \\ T_2 &= R_2 \left[G_0 (r^2 + s^2) \left(\Phi_1 - \frac{s}{r} \Phi_2 \right) - \right. \\ &\quad \left. - Q_0 \left(2s \Phi_1 - \frac{s^2 - r^2}{r} \Phi_2 \right) \right] + T_2^0; \\ T_1 &= - \left\{ G_0 \frac{s^2 + r^2}{r} \Phi_2 + Q_0 \left(\Phi_1 + \frac{s}{r} \Phi_2 \right) \right\} \operatorname{ctg} \alpha + T_1^0; \\ \varphi &= - \frac{1}{(r^2 + s^2) D_1} \left[G_0 \left(2s \Phi_1 + \frac{s^2 - r^2}{r} \Phi_2 \right) - \right. \\ &\quad \left. - Q_0 \left(\Phi_1 + \frac{s}{r} \Phi_2 \right) \right]. \end{aligned} \right\} (538)$$

As an example, we consider the problem of calculation of joining a cylindrical tank to a spherical bottom. The stressed and deformed states in the butt zone of the section are determined: in the bottom, by Eq. (538); in the cylindrical part of the container, by Eq. (492), with $\gamma=0$.

We designate all parameters of the cylindrical part of the container by exponent I and all quantities which relate to the bottom by exponent II.

From the deformation compatibility condition, the following values of initial parameters G_0 , P_0 can be found, which determine the forces

of interaction of the cylindrical part of the container with the bottom:

$$\left. \begin{aligned}
 P_0 &= P \frac{(s_1 d + s_{11}) \left[\frac{1 - \nu_{11}^I}{\sin \alpha} - (2 - \nu_1^I) e - \right.}{4 [s_1^2 e d + s_1 s_{11} (e + d \sin \alpha) +} \rightarrow \\
 &\quad \left. - 2 s_{11} \cos \alpha \right] + (k_{11}^2 - k_1^2) \frac{s \cos \alpha}{2}}{+ s_{11}^2 \sin \alpha + k_{11}^2 (d - \sin \alpha) - k_1^2 e (d \sin \alpha)} ; \\
 G_0 &= P \frac{s (s_1 e + s_{11} \sin \alpha) \cos \alpha +}{4 [s_1^2 e d + s_1 s_{11} (e + d \sin \alpha) + s_{11}^2 \sin \alpha} \rightarrow \\
 &\quad + \left[\frac{(2 - \nu_1^I) R}{2} + s_{11} \sin \alpha - \frac{1 - \nu_{11}^I}{2 \sin \alpha} \right] (d - \sin \alpha)}{+ k_{11}^2 (d - \sin \alpha) - k_1^2 e (d \sin \alpha)} ,
 \end{aligned} \right\} \quad (539)$$

where

$$\frac{E_{11}^I \delta_{11}}{E_1^I \delta_1} = e; \quad \frac{D_{11}^I k_{11}^2}{D_1^I k_1^2} = d. \quad (540)$$

In order to decrease the end effects which arise in the butt sections of the containers, it is advisable to make them as smooth and continuous as possible. Smoothness and continuity are particularly important in the butt section of a container where the cylindrical part is joined to the bottom. However, continuity of the curvature of the meridian cannot be preserved while ensuring smoothness of the contours in this section. /162

In the smoothest and most continuous coupling of the bottom to the cylindrical part of the container, except for the bending moment in the butt section, the initial thrusting forces disappear. With sharp breaks in coupling, they can lead to local collapse of the butt section of the container as a result of loss of stability. Examples of smooth nonthrust bottoms are ellipsoidal, box and hemispherical bottoms. If, for some reason, a smooth coupling cannot be achieved, the sections where there is a break in continuity should be reinforced with rings.

50. Flattened Laminated Spherical Shell Subjected to Concentrated Forces Applied at the Poles

We consider the problem of calculation of a thin laminated spherical shell loaded with concentrated forces at the poles (Fig. 72).

For isotropic spherical shells, such studies have been conducted in [7, 18]. It is evident that laminated spherical shells which are subjected to concentrated forces do not need to be anisotropic. More than that, if the point of application of the concentrated forces is not fixed, the most nearly optimum structure of the material is a full strength structure. Just such a case will be considered. /163

Parameter c^2 in Eq. (527), which describe the stressed and deformed states of axisymmetrically loaded shells of rotation near the pole, equals unity. Consequently, the basic differential equation of the

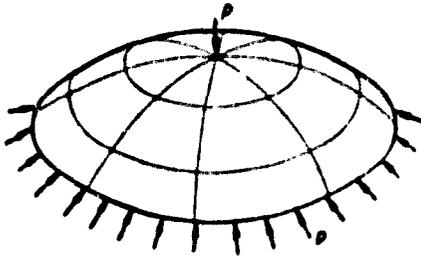


Fig. 72. Spherical shell subjected to concentrated forces.

problem under consideration has the form

$$\frac{d^2 U}{d\xi^2} + \frac{1}{\xi} \frac{dU}{d\xi} - [\omega^2 + \frac{1}{\xi^2} - i\sqrt{k^4 - \omega^4}] U = 0. \quad (541)$$

The elastic forces, bending moments and radial movements of a full strength spherical shell are determined through the stress and deformation functions by the following expressions

$$\left. \begin{aligned} Q_1 &= \frac{V}{R}; & T_1 &= -\frac{V}{\xi}; & T_2 &= -\frac{dV}{d\xi}; \\ G_1 &= D \left(\frac{\partial \varphi}{\partial \xi} + \nu \frac{\varphi}{\xi} \right); & G_2 &= -D \left(\frac{\varphi}{\xi} + \nu \frac{d\varphi}{d\xi} \right); \\ \Delta r &= -\frac{\xi}{E\delta} \left(\frac{dV}{d\xi} - \nu \frac{V}{\xi} \right). \end{aligned} \right\} \quad (524)$$

We make the following substitution of variables

$$\zeta = \frac{\xi^2}{\sqrt{\omega^2 - i\sqrt{k^4 - \omega^4}}}. \quad (543)$$

We then obtain

$$\left. \begin{aligned} \frac{dU}{d\zeta} &= \frac{\sqrt{\omega^2 - i\sqrt{k^4 - \omega^4}}}{i} \frac{dU}{d\xi}; \\ \frac{d^2 U}{d\zeta^2} &= -[\omega^2 - i\sqrt{k^4 - \omega^4}] \frac{d^2 U}{d\xi^2}. \end{aligned} \right\} \quad (544)$$

Differential Eq. (541) takes the canonical form

$$\frac{d^2 U}{d\zeta^2} + \frac{1}{\zeta} \frac{dU}{d\zeta} + \left(1 - \frac{1}{\zeta^2}\right) U = 0. \quad (545)$$

The solution of Eq. (545) is written in first order Bessel functions of the first and second kinds

$$U = C_1 J_1(\xi) + C_2 Y_1(\xi). \quad (546)$$

The deformed and stressed states of the shell near the pole are described by the second term, since they should decrease with increase in the argument. Consequently, $C_1 \neq 0$ must be set.

By switching to initial argument ζ , which is the length of an arc along the meridian, we obtain

$$U = C_2 Y_1[(r+is)\zeta], \quad (547)$$

where C_2 is the complex integration constant.

By separating the real and imaginary parts, we obtain

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$$U = (\bar{A} + \bar{B}i)(\operatorname{Re} Y_1 + i \operatorname{Im} Y_1), \quad (548)$$

and, consequently

$$\left. \begin{aligned} \operatorname{Re} U &= \bar{A} \operatorname{Re} Y_1 - \bar{B} \operatorname{Im} Y_1; \\ \operatorname{Im} U &= \bar{B} \operatorname{Re} Y_1 + \bar{A} \operatorname{Im} Y_1. \end{aligned} \right\} \quad (549)$$

In accordance with Eq. (531), the deformation and stress functions are determined by the expressions

$$\left. \begin{aligned} \varphi &= -\frac{\bar{A} \operatorname{Im} Y_1 + \bar{B} \operatorname{Re} Y_1}{2rsRD}; \\ V &= \bar{A} \left(\operatorname{Re} Y_1 + \frac{s^2 - r^2}{2rs} \operatorname{Im} Y_1 \right) - \\ &\quad - \bar{B} \left(\operatorname{Im} Y_1 - \frac{s^2 - r^2}{2rs} \operatorname{Re} Y_1 \right). \end{aligned} \right\} \quad (550)$$

or, after introduction of new constants

$$\left. \begin{aligned} \varphi &= -\frac{A \operatorname{Im} Y_1 + B \operatorname{Re} Y_1}{DR}; \\ V &= A [2rs \operatorname{Re} Y_1 + (s^2 - r^2) \operatorname{Im} Y_1] - \\ &\quad - B [2rs \operatorname{Im} Y_1 - (s^2 - r^2) \operatorname{Re} Y_1]. \end{aligned} \right\} \quad (551)$$

We find integration constants A, B from the boundary conditions at the pole of the shell, namely, in the limit as $\zeta \rightarrow 0$, the following conditions should be satisfied

$$\phi = 0; \quad 2\pi\zeta V = -PR. \quad (552)$$

Since $Y_1(z)_{\zeta \rightarrow 0} \approx -\frac{2}{\pi} \cdot \frac{1}{z}$, we obtain

$$\operatorname{Re} Y_1|_{\zeta \rightarrow 0} = -\frac{2r}{\pi k^2 \zeta}; \quad \operatorname{Im} Y_1|_{\zeta \rightarrow 0} = \frac{2s}{\pi k^2 \zeta}. \quad (553)$$

By substituting Eq. (553) in boundary conditions (552), we obtain

$$A = \frac{PR}{8s}; \quad B = \frac{PR}{8r}. \quad (554)$$

Thus, the deformation and stress functions take the following final form

$$\left. \begin{aligned} \varphi &= -\frac{P}{8rsD} (s \operatorname{Re} Y_1 + r \operatorname{Im} Y_1); \\ V &= \frac{Pk^2R}{8rs} (s \operatorname{Re} Y_1 - r \operatorname{Im} Y_1). \end{aligned} \right\} \quad (555)$$

For a full strength spherical shell near the pole, according to Eq. (509), /165

$$\left. \begin{aligned} \frac{du}{da} + w &= \frac{1}{E\delta} \left(v \frac{dV}{da} - \frac{V}{a} \right); \\ \frac{u}{a} + w &= \frac{1}{E\delta} \left(v \frac{V}{a} - \frac{dV}{da} \right). \end{aligned} \right\} \quad (556)$$

By subtracting the second equation from the first, we obtain

$$\frac{du}{da} - \frac{u}{a} = \frac{1+v}{E\delta} \left(\frac{dV}{da} - \frac{V}{a} \right). \quad (557)$$

Consequently, the movement of points along the meridian and the deflection of the shell are determined by the following formulas

$$\left. \begin{aligned} u &= \frac{1+v}{E\delta} V; \\ w &= -\frac{R}{E\delta} \left(\frac{dV}{a^2} + \frac{V}{a} \right). \end{aligned} \right\} \quad (558)$$

The following rule of differentiation of a Bessel function must be taken into account in determining the forces and movements:

$$\left. \begin{aligned} \frac{d}{d\zeta} (\operatorname{Re} Y_1) &= r \operatorname{Re} Y_0 - s \operatorname{Im} Y_0 - \frac{\operatorname{Re} Y_1}{\zeta}; \\ \frac{d}{d\zeta} (\operatorname{Im} Y_1) &= s \operatorname{Re} Y_0 + r \operatorname{Im} Y_0 - \frac{\operatorname{Im} Y_1}{\zeta}. \end{aligned} \right\} \quad (559)$$

The deflection of the shell at the point of application of the concentrated force

$$w = \frac{Pk^4 R^2}{4E\delta \gamma \sqrt{k^4 - \omega^4}} \operatorname{Im} Y_0. \quad (560)$$

The elastic forces and moments which arise in the shell are determined by the expressions

$$\left. \begin{aligned} Q_1 &= \frac{Pk^2}{8rs} (s \operatorname{Re} Y_1 - r \operatorname{Im} Y_1); \\ G_1 &= \frac{P}{8rs} \left\{ 2rs \operatorname{Re} Y_0 - (s^2 - r^2) \operatorname{Im} Y_0 - \right. \\ &\quad \left. - \frac{1-v}{\zeta} [s \operatorname{Re} Y_1 + r \operatorname{Im} Y_1] \right\}; \\ T_1 &= -\frac{Pk^2 R}{8rs} (s \operatorname{Re} Y_1 - r \operatorname{Im} Y_1) \frac{1}{\zeta} - \frac{P}{2\pi R} \frac{1}{\sin^2 \alpha}; \\ T_2 &= \frac{Pk^4 R}{8rs} \left[k^2 \operatorname{Im} Y_0 + \frac{1}{\zeta} (s \operatorname{Re} Y_1 - r \operatorname{Im} Y_1) \right] + \\ &\quad + \frac{P}{2\pi R} \frac{1}{\sin^2 \alpha}. \end{aligned} \right\} \quad (561)$$

For an axisymmetrically loaded orthotropic shell of rotation, differential Eq. (527) near the pole can be reduced to canonical form by the same substitution as for a spherical shell

$$\frac{d^2U}{d\xi^2} + \frac{1}{\xi} \frac{dU}{d\xi} + \left(1 - \frac{n^2}{\xi^2}\right)U = 0. \tag{562}$$

The solution of this equation is expressed through Bessel functions of the first and second kinds of order $n=c$, which is determined by the type of pole of the shell and the nature of the anisotropy.

The solution of differential Eq. (562) which decreases with increase in the argument of ξ has the form

$$U = c_2 Y_n[(r+si)\xi]. \tag{563}$$

51. Calculation of Temperature Compensated Pipe

For compensation of temperature deformation of long cylindrical pipes which operate under uniform internal or external pressure, small annular toroidal corrugations are made very often (Fig. 73).

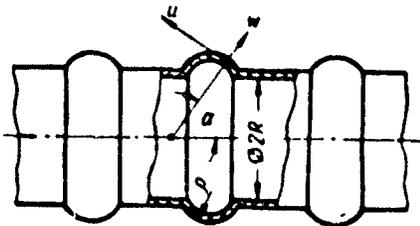


Fig. 73. Pipe with external corrugations.

Within one corrugation, $\alpha_0 \leq \alpha \leq \pi - \alpha_0$. We will assume the radius of curvature of the corrugation to be smaller than the pipe radius, and shell thickness δ to be smaller than corrugation radius ρ .

The cross section radius of the corrugated portion

$$r = R + \rho(\sin \alpha - \sin \alpha_0). \tag{564}$$

According to Eq. (504), (505), in axisymmetric loading, the elastic forces and moments are connected with the deformed state of the shell by the following expressions

$$\left. \begin{aligned} T_1 &= B_1 \left(\frac{u' + w}{\rho} + \nu_2 \frac{u \cos \alpha + w \sin \alpha}{r} \right); \\ T_2 &= B_2 \left(\frac{u \cos \alpha + w \sin \alpha}{r} + \nu_1 \frac{u' + w}{\rho} \right); \\ G_1 &= -D_1 \left(\frac{\varphi'}{\rho} + \nu_2 \frac{\varphi \cos \alpha}{r} \right); \\ G_2 &= -D_2 \left(\frac{\varphi \cos \alpha}{r} + \nu_1 \frac{\varphi}{\rho} \right); \\ Q_1 &= -K_1 \left(\varphi + \frac{w' - u}{\rho} \right). \end{aligned} \right\} \tag{565}$$

We present the forces, bending moments and movements in the form /167 of expansions by small parameter ρ/R

$$\left. \begin{aligned}
 T_1 &= \sum_{n=0}^{\infty} T_1^{(n)} \left(\frac{\rho}{R} \right)^n; \\
 T_2 &= \sum_{n=0}^{\infty} T_2^{(n)} \left(\frac{\rho}{R} \right)^n; \\
 Q_1 &= \sum_{n=0}^{\infty} Q_n \left(\frac{\rho}{R} \right)^n; \\
 G_1 &= \sum_{n=0}^{\infty} G_1^{(n)} \left(\frac{\rho}{R} \right)^n; \\
 G_2 &= \sum_{n=0}^{\infty} G_2^{(n)} \left(\frac{\rho}{R} \right)^n; \\
 u &= \sum_{n=0}^{\infty} u_n \left(\frac{\rho}{R} \right)^n; \\
 \varphi &= \sum_{n=0}^{\infty} \varphi_n \left(\frac{\rho}{R} \right)^n; \\
 w &= \sum_{n=0}^{\infty} w_n \left(\frac{\rho}{R} \right)^n.
 \end{aligned} \right\} (566)$$

In accordance with Eq. (565), the coefficients of the expansion are determined by the following expressions

$$\left. \begin{aligned}
 T_1^{(n)} &= \frac{B_1}{\rho} \left[u_n' + w_n + v_2 \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \times \right. \\
 &\quad \left. \times (u_j \cos \alpha + w_j \sin \alpha) \right]; \\
 T_2^{(n)} &= \frac{B_2}{\rho} \left[v_2 (u_n' + w_n) + \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \times \right. \\
 &\quad \left. \times (u_j \cos \alpha + w_j \sin \alpha) \right]; \\
 Q_n &= -\frac{K_1}{\rho} (\rho \varphi_n + w_n' - u_n); \\
 G_1^{(n)} &= -\frac{D_1}{\rho^2} \left[\varphi_n' + v_2 \cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \varphi_j \right]; \\
 G_2^{(n)} &= -\frac{D_2}{\rho^2} \left[v_1 \varphi_n' + \cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \varphi_j \right]
 \end{aligned} \right\} (567)$$

for $n=0, 1, 2, 3, \dots$, where

$$\eta = \sin \alpha - \sin \alpha_0. \quad (568)$$

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By substituting Eq. (567) in the equilibrium equations, we obtain

$$\left. \begin{aligned} (rT_1)' - T_2 q \cos \alpha - Q_1 r &= 0; \\ (rQ_1)' + T_1 r + T_2 q \sin \alpha &= prq; \\ (rG_1)' - G_2 q \cos \alpha - Q_1 q r &= 0; \end{aligned} \right\} \quad (569)$$

a recurrent sequence of systems of differential equations can be obtained for solution of the problem considered.

For $n=0$,

$$\left. \begin{aligned} \frac{dT_1^0}{d\alpha} - Q_0 &= 0; \\ \frac{dQ_0}{d\alpha} + T_1^0 &= prq; \\ \frac{dG_1^0}{d\alpha} - qQ_0 &= 0. \end{aligned} \right\} \quad (570)$$

For $n \geq 1$,

$$\left. \begin{aligned} \frac{dT_1^n}{d\alpha} - Q_n &= -\cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (T_1^{(j)} - T_2^{(j)}); \\ \frac{dQ_n}{d\alpha} + T_1^{(n)} &= -\sum_{j=0}^{n-1} (-\eta)^{n-j-1} (Q_j \cos \alpha + T_2^{(j)} \sin \alpha); \\ \frac{dG_1^{(n)}}{d\alpha} - qQ_n &= -\cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (G_1^{(j)} - G_2^{(j)}). \end{aligned} \right\} \quad (571)$$

The deformed state of the shell is determined by solutions of the following differential equations

$$u_n' = -\frac{q^2}{D_1} G_1^{(n)} - v_2 \cos \alpha \sum_{j=0}^{n-1} (-\eta)^{n-j-1} \varphi_j; \quad (572)$$

$$\begin{aligned} w_n' + w_n &= \frac{q}{B_1} T_1^{(n)} - \frac{q}{K_1} Q_n' - q\varphi_n' - \\ &- v_2 \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (u_j \cos \alpha + w_j \sin \alpha); \end{aligned} \quad (573)$$

$$u_n' = \frac{q}{B_1} T_1^{(n)} - u_n - v_2 \sum_{j=0}^{n-1} (-\eta)^{n-j-1} (u_j \cos \alpha + w_j \sin \alpha) \quad (574)$$

for $n=0, 1, 2, \dots$

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There are differential equations of the (573) type for determination of forces $T_1^{(n)}$, Q_n and, since integration of Eq. (572), (574) presents no fundamental difficulties, the problem of calculation of cylindrical pipes with transverse corrugations is reduced to the solu-

tion of second order differential equations of the type

$$y'' + y = f(\alpha). \quad (575)$$

If the right side of Eq. (575) is presented in the form of the expansion in trigonometric series

$$f(\alpha) = \sum_{m=0}^{\infty} (A_m \cos m\alpha + B_m \sin m\alpha), \quad (576)$$

where

$$A_m = \frac{2}{\pi} \int_0^{\pi} f(\alpha) \cos m\alpha \, d\alpha; \quad B_m = \frac{2}{\pi} \int_0^{\pi} f(\alpha) \sin m\alpha \, d\alpha; \quad (577)$$

the general solution of differential Eq. (575) is in the form

$$y(\alpha) = C_1 \cos \alpha + C_2 \sin \alpha + A_0 + \alpha (A_1 \sin \alpha - B_1 \cos \alpha) - \sum_{m=2}^{\infty} \left(\frac{A_m}{m^2 - 1} \cos m\alpha + \frac{B_m}{m^2 - 1} \sin m\alpha \right). \quad (578)$$

In the general case, the solution of Eq. (575) can be written in the form

$$y(\alpha) = C_1 \cos \alpha + C_2 \sin \alpha + \sin \alpha \int_0^{\alpha} f(\xi) \cos \xi \, d\xi - \cos \alpha \int_0^{\alpha} f(\xi) \sin \xi \, d\xi. \quad (579)$$

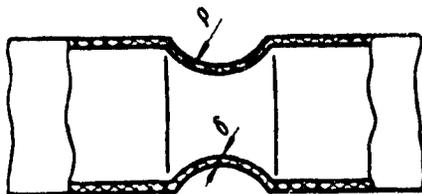


Fig. 74. Pipe with inward corrugations.

An advantage of the proposed method of calculation is that all the relationships obtained remain valid for negative curvature corrugations (Fig. 74), if the sign of ρ is changed. /170

In particular, compensators with corrugation of alternating sign curvature which are continuously and smoothly joined together can be considered. In this case, it is not strictly possible to speak of pipe with corrugations, since such shells are more like bellows. However, in view of the absence of annular or conical springs, the rigidity of such degenerate bellows will be very substantial, approximately $(R/\rho)^3$ times greater than the rigidity of conventional noncompacted bellows.

52. Thermoelastic Stresses Generated in Orthotropic Shells of Rotation by Axisymmetric Heating

In the use of shells made of laminated plastics, it must be kept

in mind that, at comparatively low temperatures, the change in thermo-physical properties of the material must be taken into account. This is the basic difference of heat problems for laminated anisotropic shells from the corresponding problems for isotropic metal shells, which usually are solved on the assumption that the elastic constants are independent of temperature.

We suppose that axisymmetric heating of a shell of rotation occurs to a temperature at which the Duhamel-Neumann thermoelastic hypothesis remains valid, and that the creep of the material can be disregarded. With such assumptions, problems were solved in [10] without taking interlayer shearing into account.

We will assume that the shell material is orthotropic and that the principal axes of anisotropy coincide with the lines of curvature of the mean surface of the shell, i.e., they coincide with the coordinate directions at each point of the surface. Since, in an orthotropic shell under tension and compression in the principal directions of anisotropy, the latter remain the principal axes of anisotropy, we will consider that heating does not distort the angles between the axes of elastic symmetry of the material.

If the standard thermoelastic hypotheses which established the connection of the thermoelastic stresses with deformations are retained, the relationship of elasticity in heating to temperature t can be written in the form

$$\left. \begin{aligned} e_1 &= \frac{\sigma_1}{E_1} - \frac{\nu_2}{E_2} \sigma_2 + \alpha_1 t; \\ e_2 &= \frac{\sigma_2}{E_2} - \frac{\nu_1}{E_1} \sigma_1 + \alpha_2 t. \end{aligned} \right\} \quad (580)$$

where e_1, e_2 are the components of the total deformation; α_1, α_2 are the coefficients of linear expansion of the material in the axial and annular directions.

Elasticity relationships (580) can be presented for the stress components in the following form /171

$$\left. \begin{aligned} \sigma_1 &= c_{11}e_1 + c_{12}e_2 - \beta_1 t; \\ \sigma_2 &= c_{12}e_1 + c_{22}e_2 - \beta_2 t. \end{aligned} \right\} \quad (581)$$

where

$$\left. \begin{aligned} c_{11} &= \bar{E}_1; & c_{22} &= \bar{E}_2; & c_{12} &= \nu_2 \bar{E}_1 = \nu_1 \bar{E}_2, \\ \beta_1 &= c_{11}(\alpha_1 + \nu_2 \alpha_2); \\ \beta_2 &= c_{22}(\alpha_2 + \nu_1 \alpha_1). \end{aligned} \right\} \quad (582)$$

In Eq. (580), (581), thermoelastic "constants" $E_1, E_2, \nu_1, \nu_2, c_{11}, c_{12}, c_{22}, \alpha_1, \alpha_2, \beta_1, \beta_2$ depend on shell temperature $t=t(\zeta, z)$, i.e., in the general case, they are functions of coordinates (ζ, z) .

By virtue of the rectilinear elements hypothesis, according to Eq. (1),

$$\left. \begin{aligned} \sigma_1 &= c_{11} \epsilon_1 + c_{12} \epsilon_2 + z (c_{11} \kappa_1^e + c_{12} \kappa_2^e) - \beta_1 t; \\ \sigma_2 &= c_{12} \epsilon_1 + c_{22} \epsilon_2 + z (c_{12} \kappa_1^e + c_{22} \kappa_2^e) - \beta_2 t. \end{aligned} \right\} \quad (583)$$

By reducing stresses (583) to the statically equivalent system of elastic forces and moments, we obtain the basic relationships which connect the deformed, stressed state of a laminated orthotropic shell

$$\left. \begin{aligned} T_1 &= B_{11}(\zeta) \epsilon_1 + B_{12}(\zeta) \epsilon_2 + A_{11}(\zeta) \kappa_1^e + A_{12}(\zeta) \kappa_2^e - N_1(\zeta); \\ T_2 &= B_{12}(\zeta) \epsilon_1 + B_{22}(\zeta) \epsilon_2 + A_{12}(\zeta) \kappa_1^e + A_{22}(\zeta) \kappa_2^e - N_2(\zeta); \end{aligned} \right\} \quad (584)$$

$$\left. \begin{aligned} G_1 &= - [A_{11}(\zeta) \epsilon_1 + A_{12}(\zeta) \epsilon_2 + D_{11}(\zeta) \kappa_1^e + \\ &\quad + D_{12}(\zeta) \kappa_2^e - M_1(\zeta)]; \\ G_2 &= - [A_{12}(\zeta) \epsilon_1 + A_{22}(\zeta) \epsilon_2 + D_{12}(\zeta) \kappa_1^e + \\ &\quad + D_{22}(\zeta) \kappa_2^e - M_2(\zeta)]; \end{aligned} \right\} \quad (586)$$

where

$$\left. \begin{aligned} B_{ij}(\zeta) &= \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) dz; \\ A_{ij}(\zeta) &= \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) z dz; \\ D_{ij}(\zeta) &= \int_{-\delta/2}^{\delta/2} c_{ij}(\zeta, z) z^2 dz \end{aligned} \right\} \quad (586)$$

for $i, j=1, 2$;

$$\left. \begin{aligned} N_j &= \int_{-\delta/2}^{\delta/2} \beta_i(\zeta, z) t(\zeta, z) dz; \\ M_j &= \int_{-\delta/2}^{\delta/2} \beta_i(\zeta, z) t(\zeta, z) z dz \end{aligned} \right\} \quad (586)$$

for $j=1, 2$.

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By solving Eq. (584) for the deformation components and by substituting the resulting values in Eq. (585), we obtain

$$\left. \begin{aligned} \epsilon_1 &= \frac{B_{12}}{\Delta} \frac{dV}{d\zeta} - \frac{B_{22}}{\Delta} \frac{V \operatorname{ctg} \alpha}{R_2} - F_{11} \kappa_1^e - F_{12} \kappa_2^e + \\ &\quad + \frac{B_{22} N_1 - B_{12} N_2}{\Delta}; \\ \epsilon_2 &= \frac{B_{12}}{\Delta} \frac{V \operatorname{ctg} \alpha}{R_2} - \frac{B_{11}}{\Delta} \frac{dV}{d\zeta} - F_{21} \kappa_1^e - F_{22} \kappa_2^e + \\ &\quad + \frac{B_{11} N_2 - B_{12} N_1}{\Delta}; \\ G_1 &= -\bar{D}_{11} \kappa_1^e - \bar{D}_{12} \kappa_2^e + F_{11} \frac{V \operatorname{ctg} \alpha}{R_2} + F_{21} \frac{dV}{d\zeta} + \bar{M}_1(\zeta); \\ G_2 &= -\bar{D}_{12} \kappa_1^e - \bar{D}_{22} \kappa_2^e + F_{12} \frac{V \operatorname{ctg} \alpha}{R_2} + F_{22} \frac{dV}{d\zeta} + \bar{M}_2(\zeta). \end{aligned} \right\} \quad (587)$$

where

$$\begin{aligned}
& \Delta = B_{11}B_{22} - B_{12}^2; \\
F_{11} &= \frac{A_{11}B_{22} - A_{12}B_{12}}{\Delta}; \quad F_{12} = \frac{A_{12}B_{22} - A_{22}B_{12}}{\Delta}; \\
F_{21} &= \frac{A_{12}B_{11} - A_{11}B_{12}}{\Delta}; \quad F_{22} = \frac{A_{22}B_{11} - A_{12}B_{12}}{\Delta}; \\
\bar{D}_{11} &= D_{11} - \frac{A_{11}^2 B_{22} + A_{12}^2 B_{11} - 2A_{11}A_{12}B_{12}}{\Delta}; \\
\bar{D}_{22} &= D_{22} - \frac{A_{22}^2 B_{11} + A_{12}^2 B_{22} - 2A_{12}A_{22}B_{12}}{\Delta}; \\
\bar{D}_{12} &= D_{12} - \frac{A_{12}(A_{11}B_{22} + A_{22}B_{11}) - B_{12}(A_{11}A_{22} + A_{12}^2)}{\Delta}; \\
\bar{M}_1(\zeta) &= M_1(\zeta) - F_{11}N_1 - F_{21}N_2; \\
\bar{M}_2(\zeta) &= M_2(\zeta) - F_{12}N_1 - F_{22}N_2.
\end{aligned} \tag{588}$$

For determination of the stress and deformation functions, two equations remained unused: the continuity of deformation equation and /173 one equilibrium Eq. (506):

$$\left. \begin{aligned}
\left(\varepsilon_2 \frac{R_2}{R_1} - \varepsilon_1 \right) \operatorname{ctg} \alpha + \frac{d}{d\zeta} (R_2 \varepsilon_2) &= -\frac{V}{K_1 R_2} - \varphi; \\
\frac{d(G_1 R_2 \sin \alpha)}{d\zeta} - G_2 \cos \alpha &= V \sin \alpha.
\end{aligned} \right\} \tag{589}$$

Since the system of equations which is obtained after substituting Eq. (587) in Eq. (589) hardly seems suitable for practical calculations, it is more advisable to find a partial solution by one of the available approximate methods. In this case, terms dependent on N_1, M_1 ($i=1, 2$) appear as the heat load. Solution of the homogeneous system of equations can be obtained approximately, based on the technical equations of axisymmetrically loaded shells of rotation [14].

We consider the case of axisymmetric quite smoothly changing heat with length, in which the thermoelastic constants change considerably more slowly than the stress and deformation functions. Then, by obtaining the equations, the coefficients dependent on the thermoelastic constants and shell geometry can be considered constant and corresponding to the shell section under consideration. Besides, lower order derivatives can be disregarded compared with higher order derivatives.

With such assumptions, system of Eq. (589) takes the form

$$\left. \begin{aligned}
\frac{d\varepsilon_2}{d\zeta} &= -\frac{V}{K_1 R_2^2} - \frac{\varphi}{R_2}; \\
\frac{dG_1}{d\zeta} &= \frac{V}{R_2},
\end{aligned} \right\} \tag{590}$$

and Eq. (587), (588) are simplified:

$$\left. \begin{aligned} e_2 &= -\frac{B_{11}}{\Delta} \frac{dV}{d\zeta} - F_{21} \frac{d\varphi}{d\zeta}; \\ G_1 &= -\bar{D}_{11} \frac{d\varphi}{d\zeta} + F_{21} \frac{dV}{d\zeta}. \end{aligned} \right\} \quad (591)$$

By substituting the latter expressions in Eq. (590), the following can be obtained

$$\left. \begin{aligned} V^{IV} - 2Q^2 V'' + k^4 V &= 0; \\ \varphi &= \frac{F_{21}}{\bar{D}_{11}} V, \end{aligned} \right\} \quad (592)$$

where

$$\begin{aligned} 2Q^2 &= \frac{\bar{D}_{11} \Delta}{K_1 R_0^2 (B_{11} \bar{D}_{11} + F_{21}^2 \Delta)}; \\ k^4 &= \frac{\Delta}{(B_{11} \bar{D}_{11} + F_{21}^2 \Delta) R_0^2}. \end{aligned} \quad (593)$$

Differential Eq. (592) is analogous to Eq. (459), which describes 174 the end effects in a laminated cylindrical shell and, consequently, its solution, which disappears at infinity,

$$V = C_1 \phi_1(\zeta) + C_2 \phi_2(\zeta), \quad (594)$$

where ϕ_1, ϕ_2 are degenerate Vlasov functions satisfied by Eq. (465).

For each section where the thermal end effect is investigated, the corresponding values of integration constants C_1, C_2 are used. The boundary conditions are written for a general solution, i.e., for the solution of Eq. (594) and some partial solution of nonhomogeneous system of Eq. (587) and (589).

The problem of calculation of the temperature stresses in axisymmetrically heated shells of rotation is significantly simplified in two cases frequently encountered in practice.

1. The thermoelastic constants do not change upon heating. In this case, basic relationships (584) and (585) take the form

$$\left. \begin{aligned} T_1 &= B_{11} e_1 + B_{12} e_2 - N_1; \\ T_2 &= B_{12} e_1 + B_{22} e_2 - N_2; \\ G_1 &= -D_{11} \kappa_1' - D_{12} \kappa_2' + M_1; \\ G_2 &= -D_{12} \kappa_1' - D_{22} \kappa_2' + M_2; \end{aligned} \right\} \quad (595)$$

i.e., we have the conventional problem of an orthotropic shell in the presence of an axisymmetric heat load, which is characterized by parameters N_1, M_1 ($i=1, 2$).

2. Uniform heating of shell. In uniform axisymmetric heating of a shell of rotation, the elastic constants and coefficients of linear expansion of the material change. However, they remain constant at the corresponding shell temperature. In this case, the basic elasticity relationships take the following form:

$$\left. \begin{aligned} T_1 &= B_{11}^{(t)} \epsilon_1 + B_{13}^{(t)} \epsilon_3 - \beta_1 \delta t; \\ T_3 &= B_{13}^{(t)} \epsilon_1 + B_{33}^{(t)} \epsilon_3 - \beta_3 \delta t; \\ G_1 &= -D_{11}^{(t)} \kappa_1^e - D_{13}^{(t)} \kappa_3^e; \\ G_3 &= -D_{13}^{(t)} \kappa_1^e - D_{33}^{(t)} \kappa_3^e. \end{aligned} \right\} \quad (596)$$

where exponent t indicates that the rigidity coefficients for temperature t are used, i.e., we again approach the conventional problem for an orthotropic shell with an extremely simple axisymmetric heat load.

53. Basic Relationships and Differential Equations of Anisotropic Cylindrical Shell

We consider a circular cylindrical shell made of a laminated anisotropic plastic with elastic constants c_{ij} , a_{ij} .

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Let R be the cross section radius of the shell and δ be the wall thickness. We select dimensionless orthogonal coordinates α , β as the curvilinear Gaussian coordinates on the mean surface of the shell, of which α defines the relative distance along the generatrix of the shell and β defines the circumferential angle (Fig. 75).

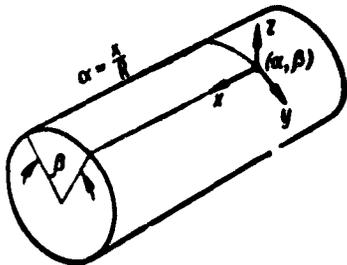


Fig. 75. Dimensionless coordinates on surface of cylindrical shell.

For convenience, we choose the Lamé parameters to be the same and equal to the shell radius, i.e., $A_1=A_2=R$. The principal radii of curvature of the mean surface of the circular cylindrical shell are $R_1 \rightarrow \infty$; $R_2=R$.

In the coordinate system selected, equilibrium Eq. (17) of a circular cylindrical shell are presented in the form

$$\left. \begin{aligned} \frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= -RX; \\ \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} - Q_2 &= -RY; \\ \frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} + T_2 &= RZ; \\ \frac{\partial G_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} &= RQ_1; \\ \frac{\partial G_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} &= RQ_2. \end{aligned} \right\} \quad (597)$$

The positive directions of the forces and moments in normal sections of the shell are shown in Fig. 1. /176

The components of the relative deformation and shear of the mean surface of a circular cylindrical shell, the effective changes in curvature and torsion, as well as the interlayer shearing along the coordinate axes, in accordance with Eq. (10)-(12), are determined by the following expressions

$$\left. \begin{aligned} \epsilon_1 &= \frac{1}{R} \frac{\partial u}{\partial \alpha}; \quad \epsilon_2 = \frac{1}{R} \left(\frac{\partial v}{\partial \beta} + w \right); \\ \omega &= \frac{1}{R} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right); \\ \kappa_1^c &= \frac{1}{R} \frac{\partial \varphi}{\partial \alpha}; \quad \kappa_2^c = \frac{1}{R} \frac{\partial \psi}{\partial \beta}; \\ 2\kappa_3^c &= \frac{1}{R} \left(\frac{\partial \varphi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right); \\ \gamma_1 &= \varphi + \frac{1}{R} \frac{\partial w}{\partial \alpha}; \quad \gamma_2 = \psi + \frac{1}{R} \frac{\partial w}{\partial \beta} - \frac{v}{R}. \end{aligned} \right\} \quad (598)$$

By substituting the values of $\epsilon_1, \epsilon_2, \omega, \kappa_1^e, \kappa_2^e, \kappa_3^e, \gamma_1, \gamma_2$ in Eq. (3)-(5), we obtain the basic elasticity relationships of a circular anisotropic cylindrical shell

$$\left. \begin{aligned} T_1 &= \frac{1}{R} \left[B_{11} \frac{\partial u}{\partial \alpha} + B_{12} \left(\frac{\partial v}{\partial \beta} + w \right) + B_{13} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right) \right]; \\ T_2 &= \frac{1}{R} \left[B_{12} \frac{\partial u}{\partial \alpha} + B_{22} \left(\frac{\partial v}{\partial \beta} + w \right) + B_{23} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right) \right]; \\ S &= \frac{1}{R} \left[B_{13} \frac{\partial u}{\partial \alpha} + B_{23} \left(\frac{\partial v}{\partial \beta} + w \right) + B_{33} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right) \right]; \\ Q_1 &= -\frac{K_1}{R} \left(R\varphi + \frac{\partial w}{\partial \alpha} \right); \\ Q_2 &= -\frac{K_2}{R} \left(R\psi + \frac{\partial w}{\partial \beta} - v \right); \\ G_1 &= -\frac{1}{R} \left[D_{11} \frac{\partial \varphi}{\partial \alpha} + D_{12} \frac{\partial \varphi}{\partial \beta} + D_{13} \left(\frac{\partial \varphi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right) \right]; \\ G_2 &= -\frac{1}{R} \left[D_{12} \frac{\partial \varphi}{\partial \alpha} + D_{22} \frac{\partial \varphi}{\partial \beta} + D_{23} \left(\frac{\partial \varphi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right) \right]; \\ H &= -\frac{1}{R} \left[D_{13} \frac{\partial \varphi}{\partial \alpha} + D_{23} \frac{\partial \varphi}{\partial \beta} + D_{33} \left(\frac{\partial \varphi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right) \right]. \end{aligned} \right\} \quad (599)$$

Shell rigidity parameters B_{ij}, D_{ij} ($i, j=1, 2, 3$), K_1, K_2 are determined through the elastic constants of the material and the thickness of the shell by Eq. (6), (7).

By substituting elasticity relationships (599) in equilibrium Eq. 177 (597), we obtain the following system of differential equations of a cylindrical anisotropic shell in movements

$$\left. \begin{aligned} \delta_{11} u + \delta_{12} v + \delta_{13} w + \delta_{14} \varphi + \delta_{15} \psi &= -R^2 X; \\ \delta_{12} u + \delta_{22} v + \delta_{23} w + \delta_{24} \varphi + \delta_{25} \psi &= -R^2 Y; \\ \delta_{13} u + \delta_{23} v + \delta_{33} w + \delta_{34} \varphi + \delta_{35} \psi &= R^2 Z; \\ \delta_{14} u + \delta_{24} v + \delta_{34} w + \delta_{44} \varphi + \delta_{45} \psi &= 0; \\ \delta_{15} u + \delta_{25} v + \delta_{35} w + \delta_{45} \varphi + \delta_{55} \psi &= 0, \end{aligned} \right\} \quad (600)$$

where designations are introduced for linear differential operators δ_{ij} in partial derivatives up to the second order:

$$\left. \begin{aligned} \delta_{11} &= B_{11} \frac{\partial^2}{\partial \alpha^2} + 2B_{12} \frac{\partial^2}{\partial \alpha \partial \beta} + B_{22} \frac{\partial^2}{\partial \beta^2}; \\ \delta_{12} &= B_{12} \frac{\partial^2}{\partial \alpha^2} + (B_{12} + B_{23}) \frac{\partial^2}{\partial \alpha \partial \beta} + B_{23} \frac{\partial^2}{\partial \beta^2}; \\ \delta_{13} &= B_{12} \frac{\partial}{\partial \alpha} + B_{23} \frac{\partial}{\partial \beta}; \\ \delta_{22} &= B_{22} \frac{\partial^2}{\partial \alpha^2} + 2B_{23} \frac{\partial^2}{\partial \alpha \partial \beta} + B_{33} \frac{\partial^2}{\partial \beta^2} - K_2; \\ \delta_{23} &= B_{23} \frac{\partial}{\partial \alpha} + (B_{23} + K_2) \frac{\partial}{\partial \beta}; \\ \delta_{33} &= -\left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} - B_{22} \right); \end{aligned} \right\} \quad (601)$$

$$\begin{aligned}
\delta_{24} &= -K_1 R \frac{\partial}{\partial \alpha}; \\
\delta_{25} &= -K_2 R \frac{\partial}{\partial \beta}; \\
\delta_{26} &= K_3 R; \\
\delta_{14} = \delta_{15} = \delta_{16} &= 0; \\
\delta_{44} &= D_{11} \frac{\partial^2}{\partial \alpha^2} + 2D_{12} \frac{\partial^2}{\partial \alpha \partial \beta} + D_{22} \frac{\partial^2}{\partial \beta^2} - K_1 R^2; \\
\delta_{45} &= D_{12} \frac{\partial^2}{\partial \alpha^2} + (D_{13} + D_{22}) \frac{\partial^2}{\partial \alpha \partial \beta} + D_{23} \frac{\partial^2}{\partial \beta^2}; \\
\delta_{46} &= D_{22} \frac{\partial^2}{\partial \alpha^2} + 2D_{23} \frac{\partial^2}{\partial \alpha \partial \beta} + D_{33} \frac{\partial^2}{\partial \beta^2} - K_2 R^2.
\end{aligned} \tag{601}$$

System of differential Eq. (600) relative to linear differential operators δ_{ij} is symmetrical. This property of symmetry of the differential equations in movements of isotropic and anisotropic shells was noted in [1, 4].

System of Eq. (600) can be reduced to one resolving differential equation of the tenth order relative to stress functions $\phi(\alpha, \beta)$, through which all quantities which define the stressed and deformed states of the shell can be expressed. /178

We will consider system of differential Eq. (600) as a complete system of algebraic equations relative to movement components u, v, w and deformation functions ϕ, ψ , with constant coefficients δ_{ij} . The determinant of this system of equations

$$\Delta = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} & 0 & 0 \\ \delta_{12} & \delta_{22} & \delta_{23} & 0 & \delta_{25} \\ \delta_{13} & \delta_{23} & \delta_{33} & \delta_{34} & \delta_{35} \\ 0 & 0 & \delta_{34} & \delta_{44} & \delta_{45} \\ 0 & \delta_{25} & \delta_{35} & \delta_{45} & \delta_{55} \end{vmatrix} = \delta_{11}\Delta_{11} + \delta_{12}\Delta_{12} + \delta_{13}\Delta_{13} \tag{602}$$

is a differential operator in partial derivatives up to the tenth order, with coefficients dependent on B_{ij}, D_{ij}, K_1, K_2 .

We designate the minors of the determinant, which also are differential operators in partial derivatives by Δ_{ij} ($i, j=1, 2, 3$).

Thus, for example

$$\Delta_{11} = \begin{vmatrix} \delta_{22} & \delta_{23} & 0 & \delta_{25} \\ \delta_{23} & \delta_{33} & \delta_{34} & \delta_{35} \\ 0 & \delta_{34} & \delta_{44} & \delta_{45} \\ \delta_{25} & \delta_{35} & \delta_{45} & \delta_{55} \end{vmatrix} = \delta_{22}\delta_{33}\delta_{44}\delta_{55} + 2\delta_{22}\delta_{34}\delta_{35}\delta_{45} - \tag{603}$$

$$\begin{aligned}
& -\delta_{22}\delta_{44}\delta_{55}^2 - \delta_{22}\delta_{33}\delta_{44}^2 - \delta_{22}\delta_{33}\delta_{55}^2 - \delta_{44}\delta_{55}\delta_{22}^2 - 2\delta_{22}\delta_{33}\delta_{44}\delta_{55} + \\
& + 2\delta_{22}\delta_{33}\delta_{44}\delta_{55} + \delta_{22}^2\delta_{44}^2 - \delta_{22}\delta_{44}\delta_{55}^2 - \delta_{22}^2\delta_{55}^2; \tag{603}
\end{aligned}$$

$$\begin{aligned}
\Delta_{12} = & - \begin{vmatrix} \delta_{12} & \delta_{22} & 0 & \delta_{25} \\ \delta_{13} & \delta_{23} & \delta_{24} & \delta_{25} \\ 0 & \delta_{34} & \delta_{44} & \delta_{45} \\ 0 & \delta_{25} & \delta_{45} & \delta_{55} \end{vmatrix} = \delta_{12}\delta_{44}\delta_{55}^2 + \delta_{12}\delta_{23}\delta_{45}^2 + \delta_{12}\delta_{33}\delta_{55}^2 + \\
& + \delta_{12}\delta_{25}\delta_{44}\delta_{55} + \delta_{12}\delta_{25}\delta_{34}\delta_{45} - \delta_{12}\delta_{22}\delta_{44}\delta_{55} - 2\delta_{12}\delta_{34}\delta_{25}\delta_{45} - \\
& - \delta_{12}\delta_{25}\delta_{35}\delta_{44} - \delta_{12}\delta_{22}\delta_{45}^2; \tag{604}
\end{aligned}$$

$$\begin{aligned}
\Delta_{13} = & \begin{vmatrix} \delta_{12} & \delta_{22} & 0 & \delta_{25} \\ \delta_{13} & \delta_{23} & \delta_{34} & \delta_{35} \\ 0 & 0 & \delta_{44} & \delta_{45} \\ 0 & \delta_{25} & \delta_{45} & \delta_{55} \end{vmatrix} = \delta_{12}\delta_{23}\delta_{44}\delta_{55}^2 + \delta_{12}\delta_{25}\delta_{34}\delta_{45} - \\
& - \delta_{12}\delta_{25}\delta_{35}\delta_{44} - \delta_{12}\delta_{22}\delta_{45}^2 - \delta_{12}\delta_{22}\delta_{44}\delta_{55} + \delta_{12}\delta_{44}\delta_{55}^2 + \delta_{12}\delta_{22}\delta_{45}^2; \tag{605}
\end{aligned}$$

$$\begin{aligned}
\Delta_{14} = & - \begin{vmatrix} \delta_{12} & \delta_{22} & \delta_{23} & \delta_{25} \\ \delta_{13} & \delta_{23} & \delta_{33} & \delta_{35} \\ 0 & 0 & \delta_{34} & \delta_{45} \\ 0 & \delta_{25} & \delta_{35} & \delta_{55} \end{vmatrix} = \delta_{12}\delta_{25}\delta_{34}\delta_{35} + \delta_{12}\delta_{23}\delta_{35}\delta_{45} + \\
& + \delta_{12}\delta_{22}\delta_{34}\delta_{55} + \delta_{12}\delta_{23}\delta_{25}\delta_{45} - \delta_{12}\delta_{22}\delta_{34}\delta_{55} - \delta_{12}\delta_{25}\delta_{23}\delta_{45} - \\
& - \delta_{12}\delta_{34}\delta_{25}^2 - \delta_{12}\delta_{22}^2\delta_{35}\delta_{45}; \tag{606}
\end{aligned}$$

$$\begin{aligned}
\Delta_{15} = & \begin{vmatrix} \delta_{12} & \delta_{22} & \delta_{23} & 0 \\ \delta_{13} & \delta_{23} & \delta_{33} & \delta_{34} \\ 0 & 0 & \delta_{34} & \delta_{44} \\ 0 & \delta_{25} & \delta_{35} & \delta_{45} \end{vmatrix} = \delta_{12}\delta_{23}\delta_{34}\delta_{45} + \\
& + \delta_{12}\delta_{25}\delta_{33}\delta_{44} - \delta_{12}\delta_{25}\delta_{34}^2 - \delta_{12}\delta_{23}\delta_{35}\delta_{44} - \delta_{12}\delta_{22}\delta_{34}\delta_{45} - \\
& - \delta_{12}\delta_{23}\delta_{25}\delta_{44} + \delta_{12}\delta_{22}\delta_{35}\delta_{44} \tag{607}
\end{aligned}$$

and the like.

We will seek a solution of system of Eq. (600) in the form of the following linear combination of functions F_1, F_2, F_3, F_4, F_5 :

$$\left. \begin{aligned}
u &= \Delta_{11}F_1 + \Delta_{12}F_2 + \Delta_{13}F_3 + \Delta_{14}F_4 + \Delta_{15}F_5; \\
v &= \Delta_{12}F_1 + \Delta_{22}F_2 + \Delta_{23}F_3 + \Delta_{24}F_4 + \Delta_{25}F_5; \\
w &= \Delta_{13}F_1 + \Delta_{23}F_2 + \Delta_{33}F_3 + \Delta_{34}F_4 + \Delta_{35}F_5; \\
\varphi &= \Delta_{14}F_1 + \Delta_{24}F_2 + \Delta_{34}F_3 + \Delta_{44}F_4 + \Delta_{45}F_5; \\
\psi &= \Delta_{15}F_1 + \Delta_{25}F_2 + \Delta_{35}F_3 + \Delta_{45}F_4 + \Delta_{55}F_5.
\end{aligned} \right\} \tag{608}$$

By substituting Eq. (608) in system of Eq. (600), the latter can be reduced to the following canonical form

$$\left. \begin{aligned} \Delta F_1 &= -R^2 X; \\ \Delta F_2 &= -R^2 Y; \\ \Delta F_3 &= R^2 Z; \\ \Delta F_4 &= \Delta F_5 = 0. \end{aligned} \right\} \quad (609)$$

Thus, the problem of calculation of a circular cylindrical shell was reduced to solution of a nonhomogeneous differential equation in partial derivatives of the tenth order.

If any partial solution of system of Eq. (609) is designated F_1^* , F_2^* , F_3^* , the partial solution of system of Eq. (600) can be written in the form

$$\left. \begin{aligned} u_* &= \Delta_{11} F_1^* + \Delta_{12} F_2^* + \Delta_{13} F_3^*; \\ v_* &= \Delta_{12} F_1^* + \Delta_{22} F_2^* + \Delta_{23} F_3^*; \\ w_* &= \Delta_{13} F_1^* + \Delta_{23} F_2^* + \Delta_{33} F_3^*; \\ \varphi_* &= \Delta_{14} F_1^* + \Delta_{24} F_2^* + \Delta_{34} F_3^*; \\ \psi_* &= \Delta_{15} F_1^* + \Delta_{25} F_2^* + \Delta_{35} F_3^*. \end{aligned} \right\} \quad (610)$$

For the types of external surface load most frequently encountered in engineering practice, it is simpler to determine a partial solution directly from the solution of system of Eq. (600).

The general solution of system of Eq. (609) equals the sum of partial solution (610) or a solution found by another method and a solution of the homogeneous equation

$$\Delta F = 0. \quad (611)$$

If it is set that $F_1 = F$ and $F_2 = F_3 = F_4 = F_5 = 0$, the general solution of system of Eq. (600) is presented in the form

$$\left. \begin{aligned} u &= u_* + \Delta_{11} F; \\ v &= v_* + \Delta_{12} F; \\ w &= w_* + \Delta_{13} F; \\ \varphi &= \varphi_* + \Delta_{14} F; \\ \psi &= \psi_* + \Delta_{15} F. \end{aligned} \right\} \quad (612)$$

54. Equations of Technical Theory of Orthotropic Cylindrical Shell

The theory of flattened shells developed by V.Z. Vlasov, which is used for calculation of strength, stability and vibrations has become widespread in engineering practice [14]. This theory rests on certain

assumptions which permit significant simplification of the differential equations of general moment theory.

Following Vlasov, we will assume:

1. annular interlayer shearing γ_2 depends negligibly on annular movements v ;

2. in the equations of equilibrium of annular forces, cutting force Q_2 can be disregarded.

Based on these hypotheses, for a circular orthotropic cylindrical shell, the principal axes of anisotropy of which coincide with the α, β coordinate axes (see Fig. 75), we obtain the equilibrium equations /181

$$\left. \begin{aligned} \frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= -RX; \\ \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= -RY; \end{aligned} \right\} \quad (613)$$

$$\left. \begin{aligned} \frac{\partial Q_1}{\partial \alpha} + \frac{\partial Q_2}{\partial \beta} + T_2 &= RZ; \\ \frac{\partial G_1}{\partial \alpha} + \frac{\partial H}{\partial \beta} &= RQ_1; \\ \frac{\partial G_2}{\partial \beta} + \frac{\partial H}{\partial \alpha} &= RQ_2. \end{aligned} \right\} \quad (614)$$

Geometric relationships:

$$\left. \begin{aligned} \epsilon_1 &= \frac{1}{R} \frac{\partial u}{\partial \alpha}; \quad \epsilon_2 = \frac{1}{R} \left(\frac{\partial v}{\partial \beta} + w \right); \quad \omega = \frac{1}{R} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right); \\ \kappa_1^a &= \frac{1}{R} \frac{\partial \varphi}{\partial \alpha}; \quad \kappa_2^c = \frac{1}{R} \frac{\partial \psi}{\partial \beta}; \quad 2\kappa_3^c = \frac{1}{R} \left(\frac{\partial \varphi}{\partial \beta} + \frac{\partial \psi}{\partial \alpha} \right); \\ \gamma_1 &= \varphi + \frac{1}{R} \frac{\partial w}{\partial \alpha}; \quad \gamma_2 = \psi + \frac{1}{R} \frac{\partial w}{\partial \beta}. \end{aligned} \right\} \quad (615)$$

Elasticity relationships:

$$\left. \begin{aligned} T_1 &= \frac{B_1}{R} \left[\frac{\partial u}{\partial \alpha} + \nu_2 \left(\frac{\partial v}{\partial \beta} + w \right) \right]; \\ T_2 &= \frac{B_2}{R} \left[\frac{\partial v}{\partial \beta} + w + \nu_1 \frac{\partial u}{\partial \alpha} \right]; \\ S &= \frac{B_3}{R} \left(\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha} \right); \\ Q_1 &= -K_1 \left(\varphi + \frac{1}{R} \frac{\partial w}{\partial \alpha} \right); \\ Q_2 &= -K_2 \left(\psi + \frac{1}{R} \frac{\partial w}{\partial \beta} \right); \\ G_1 &= -\frac{D_1}{R} \left(\frac{\partial \varphi}{\partial \alpha} + \nu_2 \frac{\partial \psi}{\partial \beta} \right); \end{aligned} \right\} \quad (616)$$

$$\left. \begin{aligned} G_2 &= -\frac{D_3}{R} \left(\frac{\partial \Psi}{\partial \beta} + \nu_1 \frac{\partial \Phi}{\partial \alpha} \right); \\ H &= -\frac{D_2}{R} \left(\frac{\partial \Phi}{\partial \beta} + \frac{\partial \Psi}{\partial \alpha} \right). \end{aligned} \right\} \quad (616)$$

where rigidity parameters $B_1, B_2, B_3, D_1, D_2, D_3, K_1, K_2$ are determined through elastic constants c_{ij} of the material and shell thickness δ by Eq. (7), (27).

Homogeneous Eq. (613) are identically satisfied, if stress function $\Phi(\alpha, \beta)$ is introduced in accordance with the expressions /183

$$T_1 = \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2}; \quad T_2 = \frac{\partial^4 \Phi}{\partial \alpha^4}; \quad S = -\frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2}. \quad (617)$$

By substituting Eq. (616) in equilibrium Eq. (614), we obtain

$$\left. \begin{aligned} D_1 \frac{\partial^2 \Phi}{\partial \alpha^2} + D_3 \frac{\partial^2 \Phi}{\partial \beta^2} + C \frac{\partial^2 \Psi}{\partial \alpha \partial \beta} &= K_1 R^2 \left(\Phi + \frac{1}{R} \frac{\partial w}{\partial \alpha} \right); \\ D_3 \frac{\partial^2 \Psi}{\partial \alpha^2} + D_2 \frac{\partial^2 \Psi}{\partial \beta^2} + C \frac{\partial^2 \Phi}{\partial \alpha \partial \beta} &= K_2 R^2 \left(\Psi + \frac{1}{R} \frac{\partial w}{\partial \beta} \right); \end{aligned} \right\} \quad (618)$$

$$\frac{\partial^4 \Phi}{\partial \alpha^4} = K_1 \frac{\partial \Phi}{\partial \alpha} + K_2 \frac{\partial \Psi}{\partial \beta} + \frac{1}{R} \left(K_1 \frac{\partial^2 w}{\partial \alpha^2} + K_2 \frac{\partial^2 w}{\partial \beta^2} \right) + RZ. \quad (619)$$

The last missing equation gives the deformation compatibility condition

$$\frac{\partial^2 \epsilon_1}{\partial \beta^2} - \frac{\partial^2 \omega}{\partial \alpha \partial \beta} + \frac{\partial^2 \epsilon_2}{\partial \alpha^2} = \frac{1}{R} \frac{\partial^2 w}{\partial \alpha^2}. \quad (620)$$

System of Eq. (618) is equivalent to the following system of differential equations

$$\left. \begin{aligned} m_2(\Phi) &= \frac{D_3 K_1}{R^3} \frac{\partial^2 w}{\partial \alpha^2} + \frac{(D_3 K_1 - C K_2)}{R^3} \frac{\partial^2 w}{\partial \alpha \partial \beta^2} - \frac{K_1 K_2}{R} \frac{\partial w}{\partial \alpha}; \\ m_2(\Psi) &= \frac{D_2 K_2}{R^3} \frac{\partial^2 w}{\partial \beta^2} + \frac{(D_1 K_2 - C K_1)}{R^3} \frac{\partial^2 w}{\partial \alpha^2 \partial \beta} - \frac{K_1 K_2}{R} \frac{\partial w}{\partial \beta}. \end{aligned} \right\} \quad (621)$$

where there is introduced the differential operator in partial derivatives

$$\begin{aligned} m_2(\) &= \frac{D_1 D_3}{R^4} \frac{\partial^4}{\partial \alpha^4} + \frac{(D_1 D_2 + D_3^2 - C^2)}{R^4} \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \\ &+ \frac{D_2 D_3}{R^4} \frac{\partial^4}{\partial \beta^4} - \frac{(D_1 K_2 + D_3 K_1)}{R^3} \frac{\partial^4}{\partial \alpha^2} - \frac{(D_2 K_1 + D_3 K_2)}{R^3} \frac{\partial^4}{\partial \beta^2} + K_1 K_2. \end{aligned} \quad (622)$$

From Eq. (620), with Eq. (617) taken into account, an expression for deflection of the shell can be obtained through the stress function

$$\frac{B_2(B_1B_2 - B_{12}^2)}{R} w - B_1B_2 \frac{\partial^4 \Phi}{\partial \alpha^4} + (B_1B_2 + B_2^2 - A^2) \frac{\partial^4 \Phi}{\partial \alpha^2 \partial \beta^2} + B_2B_3 \frac{\partial^4 \Phi}{\partial \beta^4}, \quad (623)$$

where

$$A = B_{12} + B_3, \quad (624)$$

or, in operator form

$$w = m_1(\Phi), \quad (625)$$

and here,

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$$m_1(\) = \frac{R}{B_2(B_1B_2 - B_{12}^2)} \times [B_1B_2 \frac{\partial^4}{\partial \alpha^4} + (B_1B_2 + B_2^2 - A^2) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + B_2B_3 \frac{\partial^4}{\partial \beta^4}]. \quad (626)$$

In this manner, the system of differential equations of the laminated orthotropic cylindrical shell has the form

$$\left. \begin{aligned} m_2(\varphi) &= \left(\frac{D_2K_1}{R^2} \frac{\partial^2 w}{\partial \alpha^2} + \frac{D_2K_1 - CK_2}{R^2} \frac{\partial^2 w}{\partial \alpha \partial \beta^2} - \frac{K_1K_2}{R} \frac{\partial w}{\partial \alpha} \right); \\ m_2(\psi) &= \left(\frac{D_2K_2}{R^2} \frac{\partial^2 w}{\partial \beta^2} + \frac{D_1K_2 - CK_1}{R^2} \frac{\partial^2 w}{\partial \alpha^2 \partial \beta} - \frac{K_1K_2}{R} \frac{\partial w}{\partial \beta} \right); \\ \frac{\partial^4 \Phi}{\partial \alpha^4} &= \left(\frac{K_1}{R} \frac{\partial^2}{\partial \alpha^2} + \frac{K_2}{R} \frac{\partial^2}{\partial \beta^2} \right) m_1(\Phi) + K_1 \frac{\partial \varphi}{\partial \alpha} + K_2 \frac{\partial \psi}{\partial \beta} + RZ; \\ w &= m_1(\Phi). \end{aligned} \right\} \quad (627)$$

System of Eq. (627) can be reduced to one resolving equation relative to stress functions $\Phi(\alpha, \beta)$

$$\frac{\partial^4}{\partial \alpha^4} m_2(\Phi) = \frac{1}{R} \left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} \right) m_1 m_2(\Phi) + K_1 \frac{\partial}{\partial \alpha} m_2(\varphi) + K_2 \frac{\partial}{\partial \beta} m_2(\psi) + Rm_2(Z). \quad (628)$$

We represent operator m_2 in the form of the sum of operators

$$m_2(\) = m_2^{(1)}(\) - m_2^{(2)}(\) + K_1K_2, \quad (629)$$

where

$$\left. \begin{aligned} m_2^{(1)}(\) &= \frac{1}{R^4} \times \\ &\times \left[D_1D_3 \frac{\partial^4}{\partial \alpha^4} + (D_1D_2 + D_3^2 - C^2) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + D_2D_3 \frac{\partial^4}{\partial \beta^4} \right]; \\ m_2^{(2)}(\) &= \frac{1}{R^2} \times \\ &\times \left[(D_1K_2 + D_2K_1) \frac{\partial^2}{\partial \alpha^2} + (D_2K_1 + D_3K_2) \frac{\partial^2}{\partial \beta^2} \right]. \end{aligned} \right\} \quad (630)$$

Consequently, Eq. (628) is presented in the form

$$\begin{aligned}
& \frac{1}{R} \left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} \right) m_1 m_2^{(1)}(\Phi) - \\
& - \frac{1}{R} \left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} \right) m_1 m_2^{(2)}(\Phi) + \\
& + \frac{K_1 K_2}{R} \left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} \right) m_1(\Phi) - \frac{\partial^4}{\partial \alpha^4} m_2^{(1)}(\Phi) + \\
& + \frac{\partial^4}{\partial \alpha^4} m_2^{(2)}(\Phi) - K_1 K_2 \frac{\partial^4 \Phi}{\partial \alpha^4} + \\
& + \frac{K_1}{K^2} \left[D_3 K_1 \frac{\partial^4}{\partial \alpha^4} + (D_2 K_1 - C K_2) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} \right] m_1(\Phi) - \\
& - \frac{K_1^2 K_2}{R} \frac{\partial^4}{\partial \alpha^4} m_1(\Phi) + \frac{K_2}{K^2} \left[D_3 K_1 \frac{\partial^4}{\partial \beta^4} + (D_1 K_2 - C K_1) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} \right] \times \\
& \times m_1(\Phi) - \frac{K_1 K_1}{R} \frac{\partial^4}{\partial \beta^4} m_1(\Phi) = -R m_2(Z), \tag{631}
\end{aligned}$$

or, after reduction,

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$$\begin{aligned}
& \frac{1}{R} \left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} \right) m_1 m_2^{(1)}(\Phi) - \\
& - \frac{\partial^4}{\partial \alpha^4} m_2^{(1)}(\Phi) - \frac{K_1 K_2}{K^2} m_1 m_2^{(3)}(\Phi) + \\
& + \frac{\partial}{\partial \alpha^4} m_2(\Phi) - K_1 K_2 \frac{\partial^4 \Phi}{\partial \alpha^4} = -R m_2(Z), \tag{632}
\end{aligned}$$

where

$$m_2^{(3)} = \left[D_1 \frac{\partial^4}{\partial \alpha^4} + 2(C + D) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + D_2 \frac{\partial^4}{\partial \beta^4} \right]. \tag{633}$$

In developed form, differential Eq. (632) has the form

$$\begin{aligned}
& A_{10,0} \frac{\partial^{10} \Phi}{\partial \alpha^{10}} + A_{8,2} \frac{\partial^{10} \Phi}{\partial \alpha^8 \partial \beta^2} + A_{6,4} \frac{\partial^{10} \Phi}{\partial \alpha^6 \partial \beta^4} + \\
& + A_{4,6} \frac{\partial^{10} \Phi}{\partial \alpha^4 \partial \beta^6} + A_{2,8} \frac{\partial^{10} \Phi}{\partial \alpha^2 \partial \beta^8} + A_{0,10} \frac{\partial^{10} \Phi}{\partial \beta^{10}} - \\
& - \left(A_{8,0} \frac{\partial^8 \Phi}{\partial \alpha^8} + A_{6,2} \frac{\partial^8 \Phi}{\partial \alpha^6 \partial \beta^2} + A_{4,4} \frac{\partial^8 \Phi}{\partial \alpha^4 \partial \beta^4} + \right. \\
& + A_{2,6} \frac{\partial^8 \Phi}{\partial \alpha^2 \partial \beta^6} + A_{0,8} \frac{\partial^8 \Phi}{\partial \beta^8} \left. \right) + A_{20} \frac{\partial^8 \Phi}{\partial \alpha^8} + \\
& + A_{02} \frac{\partial^8 \Phi}{\partial \alpha^4 \partial \beta^4} - A_{00} \frac{\partial^8 \Phi}{\partial \alpha^4} = -M m_2(Z), \tag{634}
\end{aligned}$$

where the coefficients are determined through the rigidity parameters /185
of the shell by the following expressions

$$\begin{aligned}
 A_{10,0} &= B_1 B_3 D_1 D_2 K_1; \\
 A_{8,2} &= K_2 B_1 B_3 D_1 D_3 + K_1 [B_1 B_3 (D_1 D_2 + D_3^2 - C^2) + \\
 &\quad + D_1 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
 A_{6,4} &= K_1 [B_1 B_3 D_2 D_3 + B_2 B_3 D_1 D_3 + (D_1 D_2 + D_3^2 - C^2) \times \\
 &\quad \times (B_1 B_2 + B_3^2 - A^2)] + K_2 [B_1 B_3 (D_1 D_2 + D_3^2 - C^2) + \\
 &\quad + D_1 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
 A_{4,6} &= K_2 [B_1 B_3 D_2 D_3 + B_2 B_3 D_1 D_3 + \\
 &\quad + (D_1 D_2 + D_3^2 - C^2) (B_1 B_2 + B_3^2 - A^2)] + \\
 &+ K_1 [B_2 B_3 (D_1 D_2 + D_3^2 - C^2) + D_2 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
 A_{2,8} &= K_1 B_2 B_3 D_2 D_3 + K_2 [B_2 B_3 (D_1 D_2 + D_3^2 - C^2) + \\
 &\quad + D_2 D_3 (B_1 B_2 + B_3^2 - A^2)]; \\
 A_{0,10} &= B_2 B_3 D_2 D_3 K_2; \\
 A_{8,0} &= B_1 B_3 D_1 K_1 K_2 R^2 + B_3 D_1 D_3 (B_1 B_2 - B_{12}^2); \\
 A_{6,2} &= K_1 K_2 R^2 [D_1 (B_1 B_2 + B_3^2 - A^2) + 2B_1 B_3 (C + D_3)] + \\
 &\quad + B_3 (B_1 B_2 - B_{12}^2) (D_1 D_2 + D_3^2 - C^2); \\
 A_{4,4} &= K_1 K_2 R^2 [B_1 B_3 D_2 + B_2 B_3 D_1 + \\
 &\quad + 2(C + D_3) (B_1 B_2 + B_3^2 - A^2)] + B_3 D_2 D_3 (B_1 B_2 - B_{12}^2); \\
 A_{2,6} &= K_1 K_2 R^2 [D_2 (B_1 B_2 + B_3^2 - A^2) + \\
 &\quad + 2B_2 B_3 (C + D_3)]; \\
 A_{0,8} &= B_2 B_3 D_2 K_1 K_2 R^2; \\
 A_{2,0} &= B_3 (B_1 B_2 - B_{12}^2) R^2 (D_1 K_2 + D_3 K_1); \\
 A_{0,2} &= B_3 (B_1 B_2 - B_{12}^2) R^2 (D_2 K_1 + D_3 K_2); \\
 A_{0,0} &= B_3 (B_1 B_2 - B_{12}^2) K_1 K_2 R^4; \\
 M &= B_3 (B_1 B_2 - B_{12}^2) R^5.
 \end{aligned}
 \tag{635}$$

55. Equations of Technical Theory of Orthotropic Shell in Movements

The equations of the technical theory of a cylindrical shell can be presented in movements, as was done in Section 53 for the general case of anisotropy.

For an orthotropic shell, the principal axes of anisotropy of which coincide with the coordinate axes, the equations in movements can be presented in the form /186

$$\left. \begin{aligned}
 \delta_{11} u + \delta_{12} v + \delta_{13} w + \delta_{14} \varphi + \delta_{15} \psi &= -R^2 X; \\
 \delta_{12} u + \delta_{22} v + \delta_{23} w + \delta_{24} \varphi + \delta_{25} \psi &= -R^2 Y; \\
 \delta_{13} u + \delta_{23} v + \delta_{33} w + \delta_{34} \varphi + \delta_{35} \psi &= R^2 Z; \\
 \delta_{14} u + \delta_{24} v + \delta_{34} w + \delta_{44} \varphi + \delta_{45} \psi &= 0; \\
 \delta_{15} u + \delta_{25} v + \delta_{35} w + \delta_{45} \varphi + \delta_{55} \psi &= 0.
 \end{aligned} \right\} \quad (636)$$

where the linear differential operators are determined by the expressions

$$\left. \begin{aligned}
 \delta_{11} &= B_1 \frac{\partial^2}{\partial \alpha^2} + B_2 \frac{\partial^2}{\partial \beta^2}; \\
 \delta_{12} &= (B_{12} + B_2) \frac{\partial^2}{\partial \alpha \partial \beta}; \\
 \delta_{22} &= B_3 \frac{\partial^2}{\partial \alpha^2} + B_2 \frac{\partial^2}{\partial \beta^2}; \\
 \delta_{13} &= B_{13} \frac{\partial}{\partial \alpha}; \\
 \delta_{23} &= B_3 \frac{\partial}{\partial \beta}; \\
 \delta_{33} &= -\left(K_1 \frac{\partial^2}{\partial \alpha^2} + K_2 \frac{\partial^2}{\partial \beta^2} - B_2 \right); \\
 \delta_{34} &= -K_1 R \frac{\partial}{\partial \alpha}; \\
 \delta_{35} &= -K_2 R \frac{\partial}{\partial \beta}; \\
 \delta_{44} &= D_1 \frac{\partial^2}{\partial \alpha^2} + D_2 \frac{\partial^2}{\partial \beta^2} - K_1 R^2; \\
 \delta_{45} &= (D_{12} + D_2) \frac{\partial^2}{\partial \alpha \partial \beta}; \\
 \delta_{55} &= D_3 \frac{\partial^2}{\partial \alpha^2} + D_2 \frac{\partial^2}{\partial \beta^2} - K_2 R^2; \\
 \delta_{15} &= \delta_{18} = \delta_{24} = \delta_{25} = 0.
 \end{aligned} \right\} \quad (637)$$

If any partial solution of system of Eq. (636) is designated by u_* , v_* , w_* , φ_* , ψ_* , the general solution can be presented in the following form:

$$\left. \begin{aligned}
 u &= u_* + \Delta_{11} \Phi; \\
 v &= v_* + \Delta_{12} \Phi; \\
 w &= w_* + \Delta_{13} \Phi; \\
 \varphi &= \varphi_* + \Delta_{14} \Phi; \\
 \psi &= \psi_* + \Delta_{15} \Phi,
 \end{aligned} \right\} \quad (638)$$

where Δ_{1j} ($j=1, 2, 3, 4, 5$) is determined by the expressions

$$\begin{aligned}
\Delta_{11} = & -B_2 D_1 D_3 K_1 \frac{\partial^4}{\partial \alpha^4} - [K_1 \{B_2 (D_1 D_2 - C^2 + D_2^2) + B_2 D_1 D_3\} + \\
& + K_2 B_2 D_1 D_3] \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} - [K_1 \{B_2 D_2 D_3 + B_2 (D_1 D_2 + D_2^2 - C^2)\} + \\
& + K_2 \{B_2 D_1 D_3 + B_2 (D_1 D_2 + D_2^2 - C^2)\}] \frac{\partial^4}{\partial \alpha^2 \partial \beta^4} - [K_1 B_2 D_2 D_3 + \\
& + K_2 \{B_2 D_2 D_3 + B_2 (D_1 D_2 + D_2^2 - C^2)\}] \frac{\partial^4}{\partial \alpha^4 \partial \beta^4} - B_2 D_2 D_3 K_2 \frac{\partial^4}{\partial \beta^4} + \\
& + B_2 D_1 (B_2 D_3 + K_1 K_2 R^2) \frac{\partial^6}{\partial \alpha^6} + \{B_2 B_3 (D_1 D_2 + D_2^2 - C^2) + \\
& + K_1 K_2 R^2 [B_2 D_1 + 2B_2 (C + D_3)]\} \frac{\partial^6}{\partial \alpha^4 \partial \beta^2} - \{B_2 B_3 D_2 D_3 + \\
& + K_1 K_2 R^2 [B_2 D_2 + 2B_2 (C + D_3)]\} \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} + B_2 D_2 K_1 K_2 R^2 \frac{\partial^6}{\partial \beta^6} - \\
& - B_2 B_3 R^2 (D_2 K_1 + D_3 K_1) \frac{\partial^6}{\partial \alpha^4} - B_2 B_3 R^2 (K_2 D_2 + K_1 D_2) \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} + \\
& + B_2 B_3 K_1 K_2 R^4 \frac{\partial^6}{\partial \alpha^2}; \tag{639}
\end{aligned}$$

$$\begin{aligned}
\Delta_{12} = & A D_1 D_3 K_1 \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} + A [K_1 (D_1 D_2 + D_2^2 - C^2) + K_2 D_1 D_3] \frac{\partial^6}{\partial \alpha^2 \partial \beta^2} + \\
& + A [K_1 D_2 D_3 + K_2 (D_1 D_2 + D_2^2 - C^2)] \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} + A D_2 D_3 K_2 \frac{\partial^6}{\partial \alpha \partial \beta^5} - \\
& - (B_2 B_3 D_1 D_3 + A D_1 K_1 K_2 R^2) \frac{\partial^6}{\partial \alpha^4 \partial \beta} - [B_2 B_3 (D_1 D_2 + D_2^2 - C^2) + \\
& + 2A (C + D_3) K_1 K_2 R^2] \frac{\partial^6}{\partial \alpha^2 \partial \beta^3} - (B_2 B_3 D_2 D_3 + A D_2 K_1 K_2 R^2) \frac{\partial^6}{\partial \alpha \partial \beta^5} + \\
& + B_2 B_3 R^2 (K_2 D_1 + K_1 D_2) \frac{\partial^6}{\partial \alpha^2 \partial \beta} + B_2 B_3 R^2 (K_2 D_2 + K_1 D_2) \frac{\partial^6}{\partial \alpha \partial \beta^3} - \\
& - B_2 B_3 K_1 K_2 R^4 \frac{\partial^6}{\partial \alpha \partial \beta}; \tag{640}
\end{aligned}$$

$$\begin{aligned}
\Delta_{13} = & -B_{12} B_3 D_1 D_3 \frac{\partial^7}{\partial \alpha^7} - [B_{12} B_3 (D_1 D_2 + D_2^2 - C^2) - \\
& - B_2 B_3 D_1 D_3] \frac{\partial^7}{\partial \alpha^5 \partial \beta^2} - [B_2 B_3 (D_1 D_2 + D_2^2 - C^2) + \\
& + B_{12} B_3 D_2 D_3] \frac{\partial^7}{\partial \alpha^2 \partial \beta^4} + B_2 B_3 D_2 D_3 \frac{\partial^7}{\partial \alpha \partial \beta^6} + B_{12} B_3 (K_2 D_1 + K_1 D_2) R^2 \times \\
& \times \frac{\partial^6}{\partial \alpha^6} - B_3 R^2 [K_1 (B_2 D_3 - B_{12} D_2) + K_2 (B_2 D_1 - B_{12} D_3)] \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} - \\
& - B_2 B_3 R^2 (K_2 D_3 + K_1 D_2) \frac{\partial^6}{\partial \alpha \partial \beta^6} - B_{12} B_3 K_1 K_2 R^4 \frac{\partial^6}{\partial \alpha^3} + \\
& + B_2 B_3 K_1 K_2 R^4 \frac{\partial^6}{\partial \alpha \partial \beta^2}; \tag{641}
\end{aligned}$$

$$\begin{aligned}
\Delta_{14} = & -B_{12} B_3 D_3 K_1 R \frac{\partial^6}{\partial \alpha^6} + B_3 R [K_1 (B_2 D_3 - B_{12} D_2) + B_{12} C K_2] \times \\
& \times \frac{\partial^6}{\partial \alpha^4 \partial \beta^2} + B_2 B_3 R (D_2 K_1 - C K_2) \frac{\partial^6}{\partial \alpha^2 \partial \beta^4} + B_{12} B_3 K_1 K_2 R^2 \frac{\partial^6}{\partial \alpha^4} - \\
& - B_2 B_3 K_1 K_2 R^2 \frac{\partial^6}{\partial \alpha^2 \partial \beta^2}; \tag{642}
\end{aligned}$$

$$\begin{aligned} \Delta_{11} = & B_1 B_2 (K_1 C - K_2 D_1) R \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} - B_2 R (K_1 B_2 C - K_2 \times \\ & \times (B_2 D_1 - B_1 D_2)) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + B_1 B_2 K_1 K_2 R^2 \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} - \\ & - B_1 B_2 K_1 K_2 R^2 \frac{\partial^4}{\partial \alpha \partial \beta^3}, \end{aligned} \quad (643)$$

and function ϕ is the solution of the homogeneous differential equation /188 of the tenth order $\Delta\phi=0$.

The elastic forces and moments are determined by Eq. (616). Ten random integration constants C_1 ($i=1, 2, \dots, 10$) are determined from boundary conditions of the type of (22), (23), of which five are at each end of the shell. If a partial solution must be found, in accordance with Eq. (609), the nonhomogeneous equation has the form

$$\Delta\phi = R^2 Z. \quad (644)$$

56. A Few Words on Integration of Equations of Technical Theory of Orthotropic Cylindrical Shell

The differential equations obtained in the preceding section can be used for solution of various engineering problems associated with the calculation of orthotropic laminated cylindrical shells.

It is convenient to present the integrals of resolving homogeneous Eq. (644) in the form of trigonometric series by coordinates α or β , depending on whether the shell is open or closed.

We begin with consideration of a closed cylindrical shell. We will seek resolving function ϕ in the form

$$\Phi(\alpha, \beta) = \sum_{n=0}^{\infty} (\Phi_n^{(1)} \cos n\beta + \Phi_n^{(2)} \sin n\beta), \quad (645)$$

where coefficients $\phi_n^{(1)}$, $\phi_n^{(2)}$ are functions of longitudinal coordinate α alone.

By substituting Eq. (645) in homogeneous differential Eq. (634), /189 we obtain

$$\sum_{n=0}^{\infty} [\Delta_n (\Phi_n^{(1)}) \cos n\beta + \Delta_n (\Phi_n^{(2)}) \sin n\beta] = 0, \quad (646)$$

where differential operator Δ_n is determined from the expression

$$\begin{aligned}
\Delta_n = & A_{10,0} \frac{d^{10}\Phi}{da^{10}} - (A_{8,2}n^2 + A_{8,0}) \frac{d^8\Phi}{da^8} + \\
& + (A_{6,4}n^4 + A_{6,2}n^2 + A_{20}) \frac{d^6\Phi}{da^6} - \\
& - (A_{4,6}n^6 + A_{4,4}n^4 + A_{02}n^2 + A_{00}) \frac{d^4\Phi}{da^4} + \\
& + n^6 (A_{2,6}n^2 + A_{2,6}) \frac{d^2\Phi}{da^2} - n^8 (A_{0,10}n^2 + A_{0,8}) \Phi.
\end{aligned} \tag{647}$$

Coefficients A_{ij} are determined by the geometric dimensions of the shell and the elastic constants of the material by Eq. (635).

In order to satisfy Eq. (646), the solution of ordinary differential equation with constant coefficients

$$\Delta_n(\Phi) = 0 \tag{648}$$

must be used as functions $\phi_n^{(1)}$, $\phi_n^{(2)}$.

After finding resolving function Φ , components of movement u , v , w and deformation functions ϕ , ψ are determined by Eq. (638), and the elastic forces and moments by Eq. (616).

It is easy to determine that all the quantities listed are determined by equations of the type of (646), i.e., they are the sum of two states, one of which is symmetrical relative to initial generatrix $\beta=0$ and the other is skew symmetrical. We agree to call the stressed and deformed states of the cylindrical shell which are described by function $\phi_n^{(1)}$ symmetrical and the stressed and deformed states described by functions $\phi_n^{(2)}$ skew symmetrical.

In the symmetrical state,

$$\Phi(\alpha, \beta) = \sum_{n=0}^{\infty} \Phi_n^{(1)}(\alpha) \cos n \beta, \tag{649}$$

and, consequently, the following expressions can be obtained:

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$$\left. \begin{aligned}
u = \sum_{n=0}^{\infty} U_n(\alpha) \cos n \beta; & \quad v = \sum_{n=1}^{\infty} V_n(\alpha) \sin n \beta; \\
w = \sum_{n=0}^{\infty} W_n(\alpha) \cos n \beta; & \quad \psi = \sum_{n=1}^{\infty} \psi_n(\alpha) \sin n \beta;
\end{aligned} \right\} \tag{650}$$

$$\left. \begin{aligned}
\varphi &= \sum_{n=0}^{\infty} \varphi_n(\alpha) \cos n \beta; & S &= \sum_{n=1}^{\infty} S_n(\alpha) \sin n \beta; \\
T_1 &= \sum_{n=0}^{\infty} T_{1n}(\alpha) \cos n \beta; & H &= \sum_{n=1}^{\infty} H_n(\alpha) \sin n \beta; \\
T_2 &= \sum_{n=0}^{\infty} T_{2n}(\alpha) \cos n \beta; & Q_2 &= \sum_{n=1}^{\infty} Q_{2n}(\alpha) \sin n \beta. \\
G_1 &= \sum_{n=0}^{\infty} G_{1n}(\alpha) \cos n \beta; & G_2 &= \sum_{n=0}^{\infty} G_{2n}(\alpha) \cos n \beta; \\
Q_1 &= \sum_{n=0}^{\infty} Q_{1n}(\alpha) \cos n \beta.
\end{aligned} \right\} \quad (650)$$

In the skew symmetrical stressed, deformed state of the shell,

$$\Phi(\alpha, \beta) = \sum_{n=1}^{\infty} \Phi_n^{(2)}(\alpha) \sin n \beta, \quad (651)$$

and, correspondingly, for the elastic forces, moments and deformations, we obtain

$$\left. \begin{aligned}
u &= \sum_{n=1}^{\infty} U_n(\alpha) \sin n \beta; & v &= \sum_{n=0}^{\infty} V_n(\alpha) \cos n \beta; \\
w &= \sum_{n=1}^{\infty} W_n(\alpha) \sin n \beta; & \psi &= \sum_{n=0}^{\infty} \psi_n(\alpha) \cos n \beta; \\
\varphi &= \sum_{n=1}^{\infty} \varphi_n(\alpha) \sin n \beta; & S &= \sum_{n=0}^{\infty} S_n(\alpha) \cos n \beta. \\
T_1 &= \sum_{n=1}^{\infty} T_{1n}(\alpha) \sin n \beta; & T_2 &= \sum_{n=1}^{\infty} T_{2n}(\alpha) \sin n \beta; \\
H &= \sum_{n=0}^{\infty} H_n(\alpha) \cos n \beta; & G_1 &= \sum_{n=1}^{\infty} G_{1n}(\alpha) \sin n \beta; \\
Q_2 &= \sum_{n=0}^{\infty} Q_{2n}(\alpha) \cos n \beta; & G_2 &= \sum_{n=1}^{\infty} G_{2n}(\alpha) \sin n \beta; \\
Q_1 &= \sum_{n=1}^{\infty} Q_{1n}(\alpha) \sin n \beta.
\end{aligned} \right\} \quad (652)$$

The engineering theory form of solution of Eq. (645) is used in /191 calculation of closed cylindrical shells, since the condition of periodicity of coordinate β is automatically fulfilled in this case. The ten random integration constants of Eq. (648) are determined from the boundary conditions at the curvilinear ends $\alpha=0$, $\alpha=\alpha_1$ (Fig. 76).

It was assumed that the boundary conditions, as in many engineering problems, are simple, i.e., of canonical form. In this case, resolving Eq. (648) can be integrated and the integration constants found independently for each harmonic n and separately for the symmetrical and skew symmetrical states of the shell.

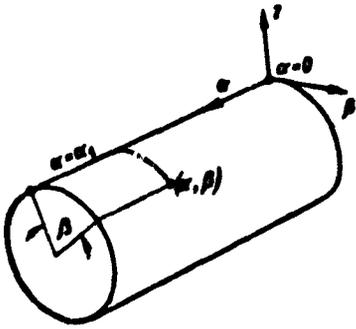


Fig. 76. For writing down boundary conditions.

We now consider another method of integration of resolving Eq. (648), which is used in calculations of open cylindrical hinge supported panels with curvilinear ends $\alpha=0$, $\alpha=\alpha_1$ (Fig. 77).

In this case, the stress function can be presented in the form

$$\Psi(u, \beta) = \sum_{m=1}^{\infty} \Psi_m(\beta) \sin \lambda_m u, \quad (653)$$

where

$$\lambda_m = \frac{m\pi}{\alpha_1}. \quad (654)$$

After substitution of Eq. (653) in Eq. (648), we obtain the following differential equation for coefficients of expansion $\Psi_m(\beta)$

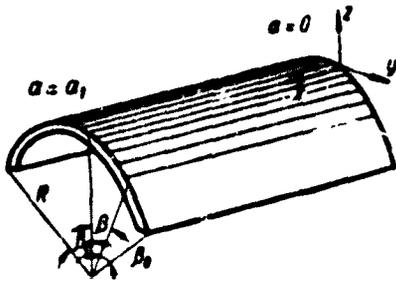


Fig. 77. Cylindrical panel.

$$\begin{aligned} & A_{0,10} \frac{d^{10}\Psi_m}{d\beta^{10}} - (A_{2,8} \lambda_m^2 + A_{0,8}) \frac{d^8\Psi_m}{d\beta^8} + \\ & + \lambda_m^2 (A_{4,6} \lambda_m^2 + A_{2,6}) \frac{d^6\Psi_m}{d\beta^6} - \lambda_m^4 (A_{0,4} \lambda_m^2 + A_{4,4}) \frac{d^4\Psi_m}{d\beta^4} + \\ & + \lambda_m^6 (A_{8,2} \lambda_m^6 + A_{6,2} \lambda_m^2 + A_{0,2}) \frac{d^2\Psi_m}{d\beta^2} - \lambda_m^8 (A_{10,0} \lambda_m^8 + \\ & + A_{8,0} \lambda_m^6 + A_{2,0} \lambda_m^2 + A_{0,0}) \Psi_m = 0 \end{aligned} \quad (655)$$

for $m=1, 2, \dots$

The components of deformation, elastic forces and moments can be obtained in trigonometric series of the type of (653) /192

$$\left. \begin{aligned} v &= \sum_{m=1}^{\infty} V_m(\beta) \sin \lambda_m u; & u &= \sum_{m=0}^{\infty} U_m(\beta) \cos \lambda_m u; \\ w &= \sum_{m=1}^{\infty} W_m(\beta) \sin \lambda_m u; & \varphi &= \sum_{m=0}^{\infty} \varphi_m(\beta) \cos \lambda_m u; \\ \Psi &= \sum_{m=1}^{\infty} \Psi_m(\beta) \sin \lambda_m u; & S &= \sum_{m=0}^{\infty} S_m(\beta) \cos \lambda_m u; \\ T_1 &= \sum_{m=1}^{\infty} T_{1m}(\beta) \sin \lambda_m u; & H &= \sum_{m=0}^{\infty} H_m(\beta) \cos \lambda_m u; \\ T_2 &= \sum_{m=1}^{\infty} T_{2m}(\beta) \sin \lambda_m u; & Q_1 &= \sum_{m=0}^{\infty} Q_{1m}(\beta) \cos \lambda_m u; \\ G_1 &= \sum_{m=1}^{\infty} G_{1m}(\beta) \sin \lambda_m u; & Q_2 &= \sum_{m=1}^{\infty} Q_{2m}(\beta) \sin \lambda_m u; \\ G_2 &= \sum_{m=1}^{\infty} G_{2m}(\beta) \sin \lambda_m u. \end{aligned} \right\} \quad (656)$$

It is inexpedient to present the expressions for the coefficients of expansion because of bulk.

The boundary conditions at the curvilinear edges of the panel $\alpha=0$, $\alpha=\alpha_1$ are satisfied. As was done earlier, it can be shown that the boundary conditions on the rectilinear ends $\beta=\pm\beta_1$ are satisfied separately for each term of the expansion, if the boundary conditions are simple, as is the case in the majority of practical problems.

Methods which use expansions in trigonometric series are widely used in engineering practice. They are presented in the monograph of A.L. Gol'denveyzer [8].

Thus, the problem of calculation of closed and open cylindrical shells made of laminated orthotropic materials was reduced to solution of ordinary tenth order differential equations with constant coefficients.

Consequently, the diversity of the solutions is determined by the 193 roots of the characteristic equations

$$A_{10,0}x^{10} - (A_{8,2}n^2 + A_{8,0})x^8 + (A_{6,4}n^4 + A_{6,2}n^2 + A_{60})x^6 - (A_{4,6}n^6 + A_{4,4}n^4 + A_{02}n^2 + A_{00})x^4 + n^2(A_{2,8}n^2 + A_{2,6})x^2 - n^2(A_{0,10}n^2 + A_{0,8}) = 0; \quad (657)$$

$$A_{0,10}x^{10} - (A_{2,8}\lambda_m^2 + A_{0,8})x^8 + \lambda_m^2(A_{4,6}\lambda_m^2 + A_{2,6})x^6 - \lambda_m^2(A_{6,4}\lambda_m^2 + A_{4,4})x^4 + \lambda_m^2(A_{8,2}\lambda_m^2 + A_{6,2}\lambda_m^2 + A_{02})x^2 - \lambda_m^2(A_{10,0}\lambda_m^2 + A_{8,0}\lambda_m^2 + A_{20}\lambda_m^2 + A_{00}) = 0, \quad (658)$$

where coefficients A_{1j} are determined through the geometric dimensions of the shell and the elastic constants of the material by Eq. (635). It is not expedient to carry out analysis of the roots of Eq. (657) and (658) for an orthotropic shell in general form. It is more efficient to study them for specific problems.

57. Transverse Vibrations of Orthotropic Cylindrical Shell

As an example, we consider the natural transverse vibrations of a laminated cylindrical shell made of an orthotropic material. We will assume that the principal axes of anisotropy coincide with the coordinate axes.

In dynamic problems of the theory of shells, the components of the external load equal the corresponding components of the inertial forces, i.e.,

$$X = -\rho\delta \frac{\partial^2 u}{\partial t^2}; \quad Y = -\rho\delta \frac{\partial^2 v}{\partial t^2}; \quad Z = -\rho\delta \frac{\partial^2 w}{\partial t^2}. \quad (659)$$

Since the flexural rigidity of a cylindrical shell is considerably less than the rigidity during its deformation in the mean plane and shearing deformations are small, we will disregard the tangential and shearing components of the inertial forces in study of transverse vibrations. In accordance with Eq. (634), the problem of transverse vibrations of a laminated orthotropic cylindrical shell is reduced to solution of the following differential equation

$$\begin{aligned}
 & A_{10,0} \frac{\partial^4 w}{\partial \alpha^4} + A_{8,2} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + A_{6,4} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^4} + A_{4,6} \frac{\partial^4 w}{\partial \alpha^4 \partial \beta^2} + \\
 & + A_{2,8} \frac{\partial^4 w}{\partial \alpha^4 \partial \beta^4} + A_{0,10} \frac{\partial^4 w}{\partial \beta^{10}} - \left(A_{8,0} \frac{\partial^2 w}{\partial \alpha^2} + A_{6,2} \frac{\partial^2 w}{\partial \alpha^2 \partial \beta^2} + \right. \\
 & \quad \left. + A_{4,4} \frac{\partial^2 w}{\partial \alpha^4 \partial \beta^2} + A_{2,6} \frac{\partial^2 w}{\partial \alpha^2 \partial \beta^4} + A_{0,8} \frac{\partial^2 w}{\partial \beta^8} \right) + \\
 & + \frac{\partial^2}{\partial \alpha^2} \left(A_{20} \frac{\partial^2 w}{\partial \alpha^2} + A_{02} \frac{\partial^2 w}{\partial \beta^2} \right) - A_{00} \frac{\partial^4 w}{\partial \alpha^4} = M Q \frac{\partial^2}{\partial t^2} m_1 m_n (w),
 \end{aligned} \tag{660}$$

where

$$M Q = (B_1 B_2 - B_{12}^2) B_3 \rho \delta R^4, \tag{661}$$

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and coefficients A_{ij} are determined by Eq. (635).

If it is assumed that the natural vibrations of a cylindrical shell are harmonic with frequency ω_{mn} , with hinge support of the ends of the shell, the form of the vibrations can be assigned in the form

$$w(\alpha, \beta; t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \bar{C}_m \sin \lambda_m \alpha \cos n \beta \sin \omega_{mn} t, \tag{662}$$

where

$$\lambda_m = \frac{\pi R m}{l}. \tag{663}$$

By substituting Eq. (662) in differential Eq. (660), we obtain the following expression for determination of the frequency of natural transverse vibrations of an orthotropic cylindrical shell:

$$\rho \delta R^4 \omega_{mn}^2 = \frac{P_1(\lambda_m, n) + P_2(\lambda_m, n) + P_3(\lambda_m, n)}{Q_1(\lambda_m, n) [Q_2(\lambda_m, n) + Q_3(\lambda_m, n)]} \tag{664}$$

for $m, n=1, 2, 3, \dots$, where the following designations are introduced for polynomials $P_1(\lambda_m, n), Q_1(\lambda_m, n)$

$$\begin{aligned}
 P_1(\lambda_m, n) &= A_{10,0} \lambda_m^{10} + A_{8,2} \lambda_m^8 n^2 + A_{6,4} \lambda_m^6 n^4 + \\
 & \quad + A_{4,6} \lambda_m^4 n^6 + A_{2,8} \lambda_m^2 n^8 + A_{0,10} n^{10}; \\
 P_2(\lambda_m, n) &= A_{8,0} \lambda_m^8 + A_{6,2} \lambda_m^6 n^2 + A_{4,4} \lambda_m^4 n^4 + \\
 & \quad + A_{2,6} \lambda_m^2 n^6 + A_{0,8} n^8;
 \end{aligned} \tag{665}$$

$$\begin{aligned}
P_2(\lambda_m, n) &= \lambda_m^6 (A_{20}\lambda_m^2 + A_{02}n^2 + A_{00}); \\
Q_1(\lambda_m, n) &= B_1B_2\lambda_m^6 + B_2(B_1B_2 + \\
&\quad + B_2 - A^2)\lambda_m^2 n^2 + B_2B_2n^6; \\
Q_2(\lambda_m, n) &= \frac{1}{R^4} [D_1D_2\lambda_m^6 + (D_1D_2 + \\
&\quad + D_2^2 - C^2)\lambda_m^2 n^2 + D_1D_2n^6]; \\
Q_3(\lambda_m, n) &= \frac{1}{R^2} [(K_1D_1 + K_1D_2)\lambda_m^6 + \\
&\quad + (K_1D_2 + K_2D_2)n^2] + K_1K_2.
\end{aligned} \tag{665}$$

If the shell is stretched by internal forces T_1^0 , T_2^0 , it is easy to show that the frequency of the natural vibrations increases and that the frequency of the natural oscillations decreases under precompression. /195

We now consider the problem of forced vibrations of a cylindrical shell as a result of a radial load which changes harmonically over time with frequency ω , i.e., load

$$Z(\alpha, \beta, t) = p(\alpha, \beta) \sin \omega t. \tag{666}$$

We will consider regular loads $p(\alpha, \beta)$, which can be presented in the form of the uniformly converging trigonometric series

$$p(\alpha, \beta) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} \sin \lambda_m \alpha \cos n \beta, \tag{667}$$

where

$$a_{mn} = \frac{2R}{\pi l} \int_0^{\alpha_1} \int_0^{2\pi} p(\alpha, \beta) \sin \lambda_m \alpha \cos n \beta \, d\alpha \, d\beta. \tag{668}$$

In the presence of perturbing forces, the equation of the transverse vibrations of an orthotropic cylindrical shell can be presented in the form

$$\begin{aligned}
&A_{10,0} \frac{\partial^{10} w}{\partial \alpha^{10}} + A_{8,2} \frac{\partial^{10} w}{\partial \alpha^8 \partial \beta^2} + A_{6,4} \frac{\partial^{10} w}{\partial \alpha^6 \partial \beta^4} + A_{4,6} \frac{\partial^{10} w}{\partial \alpha^4 \partial \beta^6} + \\
&+ A_{2,8} \frac{\partial^{10} w}{\partial \alpha^2 \partial \beta^8} + A_{0,10} \frac{\partial^{10} w}{\partial \beta^{10}} - (A_{3,0} \frac{\partial^8 w}{\partial \alpha^3} + A_{6,2} \frac{\partial^8 w}{\partial \alpha^2 \partial \beta^2} + \\
&+ A_{4,4} \frac{\partial^8 w}{\partial \alpha^4 \partial \beta^4} + A_{2,6} \frac{\partial^8 w}{\partial \alpha^2 \partial \beta^6} + A_{0,8} \frac{\partial^8 w}{\partial \beta^8}) + \frac{\partial^4}{\partial \alpha^4} (A_{20} \frac{\partial^4 w}{\partial \alpha^2} + \\
&+ A_{02} \frac{\partial^2 w}{\partial \beta^2} - A_{00} w) = M Q \frac{\partial^2}{\partial t^2} m_1 m_2(w) - \beta_2 (B_1 B_2 - B_{12}^2) R^2 m_1 m_2(Z).
\end{aligned} \tag{669}$$

We will seek the solution of Eq. (669) in the double trigonometric series

$$w = \sin \omega t \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m \alpha \cos n \beta. \tag{670}$$

By substituting this expression in Eq. (669), we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} [P_1 + P_2 + P_3 - q\delta R^2 \omega^2 Q_1 (Q_2 + Q_3)] \sin \lambda_m a \cos n \beta - \\ = R^2 \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} a_{mn} Q_1 (Q_2 + Q_3) \sin \lambda_m a \cos n \beta, \end{aligned} \quad (671)$$

from which

$$[P_1 + P_2 + P_3 - q\delta R^2 \omega^2 Q_1 (Q_2 + Q_3)] C_{mn} = Q_1 (Q_2 + Q_3) a_{mn} \quad (672)$$

for $m, n=1, 2, 3, \dots$

According to Eq. (664),

$$\frac{P_1 + P_2 + P_3}{Q_1 (Q_2 + Q_3)} = q\delta R^2 \omega_{mn}^2. \quad (673)$$

Consequently, the following expression can be obtained for the coefficients of expansion

$$C_{mn} = \frac{1}{q\delta} \frac{a_{mn}}{\omega_{mn}^2 - \omega^2} \quad (674)$$

for $m, n=1, 2, 3, \dots$

Thus, forced transverse vibrations of a laminated orthotropic cylindrical shell occur in the form

$$w = \frac{1}{q\delta} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{a_{mn}}{\omega_{mn}^2 - \omega^2} \sin \lambda_m a \cos n \beta. \quad (675)$$

Upon coincidence of forced vibration frequency ω with frequency of natural oscillations of some tone ω_{mn} ($m, n=1, 2, \dots$), resonance vibrations arise.

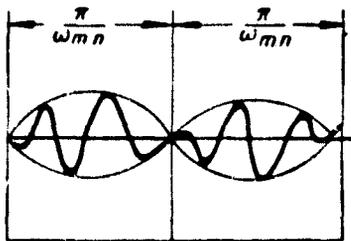


Fig. 78. Graph of beat with period π/ω_{mn} .

We consider forced vibrations of a shell, on the assumption that the natural vibrations disappear. The general solution of Eq. (669) will equal the sum of solutions (662), (670), i.e., the sum of the natural and forced vibrations

$$\begin{aligned} w = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \bar{C}_{mn} \sin \lambda_m a \cos n \beta \sin (\omega_{mn} t - \omega_0) + \\ + \sin \omega t \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} \sin \lambda_m a \cos n \beta, \end{aligned} \quad (676)$$

where C_{mn} ($m, n=1, 2, 3, \dots$) is determined by Eq. (673) and parameters C_{mn}, ω_0 are determined from the initial conditions.

Thus, for example, if the initial conditions are uniform, i.e., if $w = dw/dt = 0$ at $t=0$, the following form of vibration of the shell can be obtained

$$w = \frac{1}{\rho b} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\sin \omega t - \frac{\omega}{\omega_{mn}} \sin \omega_{mn} t \right)}{\omega_{mn}^2 - \omega^2} a_{mn} \sin \lambda_m u \cos n \beta. \quad (677)$$

If the forced vibration frequency is close to the natural vibration frequency of some tone ω_{mn} ($m, n=1, 2, 3, \dots$), so called beats occur, with oscillation period $T=2\pi/\omega_{mn}$ (Fig. 78).

CHAPTER 12. CALCULATION OF ORTHOTROPIC CYLINDRICAL SHELL SUBJECTED TO LOCALLY DISTRIBUTED AXIAL FORCES

58. Initial Hypotheses and Basic Differential Equations

Many works of domestic and foreign authors [11, 24-26] have dealt /198 with study of the strength of isotropic cylindrical shells which are subjected to locally distributed loads. Among them, the work of V.M. Darevskiy should be distinguished, in which a partial solution of the most exact cylindrical shell equations were obtained in Fourier integrals, and the convergence of the series and characteristics of the solution were studied.

A significant contribution to the development of methods of calculation of cylindrical shells for concentrated loads was made by V.Z. Vlasov [4-6]. The semimembrane theory of a cylindrical shell he developed has been widely used in engineering practice, and it has shown satisfactory correspondence with experimental results. This chapter deals with generalization of this theory for laminated orthotropic shells.

Experimental studies of thin quite long cylindrical shells shows a characteristic feature of their deformation, which is that significant annular deformation of the shell occurs compared with deformation of the generatrix as a result of concentrated radial loads. An orthogonal grid applied to the lateral surface of the shell remains nearly orthogonal after deformation but the annular lines, which bend sharply, remain almost incompressible. These characteristics of deformation, together with the results of other experimental studies, were the basis for the semimembrane theory of a cylindrical shell, two alternate versions of which were presented in the works of V.Z. Vlasov, which differed from each other in the number of initial simplifying hypotheses.

With the comparatively low resistance to shearing deformations of /199 laminated plastics taken into account, since the shear modulus of laminated plastics is at least an order of magnitude less than the shear modulus of metals, shearing deformations of the mean surface of the shell cannot be disregarded, as was done in the last alternate version of semimembrane theory. Some other simplifying assumptions can also be dropped.

Following V.Z. Vlasov, for the calculation scheme of the cylindrical shell in its calculation as a result of locally distributed axial forces, we use a three dimensional elastic system consisting of rings which are connected together by vanishingly short connecting rods which ensure the transmission of axial forces and shearing forces (Fig. 79).

Each ring in the cross section plane of the shell works both by tension or compression and by bending as well as shearing. In the longitudinal direction of the basic calculation scheme, only momentless tangential forces N , S can appear. This calculation scheme corresponds to the assumption of the membrane structure of the shell in the axial direction, i.e., the small magnitude of bending moments G_1 in cross sections and the insignificant effect of torques H .

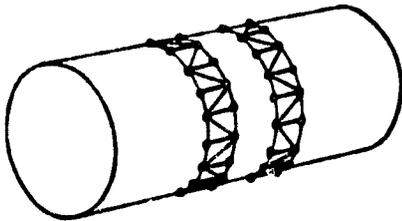


Fig. 79. Diagram of cylindrical shell for semimembrane theory calculation.

In this manner, according to the hypotheses used, the stressed state of a laminated cylindrical shell subjected to axial locally distributed forces is determined by the following parameters:

longitudinal normal force $N=N(\alpha, \beta)$;

shearing force $S=S(\alpha, \beta)$;

cutting force in axial section $Q=Q(\alpha, \beta)$;

annular bending moment $G=G(\alpha, \beta)$;

normal annular force $T=T(\alpha, \beta)$.

For convenience in writing, we use symbols here which differ somewhat from the previous symbols for the elastic forces and moments. This permits subsequent avoidance of excess piling up of symbols.

The deformed state of the shell also is described by five functions:

axial movement $u=u(\alpha, \beta)$;

annular movement $v=v(\alpha, \beta)$;

positive shell deflection toward outer normal $w=w(\alpha, \beta)$;

deformation functions $\phi(\alpha, \beta)$, $\psi(\alpha, \beta)$, which characterize 200 the interlayer shearing of the laminated shell.

The equilibrium equations have the form

$$\left. \begin{aligned} \frac{\partial N}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0; \\ \frac{\partial T}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= 0; \\ \frac{\partial Q}{\partial \beta} + T &= 0; \\ \frac{\partial G}{\partial \beta} &= RQ. \end{aligned} \right\} \quad (678)$$

We present the elasticity relationships in the form

$$\left. \begin{aligned} e_1 &= \frac{N - \nu_1 T}{E_1 \delta}; \\ e_2 &= \frac{T - \nu_2 N}{E_2 \delta}; \\ \omega &= \frac{S}{G_{12} \delta}; \end{aligned} \right\} \quad (679)$$

$$G = -\frac{D_2}{R} \left(\frac{\partial \psi}{\partial \beta} + \nu_2 \frac{\partial \varphi}{\partial \alpha} \right); \quad (680)$$

$$Q = -K_2 \left(\psi + \frac{1}{R} \frac{\partial w}{\partial \beta} - \frac{v}{R} \right). \quad (681)$$

It is easy to verify that equilibrium equations and elasticity relationships (679), (680) are identically satisfied, if the following stress function $F=F(\alpha, \beta)$ is introduced

$$\left. \begin{aligned}
 u &= \frac{R}{E_1 \delta} \left[v_1 \frac{\partial^2 F}{\partial \alpha^2 \partial \beta^2} - \frac{\partial^4}{\partial \alpha \partial \beta^3} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F \right]; \\
 v &= \frac{R}{G_{13} \delta} \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F - \frac{R}{E_1 \delta} \times \\
 &\quad \times \left[v_1 \frac{\partial^2 F}{\partial \alpha^2 \partial \beta^2} - \frac{\partial^4}{\partial \beta^4} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F \right]; \\
 w &= \frac{R}{E_2 \delta} \left[v_2 \frac{\partial^2}{\partial \alpha^2 \partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F - \frac{\partial^2 F}{\partial \alpha^4 \partial \beta^2} \right] - \\
 &\quad - \frac{R}{G_{12} \delta} \cdot \frac{\partial^2}{\partial \alpha^2 \partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F + \frac{R}{E_1 \delta} \left[v_1 \frac{\partial^2 F}{\partial \alpha^2 \partial \beta^2} - \right. \\
 &\quad \quad \left. - \frac{\partial^4}{\partial \beta^4} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F \right]; \\
 \varphi &= \frac{R^2 v_2}{D_2 (1 - v_1 v_2)} \cdot \frac{\partial^4 F}{\partial \alpha^2 \partial \beta^2}; \\
 \psi &= - \frac{R_2}{D_2 (1 - v_1 v_2)} \cdot \frac{\partial^4 F}{\partial \alpha^4}; \\
 N &= - \frac{\partial^2}{\partial \alpha^2 \partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F; \\
 T &= - \frac{\partial^2 F}{\partial \alpha^4 \partial \beta^2}; \quad S = \frac{\partial^2}{\partial \alpha^2 \partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) F; \\
 Q &= \frac{c^2 F}{\partial \alpha^4 \partial \beta^2}; \quad G = R \frac{\partial^2 F}{\partial \alpha^4 \partial \beta^2}.
 \end{aligned} \right\} \quad (682)$$

We obtain equations for determination of stress functions $F(\alpha, \beta)$, /201 if we satisfy the last elasticity relationship (681). By substituting the necessary equations (682) in Eq. (681), the following differential equation can be obtained for determination of the stress functions

$$\begin{aligned}
 &\frac{\partial^2 F}{\partial \alpha^4 \partial \beta^2} - v_1 \frac{\partial^2 F}{\partial \alpha^2 \partial \beta^2} + \frac{\partial^4}{\partial \beta^4} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) \left[(g^2 - v_2) \frac{\partial^2 F}{\partial \alpha^2} + \right. \\
 &+ \lambda^2 \frac{\partial^2 F}{\partial \beta^2} \left. \right] - v_2 \frac{\partial^2 F}{\partial \alpha^2 \partial \beta^2} - k^2 \frac{\partial^2 F}{\partial \alpha^4 \partial \beta^2} + \frac{\partial^2}{\partial \beta^2} \left(\frac{\partial^2}{\partial \beta^2} + 1 \right) \left[g^2 \frac{\partial^2 F}{\partial \alpha^2} + \right. \\
 &\quad \left. + \lambda^2 \frac{\partial^2 F}{\partial \beta^2} \right] + c^2 \frac{\partial^2 F}{\partial \alpha^4} = 0,
 \end{aligned} \quad (683)$$

where the basic elasticity parameters

$$\left. \begin{aligned}
 \lambda^2 &= \frac{E_2}{E_1}; \quad g^2 = \frac{E_2}{G_{13}}; \\
 k^2 &= \frac{6E_2}{5G_{13}}; \quad c^2 = \frac{12R^2}{\delta^2}.
 \end{aligned} \right\} \quad (684)$$

Eq. (683) is a differential equation in partial derivatives with constant coefficients, which depend on elastic parameters λ^2 , g^2 , k^2

and ν_2 and one geometric parameter c^2 . In the case of a thin isotropic shell, only two independent parameters c^2 and ν remain, since $\lambda^2=1$, $k^2=0$ and $E_2/G_{12}=2(1+\nu)$.

Because of relationships (682), differential Eq. (683) is satisfied both by stress functions $F(\alpha, \beta)$ and any of the ten functions which define the stressed and deformed states of a laminated cylindrical shell subjected to axial locally distributed loads.

59. Integration of Eq. (683) in Single Trigonometric Series and Boundary Conditions

In the case of a closed cylindrical shell, the solution of Eq. (683) /2 can be sought in the form of the single trigonometric series

$$F(\alpha, \beta) = \sum_{n=0}^{\infty} F_n(\alpha) \cos n\beta. \quad (685)$$

Coefficients of expansion $F_n(\alpha)$ are satisfied by the following ordinary differential equations with constant coefficients

$$\frac{d^4 F_n}{d\alpha^4} - 2p_n^2 \frac{d^2 F_n}{d\alpha^2} + q_n^4 F_n = 0 \quad \text{for } n = 1, 2, 3, \dots \quad (686)$$

where

$$\left. \begin{aligned} 2p_n^2 &= \frac{n^2(n^2-1)^2 g^2 - 2\nu_2 n^4(n^2-1)}{n^4 + k^2 n^2 + c^2}; \\ q_n^4 &= \frac{\lambda^2 n^4(n^2-1)^2}{n^4 + k^2 n^2 + c^2}. \end{aligned} \right\} \quad (687)$$

The solutions of differential Eq. (686) are written in V.Z. Vlasov functions, which are determined by Eq. (188) and are satisfied by Eq. (189),

$$\left. \begin{aligned} \Phi_1(\alpha) &= \text{ch } s_n \alpha \cos r_n \alpha; & \Phi_3(\alpha) &= \text{sh } s_n \alpha \cos r_n \alpha; \\ \Phi_2(\alpha) &= \text{sh } s_n \alpha \sin r_n \alpha; & \Phi_4(\alpha) &= \text{ch } s_n \alpha \sin r_n \alpha, \end{aligned} \right\} \quad (688)$$

where

$$s_n = \sqrt{\frac{q_n^2 + p_n^2}{2}}; \quad r_n = \sqrt{\frac{q_n^2 - p_n^2}{2}}. \quad (689)$$

If the elastic constants of a laminated plastic are such that the inequality

$$G_{12} < \frac{\sqrt{E_1 E_2}}{2}, \quad (690)$$

is satisfied, with sufficiently large values of $n \gg R/\delta$, parameter r_n becomes imaginary. In this case, instead of the V.Z. Vlasov functions,

hyperbolic functions should be used which are determined by the expressions

$$\left. \begin{aligned} \theta_1(\alpha) &= \operatorname{ch} s_n \alpha \operatorname{ch} r_n \alpha; & \theta_3(\alpha) &= \operatorname{sh} s_n \alpha \operatorname{ch} r_n \alpha; \\ \theta_2(\alpha) &= \operatorname{sh} s_n \alpha \operatorname{sh} r_n \alpha; & \theta_4(\alpha) &= \operatorname{ch} s_n \alpha \operatorname{sh} r_n \alpha, \end{aligned} \right\} \quad (691)$$

where

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$$s_n = \sqrt{\frac{q_n^2 + p_n^2}{2}}; \quad r_n = \sqrt{\frac{p_n^2 - q_n^2}{2}}. \quad (692)$$

We also represent the components of the stressed and deformed states of the shell by the single trigonometric series

$$\left. \begin{aligned} N(\alpha, \beta) &= \sum_{n=0}^{\infty} N_n(\alpha) \cos n \beta; \\ T(\alpha, \beta) &= \sum_{n=0}^{\infty} T_n(\alpha) \cos n \beta; \\ S(\alpha, \beta) &= \sum_{n=0}^{\infty} S_n(\alpha) \sin n \beta; \\ Q(\alpha, \beta) &= \sum_{n=0}^{\infty} Q_n(\alpha) \sin n \beta; \\ G(\alpha, \beta) &= \sum_{n=0}^{\infty} G_n(\alpha) \cos n \beta; \\ u(\alpha, \beta) &= \sum_{n=0}^{\infty} U_n(\alpha) \cos n \beta; \\ v(\alpha, \beta) &= \sum_{n=0}^{\infty} V_n(\alpha) \sin n \beta; \\ w(\alpha, \beta) &= \sum_{n=0}^{\infty} W_n(\alpha) \cos n \beta; \\ \varphi(\alpha, \beta) &= \sum_{n=0}^{\infty} \Phi_n(\alpha) \cos n \beta; \\ \psi(\alpha, \beta) &= \sum_{n=0}^{\infty} \Psi_n(\alpha) \sin n \beta. \end{aligned} \right\} \quad (693)$$

In accordance with Eq. (682), the coefficients of expansion of the elastic forces, moments, movements and deformation functions in the trigonometric series are determined by the following expressions

$$\left. \begin{aligned} U_n(\alpha) &= \frac{Rn^2}{E_2 \delta} [v_2 F_n'' + \lambda^2 (n^2 - 1) F_n']; \\ V_n(\alpha) &= \frac{Rn^2}{E_2 \delta} [g^2 (n^2 - 1) - v_2 n^2] F_n'' - \lambda^2 n^2 (n^2 - 1) F_n'; \end{aligned} \right\} \quad (694)$$

$$\left.
\begin{aligned}
w_n(\alpha) &= -\frac{Rn^2}{E_s \delta} \left\{ [2p_n^2 - g^2(n^2 - 1) + v_g(2n^2 - 1)] F_n' + \right. \\
&\quad \left. + \frac{\lambda^2 n^2 (n^2 - 1) [(k^2 + 1)n^2 + c^2]}{n^4 + k^2 n^2 + c^2} F_n \right\}; \\
\varphi_n(\alpha) &= -\frac{v_g c^2}{E_s \delta} n F_n''; \\
\Psi_n(\alpha) &= -\frac{c^2}{E_s \delta} (2p_n^2 F_n' - q_n^2 F_n); \\
N_n(\alpha) &= n^3 (n^2 - 1) F_n'; \\
S_n(\alpha) &= n^3 (n^2 - 1) F_n''; \\
Q_n(\alpha) &= -n^3 (2p_n^2 F_n' - q_n^2 F_n); \\
G_n(\alpha) &= -nR (2p_n^2 F_n' - q_n^2 F_n).
\end{aligned}
\right\} \quad (694)$$

In accordance with Eq. (688) and relationships (189), we obtain /204

$$\left.
\begin{aligned}
F_n(\alpha) &= C_1 \Phi_1(\alpha) + C_2 \Phi_2(\alpha) + C_3 \Phi_3(\alpha) + C_4 \Phi_4(\alpha); \\
F_n'(\alpha) &= C_1 (s_n \Phi_3 - r_n \Phi_4) + C_2 (s_n \Phi_4 + r_n \Phi_3) + \\
&\quad + C_3 (s_n \Phi_1 - r_n \Phi_2) + C_4 (s_n \Phi_2 + r_n \Phi_1); \\
F_n''(\alpha) &= C_1 [(s_n^2 - r_n^2) \Phi_1 - 2r_n s_n \Phi_2] + \\
&\quad + C_2 [s_n^2 - r_n^2) \Phi_3 + 2r_n s_n \Phi_4] + C_3 [(s_n^2 - r_n^2) \Phi_2 - \\
&\quad - 2r_n s_n \Phi_3] + C_4 [(s_n^2 - r_n^2) \Phi_4 + 2r_n s_n \Phi_1]; \\
F_n'''(\alpha) &= C_1 [s_n (s_n^2 - 3r_n^2) \Phi_3 + r_n (r_n^2 - 3s_n^2) \Phi_4] + \\
&\quad + C_2 [s_n (s_n^2 - 3r_n^2) \Phi_4 - r_n (r_n^2 - 3s_n^2) \Phi_3] + \\
&\quad + C_3 [s_n (s_n^2 - 3r_n^2) \Phi_1 + r_n (r_n^2 - 3s_n^2) \Phi_2] + \\
&\quad + C_4 [s_n (s_n^2 - 3r_n^2) \Phi_2 - r_n (r_n^2 - 3s_n^2) \Phi_1].
\end{aligned}
\right\} \quad (695)$$

Because of the momentless structure of the shell in the axial direction at ends $\alpha=0$, $\alpha=\alpha_1$, only two boundary conditions each can be assigned:

- a. static relative to elastic forces N_n , S_n ;
- b. kinematic relative to movement U_n , V_n ;
- c. or mixed static-kinematic relative to N_n , V_n or S_n , U_n .

The canonical form of the uniform boundary conditions can be written thus:

a. hinge supported edges, at $\alpha=0$, $V_n(0)=0$; $N_n(0)=0$; at $\alpha=\alpha_1$, $V_n(\alpha_1)=0$; $N_n(\alpha_1)=0$;

b. rigidly fastened ends, at $\alpha=0$, $U_n(0)=0$; $V_n(0)=0$; at $\alpha=\alpha_1$, /205
 $U_n(\alpha_1)=0$; $V_n(\alpha_1)=0$;

c. unsupported ends, at $\alpha=0$, $N_n(0)=0$; $S_n(0)=0$; at $\alpha=\alpha_1$, $N_n(\alpha_1)=0$; $S_n(\alpha_1)=0$.

For convenience in practical use, it is advisable to express integration constants C_1, C_2, C_3, C_4 by initial parameters $U_n(0), V_n(0), N_n(0), S_n(0)$, i.e., by the values of functions U_n, V_n, N_n, S_n at the coordinate origin ($\alpha=0$). After this, the problem of calculation of a laminated orthotropic shell subjected to concentrated and locally distributed forces is reduced to the solution of a system of two algebraic equations for two unknown initial functions, since two of them must be assigned.

It is easy to obtain the following expressions by Eq. (694):

$$\left. \begin{aligned}
 \frac{E_2 \delta}{R} U_n(\alpha) &= n^2 [C_1 (f_1 \Phi_3 + f_2 \Phi_4) + C_2 (f_1 \Phi_4 - f_2 \Phi_3) + \\
 &\quad + C_3 (f_1 \Phi_1 + f_2 \Phi_2) + C_4 (f_1 \Phi_2 - f_2 \Phi_1)]; \\
 \frac{E_2 \delta}{R} V_n(\alpha) &= n^2 [C_1 (h_1 \Phi_1 - h_2 \Phi_2) + C_2 (h_1 \Phi_2 + h_2 \Phi_1) + \\
 &\quad + C_3 (h_1 \Phi_3 - h_2 \Phi_4) + C_4 (h_1 \Phi_4 + h_2 \Phi_3)]; \\
 N_n(\alpha) &= n^2 (n^2 - 1) [C_1 [(s_n^2 - r_n^2) \Phi_1 - 2r_n s_n \Phi_2] + \\
 &\quad + C_2 [(s_n^2 - r_n^2) \Phi_2 + 2r_n s_n \Phi_1] + C_3 [(s_n^2 - r_n^2) \Phi_3 - 2r_n s_n \Phi_4] + \\
 &\quad + C_4 [(s_n^2 - r_n^2) \Phi_4 + 2r_n s_n \Phi_3]]; \\
 S_n(\alpha) &= n^2 (n^2 - 1) [C_1 [s_n (s_n^2 - 3r_n^2) \Phi_3 + r_n (r_n^2 - 3s_n^2) \Phi_4] + \\
 &\quad + C_2 [s_n (s_n^2 - 3r_n^2) \Phi_4 - r_n (r_n^2 - 3s_n^2) \Phi_3] + \\
 &\quad + C_3 [s_n (s_n^2 - 3r_n^2) \Phi_1 + r_n (r_n^2 - 3s_n^2) \Phi_2] + \\
 &\quad + C_4 [s_n (s_n^2 - 3r_n^2) \Phi_2 - r_n (r_n^2 - 3s_n^2) \Phi_1]];
 \end{aligned} \right\} (697)$$

where

$$\left. \begin{aligned}
 f_1 &= s_n [v_2 (s_n^2 - 3r_n^2) + \lambda^2 (n^2 - 1)]; \\
 f_2 &= r_n [v_2 (r_n^2 - 3s_n^2) - \lambda^2 (n^2 - 1)]; \\
 h_1 &= (s_n^2 - r_n^2) [g^2 (n^2 - 1) - n^2 v_2] - \lambda^2 n^2 (n^2 - 1); \\
 h_2 &= 2r_n s_n [g^2 (n^2 - 1) - n^2 v_2]; \\
 C_1 &= \frac{1}{\lambda^2 n^2 (n^2 - 1)} \left[\frac{g^2 (n^2 - 1) - n^2 v_2}{n^2 (n^2 - 1)} N_n(0) - \right. \\
 &\quad \left. - \frac{E_2 \delta}{R n^2} V_n(0) \right]; \\
 C_2 &= -\frac{1}{2\lambda^2 s_n r_n n^2 (n^2 - 1)} \times \\
 &\quad \times \left[\frac{(s_n^2 - r_n^2) [g^2 (n^2 - 1) - n^2 v_2] - \lambda^2 n^2 (n^2 - 1)}{n^2 (n^2 - 1)} N_n(0) - \right. \\
 &\quad \left. - \frac{E_2 \delta}{R} \frac{(s_n^2 - r_n^2)}{n^2} V_n(0) \right]; \\
 C_3 &= -\frac{1}{2\lambda^2 s_n (n^2 - 1) (s_n^2 + r_n^2)} \left[\frac{E_2 \delta}{R n^2} (r_n^2 - 3s_n^2) U_n(0) + \right. \\
 &\quad \left. + \frac{\lambda^2 (n^2 - 1) - v_2 (r_n^2 - 3s_n^2)}{n^2 (n^2 - 1)} S_n(0) \right];
 \end{aligned} \right\} (698)$$

$$C_4 = \frac{1}{2\lambda^3 r_n (n^2 - 1) (r_n^2 + r_n^2)} \left[\frac{E_2 \delta}{R n^3} (s_n^2 - 3r_n^2) U_n(0) - \frac{\lambda^2 (n^2 - 1) + \nu_2 (s_n^2 - 3r_n^2)}{n^2 (n^2 - 1)} S_n(0) \right] \quad (698)$$

60. Girder Analogies and Initial Parameters Method

Basic differential Eq. (686) of the semimembrane theory of an orthotropic laminated cylindrical shell and boundary conditions (696) are similar to the corresponding equations and boundary conditions for girders lying on a solid elastic base. /206

The analogy is that static quantities $N_n(\alpha)$ and $S_n(\alpha)$ in the bending theory of girders correspond to the bending moment and cutting force, and movement components $V_n(\alpha)$, $U_n(\alpha)$ correspond to the deflection of the elastic axis of the girder and the angle of rotation of an element of this axis. The analogy goes still further, namely, at $n=0$ and $r=1$, differential Eq. (686) of the semimembrane theory changes to a differential equation of bending of the girder, i.e., it describes the deformed state which corresponds to the flat section principle, and members $n \geq 2$ describe the deformed state which develops as a result of self balancing loads, when there is warping of the cross sections of the shell.

Thus, if a locally distributed axial load applied to the cylindrical shell in section $\alpha=\xi$ (Fig. 80) is represented in the form of the trigonometric series

$$P = \frac{P}{2\pi R} + \frac{P}{\pi R} \cos \beta + \sum_{n=2}^{\infty} a_n \cos n \beta, \quad (699)$$

the solution which corresponds to the first two terms $n=0$ and $n=1$ can be found, by considering the shell as a girder with the corresponding support fastenings. The term $n=0$ represents an axisymmetric load uniformly distributed in section $\alpha=\xi$. The second term ($n=1$) represents an axial load distributed over section $\alpha=\xi$ by the $\cos \beta$ law, i.e., a bending moment applied in this section of the shell. Both of these cases of loading apply to the simplest problems of the strength of materials. /207

The remaining terms of the series $a_n \cos n\beta$ define self balancing axial loads applied in section $\alpha=\xi$. We will call such self balancing loads n -th order harmonic forces. They cause deformation of the outline and warping of the cross section of the shell.

For $n \geq 2$, the basic forces and generalized movements can be presented in the following form (subscript n is omitted for convenience):

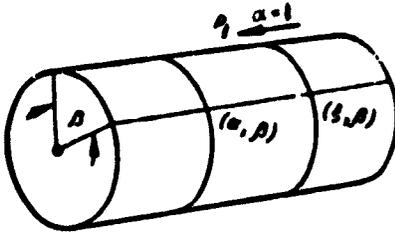


Fig. 80. Cylindrical shell subjected to concentrated axial force.

$$\left. \begin{aligned}
 N(a) &= K_{NN}(a)N(0) + K_{NS}(a)S(0) + K_{NV}(a)V(0) + \\
 &\quad + K_{NU}(a)U(0) - K_{NN}(a-\xi)a_n; \\
 S(a) &= K_{SN}(a)N(0) + K_{SS}(a)S(0) + K_{SV}(a)V(0) + \\
 &\quad + K_{SU}(a)U(0) - K_{SN}(a-\xi)a_n; \\
 V(a) &= K_{VN}(a)N(0) + K_{VS}(a)S(0) + K_{VV}(a)V(0) + \\
 &\quad + K_{VU}(a)U(0) - K_{VN}(a-\xi)a_n; \\
 U(a) &= K_{UN}(a)N(0) + K_{US}(a)S(0) + K_{UV}(a)V(0) + \\
 &\quad + K_{UU}(a)U(0) - K_{UN}(a-\xi)a_n.
 \end{aligned} \right\} (700)$$

In Eq. (700), generalized movements $U(a)$, $V(a)$, $U(0)$, $V(0)$ have the dimensionality of forces. These are components of movement multiplied by rigidity parameter $E_2 \delta / R$.

Coefficients $K_{NN}(a)$, $K_{NS}(a)$, . . . , $K_{UU}(a)$ are effect functions, /208
 which can be determined by the following expressions

$$\left. \begin{aligned}
 K_{NN}(a) &= \Phi_1 - \frac{1}{2\lambda^2 sr n^2 (n^2 - 1)} \{ (s^2 + r^2) [g^2 (n^2 - 1) - n^2 v_2] - \\
 &\quad - \lambda^2 n^2 (n^2 - 1) (s^2 - r^2) \} \Phi_2; \\
 K_{NS}(a) &= \frac{n}{2\lambda^2 (n^2 - 1)} \left\{ \frac{1}{s} [\lambda^2 (n^2 - 1) - (s^2 + r^2) v_2] \Phi_3 + \right. \\
 &\quad \left. + \frac{1}{r} [\lambda^2 (n^2 - 1) + (s^2 + r^2) v_2] \Phi_4 \right\}; \\
 K_{NV}(a) &= \frac{(s^2 + r^2)^2}{2\lambda^2 sr n} \Phi_2; \\
 K_{NU}(a) &= \frac{s^2 + r^2}{2\lambda^2 rs} (r \Phi_3 - s \Phi_4); \\
 K_{SN}(a) &= -\frac{1}{2\lambda^2 n^2 (n^2 - 1) rs} \{ |r [(s^2 + r^2)^2 [g^2 (n^2 - 1) - \\
 &\quad - n^2 v_2] + \lambda^2 n^2 (n^2 - 1) (r^2 - 3s^2)] \Phi_3 + s [(s^2 + r^2)^2 [g^2 (n^2 - 1) - \\
 &\quad - n^2 v_2] - \lambda^2 n^2 (n^2 - 1) (s^2 - 3r^2)] \Phi_4 \}; \\
 K_{SS}(a) &= \Phi_1 + \frac{1}{2\lambda^2 sr (n^2 - 1)} \{ (s^2 - r^2) \lambda^2 (n^2 - 1) + \\
 &\quad + (s^2 + r^2)^2 v_2 \} \Phi_2; \\
 K_{SV}(a) &= \frac{(s^2 + r^2)^2}{2\lambda^2 n^2 rs} (r \Phi_3 + s \Phi_4); \\
 K_{SU}(a) &= -\frac{(s^2 + r^2)^2}{2\lambda^2 sr n} \Phi_2; \\
 K_{VN}(a) &= \frac{1}{2\lambda^2 sr n^2 (n^2 - 1)^2} \{ [g^2 (n^2 - 1) - n^2 v_2] \times \\
 &\quad \times [2\lambda^2 n^2 (n^2 - 1) (s^2 - r^2) - (s^2 + r^2) [g^2 (n^2 - 1) - n^2 v_2]] - \\
 &\quad - \lambda^4 n^4 (n^2 - 1)^2 \} \Phi_2;
 \end{aligned} \right\} (701)$$

$$\begin{aligned}
 K_{VS}(a) &= \frac{1}{2\lambda^2(n^2-1)(s^2+r^2)sr} \{ |r| (s^2+r^2) [g^2(n^2-1) - \\
 &\quad - n v_2] [\lambda^2(n^2-1) - (s^2+r^2) v_2] + \lambda^2 n^2 (n^2-1) \times \\
 &\quad \times [\lambda^2(n^2-1) - (r^2-3s^2) v_2] \Phi_2 + s [(s^2+r^2) [g^2(n^2-1) - \\
 &\quad - n^2 v_2] [\lambda^2(n^2-1) + (s^2+r^2) v_2] - \lambda^2 n^2 (n^2-1) \times \\
 &\quad \times [\lambda^2(n^2-1) + (s^2-3r^2) v_2] \Phi_4 \}; \\
 K_{VV}(a) &= \Phi_1 + \frac{1}{2\lambda^2 sr n^2 (n^2-1)} [(s^2+r^2) [g^2(n^2-1) - n^2 v_2] - \\
 &\quad - \lambda^2 n^2 (n^2-1) (s^2-r^2)] \Phi_2; \\
 K_{VU}(a) &= \frac{1}{2\lambda^2 n (n^2-1) (s^2+r^2) sr} \{ |r| (s^2+r^2)^2 [g^2(n^2-1) - \\
 &\quad - n^2 v_2] + \lambda^2 n^2 (n^2-1) (r^2-3s^2) \Phi_2 - s [(s^2+r^2)^2 \times \\
 &\quad \times [g^2(n^2-1) - n^2 v_2] - \lambda^2 n^2 (n^2-1) (s^2-3r^2)] \Phi_4 \}; \\
 K_{UN}(a) &= \frac{1}{2\lambda^2 r n^2 (n^2-1)^2} \{ |r| (s^2+r^2) [g^2(n^2-1) - n^2 v_2] \times \\
 &\quad \times [\lambda^2(n^2-1) - (s^2+r^2) v_2] + \lambda^2 n^2 (n^2-1) [\lambda^2(n^2-1) - \\
 &\quad - (r^2-3s^2) v_2] \Phi_2 - s [(s^2+r^2) [g^2(n^2-1) - n^2 v_2] \times \\
 &\quad \times [\lambda^2(n^2-1) + (s^2+r^2) v_2] - \lambda^2 n^2 (n^2-1) [\lambda^2(n^2-1) + \\
 &\quad + (s^2-3r^2) v_2] \Phi_4 \}; \\
 K_{US}(a) &= \frac{n}{2\lambda^2 sr (n^2-1)} [(s^2+r^2)^2 v_2 + \lambda^2 (n^2-1) \times \\
 &\quad \times [\lambda^2(n^2-1)^2 + 2(s^2-r^2) v_2] \Phi_2; \\
 K_{UV}(a) &= -\frac{(s^2+r^2)}{2\lambda^2 r n (n^2-1)} [r [\lambda^2(n^2-1) - (s^2+r^2) v_2] \Phi_2 - \\
 &\quad - s [\lambda^2(n^2-1) + (s^2+r^2) v_2] \Phi_4]; \\
 K_{UU}(a) &= \Phi_1 - \frac{1}{2\lambda^2 sr (n^2-1)} [\lambda^2(n^2-1) (s^2-r^2) + \\
 &\quad + (s^2+r^2) v_2] \Phi_2.
 \end{aligned}
 \tag{701}$$

where $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ are Vlasov functions, determined by Eq. (188) and /209 which are satisfied by differential Eq. (189).

Vlasov function tables are presented in the Appendices.

Parameters r_n, s_n , which are included in the values of the arguments and appear during differentiation, are determined by Eq. (689).

With a negative value of the argument, functions $K_{ij}(a)$ revert to zero. Initial parameters $U(0), V(0), N(0), S(0)$ play the part of integration constant, and are determined from the boundary conditions.

Laminated orthotropic cylindrical shells can be divided into three classes by the nature of transmission of axial locally distributed forces: long shells; medium length shells; short shells.

We will understand long cylindrical shells to be cylindrical shells for which the semimembrane theory presented above is valid, and an axial load applied to one end is transmitted uniformly distributed through the other end.

We classify laminated cylindrical shells, for which semimembrane theory remains applicable, but the stressed and deformed states which result from axial locally distributed loads depend significantly on the boundary conditions at both ends of the shell, as medium length shells. /210 Finally, we will call short laminated cylindrical shells to which semimembrane theory is inapplicable.

Damping of the stressed and deformed states over the length of a laminated cylindrical shell, which results from axial locally distributed forces, is defined by the exponent $e^{-Sn\alpha}$ and, consequently, the class of long cylindrical shells includes shells in which the following inequality is satisfied

$$\frac{L^2}{R^2} \left[\sqrt{\frac{E_2}{E_1}} + \frac{E_2}{G} \frac{(n^2-1)}{\sqrt{n^4+k^2n^2+c^2}} \right] \frac{n^2(n^2-1)}{\sqrt{n^4+k^2n^2+c^2}} > 50. \quad (702)$$

In accordance with inequality (702), the concept of length of a shell is not purely geometrical, but it depends on both the geometric dimensions and on the nature of the loading and the elastic properties of the material.

We note that, with increase in shear modulus G , increase in annular modulus of elasticity E_2 and decrease in axial modulus of elasticity E_1 , the rate of smoothing out the stressed and deformed states over the length of the shell increases. Interlayer shearing contributes to stress concentration.

The nature of loading of a shell shows up in that, beginning with some number n , inequality (702) is satisfied and, consequently, with respect to harmonic forces of sufficiently high order, all shells can be considered long. This situation is important in calculations, since it permits significant simplification of calculation formulas, beginning with a specific harmonic.

It also follows from this that, compared with the effect of low order forces, higher order harmonic forces are damped considerably more rapidly along the length of a cylindrical shell, i.e., the transmission of axial locally distributed loads is determined primarily by the first terms of the expansion in trigonometric series ($n=1, 2, 3, 4$). Calculations show that harmonic forces up to the fourth order go through a shell of elongation $L/R_0 \approx 2-3$, and that the effect of higher order harmonic forces is damped without reaching the other end of the shell.

The nature of damping of harmonic forces of various orders as a result of axial concentrated loads is shown in Fig. 81.

Since the rate of damping of harmonic forces determines the capacity of the shell to resist the effect of locally distributed axial forces, for more efficient design of such systems, it can be recommended that shell elastic parameters λ^2 and g^2 be increased if its dimensions /211

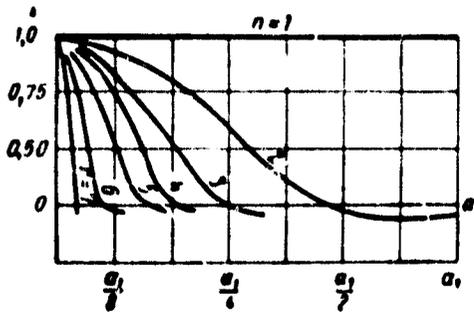


Fig. 81. Nature of damping of harmonic forces along cylindrical shell.

It is also follows from this that the installation of reinforcing rings is advisable for cylindrical shells subjected to axial loads which are not uniformly distributed over the perimeter.

As an example, we consider the calculation of a laminated cylindrical shell subjected to an axial load applied to one end.

61. Cylindrical Orthotropic Shell Subjected to Axial Locally Distributed Forces Applied to End

Let locally distributed axial forces applied to the upper end act on a cylindrical shell made of a laminated orthotropic material. We will assume that the principal axes of anisotropy coincide with the coordinate axes. We will assume the area over which each force is distributed is uniform and determined by central angle γ (Fig. 82).

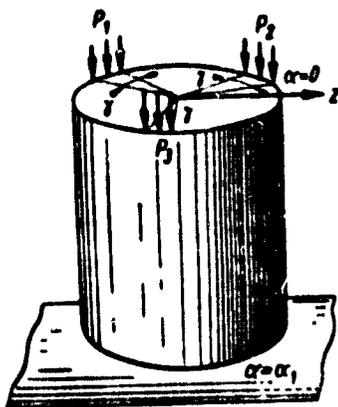


Fig. 82. Cylindrical shell subjected to locally distributed loads.

Solution of the problem is reduced to determination of two initial parameters from the boundary conditions on shell ends $\alpha=0, \alpha=\alpha_1$.

For example, let end $\alpha=0$ where the forces are applied be free of connections and end $\alpha=\alpha_1$ be rigidly fastened.

We expand load $P=p_1+p_2+p_3$ in trigonometric series

$$P = \frac{P}{2\pi R} + \frac{P}{\pi R} \cos \beta + \sum_{n=2,3,4}^{\infty} a_n \cos n \beta. \quad (703)$$

From the boundary conditions for $n \geq 2$, /212

$$\left. \begin{aligned} N_n(0) &= -a_n; & S_n(0) &= 0; \\ U_n(\alpha_1) &= 0; & V_n(\alpha_1) &= 0; \end{aligned} \right\} \quad (704)$$

we determine initial parameters $U_n(0)$, $V_n(0)$ and, further, by Eq. (694), for which $n \geq 2$, we calculate the coefficients of expansion of the elastic forces, moments and deformations.

The solution for $n=0$ corresponds to uniform compression of the shell and, consequently,

$$N_0(\alpha) = -\frac{P}{2\pi R}; \quad S_0 = Q_0 = G_0 = 0. \quad (705)$$

The $n=1$ case corresponds to bending of the cylindrical shell as a bracket, by bending moment $M=PR$ applied to the $\alpha=0$ end, i.e.,

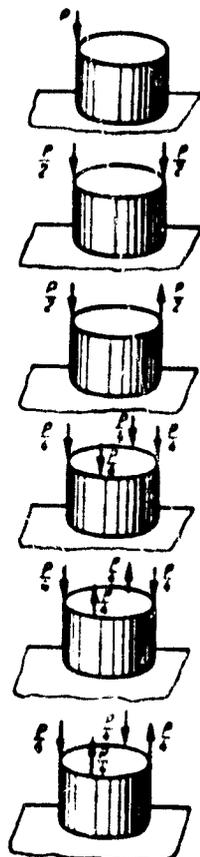
$$N_1(\alpha) = \frac{P}{\pi R} \cos \beta; \quad S_1 = Q_1 = G_1 = 0. \quad (706)$$

Further, by summing the forces, moments and movements in accordance with Eq. (693), we obtain the distribution of the stressed and deformed states over the surface of the shell.

We give the expansion in trigonometric series of the locally distributed axial loads most often encountered in engineering practice: /213

Type of load

Trigonometric series $P(\beta)$



$$\frac{P}{2\pi R} + \frac{P}{\pi R} \sum_{1, 2, 3}^{\infty} \frac{\sin n\gamma}{n\gamma} \cos n\beta$$

$$\frac{P}{2\pi R} + \frac{P}{\pi R} \sum_{2, 4, 6}^{\infty} \frac{\sin n\gamma}{n\gamma} \cos n\beta$$

$$\frac{P}{\pi R} \sum_{1, 3, 5}^{\infty} \frac{\sin n\gamma}{n\gamma} \cos n\beta$$

$$\frac{P}{2\pi R} + \frac{P}{\pi R} \sum_{4, 8, 12}^{\infty} \frac{\sin n\gamma}{n\gamma} \cos n\beta$$

$$\frac{P}{\pi R} \sum_{2, 6, 10}^{\infty} \frac{\sin n\gamma}{n\gamma} \cos n\beta$$

$$\frac{P}{\pi R} \sum_{1, 3, 5}^{\infty} \frac{1}{2} \frac{\sin n\gamma}{n\gamma} \left[\cos n\beta + (-1)^{\frac{n+3}{2}} \sin n\beta \right]$$

In the limit, as $\gamma \rightarrow 0$, we obtain an expansion of the system of concentrated axial forces in diverging trigonometric series.

We now consider transmission of an axial load to a cylindrical shell through an elastic ring (Fig. 83).

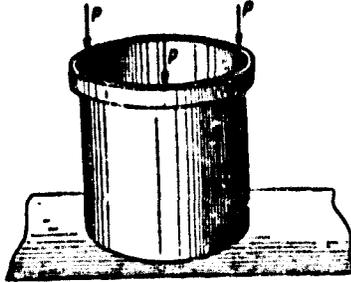


Fig. 83. Transmission of elastic forces to cylindrical shell through elastic ring.

Because of low rigidity, it is advisable to make the laminated plastic rings quite massive, i.e., solid. Therefore, we will disregard warping of the cross section of the ring.

If the force of interaction of the ring /214 and shell is designated

$$q = \sum_{n=2}^{\infty} q_n \cos n\beta, \quad (707)$$

the ring will be deformed as a result of force $P-q$ perpendicular to the plane of the ring

$$P - q = \sum_{n=2}^{\infty} (P_n - q_n) \cos n\beta, \quad (708)$$

and the shell will be compressed by distributed forces q .

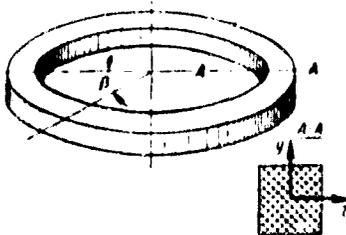


Fig. 84. Symbols for reinforcing ring calculation.

Thus, the deflection of the ring from the plane of curvature as a result of periodic loading normal to the plane of the ring must be determined. Following Grammel' [3], we consider a ring with a solid cross section, one of the principal axes of which lies in the plane of curvature. We will define the position of any section of the ring by angle β (Fig. 84).

For a ring, one of the principal axes of inertia of the cross section of which lies in the plane of curvature, deformation in the plane of the ring and bending from the plane of curvature of the ring can be considered independently.

The stressed state of the ring in bending from the plane is defined by bending moment M , torque H and cutting force Q . The deformed state of the ring is determined by deflection w and cross section warp angle θ . Rotation of an element of the elastic line of the ring around cross section radius ψ is connected with the deflection of the ring.

If $p(\beta)$ transverse harmonic load $P_n \cos n\beta$ distributed around the circumference of the ring, the equilibrium equations have the form

$$\left. \begin{aligned} \frac{dQ}{d\beta} &= -pR; \\ \frac{dH}{d\beta} + M &= 0; \\ \frac{dM}{d\beta} - H &= -QR. \end{aligned} \right\} \quad (709)$$

The elasticity relationships which connect the stressed and deformed states of the ring can be presented in the form

$$\frac{d\phi}{d\beta} + \psi = \frac{HR}{C}; \quad \frac{d\psi}{d\beta} - \phi = \frac{MR}{D_z}, \quad (710)$$

where C is the torsional rigidity of the ring; D_z is the rigidity of the ring in bending from the plane of curvature. /215

From Eq. (709) and (710), it is easy to obtain differential equations for determination of bending moments $M(\beta)$ and deflection angles $\psi(\beta)$

$$\frac{d^2M}{d\beta^2} + M = P_n R^2 \cos n\beta; \quad (711)$$

$$\frac{d^2\psi}{d\beta^2} + \psi = \frac{R}{D_z} \frac{dM}{d\beta} + \frac{RH}{C}. \quad (712)$$

By integrating differential Eq. (709), (711) and (712), the following expressions can be obtained for the elastic forces, moments, deflection angles and deflection of the ring

$$\left. \begin{aligned} M(\beta) &= C_1 \cos \beta + C_2 \sin \beta - \frac{P_n R^2}{n^2 - 1} \cos n\beta; \\ Q(\beta) &= -\frac{P_n R}{n} \sin n\beta + C_3; \\ H(\beta) &= -C_1 \sin \beta + C_2 \cos \beta + \frac{P_n R^2}{n(n^2 - 1)} \sin n\beta + C_3 R; \\ \psi(\beta) &= C_4 \cos \beta + C_5 \sin \beta + \frac{R}{2} \frac{(C + D_z)}{C D_z} \beta (C_1 \cos \beta + C_2 \sin \beta) + \\ &\quad + \frac{C_3 R^2}{C} - \frac{n^2 C + D_z}{C D_z} \frac{P_n R^2}{n(n^2 - 1)^2} \sin n\beta; \\ y(\beta) &= R(C_4 \sin \beta - C_5 \cos \beta) + \frac{C_3 R^2 \beta}{C} + C_6 + \\ &\quad + \frac{n^2 C + D_z}{C D_z} \frac{P_n R^4}{n^2 (n^2 - 1)^2} \cos n\beta. \end{aligned} \right\} \quad (713)$$

Because of the periodicity of the functions and the conditions that the ring be subjected to self balancing harmonic loads, $C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0$ and, consequently, the elastic forces and deformations of the ring are determined by the following equations

$$\left. \begin{aligned}
 Q(\beta) &= -\frac{P_n R}{n} \sin n \beta; \\
 M(\beta) &= -\frac{P_n R^2}{n^2 - 1} \cos n \beta; \\
 H(\beta) &= \frac{P_n R^2}{n(n^2 - 1)} \sin n \beta; \\
 \psi(\beta) &= -\frac{(n^2 C + D_2)}{C D_2} \frac{P_n R^2}{n(n^2 - 1)^2} \sin n \beta; \\
 \vartheta(\beta) &= -\frac{(C + D_2)}{C D_2} \frac{P_n R^2}{(n^2 - 1)^2} \cos n \beta; \\
 y(\beta) &= \frac{n^2 C + D_2}{C D_2} \frac{P_n R^4}{n^2 (n^2 - 1)^2} \cos n \beta.
 \end{aligned} \right\} (714)$$

Transmission of the axial forces to the cylindrical shell through /216 the elastic ring can now be calculated.

Deflection of the ring for each number $n \geq 2$ as a result of forces $P-q$

$$y = \frac{D_2 + n^2 C}{C D_2} \cdot \frac{(P_n - q_n) R^4}{n^2 (n^2 - 1)^2} \cos n \beta. \quad (715)$$

Axial movement of the $\alpha=0$ end of the ring as a result of distributed axial load q equals $U_n(0) \cos n \beta$.

From the condition of compatibility of the deformations of the ring and shell

$$y_n = U_n(0) \quad (716)$$

the following expression can be obtained for the initial parameter

$$U_n(0) = \frac{D_2 + n^2 C}{C D_2} \frac{R^4}{n^2 (n^2 - 1)^2} (P_n - q_n). \quad (717)$$

The remaining initial parameters are found from the boundary conditions:

on the unsupported $\alpha=0$ end, $S(0)=0$, $N(0)=-q_n$;

at the fixed $\alpha=\alpha_1$ end, $U(\alpha_1)=0$, $V(\alpha_1)=0$.

CHAPTER 13. SOME PROBLEMS OF SELECTION OF OPTIMUM STRUCTURE OF LAMINATED PLASTIC OF CYLINDRICAL SHELL

62. Initial Hypotheses, Assumptions and Relationships

The extensive use of laminated plastics in the most diverse fields of the national economy is explained by the exceptionally great diversity of their properties. Laminated plastics can have high unit strength, high chemical and biological stability, good electrical and sound insulating qualities, nonmagnetism, radiotransparency and other valuable properties, which structures of the most diverse technical or every day purposes require. /217

The technology of production and processing of laminated plastics products does not require subsequent extremely laborious mechanical working, as a result of which, moreover, there are great losses of material. Laminated plastics are easily extruded, molded at low pressures and cast. Laminated plastics products can be manufactured directly in the process of producing the material itself. The production of reinforced plastics with a given orientation of the reinforcing material can be considered the origin of the extensive use of plastics as structural materials.

Cylindrical shells produced by continuous winding of various types of reinforcing fillers are laminated elastic anisotropic systems. The nature of the anisotropy of the elastic properties of a shells depends essentially on the mutual placement and orientation of the reinforcing filler, and it can be easily regulated during manufacture. This new property of laminated plastics, controllable anisotropy, favorably distinguishes them from traditional building and structural materials. As it were, they connect laminated plastics with the structures and, consequently, this permits the creation of that structure of the material which ensures the maximum resistance to given external loads during manufacture of the shells. /218

Some problems on selection of the optimum structure of fiberglass reinforced plastics were considered in Chapter 8, where the performance of a shell in the momentless stressed state was considered.

More complex cases are studied in this chapter, when maximum shell rigidity must be ensured [14].

Since the results obtained below are generally speaking of a qualitative nature, because of difficulties associated with obtaining initial data on the elastic characteristics, we will disregard the effects of interlayer shearing on the assumption that they are the same for the entire spectrum of elastic systems considered.

We will assume that the unit layers of which a laminated shell is composed are orthotropic and uniform. We will assume the elastic properties of such a layer to be known, for example, from mechanical testing of strips unwound from the shell after manufacture.

For brevity, we will call a unit layer of the shell the "fabric,"

and its principal directions of anisotropy the warp and woof directions, with the understanding that the warp is the principal direction of anisotropy with the greater modulus of elasticity. The elastic properties of the fabric are defined by four independent parameters: moduli of elasticity in the warp and woof directions E_1, E_2 ; shear modulus G ; Poisson coefficients ν_1, ν_2 , which are connected by the known relationship

$$E_1 \nu_2 = E_2 \nu_1.$$

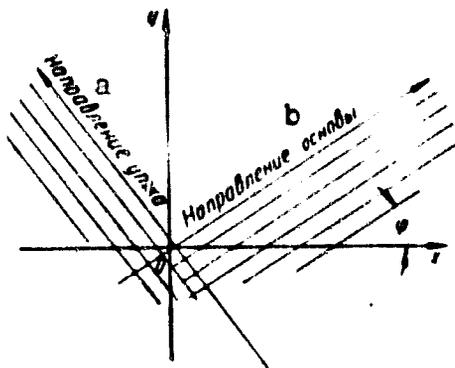


Fig. 85. Symbols for determination of elastic constants of material.

Key: a. Woof direction
b. Warp direction

If the coordinate axes do not coincide with the warp and woof directions of the fabric and are rotated relative to them by angle ϕ (Fig. 85), elastic constants c_{ij} , a_{ij} of the material are determined by Eq. (29) and (31).

In such a coordinate system, the basic elasticity relationships which connect the stressed and deformed states of the shell have the form /219

$$\left. \begin{aligned} \sigma_x &= c_{11} \epsilon_x + c_{12} \epsilon_y + c_{13} \omega_{xy}; \\ \sigma_y &= c_{12} \epsilon_x + c_{11} \epsilon_y + c_{23} \omega_{xy}; \\ \tau &= c_{13} \epsilon_x + c_{23} \epsilon_y + c_{33} \omega_{xy} \end{aligned} \right\} \quad (718)$$

or

$$\left. \begin{aligned} \epsilon_x &= a_{11} \sigma_x + a_{12} \sigma_y + a_{13} \tau; \\ \epsilon_y &= a_{12} \sigma_x + a_{11} \sigma_y + a_{23} \tau; \\ \omega_{xy} &= a_{13} \sigma_x + a_{23} \sigma_y + a_{33} \tau. \end{aligned} \right\} \quad (719)$$

We note for subsequent use that, according to Eq. (29) and (31), elastic constants $a_{13}, a_{23}, c_{13}, c_{23}$, in distinction from the remaining constants, change sign with change in sign of ϕ .

We consider a laminated shell a uniform anisotropic elastic system. Since the elastic properties of the shell are determined by the properties of the fabric and their mutual placement and orientation, after determination of the elastic properties of the shell through the elastic constants of the fabric and winding angle ϕ , that orientation direction can be selected in which the structure of the laminated plastic becomes the optimum. In addition, problems can be solved which are connected with the selection of the best initial materials for manufacture of the shells.

It is evident that, if the shell is wound so that the warp and woof directions of adjacent layers either coincide or mutually orthogonal, the elastic properties of the shell will be orthotropic. However, because the principal axes of anisotropy do not coincide with the coordinate axes, the elasticity relationships have the form

$$\left. \begin{aligned} T_1 &= B_{11} \epsilon_1 + B_{12} \epsilon_2 + B_{13} \omega; \\ T_2 &= B_{12} \epsilon_1 + B_{22} \epsilon_2 + B_{23} \omega; \\ S &= B_{13} \epsilon_1 + B_{23} \epsilon_2 + B_{33} \omega; \end{aligned} \right\} \quad (720)$$

$$\left. \begin{aligned} G_1 &= -D_{11} \kappa_1 - D_{12} \kappa_2 - 2D_{13} \kappa_3; \\ G_2 &= -D_{12} \kappa_1 - D_{22} \kappa_2 - 2D_{23} \kappa_3; \\ H &= -D_{13} \kappa_1 - D_{23} \kappa_2 - 2D_{33} \kappa_3; \end{aligned} \right\} \quad (721)$$

$$\left. \begin{aligned} \epsilon_1 &= A_{11} T_1 + A_{12} T_2 + A_{13} S; \\ \epsilon_2 &= A_{12} T_1 + A_{22} T_2 + A_{23} S; \\ \omega &= A_{13} T_1 + A_{23} T_2 + A_{33} S. \end{aligned} \right\} \quad (722)$$

If a laminated shell with a sufficiently large number of layers is made by cross bias winding of the reinforcing filler (at angle $+\phi$), it has an orthotropic structure which is symmetrical relative to the generatrix, i.e., the principal axes of anisotropy will coincide with the coordinate axes. In this case, the elasticity relationships are simplified, and they take the form /220

$$\left. \begin{aligned} T_1 &= B_{11} \epsilon_1 + B_{12} \epsilon_2; \\ T_2 &= B_{12} \epsilon_1 + B_{22} \epsilon_2; \\ S &= B_{33} \omega; \end{aligned} \right\} \quad (723)$$

$$\left. \begin{aligned} G_1 &= -D_{11} \kappa_1 - D_{12} \kappa_2; \\ G_2 &= -D_{12} \kappa_1 - D_{22} \kappa_2; \\ H &= -2D_{33} \kappa_3. \end{aligned} \right\} \quad (724)$$

where shell rigidity parameters A_{ij} , B_{ij} , D_{ij} are determined by the expressions

$$A_{ij} = \frac{a_{ij}}{\delta}; \quad B_{ij} = c_{ij} \delta; \quad D_{ij} = \frac{c_{ij} \delta^3}{12} \quad (725)$$

for $i, j=1, 2, 3, \dots$

Elastic constant a_{ij} , c_{ij} of the material depend on winding angle ϕ and elastic constants E_1 , E_2 , G , ν_1 , ν_2 of the fabric, and they are determined by Eq. (29)-(31).

The changes in curvature and warping of the mean surface of a tapered cylindrical shell are determined by the known expressions

$$\left. \begin{aligned} \kappa_1 &= -\frac{1}{R^2} \frac{\partial^2 w}{\partial \alpha^2}; \\ \kappa_2 &= -\frac{1}{R^2} \frac{\partial^2 w}{\partial \beta^2}; \\ \kappa_3 &= -\frac{1}{R^2} \frac{\partial^2 w}{\partial \alpha \partial \beta}. \end{aligned} \right\} \quad (726)$$

63. Differential Equation of Cylindrical Shell Stability

We obtain the differential equation of stability of a laminated cylindrical shell made of an orthotropic material by single thread bias winding, i.e., for the general case of anisotropy when the principal axes of anisotropy do not coincide with the coordinate axes. It is evident that the equation of stability of a shell produced by straight winding will be a partial case, with

$$a_{13} = a_{23} = c_{13} = c_{23} = 0.$$

We again use the orthogonal α, β system of dimensionless coordinates (Fig. 86) as the curvilinear Gaussian coordinates on the surface of the shell.

By substituting Eq. (720), (721) in the equations of neutral equilibrium of the shell (221)

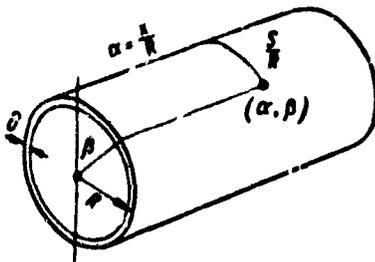


Fig. 86. Coordinate system and conventional symbols.

$$\left. \begin{aligned} \frac{\partial T_1}{\partial \alpha} + \frac{\partial S}{\partial \beta} &= 0; \\ \frac{\partial T_2}{\partial \beta} + \frac{\partial S}{\partial \alpha} &= 0; \end{aligned} \right\} \quad (727)$$

$$\frac{\partial^2 G_1}{\partial \alpha^2} + 2 \frac{\partial^2 H}{\partial \alpha \partial \beta} + \frac{\partial^2 G_2}{\partial \beta^2} + T_1 R = T_1^0 \frac{\partial^2 w}{\partial \alpha^2} + T_2^0 \frac{\partial^2 w}{\partial \beta^2} + 2S^0 \frac{\partial^2 w}{\partial \alpha \partial \beta}, \quad (728)$$

where T_1^0, T_2^0, S^0 are components of the elastic membrane forces which act on the shell before buckling, we obtain

$$\begin{aligned} B_{11} \frac{\partial^2 u}{\partial \alpha^2} + 2B_{12} \frac{\partial^2 u}{\partial \alpha \partial \beta} + B_{22} \frac{\partial^2 u}{\partial \beta^2} + B_{13} \frac{\partial^2 v}{\partial \alpha^2} + A \frac{\partial^2 v}{\partial \alpha \partial \beta} + \\ + B_{23} \frac{\partial^2 v}{\partial \beta^2} = -B_{12} \frac{\partial w}{\partial \alpha} - B_{23} \frac{\partial w}{\partial \beta}; \\ B_{12} \frac{\partial^2 u}{\partial \alpha^2} + A \frac{\partial^2 u}{\partial \alpha \partial \beta} + B_{23} \frac{\partial^2 u}{\partial \beta^2} + B_{22} \frac{\partial^2 v}{\partial \alpha^2} + 2B_{12} \frac{\partial^2 v}{\partial \alpha \partial \beta} + B_{13} \frac{\partial^2 v}{\partial \beta^2} = \\ = -B_{23} \frac{\partial w}{\partial \alpha} - B_{12} \frac{\partial w}{\partial \beta}; \end{aligned} \quad (729)$$

$$\begin{aligned} D_{11} \frac{\partial^4 w}{\partial \alpha^4} + 4D_{12} \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + 2(C + D_{22}) \frac{\partial^4 w}{\partial \alpha^2 \partial \beta^2} + 4D_{22} \frac{\partial^4 w}{\partial \alpha \partial \beta^3} + \\ + D_{23} \frac{\partial^4 w}{\partial \beta^4} + R^2 \left(B_{12} \frac{\partial u}{\partial \alpha} + B_{23} \frac{\partial u}{\partial \beta} + B_{13} \frac{\partial v}{\partial \alpha} + B_{12} \frac{\partial v}{\partial \beta} + B_{13} w \right) = \\ = R^2 \left(T_1^0 \frac{\partial^2 w}{\partial \alpha^2} + T_2^0 \frac{\partial^2 w}{\partial \beta^2} + 2S^0 \frac{\partial^2 w}{\partial \alpha \partial \beta} \right), \end{aligned} \quad (730)$$

where

$$A = B_{12} + B_{23}; \quad C = D_{12} + D_{22}. \quad (731)$$

We introduce the following differential operators in second order /222 partial derivatives

$$\left. \begin{aligned} \nabla_1^2 &= B_{11} \frac{\partial^2}{\partial \alpha^2} + A \frac{\partial^2}{\partial \alpha \partial \beta} + B_{22} \frac{\partial^2}{\partial \beta^2}; \\ \nabla_2^2 &= B_{11} \frac{\partial^2}{\partial \alpha^2} + 2B_{12} \frac{\partial^2}{\partial \alpha \partial \beta} + B_{22} \frac{\partial^2}{\partial \beta^2}; \\ \nabla_3^2 &= B_{22} \frac{\partial^2}{\partial \alpha^2} + 2B_{12} \frac{\partial^2}{\partial \alpha \partial \beta} + B_{11} \frac{\partial^2}{\partial \beta^2}. \end{aligned} \right\} \quad (732)$$

System of Eq. (729) is then presented in the following form

$$\left. \begin{aligned} \nabla_1^2 u + \nabla_1^2 v &= -B_{11} \frac{\partial w}{\partial \alpha} - B_{12} \frac{\partial w}{\partial \beta}; \\ \nabla_2^2 u + \nabla_2^2 v &= -B_{12} \frac{\partial w}{\partial \alpha} - B_{22} \frac{\partial w}{\partial \beta}. \end{aligned} \right\} \quad (733)$$

System of differential Eq. (733) is equivalent to the following system of differential equations

$$\left. \begin{aligned} \nabla_1^4 u &= (B_{11}B_{22} - B_{12}^2) \frac{\partial^2 w}{\partial \alpha^2} + (B_{12}B_{22} - B_{11}B_{12}) \frac{\partial^2 w}{\partial \alpha \partial \beta} + \\ &\quad + (B_{12}B_{22} - B_{11}^2) \frac{\partial^2 w}{\partial \alpha \partial \beta^2}; \\ \nabla_1^4 v &= (B_{12}B_{12} - B_{11}B_{22}) \frac{\partial^2 w}{\partial \alpha^2} + (B_{12}A - B_{12}B_{22} - B_{11}B_{22}) \frac{\partial^2 w}{\partial \alpha \partial \beta} + \\ &\quad + 2(B_{12}B_{22} - B_{11}B_{22}) \frac{\partial^2 w}{\partial \alpha \partial \beta^2} + (B_{12}^2 - B_{22}B_{22}) \frac{\partial^2 w}{\partial \beta^2}. \end{aligned} \right\} \quad (734)$$

where

$$\nabla_1^4 = \nabla_1^2 \nabla_2^2 - (\nabla_1^2)^2. \quad (735)$$

By expanding operator ∇_1^4 according to Eq. (732), we obtain

$$\begin{aligned} \nabla_1^4 &= (B_{11}B_{22} - B_{12}^2) \frac{\partial^4}{\partial \alpha^4} + 2(B_{12}B_{22} - B_{11}B_{12}) \frac{\partial^4}{\partial \alpha^3 \partial \beta} + \\ &\quad + (B_{11}B_{22} - B_{12}^2 + 2B_{12}B_{22} - 2B_{11}B_{22}) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \\ &\quad + 2(B_{12}B_{22} - B_{11}B_{22}) \frac{\partial^4}{\partial \alpha \partial \beta^3} + (B_{22}B_{22} - B_{12}^2) \frac{\partial^4}{\partial \beta^4}. \end{aligned} \quad (736)$$

If the following new differential operator now is introduced

$$\begin{aligned} \nabla_0^4 &= D_{11} \frac{\partial^4}{\partial \alpha^4} + 4D_{12} \frac{\partial^4}{\partial \alpha^3 \partial \beta} + 2(C + D_{22}) \frac{\partial^4}{\partial \alpha^2 \partial \beta^2} + \\ &\quad + 4D_{22} \frac{\partial^4}{\partial \alpha \partial \beta^3} + D_{22} \frac{\partial^4}{\partial \beta^4}, \end{aligned} \quad (737)$$

differential Eq. (730) can be presented in the form

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$$\nabla_0^4 w + R^2 \left(B_{11} \frac{\partial u}{\partial u} + B_{22} \frac{\partial u}{\partial \beta} + B_{33} \frac{\partial v}{\partial u} + B_{33} \frac{\partial v}{\partial \beta} + B_{11} w \right) - \\ - R^2 \left(T_1^0 \frac{\partial^2 w}{\partial u^2} + T_2^0 \frac{\partial^2 w}{\partial \beta^2} + 2S^0 \frac{\partial^2 w}{\partial u \partial \beta} \right). \quad (738)$$

By multiplying Eq. (738) by operator ∇_1^4 , the following differential equation can be obtained for the stability of a laminated cylindrical shell made of a single thread bias wound orthotropic material

$$\nabla_0^4 \nabla_1^4 w + BR^2 \frac{\partial^4 w}{\partial u^4} = R^2 \left(T_1^0 \frac{\partial^2}{\partial u^2} + T_2^0 \frac{\partial^2}{\partial \beta^2} + 2S^0 \frac{\partial^2}{\partial u \partial \beta} \right) \nabla_1^4 w, \quad (739)$$

where

$$B = B_{11}B_{22}B_{33} + 2B_{12}B_{13}B_{23} - B_{11}B_{23}^2 - B_{22}B_{13}^2 - B_{33}B_{12}^2. \quad (740)$$

Based on Eq. (9) and (725), differential operator ∇_1^4 can be presented in the form

$$\nabla_1^4 = \nabla^4 = \frac{B}{\delta} \left[a_{22} \frac{\partial^4}{\partial u^4} - 2a_{12} \frac{\partial^4}{\partial u^2 \partial \beta} + \right. \\ \left. + (2a_{12} + a_{23}) \frac{\partial^4}{\partial u^2 \partial \beta^2} - 2a_{13} \frac{\partial^4}{\partial u \partial \beta^3} + a_{11} \frac{\partial^4}{\partial \beta^4} \right]. \quad (741)$$

and, consequently, the differential equation of stability of a laminated cylindrical shell finally takes the form

$$\nabla_0^4 \nabla^4 w + R^2 \delta \frac{\partial^4 w}{\partial u^4} = R^2 \left(T_1^0 \frac{\partial^2}{\partial u^2} + T_2^0 \frac{\partial^2}{\partial \beta^2} + 2S^0 \frac{\partial^2}{\partial u \partial \beta} \right) \nabla^4 w. \quad (742)$$

where operators ∇_0^4 , ∇^4 are determined from Eq. (737) and (741), respectively.

It appears to be extremely complicated to find an exact solution of Eq. (742). However, since the magnitude of the critical load of a cylindrical shell of medium length depends little on the boundary conditions, an approximate solution can be selected in the form

$$w = w_0 \sin(\lambda u \pm n\beta), \quad (743)$$

where

$$\lambda = \frac{m \pi R}{L}. \quad (744)$$

Since the loss of stability of a cylindrical shell under external pressure and uniform axial compression is accompanied by the formation of a large number of annular waves, Eq. (743) approximately satisfies the hinge support boundary conditions at a quite large number of points ($n\beta = k\pi$).

By substituting Eq. (743) in differential equation of stability (742), the following general expression, which depends on two arbitrary parameters λ , n can be obtained for determination of the critical load:

$$\begin{aligned} & -R^2(T_1^0 \lambda^2 + T_2^0 n^2 + 2S^0 \lambda n)_{cr} \\ & D_{11} \lambda^4 + 4D_{12} \lambda^2 n + 2(C + D_{22}) \lambda^2 n^2 + 4D_{23} \lambda n^3 + D_{33} n^4 + \\ & + \frac{R^2 \lambda^2 \delta}{a_{11} \lambda^4 - 2a_{12} \lambda^2 n + (2a_{13} + a_{22}) \lambda^2 n^2 - 2a_{23} \lambda n^3 + a_{33} n^4}. \end{aligned} \quad (745)$$

64. Optimum Structure of Laminated Plastic in Cylindrical Shell Operating under Uniform External Pressure

Let a laminated cylindrical shell be subjected to uniform external pressure. In this case, on the assumption that $T_1^0 = S_1^0 = 0$ and $T_2^0 = -pR$, for determination of the critical pressure, the following calculation formulas can be obtained by Eq. (745):

a. shell made by single thread bias winding

$$p_{cr} = \frac{\delta^2}{R^2 n^2} \left[\Psi_1(n, \varphi) + \frac{\lambda^4 R^2}{\delta^2 \Psi_2(n, \varphi)} \right], \quad (746)$$

where

$$\left. \begin{aligned} \Psi_1(n, \varphi) &= [c_{11} \lambda^4 + 4c_{12} \lambda^2 n + 2(c_{13} + 2c_{22}) \lambda^2 n^2 + \\ & + 4c_{23} \lambda n^3 + c_{33} n^4] \frac{1}{12}; \\ \Psi_2(n, \varphi) &= a_{22} \lambda^4 - 2a_{23} \lambda^2 n + (2a_{12} + a_{22}) \lambda^2 n^2 - \\ & - 2a_{13} \lambda n^3 + a_{11} n^4 \end{aligned} \right\} \quad (747)$$

elastic constants c_{ij} , a_{ij} depend on bias winding angle ϕ ;

b. shell made by cross bias winding

$$p_{cr} = \frac{\delta^2}{R^2 n^2} \left[\Theta_1(n, \varphi) + \frac{\lambda^4 R^2}{\delta^2 \Theta_2(n, \varphi)} \right], \quad (748)$$

where

$$\left. \begin{aligned} \Theta_1(n, \varphi) &= \frac{1}{12} [c_{11} \lambda^4 + 2(c_{12} + 2c_{22}) \lambda^2 n^2 + c_{33} n^4]; \\ \Theta_2(n, \varphi) &= a_{22} \lambda^4 + (2a_{12} + a_{22}) \lambda^2 n^2 + a_{11} n^4. \end{aligned} \right\} \quad (749)$$

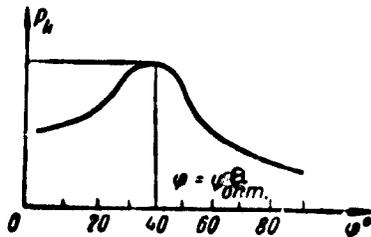
In the event $n^2 \gg \lambda^2$, the following approximate formula can be obtained

$$p_{cr} = \frac{1.75 \pi^2}{1R^2 n^2} \sqrt{(1 - \nu_1 \nu_2) B_{11} D_{33}^2}. \quad (750)$$

In accordance with Eq. (746), (748), selection of the optimum structure of the laminated plastic for a given fabric is reduced to the following simple procedure: for each value of winding angle ϕ , which

changes in the $0^\circ < \phi < 90^\circ$ interval, the lowest pressure p is determined as a function of whole number parameter n , ($m=1$). This value determines the critical external pressure of a cylindrical shell made by single thread bias or cross bias winding, i.e.,

$$p_K = p(\phi). \tag{751}$$



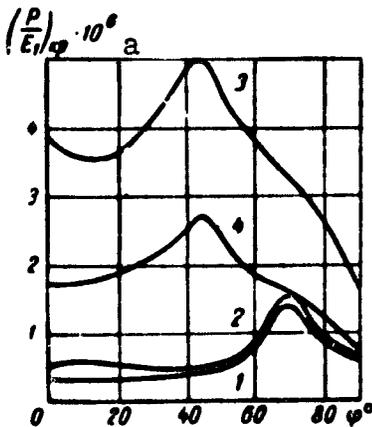
It is evident that the optimum structure of the laminated plastic is determined from the condition that the critical pressure is the highest: $p_K = (p_{cr})_{max}$ (Fig. 87).

The maximum value of function $p = p(\phi)$ determines the optimum winding angle and the maximum pressure which a cylindrical shell made by continuous winding with a given fabric can stand.

Fig. 87. p_K vs. winding angle ϕ

Key: a. $\phi = \phi_{opt}$

As calculations show, over a wide range of change of elastic constants E_1, E_2, G, ν , of the fabric and geometric dimensions R, l, δ of the shell, the most stable shells made by cross bias winding are, as a rule, shells produced by right angle winding ($\phi=0^\circ$ and $\phi=90^\circ$). Consequently, calculations by Eq. (746) will determine the optimum structure of the laminated plastic. It should be stated that the optimum winding of long cylindrical shells is straight annular winding of the warp, since such shells lose stability in the form of collapse of the cross section and, consequently, the maximum annular rigidity of the shell must be ensured.



The results of calculation to determine the optimum single thread winding angles for some fabric elastic constants and geometric dimensions of shells of medium length are presented in Table 7 and Fig. 88.

Fig. 88. Relative critical external pressure vs. single thread bias winding angle.

Key: a. $(\frac{P}{E_1})_{cr} \cdot 10^6$

It is evident from Table 7 and Fig. 88 that the winding angle in manufacture of shells by single thread bias winding significantly affects the critical external uniform pressure. The optimum winding angle evidently is determined by the elastic constants of the fabric and the geometric dimensions of the shell.

Thus, for the manufacture of cylindrical shells of medium length which operate under uniform external pressure, single thread bias winding may prove to be more expedient. The explanation of this is that single thread bias winding of medium length shells produces anisotropy of the laminated plastic which disturbs the symmetrical nature of the wave formation and forces it to buckle with wave formation at higher pressure. It also follows from this that this conclusion is only valid

for not very long shells.

Calculation formulas (746), (748) can be used for selection of the most nearly optimum fabrics, i.e., fabrics with elastic constants which provide the greatest critical pressure.

TABLE 7. CRITICAL EXTERNAL PRESSURE OF CYLINDRICAL SHELL VS. WINDING ANGLE

φ°	a Кривая 1 (рис. 88)		a Кривая 2 (рис. 88)		a Кривая 3 (рис. 88)		a Кривая 4 (рис. 88)	
	n	$\frac{10^8}{E_1} \left(\frac{a}{L_1} \right)$						
0	8	0.314	7	0.501	5	3.83	4	1.74
5	7	0.308	7	0.511	5	3.64	4	1.70
10	8	0.311	7	0.513	5	3.55	4	1.69
15	8	0.322	7	0.507	5	3.55	4	1.73
20	7	0.333	7	0.496	5	3.65	4	1.81
25	7	0.347	7	0.483	5	3.83	4	1.93
30	7	0.365	7	0.472	5	4.09	4	2.07
35	7	0.389	7	0.467	5	4.42	4	2.24
40	7	0.418	7	0.470	6	4.80	4	2.45
45	7	0.452	7	0.482	6	4.98	5	2.70
50	7	0.460	7	0.502	6	4.48	5	2.44
55	7	0.527	6	0.587	6	4.07	5	2.07
60	6	0.708	6	0.819	7	3.83	5	1.85
65	6	1.16	6	1.31	7	3.50	5	1.72
70	6	1.41	6	1.56	7	3.28	5	1.60
75	6	1.07	6	1.19	7	3.03	5	1.47
80	6	0.818	6	0.931	7	2.70	6	1.26
85	6	0.688	6	0.788	8	2.26	6	1.02
90	6	0.623	6	0.710	8	1.77	6	0.83

Key: a. Curve 1, 2, 3, 4 (Fig. 88)

65. Most Stable Laminated Cylindrical Shell under Axial Uniform Compression

Let a cylindrical shell made of laminated plastic be subjected to 227 uniform axial compression (Fig. 89). In this case, we study the question of the selection of the optimum structure of the laminated plastic which realizes the greatest carrying capacity of the shell at a given weight. Two possible types of elastic property symmetry of the laminated plastic which correspond to single thread bias and cross bias winding, should also be considered here.

According to Eq. (745), with $T_2^0 = \sigma^0 = 0$, $T_1^0 = -T_1$, the critical load of a laminated cylindrical shell under uniform axial compression is determined by the following expression

$$(T_1)_{cr} \frac{R^2 \lambda^2}{\delta^2} = \frac{1}{12} [c_{11} \lambda^4 + 4c_{12} \lambda^2 n + 2(c_{12} + 2c_{22}) \lambda^2 n^2 + 4c_{22} \lambda n^3 + c_{33} n^4] + \frac{\lambda^4 \left(\frac{R}{\delta}\right)^2}{a_{22} \lambda^4 - 2a_{23} \lambda^2 n + (2a_{12} + a_{22}) \lambda^2 n^2 - 2a_{12} \lambda n^3 + a_{11} n^4} \quad (752)$$

where λ , n are arbitrary parameters which define the form of wave formation upon loss of stability, and c_{1j} , a_{1j} ($1, j=1, 2, 3$) are the elasticity constants of the laminated plastic, which depend on the elastic characteristics of the fabric and winding angle in continuous winding with fabrics and which are determined by Eq. (29)-(31).

According to Eq. (752), for calculation of the critical axial load of a laminated cylindrical shell, the following calculation formulas can be obtained, which depend only on random wave formation parameter $\mu = n/\lambda$:

a. shell made by single thread bias winding

$$(T_1)_{cr} \frac{R \sqrt{3}}{\delta^2} = \sqrt{\frac{c_{11} + 4c_{12} \mu + 2(c_{12} + 2c_{22}) \mu^2 + 4c_{22} \mu^3 + c_{33} \mu^4}{a_{22} - 2a_{23} \mu + (2a_{12} + a_{22}) \mu^2 - 2a_{12} \mu^3 + a_{11} \mu^4}}; \quad (753)$$

b. shell made by cross bias winding

$$(T_1)_{cr} \frac{R \sqrt{3}}{\delta^2} = \sqrt{\frac{c_{11} + 2(c_{12} + 2c_{22}) \mu^2 + c_{33} \mu^4}{a_{22} + (2a_{12} + a_{22}) \mu^2 + a_{11} \mu^4}}. \quad (754)$$

Determination of the critical load by Eq. (753), (754) is again reduced to finding the minimum of the right side relative to parameter μ for a given winding angle ϕ .

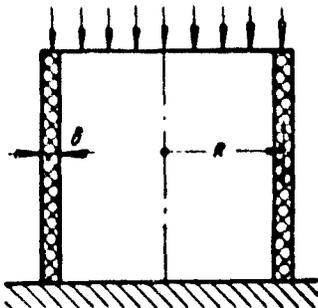


Fig. 89. Conventional symbols.

The maximum critical load determines the optimum winding angle and, consequently, the optimum structure of the laminated plastic and the upper limit of the carrying capacity of the shell which can be reached by a change in winding angle.

It should be noted that, in uniform axial compression of isotropic cylindrical shells, the critical load obtained by linear theory is in poor agreement with experimental results. For laminated shells, the correspondence of experimental data with the results of calculation of the critical load by linear theory is more satisfactory, and the scatter of the experimental data is not so great as in the case of thin isotropic shells.

Some numerical calculation results. It is known that the elastic properties of fiberglass reinforced plastic are determined primarily by the properties and orientation of the glass filler. The moduli of elasticity in the principal directions of anisotropy depend on the number of

glass fibers oriented in these directions of anisotropy. Consequently, with a given total warp and woof fabric density, the sum of the moduli of elasticity in the principal directions of anisotropy $E_1 + E_2$ remains nearly constant. Based on this, in order to investigate the effect of the shear modulus of the fabric, the initial data given in Table 8 were selected for comparative calculations.

TABLE 8. INITIAL DATA

a Упругие константы ткань	b ВАРИАНТЫ							
	I	II	III	IV	V	VI	VII	VIII
$E_1 \cdot 10^{-10}$ Н/м ²	1.5	1.5	1.5	1.5	1.5	1.5	4	2.25
$E_2 \cdot 10^{-10}$ Н/м ²	3	3	3	3	3	3	0.5	2.25
$G \cdot 10^{-10}$ Н/м ²	0.1	0.3	0.4	0.5	0.6	0.7	0.3	0.3
ν_1	0.1	0.1	0.1	0.1	0.1	0.1	0.8	0.2
ν_2	0.2	0.2	0.2	0.2	0.2	0.2	0.1	0.2

Key: a. Elastic constants of fabric
 b. Variants
 c. n/m^2

TABLE 9. CRITICAL LOAD OF CYLINDRICAL SHELL MADE BY CROSS BIAS WINDING

α°	a ВАРИАНТЫ							
	I	II	III	IV	V	VI	VII	VIII
0	0.70	1.22	1.41	1.57	1.72	1.86	1.09	1.30
5	0.72	1.24	1.43	1.59	1.74	1.87	1.10	1.32
10	0.77	1.30	1.48	1.64	1.78	1.91	1.14	1.38
15	0.85	1.39	1.57	1.72	1.85	1.96	1.17	1.48
20	0.96	1.50	—	—	1.96	2.05	—	1.62
25	1.19	1.70	1.84	1.95	2.04	2.12	1.38	1.79
30	1.42	1.87	1.98	2.06	2.13	2.19	1.43	1.98
35	1.72	2.04	2.11	2.16	2.21	2.25	1.48	2.17
40	2.02	2.17	2.20	2.23	2.26	2.29	1.51	2.31
45	2.17	2.21	2.23	2.26	2.28	2.30	1.52	2.37

Key: a. Variant

The results of calculation to determine the critical load, obtained by computer, are presented in Table 9.

For shells made by cross bias winding, critical load vs. winding angle is presented in Fig. 90.

As should have been expected, single thread bias winding extremely /229 insignificantly increases the axial critical load of a cylindrical shell.

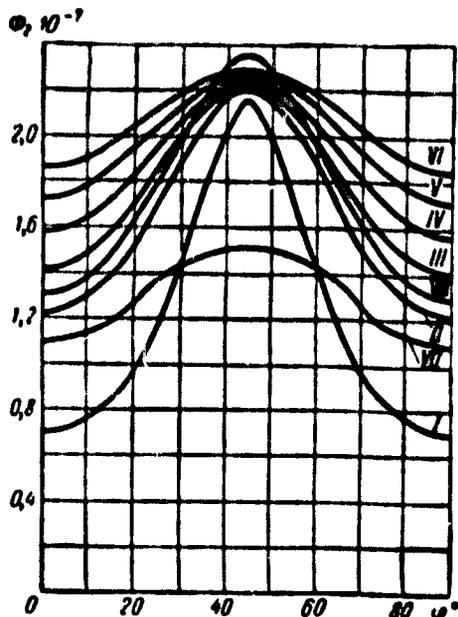


Fig. 90. Axial critical load vs. cross bias winding angle.

The explanation of this is that the most nearly optimum ratio of the moduli of elasticity in the axial and annular directions, and the shear modulus which can be obtained by change in winding angle and which should lead to a significant increase in the critical load, is associated with an increase in the degree of freedom, which is expressed by the possibility of the appearance of oblique forms of wave formation upon loss of stability. This leads to a decrease in the critical load.

It should be noted that the conduct of calculations in order to obtain recommendations on the most diverse cases of fabric elasticity property ratios and geometric shell dimensions requires a great amount of work and is hardly advisable. Evidently, it is more reasonable to carry out the calculations for given geometric dimensions and a limited number of fabrics from which the most nearly optimum must be selected, i.e., obtaining the greatest critical load for the weight of the shell. /230

General Eq. (745) permits calculation to determine critical loads in various types of uniform and simple combined loading of a shell, as well as selection of the optimum fabrics. It should be stated that the form of wave formation depends, in each case, extremely appreciably on the nature of anisotropy of the reinforced plastic. Therefore, the simplifications usually made with respect to order of magnitude in the theory of stability of thin isotropic shells should be made with great caution.

TABLE 10. CRITICAL LOAD OF CYLINDRICAL SHELLS OF OPTIMUM AND NONOPTIMUM STRUCTURE VS. SHEAR MODULUS OF FABRIC

a Намотка	b $G \cdot 10^{-10}$ н/м ²					
	0,1	0,3	0,4	0,5	0,6	0,7
Однозаходная	1.10	1.22	1.69	1.65	1.72	2.01
Перекрестная	2.17	2.21	2.23	2.26	2.28	2.30
Прямая	0.7	1.22	1.41	1.57	1.72	1.86

Key: a. Winding c. Single thread e. Right angle
 b. n/m^2 d. Cross

The critical axial load of a cylindrical shell of optimum structure made by single thread bias and cross bias windings vs. shear modulus of the fabric is presented in Table 10. The critical loads for shells made by right angle winding also are presented.

The calculation results show that the optimum structure of a laminated cylindrical shell operating in axial compression is obtained by continuous cross bias winding of full strength fabrics at angle $\phi = +45^\circ$. It should be noted that the use of nonfull strength fabrics, for example, with 1:2 anisotropy, highly insignificantly reduces the critical load of the shell. With more clearly defined anisotropy, a more intensive decrease in critical load occurs.

66. Stability of Anisotropic Shells of Rotation as a Result of Uniform Pressure

Let a double curvature anisotropic shell be exposed to forces T_1^0 , T_2^0 , S^0 . Then, by using the theory of tapered shells, the following system of differential equations can be obtained, which describe the local form of loss of stability /231

$$\left. \begin{aligned} L_1(w) = -k_2 \frac{\partial^2 \Phi}{\partial x^2} - k_1 \frac{\partial^2 \Phi}{\partial y^2} + T_1^0 \frac{\partial^2 w}{\partial x^2} + T_2^0 \frac{\partial^2 w}{\partial y^2} + 2S^0 \frac{\partial^2 w}{\partial x \partial y}; \\ L_2(\Phi) = k_2 \frac{\partial^2 w}{\partial x^2} + k_1 \frac{\partial^2 w}{\partial y^2}, \end{aligned} \right\} \quad (755)$$

where $L_1()$, $L_2()$ designate differential operators in up to fourth order partial derivatives

$$\left. \begin{aligned} L_1(\) = \frac{\delta^2}{12} \left[c_{11} \frac{\partial^4}{\partial x^4} + 4c_{12} \frac{\partial^4}{\partial x^2 \partial y^2} + \right. \\ \left. + 2(c_{13} + 2c_{23}) \frac{\partial^4}{\partial x^2 \partial y^2} + 4c_{23} \frac{\partial^4}{\partial x \partial y^3} + c_{22} \frac{\partial^4}{\partial y^4} \right]; \\ L_2(\) = \frac{1}{6} \left[a_{22} \frac{\partial^4}{\partial x^4} - 2a_{23} \frac{\partial^4}{\partial x^2 \partial y^2} + \right. \\ \left. + (2a_{12} + a_{33}) \frac{\partial^4}{\partial x^2 \partial y^2} - 2a_{13} \frac{\partial^4}{\partial x \partial y^3} + a_{11} \frac{\partial^4}{\partial y^4} \right]. \end{aligned} \right\} \quad (756)$$

System of Eq. (755) is equivalent to the following differential equation

$$LL_2(w) + \left(k_2 \frac{\partial^2}{\partial x^2} + k_1 \frac{\partial^2}{\partial y^2} \right) w = \left(T_1^0 \frac{\partial^2}{\partial x^2} + T_2^0 \frac{\partial^2}{\partial y^2} + 2S^0 \frac{\partial^2}{\partial x \partial y} \right) L_2(w). \quad (757)$$

In the case of shells of rotation subjected to external uniform pressure,

$$T_1^0 = -\frac{pR_2}{2}; \quad T_2^0 = -\frac{pR_1}{2} \left(2 - \frac{R_2}{R_1} \right), \quad (758)$$

and, consequently, stability Eq. (757) takes the form

$$L_1 L_2 (w) + \left(k_2 \frac{\partial^2}{\partial x^2} + k_1 \frac{\partial^2}{\partial y^2} \right) w = -\frac{p R_2}{2} \left[\frac{\partial^2}{\partial x^2} + \left(2 - \frac{R_2}{R_1} \right) \frac{\partial^2}{\partial y^2} \right] L_2 (w). \quad (759)$$

By substituting the solution in the form $w = w_0 \cos (\lambda x + \eta y)$ in Eq. (759), we obtain

$$\frac{6pR_2}{\delta^3} = \frac{1}{\lambda^2 + \left(2 - \frac{R_2}{R_1} \right) \eta^2} \left(c_{11} \lambda^4 + 4c_{12} \lambda^2 \eta + 2(c_{13} + 2c_{22}) \lambda^2 \eta^2 + \right. \\ \left. + 4c_{23} \lambda \eta^3 + c_{22} \eta^4 + \right. \\ \left. + \frac{12(k_2 \lambda^2 + k_1 \eta^2)^2}{\delta^3 [a_{22} \lambda^4 - 2a_{23} \lambda^2 \eta + (2a_{12} + a_{22}) \lambda^2 \eta^2 - 2a_{13} \lambda \eta^3 + a_{11} \eta^4]} \right). \quad (760)$$

Instead of arbitrary parameters λ , η , we introduce new parameters u , z according to the expressions

$$u = \lambda^2 + \eta^2; \quad uz = k_2 \lambda^2 + k_1 \eta^2. \quad (761)$$

It is easy to show that there are the following inequalities: /232

$$\left. \begin{aligned} k_1 < z \leq k_2, & \quad \text{if } k_2 > k_1, \\ k_2 \leq z \leq k_1, & \quad \text{if } k_2 < k_1. \end{aligned} \right\} \quad (762)$$

By substitution of new parameters u , z for parameters λ , η in Eq. (760), the following formula can be obtained for determination of the critical external pressure

$$\frac{6pR_2}{\delta^3} = \frac{1}{2 - R_2/z} \left[\frac{u}{(k_2 - k_1)^2} f_1(z) + \frac{z^2 (k_2 - k_1)^2}{u f_2(z)} \right], \quad (763)$$

where homogeneous quadratic functions

$$\left. \begin{aligned} f_1(z) &= c_{11} (z - k_1)^2 + 4c_{12} |z - k_1|^{\frac{3}{2}} |z - k_2|^{\frac{1}{2}} + \\ &+ 2(c_{13} + 2c_{22}) |z - k_1| |z - k_2| + 4c_{23} |z - k_1|^{\frac{1}{2}} |z - k_2|^{\frac{3}{2}} + \\ &+ c_{22} (z - k_2)^2; \\ f_2(z) &= \frac{\delta^2}{12} \left(a_{22} (z - k_1)^2 - 2a_{23} |z - k_1|^{\frac{3}{2}} |z - k_2|^{\frac{1}{2}} + \right. \\ &+ (2a_{12} + a_{22}) |z - k_1| |z - k_2| - \\ &\left. - 2a_{13} |z - k_1|^{\frac{1}{2}} |z - k_2|^{\frac{3}{2}} + a_{11} (z - k_2)^2 \right). \end{aligned} \right\} \quad (764)$$

The right side of Eq. (763) has the least value for parameter u at

$$u = \frac{(k_2 - k_1)^2 z}{V f_1(z) f_2(z)} \quad (765)$$

and it is determined by the expression

$$\frac{6pR_2}{\delta^3} = \frac{2z}{2 - R_2 z} \sqrt{\frac{f_1(z)}{f_2(z)}}. \quad (766)$$

At the poles of a shell of rotation, when they are spherical points, i.e., when $R_1 = R_2 = R$, the critical pressure is determined by the expression

$$\frac{pR^2 \sqrt{3}}{2\delta^3} = \sqrt{\frac{c_{11} + 4c_{12}\mu + 2(c_{13} + 2c_{20})\mu^2 + 4c_{23}\mu^3 + c_{33}\mu^4}{a_{22} - 2a_{23}\mu + (2a_{12} + a_{23})\mu^2 - 2a_{13}\mu^3 + a_{11}\mu^4}}; \quad (767)$$

consequently, the critical external pressure on a spherical shell is determined exactly by the same expressions as the critical axial load of a cylindrical shell.

In the general case, the critical external pressure is found by Eq. (766), by minimization of the right side relative to parameter z , which inequalities (762) satisfy.

67. Hypotheses and Basic Relationships Used

Despite the fact that three ply structures with elastic fillers /233 have a whole set of valuable qualities which are necessary to products in the most diverse fields of the national economy, their use until recently was extremely limited, because of difficulties of high quality manufacture which ensured complete reliability of such structures in operation. With the appearance and extensive use of synthetic materials and reinforced plastics and resins, the possibilities of the development of reliable effective three ply structures with light elastic fillers increased sharply. Synthetic cements produced on phenol-formaldehyde and epoxy resin bases permit reliable joining of bearing layers with honeycomb or foam plastic fillers. More than that, in the manufacture of three ply structures, including metal structures, the cement compounds are generally speaking the only possible ones.

In many engineering applications, namely, when great flexural rigidity is required, the structure of three ply shells permits elimination of the basic structural defect of plastics, comparatively large yielding to deformation. Therefore, highly improved nonmetal three ply structures can be produced, especially with the use of fiberglass reinforced plastics.

The basic types of three ply structures are presented in Fig. 91.

Polyvinyl chloride, polystyrene or polyurethane foam plastics, glass honeycomb plastics, foam plastics, cork and balsa, corrugated or hollow thin walled elements and other light elastic materials can be used as fillers of three ply plates and shells.

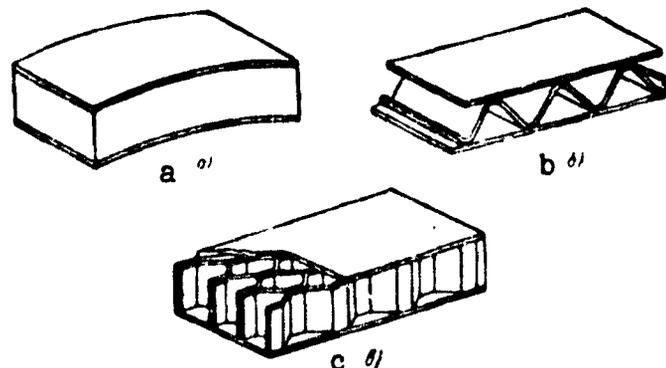


Fig. 91. Types of three ply shells with elastic fillers: a. foam plastic; b. corrugated; c. honeycomb.

Three ply plates and shells /234 with fiberglass reinforced plastic bearing layers are anisotropic. Anisotropy of the elastic properties also arises in the use of asymmetrical honeycomb plastics, corrugated fillers or other anisotropic materials.

It should be noted that a rigorous solution of problems connected with the calculation of three ply plates and shells is an extremely complex problem. Therefore, to obtain visible calculation formulas and calculation methods accessible to engineering analysis, various simplifying assumptions and hypotheses must be introduced. Reference to modern computers does not change the situation, since the entire spectrum of problems encountered in engineering cannot be specified and

programed beforehand. However, of course, this does not exclude and does not reduce the great value of exact methods of calculation.

The total number of works which deal with the calculation of three ply plates and shells is extremely large. However, orthotropic three ply shells have been investigated less thoroughly.

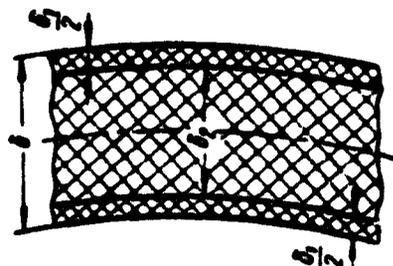


Fig. 92. Normal section of three ply shell with symbols used.

We consider thin three ply plates and shells with light elastic fillers which are symmetrical relative to the mean surface of the structure (Fig. 92). It is assumed that the materials of which the shells are made are orthotropic and that their principal directions of anisotropy coincide with the coordinate directions on the surface of the shell.

In the discussion, we limit ourselves to a class of shells in which the loads acting on the mean surface are entirely absorbed by the bearing layers ($E_1' \delta_2 < E_1 \delta_1$).

We use the hypothesis of a rectilinear element, i.e., we will assume that rectilinear elements which are normal before deformation of the mean surface of the shell remain rectilinear after deformation but, generally, not the normal deformed mean surface.

According to this hypothesis, the stresses in the bearing layers and the filler layer are determined by the following expressions:

in bearing layers $-\frac{\delta_2}{2} \leq |z| \leq \frac{\delta_2}{2}$

$$\left. \begin{aligned} \sigma_1^{(1)} &= \bar{E}_1 [e_1 + \nu_2 e_2 + z (\kappa_1^0 + \nu_2 \kappa_2^0)]; \\ \sigma_2^{(1)} &= \bar{E}_2 [e_2 + \nu_1 e_1 + z (\kappa_2^0 + \nu_1 \kappa_1^0)]; \\ \tau^{(1)} &= G (\omega + 2z \kappa_3^0); \end{aligned} \right\} \quad (768)$$

in the filler layer, $\frac{\delta_2}{2} \leq z \leq \frac{\delta_2}{2}$

$$\left. \begin{aligned} \sigma_{(1)}^{(2)} &= \bar{E}_1^{(2)} [e_1 + \nu_2^{(2)} e_2 + z (\kappa_1^0 + \nu_2^{(2)} \kappa_2^0)]; \\ \sigma_2^{(2)} &= \bar{E}_2^{(2)} [e_2 + \nu_1^{(2)} e_1 + z (\kappa_2^0 + \nu_1^{(2)} \kappa_1^0)]; \\ \tau^{(2)} &= G^{(2)} (\omega + 2z \kappa_3^0); \end{aligned} \right\} \quad (769)$$

the subscript "2" means that the corresponding value refers to the filler layer.

If the system of stresses which develops in normal sections of the shell is replaced by the statically equivalent system of elastic forces and moments and the effect of transverse compression of the fill-

er layer is disregarded, the following basic elasticity relationships can be obtained which connect the stressed and deformed states of three ply plates and shells

$$\left. \begin{aligned} T_1 &= B_1 (\epsilon_1 + \nu_2 \epsilon_2); \\ T_2 &= B_2 (\epsilon_2 + \nu_1 \epsilon_1); \\ S &= B_3 \omega; \\ G_1 &= -D_1 (\kappa_1^e + \nu_2 \kappa_2^e); \\ G_2 &= -D_2 (\kappa_2^e + \nu_1 \kappa_1^e); \\ H &= -D_3 \kappa_3^e; \\ Q_1 &= -K_1 \gamma_1; \\ Q_2 &= -K_2 \gamma_2. \end{aligned} \right\} \quad (770)$$

where the rigidity parameters of the shell are determined by the expression /236

$$\left. \begin{aligned} B_1 &= \bar{E} \delta_1; \quad B_2 = \bar{E}_2 \delta_2; \quad B_3 = G \delta_1; \\ D_1 &= \frac{\bar{E}_1 (\delta_1^e - \delta_2^e)}{12}; \quad D_2 = \frac{\bar{E}_2 (\delta_2^e - \delta_1^e)}{12}; \quad D_3 = \frac{G (\delta_1^e - \delta_2^e)}{12}; \\ K_1 &= \frac{2G_{13} (\delta_1^e - \delta_2^e)}{3\delta_1 (\delta_1 + \delta_2)}; \quad K_2 = \frac{2G_{23} (\delta_2^e - \delta_1^e)}{3\delta_2 (\delta_1 + \delta_2)}. \end{aligned} \right\} \quad (771)$$

here, G_{13} , G_{23} are the transverse shear moduli of the filler.

For separate fillers, G_{13} , G_{23} should be understood to be the reduced shear moduli which are determined either from some theoretical considerations or experimentally.

Components of the deformed state of a three ply shell ϵ_1 , κ_1^e , γ_1 are determined from Eq. (10)-(12).

Since the system of stresses was reduced to the statically equivalent system of elastic forces and moments in the mean surface of the shell, the equilibrium equations are written in the form of (17), with boundary conditions of the canonical type of (23).

In this manner, all results obtained in the preceding sections for the forces and moments of deformation and movements of orthotropic laminated plates and shells are completely valid for three ply plates and shells with light elastic fillers, if the rigidity parameters are in accordance with Eq. (771).

The difference in calculations of three ply shells appears only in calculation of the stresses, which are determined by Eq. (768) and (769).

It is easy to see that the results will be valid for three ply plates and shells, the bearing layers of which are made of various ma-

terials and have varied thickness, if $\delta_2 \gg \delta_1$ and the conditions of elastic symmetry through the shell are satisfied, i.e., if there are the relationships

$$\begin{aligned} E_1^* \delta_1^* - E_2^* \delta_2^*; \quad E_1^* \delta_1^* - E_2^* \delta_2^*; \quad G^* \delta_1^* - G^* \delta_2^*; \\ v_1^* = v_2^*; \quad v_1^* = v_2^*. \end{aligned} \quad (772)$$

Because the filler may have extremely low transverse strength, the normal transverse stresses must also be determined. The following can be obtained for them:

$$\sigma_z = \frac{3(\delta_2^* - \delta_1^*)}{2(\delta_2^* + \delta_1^*)} \left(\frac{G_1}{R_1} + \frac{G_2}{R_2} \right). \quad (773)$$

68. Boundary Conditions and Estimate of Error of the Theory As Applied to Three Ply Plates and Shells

The random constants which occur in integration of the differential equations are determined from the boundary conditions. As has been noted, five independent parameters which define the deformed state of laminated shells, even in the case of homogeneous canonical boundary conditions, increase the diversity of types of support fastenings to a considerable extent. /237

For plates, the system of differential equations breaks down into two, one of which describes the planar stressed state of the plate and the other describes the bending of the mean surface. The boundary conditions are broken down correspondingly. In the case of bending of three ply plates, boundary conditions (22) can give the following graphic static geometric interpretation:

DIAGRAMS OF SUPPORT FASTENINGS OF THREE PLY PLATES

Boundary conditions	Support Fastening diagram
$G = H = Q = 0$	
$w = \varphi = \psi = 0$	
$w = G = H = 0$	
$w = G = \psi = 0$	

$$w - \varphi - H = 0$$

$$Q - \varphi - H = 0$$

$$Q - \varphi - \psi = 0$$

$$Q - G - \psi = 0$$



The first type of boundary condition corresponds to an unsupported end, the second corresponds to rigid fastening, and the remaining types of boundary conditions correspond to various cases of hinge support.

We now estimate the errors allowed by the initial hypotheses and assumptions.¹ It is evident that the hypothesis of the rectilinear element does not take into account the bending of the bearing layers relative to the mean surface itself which occurs in transverse shearing in the filler layer. /238

A change of curvature and torsion of the mean surface of the bearing layers due to shearing of the filler layer are determined by the following expressions

$$\left. \begin{aligned} \kappa_1^c &= -\frac{1}{K_1} \frac{\partial Q_1}{\partial x}; & \kappa_2^c &= -\frac{1}{K_2} \frac{\partial Q_2}{\partial y}; \\ \kappa_3^c &= -\frac{1}{2} \left(\frac{1}{K_1} \frac{\partial Q_1}{\partial y} + \frac{1}{K_2} \frac{\partial Q_2}{\partial x} \right). \end{aligned} \right\} \quad (774)$$

Consequently, the rectilinear element hypotheses adopted for the entire set of shells are equivalent to disregard of the following quantities in the equilibrium equations, compared with cutting forces Q_1, Q_2 :

$$\left. \begin{aligned} \frac{D_1}{K_1} \frac{\partial^2 Q_1}{\partial x^2} + \frac{D_2}{K_2} \frac{\partial^2 Q_2}{\partial y^2} + \frac{C}{K_2} \frac{\partial^2 Q_2}{\partial x \partial y}; \\ \frac{D_1}{K_2} \frac{\partial^2 Q_2}{\partial y^2} + \frac{D_2}{K_1} \frac{\partial^2 Q_1}{\partial x^2} + \frac{C}{K_1} \frac{\partial^2 Q_1}{\partial x \partial y}, \end{aligned} \right\} \quad (775)$$

i.e., in determination of the deformed state of the shell, error ζ_1

¹In the end effect zones, the errors permitted by the initial assumptions will be larger.

compared with unity is committed:

$$\zeta_1 = \max \left(\frac{E_1 \delta_1^0}{G_{12} a^2 \delta_2}, \frac{E_2 \delta_1^0}{G_{23} a^2 \delta_2} \right), \quad (776)$$

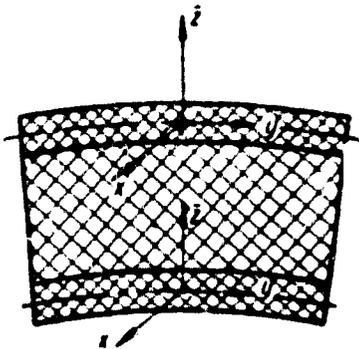
where a is the characteristic plan dimension of the shell.

In determination of the stressed state, a somewhat larger error is committed, namely, ζ_2 compared with unity:

$$\zeta_2 = \max \left(\frac{E_1 \delta_1 \delta_2}{G_{12} a^2}; \frac{E_2 \delta_1 \delta_2}{G_{23} a^2} \right). \quad (777)$$

Since usually $\delta_2/\delta_1 \approx 0.1-0.3$ in three ply shells, according to the rectilinear element hypothesis, the deformed state is defined with sufficient accuracy. The error in determination of the stressed state is somewhat higher. However, as needed, when error (777) is indeterminate, 23 the stress can be refined if the accuracy in determination of the deformed state is sufficient.

Refinement of the stresses is carried out by Eq. (774), by which the bending of the bearing layers due to shear in the filler layer is determined. If a coordinate system \bar{x} in the mean surface of the bearing layers is selected (Fig. 93) and \bar{z} designates the distance of the fibers from the mean surface of the bearing layer, the components of the deformation which originate in the bearing layers due to transverse shear in the filler layer are determined by the expressions



$$\left. \begin{aligned} \epsilon_1 &= \bar{z} (\kappa_1^c + \nu_1 \kappa_2^c); \\ \epsilon_2 &= \bar{z} (\kappa_2^c + \nu_2 \kappa_1^c); \\ \epsilon_{12} &= 2 \bar{z} \kappa_3^c. \end{aligned} \right\} \quad (778)$$

Consequently, the maximum additional stresses which were disregarded are determined by the following expressions

$$\left. \begin{aligned} \sigma_{1 \max} &= \bar{E}_1 \frac{\delta_1}{4} (\kappa_1^c + \nu_1 \kappa_2^c); \\ \sigma_{2 \max} &= \bar{E}_2 \frac{\delta_1}{4} (\kappa_2^c + \nu_2 \kappa_1^c); \\ \tau_{\max} &= G \frac{\delta_1}{2} \kappa_3^c. \end{aligned} \right\} \quad (779)$$

Fig. 93. Symbols in stressed state refinement.

Finally, if the shell is such that errors in determination of the deformed state are inadmissible, theories must be used which are based on less rigid hypotheses. However, the complexity of solution of engineering problems increases significantly here.

It was assumed in the initial hypotheses that the effect of transverse deformations of the filler could be disregarded. It is easy to show that the error of such an assumption has the value ζ_3 compared with unity:

$$\zeta_3 = \max\left(\frac{E_1 \delta_1 \delta_2}{E_2 R^2}; \frac{E_1 \delta_1 \delta_2}{E_2 R^2}\right), \quad (780)$$

where E_2 is the transverse modulus of elasticity of the filler.

In accordance with Eq. (780), in the case of very soft fillers, i.e., filler which satisfy the condition $E_2/E_1 \sim \delta_1 \delta_2/R^2$, the effect of transverse deformations must be taken into account. /240

69. Differential Equation of Symmetrical Form of Loss of Stability

In Chapter 5, solutions of some problems of stability of laminated plates were obtained which, with the abovementioned stipulations, are valid for three ply plates with an elastic filler. The forms of loss of stability which were considered there are characterized by distortion of the mean plane of the plate, i.e., they are asymmetric relative to the mean plane.

As the calculation formulas obtained show, the critical loads in the asymmetric form of loss of stability increase with increase in thickness of the filler layer. However, this relationship occurs up to a certain thickness, beginning at which a further increase in thickness of the filler layer in order to increase the critical load is useless, since the possibility of loss of stability in a fundamentally different form appears (symmetrical wrinkling of the bearing layers occurs relative to the mean plane). The critical load of the symmetrical form of loss of stability depends little on filler layer thickness, and such a form of instability is characteristic only of three ply plates and shells with elastic fillers, although it is found in laminated structures in the form of flaking.

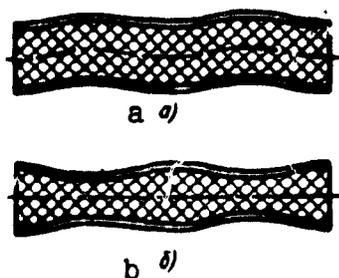


Fig. 94. Forms of loss of stability of three ply shells with elastic filler:
a. skew symmetric;
b. symmetric.

The forms of loss of stability which can occur in three ply plates and shells with light elastic fillers are shown in Fig. 94.

We obtain the differential equation of the symmetrical form of loss of stability by using the energy method. The total potential energy of the plate is made up of the potential energy of bending of the bearing layers, the potential energy of the filler and the work of external forces, and it is determined by the following expression [13]:

$$U = \iint \Phi(w_{xx}, w_{yy}, w_{xy}, w_x, w_y, w) dx dy, \quad (781)$$

where

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$$\begin{aligned} & \Phi(w_{xx}, w_{yy}, w_{xy}, w_x, w_y, w) = \\ & D_1^{(1)} w_{xx}^2 + 2D_{12}^{(1)} w_{xx} w_{yy} + D_2^{(1)} w_{yy}^2 + 4D_3^{(1)} w_{xy}^2 + \\ & + \frac{1}{2} (T_1^0 w_x^2 + T_2^0 w_y^2 + 2S_0 w_x w_y) + \\ & + \frac{\sinh 2\xi - 2\xi}{8\xi^3 \cosh^2 \xi} (K_1 w_x^2 + K_2 w_y^2) + \frac{F_z}{2\delta_2} \frac{\xi (\sinh 2\xi + 2\xi)}{\sinh^2 \xi} w^2, \end{aligned} \quad (782)$$

here $D_1^{(1)}$, $D_2^{(1)}$, $D_{12}^{(1)}$, $D_3^{(1)}$ are the rigidity parameters of the bearing layers of the plate;

$$\left. \begin{aligned} D_1^{(1)} &= \frac{E_1 \delta_1^3}{96(1-\nu_1\nu_2)}; \\ D_2^{(1)} &= \frac{E_2 \delta_2^3}{96(1-\nu_1\nu_2)}; \\ D_3^{(1)} &= \frac{G \delta_1^2}{96}; \\ D_{12}^{(1)} &= \nu_2 D_1^{(1)} - \nu_1 D_2^{(1)}; \\ K_1 &= G_{13} \delta_2; K_2 = G_{23} \delta_2; \end{aligned} \right\} \quad (783)$$

E_z is the reduced modulus of normal transverse elasticity of the filler;

$$E_z = \frac{a_{11} a_{22} - a_{12}^2}{a_{11} a_{22} a_{33} + 2a_{12} a_{13} a_{23} - a_{11} a_{23}^2 - a_{22} a_{13}^2 - a_{33} a_{12}^2}, \quad (784)$$

a_{ij} are the elasticity constants of an orthotropic filler as a three dimensional body; ξ is a random parameter proportional to filler layer thickness δ_2 , determinable from the condition of the minimum critical load.

Because the potential energy of the plate has a minimum in the equilibrium position, deflection of the bearing layers in symmetrical wrinkling w should satisfy the following differential equation:

$$\frac{\partial^2}{\partial x^2} \Phi_{w_{xx}} + \frac{\partial^2}{\partial x \partial y} \Phi_{w_{xy}} + \frac{\partial^2}{\partial y^2} \Phi_{w_{yy}} - \frac{\partial}{\partial x} \Phi_{w_x} - \frac{\partial}{\partial y} \Phi_{w_y} + \Phi_{w_{zz}} = 0. \quad (785)$$

By expanding Eq. (785) according to Eq. (782), the following differential equation can be obtained, which describes the symmetrical form of loss of stability of three ply plates, i.e., the stability loss phenomenon associated with wrinkling of the bearing layers: 242

$$\begin{aligned}
 & D_1^{(1)} \frac{\partial^4 w}{\partial x^4} + 2(D_1^{(1)} + 2D_2^{(1)}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + \\
 & + D_2^{(1)} \frac{\partial^4 w}{\partial y^4} - \frac{\varphi_1(\xi)}{4} \left(K_1 \frac{\partial^2 w}{\partial x^2} + K_2 \frac{\partial^2 w}{\partial y^2} \right) + \frac{E_1}{\delta_1} \varphi_1(\xi) w = \\
 & = T_1^0 \frac{\partial^2 w}{\partial x^2} + 2S_0 \frac{\partial^2 w}{\partial x \partial y} + T_2^0 \frac{\partial^2 w}{\partial y^2}.
 \end{aligned} \tag{786}$$

where, for convenience in practical use, functions ϕ_1, ϕ_2 of random parameter ξ are introduced:

$$\varphi_1(\xi) = \frac{\xi (\operatorname{sh} 2\xi + 2\xi)}{\operatorname{sh}^2 \xi}; \quad \varphi_2(\xi) = \frac{\operatorname{sh} 2\xi - 2\xi}{\xi \operatorname{sh}^2 \xi}. \tag{787}$$

TABLE 11. VALUES OF FUNCTIONS ϕ_1, ϕ_2, ϕ_3

ξ	$\varphi_1(\xi)$	$\varphi_2(\xi)$	$\varphi_3(\xi)$	ξ	$\varphi_1(\xi)$	$\varphi_2(\xi)$	$\varphi_3(\xi)$
0	4.000	1.333	2.000	2.6	5.560	0.733	2.358
0.1	4.000	1.333	2.000	2.7	5.716	0.711	2.391
0.2	4.000	1.333	2.000	2.8	5.874	0.690	2.424
0.3	4.001	1.316	2.000	2.9	6.042	0.670	2.458
0.4	4.002	1.305	2.004	3	6.206	0.650	2.491
0.5	4.007	1.290	2.002	3.1	6.382	0.631	2.526
0.6	4.013	1.273	2.002	3.2	6.556	0.613	2.560
0.7	4.019	1.252	2.005	3.3	6.738	0.597	2.596
0.8	4.033	1.229	2.008	3.4	6.915	0.580	2.639
0.9	4.049	1.201	2.012	3.5	7.103	0.565	2.685
1.0	4.074	1.178	2.018	3.6	7.290	0.551	2.700
1.1	4.105	1.150	2.026	3.7	7.478	0.537	2.734
1.2	4.142	1.121	2.035	3.8	7.669	0.523	2.769
1.3	4.187	1.092	2.046	3.9	7.856	0.510	2.803
1.4	4.244	1.062	2.060	4	8.046	0.498	2.836
1.5	4.307	1.032	2.075	4.1	8.239	0.486	2.870
1.6	4.381	1.002	2.093	4.2	8.434	0.475	2.904
1.7	4.459	0.971	2.112	4.3	8.637	0.464	2.939
1.8	4.549	0.942	2.133	4.4	8.824	0.454	2.970
1.9	4.651	0.914	2.157	4.5	9.022	0.444	3.004
2.0	4.759	0.886	2.182	4.6	9.220	0.434	3.036
2.1	4.872	0.858	2.207	4.7	9.414	0.425	3.068
2.2	4.958	0.831	2.227	4.8	9.612	0.416	3.100
2.3	5.129	0.805	2.265	4.9	9.811	0.408	3.132
2.4	5.264	0.780	2.294	5	10.010	0.400	3.164
2.5	5.410	0.756	2.326				

The values of functions ϕ_1, ϕ_2 are presented in Table 11, where /243
the value of function $\phi_3(\xi)$ also is given, which can be determined by
the expression

$$\varphi_3(\xi) = 1 - \overline{\varphi_1(\xi)}. \tag{788}$$

AS $\xi \rightarrow \infty$

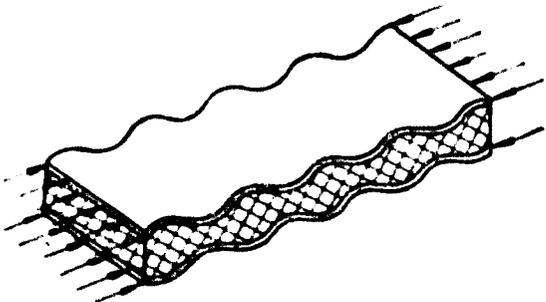
$$\left. \begin{aligned} \varphi_1(\xi) &\sim 2\xi, \\ \varphi_2(\xi) &\sim \frac{2}{\xi}, \\ \varphi_3(\xi) &\sim \sqrt{2\xi}. \end{aligned} \right\} \quad (789)$$

70. Some Problems of Determination of Critical Loads of Symmetrical Form of Stability Loss

As an illustration, we consider some simplest examples of determination of critical loads in which symmetrical wrinkling of the bearing layers occurs relative to the mean plane.

Cylindrical wrinkling of bearing layers of three ply strip. Let a rectangular three ply plate with two opposite unsupported ends be compressed uniformly by distributed forces T_1^0 in the direction of the unsupported ends. It is evident that wrinkling of the bearing layers will occur in only one direction (Fig. 95) and, consequently, differential equation of stability (786) takes the form

$$D_1^{(1)} \frac{d^4 w}{dx^4} - \frac{1}{4} \varphi_3(\xi) K_1 \frac{d^2 w}{dx^2} + \frac{E_2}{\delta_2} \varphi_1(\xi) w = -T_1^0 \frac{d^2 w}{dx^2}. \quad (790)$$



With hinge support of ends $x=0$, $x=l$, the solution can be sought in the form $w=A \sin m\pi x/l$. By substituting this solution in Eq. (790), the following expression can be obtained for determination of the critical load:

$$T_1^0 = \left(\frac{m\pi}{l}\right)^2 D_1^{(1)} + \frac{E_2 \varphi_1(\xi)}{\delta_2 \left(\frac{m\pi}{l}\right)^2} + \frac{K_1}{4} \varphi_3(\xi). \quad (791)$$

Fig. 95. Stability of three ply panels in longitudinal compression.

The right side of Eq. (791) will have the smallest value at 244

$$\frac{m\pi}{l} = \sqrt[4]{\frac{E_2 \varphi_1(\xi)}{\delta_2 D_1^{(1)}}}. \quad (792)$$

Consequently, the critical force is determined by the expression

$$T_{cr} = 2 \sqrt{\frac{D_1^{(1)} E_2}{\delta_2} \varphi_3(\xi) + \frac{K_1}{4} \varphi_3(\xi)}. \quad (793)$$

Parameter ξ is selected from the condition of the minimum of the right side of Eq. (793).

In the case of soft fillers, when the number of half waves in the longitudinal direction is small, it should be kept in mind that parameter ξ satisfies condition (792).

If it turns out that the minimum of the right side of Eq. (793) occurs with a value of parameter ξ which corresponds to $m < 1$, this means that only one half wave forms in buckling in the direction of compression, i.e., the critical load should be found by the formula

$$T_{cr} = \frac{\pi^2 D_1^{(1)}}{l^2} + \frac{E_2 l^2 \varphi_1(\xi)}{\pi^2 \delta_2} + \frac{K_1}{4} \varphi_2(\xi), \quad (794)$$

where parameter ξ again is determined from the condition of the minimum of the right side.

It also is of interest to obtain calculation formulas for determination of the critical force of a three ply strip with a comparatively thick filler layer.

In this case, it should be assumed that $\xi \rightarrow \infty$ and, consequently, according to Eq. (793),

$$T_{cr} = \frac{0,656}{\sqrt{1 - \nu_1^{(1)} \nu_2^{(1)}}} \sqrt{E_1^{(1)} E_2 G_{13}}. \quad (795)$$

Correspondingly, in the case of soft thick fillers, when one half wave develops along its length in loss of stability in the direction of compression, according to Eq. (794), the critical load is determined by the expression

$$T_{cr} = \frac{\pi^2 D_1^{(1)}}{l^2} + \frac{2l}{\pi} \sqrt{E_2 G_{13}}. \quad (796)$$

Stability of hinge supported plate during compression in one direction. In compression of a rectangular plate in one principal direction of anisotropy, differential equation of stability (786) takes the form

$$D_1^{(1)} \frac{\partial^4 w}{\partial x^4} + 2(D_{12}^{(1)} + 2D_3^{(1)}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2^{(1)} \frac{\partial^4 w}{\partial y^4} - \frac{\varphi_2(\xi)}{4} \left(K_1 \frac{\partial^2 w}{\partial x^2} + K_2 \frac{\partial^2 w}{\partial y^2} \right) + \frac{E_2}{\delta_2} \varphi_1(\xi) w = -T \frac{\partial^2 w}{\partial x^2}. \quad (797)$$

By substituting the solution in Eq. (797) in the form $w = A \sin \lambda x \sin \eta y$, where

$$\lambda = \frac{m \pi}{a}; \quad \eta = \frac{n \pi}{b}, \quad (798)$$

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for determination of the critical load, the following expression can be obtained:

$$T_{cr} = D_1^{(1)} \lambda^2 + \frac{D_2^{(1)} \eta^4 + \frac{K_2}{4} \varphi_2(\xi) \eta^2 + \frac{E_2}{\delta_2} \varphi_1(\xi)}{\lambda^2} + 2(D_{12}^{(1)} + 2D_{33}^{(1)}) \eta^2 + \frac{K_1}{4} \varphi_2(\xi). \quad (799)$$

It is easy to see that the smallest value of the right side of Eq. (799) occurs with the smallest value of parameter η , i.e., when, during buckling, one transverse half wave forms $n=1$, $\eta=\pi/b$.

For determination of the critical load, the following formula can be obtained

$$T_{cr} = \sqrt{D_1^{(1)} \left[D_2^{(1)} \left(\frac{\pi}{b} \right)^4 + \frac{K_2}{4} \left(\frac{\pi}{b} \right)^2 \varphi_2(\xi) + \frac{E_2}{2} \varphi_1(\xi) \right] + 2(D_{12}^{(1)} + 2D_{33}^{(1)}) \left(\frac{\pi}{b} \right)^2 + \frac{K_1}{4} \varphi_2(\xi)} \quad (800)$$

with

$$\frac{m\pi}{a} = \sqrt{\frac{E_2^{(1)}}{E_1^{(1)}} \left(\frac{\pi}{b} \right)^4 + \frac{K_2}{4D_1^{(1)}} \frac{\pi^2}{b^2} \varphi_2(\xi) + \frac{E_2}{\delta_2 D_1^{(1)}} \varphi_1(\xi)}. \quad (801)$$

Parameter ξ again is selected from the condition of the minimum of the right side of Eq. (800).

Condition (801) must be kept in mind in the case of soft fillers. If it turns out that $m < 1$, the critical load must be determined by Eq. (800) with $\lambda = \pi/a$.

Axisymmetric wrinkling of bearing layers of three ply cylindrical shell in uniform axial compression. We now consider axisymmetric buckling of the bearing layers of an orthotropic cylindrical shell in uniform axial compression (Fig. 96).

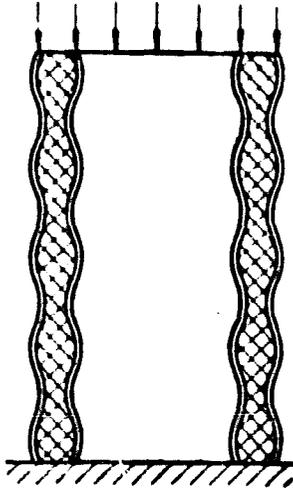
Because of symmetry, a unit strip of unit width can be considered. Because of the curvature and elasticity of the filler, the bearing layers are kept from buckling by a double elastic base as it were and, consequently, according to Eq. (790), the differential equation of stability can be written in the following form: /246

$$D_1^{(1)} \frac{d^4 w}{dx^4} - \frac{K_1}{4} \varphi_2(\xi) \frac{d^2 w}{dx^2} + \left[\frac{E_2}{\delta_2} \varphi_1(\xi) + \frac{E_2^{(1)} \delta_1}{2R^2} \right] w = -T \frac{d^2 w}{dx^2}. \quad (802)$$

By substituting the solution in Eq. (802) in the form $w = A \sin m\pi x / l$, the following expression can be obtained for determination of the critical load:

$$T_{cr} = D_1^{(1)} \left(\frac{m\pi}{l} \right)^2 + \frac{\left[\frac{E_2}{\delta_2} \varphi_1(\xi) + \frac{E_2^{(1)} \delta_1}{2R^2} \right]}{\left(\frac{m\pi}{l} \right)^2} + \frac{K_1}{4} \varphi_2(\xi). \quad (803)$$

The smallest value of the right side of Eq. (803) occurs with



$$\frac{m\pi}{l} = \sqrt[4]{\frac{E_2}{\delta_2 D_1^{(1)}} \varphi_1(\xi) + \frac{48E_2^{(1)}(1-\nu_1\nu_2)}{E_1^{(1)}R^2\delta_1^2}} \quad (804)$$

and it is determined by the expression

$$T_{cr} = 2 \sqrt{D_1^{(1)} \left[\frac{E_2}{\delta_2} \varphi_1(\xi) + \frac{E_2^{(1)} \delta_1}{2R^2} \right] + \frac{K_1}{4} \varphi_2(\xi)}. \quad (805)$$

Parameter \$\xi\$ in Eq. (805) is determined from the condition of the minimum critical force.

Fig. 96. Stability of three ply cylindrical shell in uniform axial compression.

In the case of a cylindrical shell with a quite thick filler layer, the critical load is determined by the formula

$$T_{cr} = 2 \sqrt{D_1^{(1)} \left(E_2 u + \frac{E_2^{(1)} \delta_1}{2R^2} \right) + \frac{G_{13}}{u}}, \quad (806)$$

where parameter \$u\$ is selected from the condition of the minimum of the right side or as the positive root of the equation

$$u^4 - \frac{G_{13}^2}{D_1^{(1)} E_2} u - \frac{G_{13}^2 E_2^{(1)} \delta_1}{2D_1^{(1)} E_2^2 R^2} = 0. \quad (807)$$

Appendix 1

VLASOV FUNCTIONS $\phi_1 = \text{ch } y \cdot \cos ky$

v=ps	$\lambda = \frac{v}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1	1	1	1	1	1	1	1	1
0.05	1.00119	1.00115	1.00109	1.00100	1.00088	1.00075	1.00058	1.00040	1.00017
0.10	1.00495	1.00480	1.00460	1.00440	1.00419	1.00381	1.00320	1.00253	1.00179
0.15	1.01119	1.01084	1.01027	1.00948	1.00844	1.00720	1.00571	1.00402	1.00209
0.20	1.01987	1.01925	1.01823	1.01681	1.01500	1.01273	1.01010	1.00704	1.00359
0.25	1.03108	1.03012	1.02850	1.02625	1.02334	1.01982	1.01564	1.01085	1.00540
0.30	1.04487	1.04340	1.04111	1.03782	1.03360	1.02845	1.02237	1.01538	1.00747
0.35	1.06122	1.05928	1.05503	1.05050	1.04568	1.03957	1.03218	1.02353	1.01366
0.40	1.08020	1.07701	1.07330	1.06726	1.06052	1.05209	1.04187	1.02919	1.01477
0.45	1.10184	1.09850	1.09292	1.08515	1.07515	1.06301	1.04868	1.03227	1.01373
0.50	1.12622	1.12199	1.11497	1.10510	1.09257	1.07727	1.05926	1.03861	1.01537
0.55	1.15334	1.14812	1.13770	1.12276	1.11169	1.09277	1.07052	1.04508	1.01644
0.60	1.18334	1.17699	1.16331	1.14149	1.12253	1.10448	1.08244	1.05159	1.01679
0.65	1.21620	1.20850	1.19568	1.17428	1.15497	1.13727	1.09478	1.05769	1.01610
0.70	1.25209	1.24280	1.22759	1.20629	1.17907	1.15608	1.10747	1.06345	1.01421
0.75	1.29103	1.28014	1.26203	1.23686	1.20470	1.16580	1.12031	1.06855	1.01076
0.80	1.33315	1.32035	1.31247	1.28954	1.23185	1.18629	1.13345	1.07275	1.00550
0.85	1.37852	1.36358	1.33877	1.29049	1.26044	1.20746	1.14576	1.07579	0.99805
0.90	1.42730	1.40983	1.38117	1.34123	1.29043	1.22018	1.15798	1.07741	0.98811
0.95	1.47951	1.45918	1.42627	1.38020	1.32167	1.25126	1.16951	1.07728	0.97525
1.00	1.53536	1.51233	1.47417	1.42127	1.35418	1.27356	1.18021	1.07508	0.95970
1.05	1.59494	1.56855	1.52485	1.46440	1.39180	1.29428	1.18972	1.07046	0.93040
1.10	1.65844	1.62831	1.57849	1.50959	1.42245	1.31811	1.19785	1.06310	0.91600
1.15	1.72593	1.69166	1.63501	1.55681	1.45800	1.33998	1.20420	1.05256	0.89698
1.20	1.79764	1.75876	1.69460	1.60692	1.49441	1.36127	1.20854	1.03844	0.85342
1.25	1.87367	1.82971	1.75717	1.65724	1.53143	1.38174	1.20951	1.02031	0.81423
1.30	1.95427	1.90467	1.82291	1.71039	1.56966	1.40114	1.20965	0.99771	0.76895
1.35	2.03958	1.98376	1.89185	1.76545	1.60694	1.41922	1.20564	0.97015	0.71760
1.40	2.12986	2.06714	1.96396	1.82237	1.64509	1.43564	1.19809	0.93710	0.65779
1.45	2.22520	2.15493	2.03938	1.88106	1.68323	1.45012	1.18653	0.89805	0.59072
1.50	2.32599	2.24735	2.11823	1.94154	1.72109	1.46228	1.17049	0.85242	0.51520
1.55	2.43232	2.34450	2.20045	2.00266	1.75978	1.47177	1.14947	0.79961	0.43050
1.60	2.54455	2.44663	2.28618	2.06738	1.79874	1.47822	1.12295	0.73901	0.33615
1.65	2.66282	2.55384	2.37546	2.13259	1.83175	1.48119	1.09059	0.66996	0.23132
1.70	2.78754	2.66640	2.46339	2.19922	1.86663	1.48026	1.05117	0.59180	0.11537
1.75	2.91887	2.78447	2.55015	2.26707	1.90002	1.47480	1.00468	0.50353	-0.01245
1.80	3.05725	2.90828	2.66531	2.33592	1.93163	1.46464	0.95033	0.40528	-0.15283
1.85	3.20288	3.03803	2.76939	2.40633	1.96105	1.44894	0.88733	0.29545	-0.30650
1.90	3.35621	3.17391	2.87739	2.47731	1.98802	1.42724	0.81506	0.17352	-0.47421
1.95	3.51750	3.31625	2.91750	2.54892	2.01203	1.39887	0.73266	0.03872	-0.65672
2.00	3.68722	3.46521	3.05909	2.62116	2.03272	1.36327	0.63946	-0.10986	-0.85477
2.05	3.86738	3.62108	3.20509	2.69362	2.04957	1.31969	0.53452	-0.27303	-1.06013
2.10	4.05326	3.78413	3.34873	2.76616	2.06208	1.26741	0.41704	-0.45169	-1.30053
2.15	4.25047	3.95403	3.47658	2.83855	2.06970	1.20566	0.28606	-0.64673	-1.54971
2.20	4.45782	4.13282	3.60860	2.91044	2.07200	1.13366	0.14065	-0.85913	-1.81748
2.25	4.67565	4.31907	3.74468	2.98160	2.06814	1.05059	-0.04029	-1.08978	-2.10445
2.30	4.90460	4.51360	3.88496	3.05165	2.05765	0.95526	-0.19741	-1.33970	-2.41147
2.35	5.14503	4.71682	4.02922	3.12027	2.03969	0.84700	-0.39224	-1.60989	-2.73904
2.40	5.39760	4.92896	4.17777	3.18702	2.01362	0.72474	-0.60565	-1.90131	-3.08800
2.45	5.66290	5.15044	4.33024	3.25150	1.97850	0.58741	-0.83882	-2.21497	-3.45879
2.50	5.94163	5.38157	4.48693	3.31327	1.93363	0.43380	-1.09308	-2.55195	-3.85212
2.55	6.23417	5.62271	4.64756	3.37186	1.87801	0.26279	-1.36944	-2.91321	-4.26847
2.60	6.54150	5.87428	4.81216	3.42661	1.81071	0.07311	-1.66944	-3.29976	-4.70825

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Appendix 1 continued

v = pβ	$\frac{v}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.65	6.86398	6.13660	4.98063	3.47701	1.73065	-0.43056	-1.99423	-3.74271	-5.17188
2.70	7.20271	6.41007	5.15206	3.52217	1.63076	-0.36755	-2.34525	-4.15301	-5.65980
2.75	7.55808	6.69508	5.32884	3.56225	1.52778	-0.62135	-2.72383	-4.62466	-6.17205
2.80	7.93137	6.99221	5.50837	3.59555	1.32019	-0.80947	-3.13150	-5.11966	-6.70906
2.85	8.32286	7.30164	5.69118	3.62168	1.25098	-1.20335	-3.56904	-5.61709	-7.27050
2.90	8.73395	7.62398	5.87735	3.63982	1.09831	-1.53490	-4.03923	-6.20758	-7.85077
2.95	9.16523	7.95970	6.06828	3.64889	0.91626	-1.89562	-4.54358	-6.79918	-8.46720
3.00	9.61804	8.30924	6.25810	3.64812	0.71219	-2.28737	-5.09266	-7.42379	-9.10187
3.05	10.09286	8.67300	6.45230	3.63628	0.48441	-2.71210	-5.65827	-8.08225	-9.75092
3.10	10.59134	9.05157	6.64877	3.61226	0.23122	-3.17174	-6.27241	-8.77497	-10.41120
3.15	11.11410	9.44543	6.84114	3.54787	-0.04910	-3.66827	-6.92636	-9.50295	-11.14426
3.20	11.66298	9.85612	7.04664	3.52283	-0.35877	-4.20388	-7.62215	-10.26600	-11.86808
3.25	12.23835	10.28102	7.24713	3.45464	-0.69971	-4.78971	-8.36114	-11.06636	-12.63139
3.30	12.84227	10.72392	7.44833	3.36898	-1.07330	-5.40113	-9.14532	-11.90262	-13.37535
3.35	13.47556	11.18426	7.64928	3.26101	-1.48425	-6.06743	-9.97597	-12.77553	-14.15447
3.40	14.14006	11.66257	7.84989	3.01361	-1.93244	-6.78213	-10.85519	-13.68545	-14.94879
3.45	14.83603	12.15958	8.04889	2.68088	-2.42151	-7.54769	-11.78403	-14.63202	-15.75535
3.50	15.56801	12.67556	8.24614	2.65115	-2.95411	-8.36662	-12.76488	-15.61524	-16.57216
3.55	16.33465	13.21140	8.44048	2.61710	-3.53263	-9.24202	-13.79727	-16.63464	-17.39593
3.60	17.13893	13.76773	8.63136	2.38835	-4.16066	-10.17641	-14.88730	-17.68978	-18.22415
3.65	17.98212	14.34477	8.81742	2.12851	-4.84088	-11.17270	-16.03206	-18.77966	-19.05244
3.70	18.86664	14.94367	8.99814	1.83480	-5.57684	-12.23368	-17.23481	-19.96333	-19.87783
3.75	19.79175	15.56473	9.17198	1.50480	-6.37169	-13.36262	-18.48626	-20.84664	-20.69539
3.80	20.76649	16.20671	9.33806	1.13531	-7.22934	-14.56266	-19.81351	-22.24639	-21.50063
3.85	21.78597	16.87093	9.49596	0.72379	-8.15323	-15.83678	-21.20135	-23.46251	-22.28832
3.90	22.85577	17.56755	9.64113	0.26888	-9.14765	-17.18823	-22.64746	-24.70567	-23.05322
3.95	23.97654	18.28412	9.77500	-0.23899	-10.21636	-18.62054	-24.15656	-25.97289	-23.78846
4.00	25.15252	19.02592	9.89541	-0.79740	-11.36432	-20.13682	-25.73036	-27.26153	-24.48893
4.05	26.39007	19.79378	9.99995	-1.41183	-12.59510	-21.74072	-27.36852	-28.56850	-25.14509
4.10	27.67724	20.58833	10.08984	-2.08984	-13.91437	-23.43506	-29.07299	-29.83092	-25.75109
4.15	29.01144	21.41049	10.15396	-2.82637	-15.32629	-25.22509	-30.84071	-31.22141	-26.31425
4.20	30.45215	22.26021	10.19930	-3.63189	-16.83708	-27.11242	-32.67531	-32.55825	-26.78225
4.25	31.99968	23.14539	10.22286	-4.51840	-18.45585	-29.00072	-34.58377	-34.25622	-27.19385
4.30	33.50123	24.04677	10.21372	-5.47874	-20.17387	-31.19586	-36.53787	-35.22798	-27.50393
4.35	35.13591	24.96438	10.17771	-6.52477	-22.01142	-33.39873	-38.56435	-36.54802	-27.72331
4.40	36.85189	25.95212	10.10876	-7.66079	-23.97053	-35.71425	-40.65418	-37.84981	-27.83514
4.45	38.64871	26.95090	10.00345	-8.89271	-26.05641	-38.14558	-42.80389	-39.12572	-27.82570
4.50	40.53290	27.98128	9.85854	-10.22721	-28.27652	-40.69591	-45.01232	-40.36686	-27.68368
4.55	42.50653	29.04350	9.66967	-11.67090	-30.63688	-43.36920	-47.27606	-41.56483	-27.39441
4.60	44.67849	30.13784	9.43405	-13.23076	-33.14555	-46.16886	-49.59446	-42.70945	-26.94506
4.65	46.74382	31.31717	9.14267	-14.91466	-35.80794	-49.09825	-51.96047	-43.79007	-26.31065
4.70	49.01685	32.42555	8.80116	-16.72984	-38.63368	-52.15940	-54.37392	-44.79618	-25.50435
4.75	51.39724	33.67685	8.39667	-18.68502	-41.62849	-55.35088	-56.82665	-45.71527	-24.48085
4.80	53.89291	34.84668	7.92423	-20.78880	-44.80331	-58.69229	-59.31568	-46.53374	-23.23436
4.85	56.50687	36.10803	7.38001	-23.01831	-48.16363	-62.16867	-61.83397	-47.23746	-21.74530
4.90	59.24035	37.40256	6.75706	-25.47916	-51.71910	-65.78787	-64.37441	-47.81111	-19.99656
4.95	62.11727	38.73259	6.04894	-28.08676	-55.47889	-69.55404	-66.93088	-48.31108	-17.96965
5.00	65.12517	40.09564	6.24963	-30.88247	-59.45236	-73.42258	-69.49465	-48.50659	-15.64346
5.05	68.27512	41.49276	4.35008	-33.87777	-63.64730	-77.52139	-72.05414	-48.59207	-12.99799
5.10	71.57690	42.92367	3.34535	-37.08509	-68.07572	-81.74089	-74.60239	-48.47601	-10.01227
5.15	75.03436	44.38708	2.22358	-40.51586	-72.74439	-86.10997	-77.12430	-48.13932	-6.66642
5.20	78.65823	45.88321	0.97890	-44.09400	-77.66750	-90.61803	-79.61266	-47.56003	-2.93489
5.25	82.45267	47.41136	0.40020	-48.10501	-82.85097	-95.28195	-82.04771	-46.71480	1.20155
5.30	86.48845	48.97055	-1.92328	-52.29021	-88.30866	-100.09677	-84.42003	-45.57876	5.76784

Appendix 1 continued

v = pβ	$A = \frac{q}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5.35	90.59099	51.81283	-3.60148	-56.75000	-94.04925	-105.06010	-86.70939	-44.42764	10.78655
5.40	97.95319	52.17881	-5.44449	-61.51903	-100.08549	-110.16965	-88.90203	-42.33377	16.28256
5.45	94.51992	53.82515	-7.46702	-66.59450	-106.42598	-115.42341	-90.90822	-40.16782	22.27769
5.50	104.30412	55.49706	-9.68017	-72.00180	-113.08626	-120.81621	-92.91108	-37.80121	27.57979
5.55	109.29647	57.18500	-12.73919	-77.74556	-120.05392	-126.32449	-94.67071	-34.50745	33.86160
5.60	114.56230	58.91049	-14.73709	-83.88201	-127.40232	-132.00234	-96.27717	-31.14003	43.50350
5.65	120.05493	60.36435	-17.60923	-90.39443	-135.08134	-137.35854	-97.65817	-27.17860	52.35811
5.70	125.80964	62.40272	-20.73416	-97.31680	-143.12770	-143.69061	-98.80308	-22.68429	60.60800
5.75	131.83259	64.17252	-24.12851	-104.67054	-151.54511	-149.68508	-99.69114	-17.61842	70.10921
5.80	138.14248	65.95152	-27.81148	-112.47794	-160.35530	-155.78391	-100.26170	-11.94539	80.27839
5.85	144.74585	67.73731	-31.79909	-120.76217	-169.55943	-161.96882	-100.50782	-5.62117	91.13242
5.90	151.60326	69.52556	-36.11890	-129.55094	-179.17998	-168.22513	-100.38970	1.38898	102.69858
5.95	158.92101	71.31337	-40.78558	-139.82915	-189.21854	-174.54715	-99.86289	9.13147	115.39864
6.00	166.48399	73.09368	-45.82979	-148.74309	-199.69947	-180.89052	-98.89312	17.65012	128.02690
6.05	174.41745	74.86061	-51.27131	-159.20231	-210.61570	-187.26390	-97.43203	26.93646	142.88448
6.10	182.72458	76.60900	-57.13919	-170.27840	-221.99593	-193.63923	-95.43034	37.19141	156.40324
6.15	191.41568	78.33241	-63.45994	-181.99676	-233.83714	-199.99324	-92.85568	48.32967	171.36995
6.20	200.52012	80.02277	-70.26384	-194.39276	-246.16366	-206.29763	-89.64128	60.38418	187.39280
6.25	210.04462	81.67021	-77.58048	-207.50117	-258.96850	-212.53110	-85.73662	73.47003	204.90074
6.30	220.01596	83.27077	-85.41634	-221.35561	-272.27598	-218.65997	-81.08703	87.61617	222.68165
6.35	230.74236	84.92843	-94.01549	-236.20696	-286.45174	-224.63840	-75.72658	103.00660	241.57928
6.40	241.38666	86.28648	-102.96777	-251.46525	-300.42874	-230.48348	-69.30725	119.20996	260.68616
6.45	253.11772	87.77983	-112.82901	-268.10668	-315.65198	-236.37709	-62.11614	137.09580	281.22361
6.50	264.75357	88.96289	-123.11134	-284.97725	-330.61917	-241.42368	-53.77017	155.81643	301.85191
6.55	277.61152	90.20295	-134.44401	-303.51245	-346.94439	-246.78524	-44.47161	176.24467	324.00846
6.60	290.35946	91.18137	-146.24010	-322.27367	-362.94656	-251.17514	-33.91000	197.60130	346.11653

Appendix 2

VLASOV FUNCTIONS $\phi_2 = \text{sh } y \cdot \sin ky$

$\nu = p/\beta$	$k = \frac{q}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.00025	0.00050	0.00075	0.00100	0.00125	0.00150	0.00175	0.00200	0.00225
0.10	0.00100	0.00200	0.00301	0.00401	0.00501	0.00601	0.00701	0.00801	0.00901
0.15	0.00226	0.00452	0.00677	0.00903	0.01128	0.01354	0.01578	0.01808	0.02027
0.20	0.00403	0.00805	0.01207	0.01609	0.02010	0.02410	0.02809	0.03208	0.03601
0.25	0.00631	0.01262	0.01892	0.02522	0.03149	0.03775	0.04398	0.05019	0.05636
0.30	0.00913	0.01826	0.02737	0.03645	0.04551	0.05452	0.06348	0.07238	0.08122
0.35	0.01250	0.02498	0.03743	0.04984	0.06219	0.07446	0.08664	0.09871	0.11066
0.40	0.01643	0.03282	0.04917	0.06544	0.08160	0.09763	0.11351	0.12921	0.14469
0.45	0.02093	0.04182	0.06262	0.08331	0.10382	0.12412	0.14417	0.16392	0.18335
0.50	0.02604	0.05202	0.07787	0.10353	0.12892	0.15399	0.17868	0.20293	0.22666
0.55	0.03178	0.06347	0.09499	0.12617	0.15728	0.18734	0.21713	0.24626	0.27461
0.60	0.03817	0.07621	0.11398	0.15133	0.18814	0.22427	0.25960	0.29399	0.32733
0.65	0.04525	0.09032	0.13500	0.17912	0.22247	0.26559	0.30619	0.34620	0.38474
0.70	0.05305	0.10580	0.15813	0.20964	0.26012	0.30932	0.35701	0.40295	0.44691
0.75	0.06162	0.12289	0.18346	0.24301	0.30530	0.35768	0.41231	0.46431	0.51384
0.80	0.07097	0.14149	0.21110	0.27937	0.34585	0.41011	0.47175	0.53038	0.58560
0.85	0.08116	0.16176	0.24117	0.31886	0.39423	0.46676	0.53500	0.60120	0.66214
0.90	0.09226	0.18378	0.27380	0.36161	0.44650	0.52777	0.60476	0.67687	0.74350
0.95	0.10430	0.20765	0.30913	0.40782	0.50284	0.59331	0.67844	0.75745	0.82962
1.00	0.11732	0.23348	0.34729	0.45765	0.56343	0.66356	0.75709	0.84304	0.92057
1.05	0.13140	0.26138	0.38847	0.51127	0.62843	0.73870	0.84081	0.93367	1.01624
1.10	0.14663	0.29148	0.43280	0.56891	0.69813	0.81891	0.92080	1.02044	1.11664
1.15	0.16305	0.32401	0.48051	0.63075	0.77165	0.90433	1.02413	1.13037	1.25307
1.20	0.18070	0.35880	0.53174	0.69704	0.85230	0.99531	1.12400	1.23653	1.33128
1.25	0.19971	0.39631	0.58673	0.76801	0.93727	1.09193	1.22954	1.34797	1.44535
1.30	0.22016	0.43662	0.64571	0.84389	1.02784	1.19444	1.34087	1.46468	1.56378
1.35	0.24214	0.47987	0.70888	0.92498	1.12423	1.30306	1.45814	1.58072	1.68639
1.40	0.26573	0.52627	0.77650	1.01154	1.22679	1.41802	1.58152	1.71406	1.81396
1.45	0.29106	0.57598	0.84881	1.10386	1.33570	1.53957	1.71106	1.84668	1.94351
1.50	0.31820	0.62904	0.92617	1.20228	1.45140	1.66793	1.84698	1.98457	2.07764
1.55	0.34729	0.68627	1.00877	1.30711	1.57407	1.80333	1.98033	2.12764	2.23753
1.60	0.37879	0.74791	1.09761	1.41988	1.70557	1.94768	2.14005	2.27775	2.35726
1.65	0.41185	0.81252	1.19112	1.53737	1.84183	2.09631	2.29377	2.42903	2.49778
1.70	0.44759	0.88229	1.29154	1.66355	1.98761	2.25437	2.45612	2.58706	2.64343
1.75	0.48584	0.95683	1.39855	1.79764	2.14175	2.42046	2.62522	2.74981	2.79035
1.80	0.52674	1.03635	1.51269	1.94007	2.30469	2.59488	2.80121	2.91704	2.93861
1.85	0.57048	1.12149	1.63419	2.09117	2.47672	2.77782	2.98406	3.08851	3.08749
1.90	0.61722	1.21222	1.76360	2.25150	2.65839	2.96955	3.17387	3.26394	3.23656
1.95	0.66719	1.30907	1.90131	2.42154	2.84994	3.17033	3.37049	3.44300	3.38402
2.00	0.72055	1.41237	2.04787	2.60176	3.05189	3.38038	3.57409	3.62530	3.53202
2.05	0.77755	1.52252	2.20374	2.79268	3.26459	3.59992	3.78436	3.81045	3.67665
2.10	0.83840	1.63995	2.36944	2.99484	3.48864	3.82917	4.00143	3.99789	3.81872
2.15	0.90336	1.76511	2.54559	3.20884	3.72431	4.06838	4.22496	4.18715	3.95643
2.20	0.97267	1.89846	2.73274	3.43527	3.97258	4.31769	4.45501	4.33755	4.08779
2.25	1.04665	2.04052	2.93146	3.67473	4.23261	4.57727	4.69108	4.56849	4.21548
2.30	1.12552	2.19176	3.14257	3.92781	4.50626	4.84735	4.93316	4.75913	4.33445
2.35	1.20963	2.35281	3.36653	4.19535	4.79341	5.12808	5.18076	4.94875	4.44456
2.40	1.29982	2.49953	3.60432	4.47788	5.09474	5.41955	5.43365	5.13634	4.54451
2.45	1.39491	2.70661	3.85652	4.77623	5.41057	5.72181	5.69128	5.30538	4.63268
2.50	1.49683	2.90065	4.12406	5.09106	5.74152	6.03501	5.95334	5.50145	4.70748
2.55	1.60538	3.10703	4.40761	5.42326	6.08803	6.35923	6.22099	5.67670	4.76714
2.60	1.72108	3.32648	4.70827	5.77353	6.45077	6.69433	6.48793	5.84537	4.80989

Appendix 2 continued

v = p	$A = \frac{v}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.05	1.81422	3.55078	5.02078	6.14280	6.83901	7.04041	6.75017	6.00613	4.88300
2.70	1.07547	3.80785	5.36428	6.53202	7.22644	7.39730	7.03217	6.15742	4.83050
2.75	2.11512	4.07141	5.72159	6.94105	7.44042	7.76497	7.30579	6.20760	4.81615
2.80	2.26392	4.35147	6.10003	7.37355	8.07273	8.14310	7.57925	6.42509	4.77040
2.85	2.42217	4.64880	6.50047	7.82777	8.52342	8.53160	7.85128	6.53768	4.69650
2.90	2.59058	4.96482	6.92449	8.30580	8.96352	8.93019	8.13014	6.63368	4.59238
2.95	2.76982	5.30034	7.37280	8.80858	9.48207	9.33837	8.38683	6.71076	4.45483
3.00	2.96048	5.65649	7.84730	9.33705	9.99272	9.75590	8.64752	6.76667	4.28144
3.05	3.16325	6.03461	8.36978	9.89247	10.52282	10.18152	8.90122	6.79906	4.06890
3.10	3.37898	6.43586	8.87910	10.47598	11.07401	10.61644	9.14682	6.80504	3.81162
3.15	3.60835	6.86148	9.43946	11.08862	11.64638	11.05834	9.38181	6.77217	3.51566
3.20	3.85219	7.31324	10.03170	11.73180	12.24061	11.50684	9.60469	6.72740	3.16703
3.25	4.11124	7.79229	10.65728	12.40059	12.85672	11.96108	9.81289	6.63746	2.76700
3.30	4.38681	8.30034	11.31807	13.11441	13.49551	12.42010	10.00463	6.50928	2.31159
3.35	4.67951	8.83906	12.01548	13.85634	14.15639	12.88289	10.17661	6.33932	1.79702
3.40	5.00576	9.41007	12.75213	14.63403	14.84065	13.34805	10.32715	6.12397	1.21068
3.45	5.32136	10.01552	13.52915	15.44874	15.54740	13.81410	10.45262	5.85946	0.57556
3.50	6.07247	10.65709	14.34941	16.30193	16.27978	14.27978	10.55023	5.54161	-0.13912
3.55	6.04522	11.33682	15.21444	17.19490	17.03071	14.74310	10.61643	5.16621	-0.92840
3.60	6.44142	12.05707	16.12704	18.12030	17.80729	15.20216	10.64817	4.72809	-1.70655
3.65	6.86193	12.82013	17.08890	19.08649	18.60628	15.65515	10.64106	4.22512	-2.47554
3.70	7.30881	13.62827	18.10325	20.12802	19.42871	16.09971	10.59152	3.65016	-3.24537
3.75	7.78279	14.48402	19.17192	21.19544	20.27304	16.53304	10.49520	2.99863	-4.01484
3.80	8.28613	15.39007	20.29826	22.31059	21.13598	16.95293	10.34739	2.26566	-4.78962
3.85	8.82036	16.34947	21.48442	23.47485	22.02788	17.35631	10.14332	1.44556	-5.56382
3.90	9.38731	17.36476	22.73933	24.68989	22.93704	17.73957	9.87817	0.53368	-6.34201
3.95	9.98866	18.43931	24.04988	25.95702	23.86584	18.10004	9.54608	-0.37789	-7.12813
4.00	10.62724	19.57670	25.43530	27.27818	24.81472	18.43325	9.14185	-1.59291	-7.92763
4.05	11.30444	20.77089	26.89297	28.65540	25.78694	18.73579	8.65996	-2.81880	-8.74040
4.10	12.02282	22.05284	28.42725	30.08977	26.70443	19.00288	8.09152	-4.16143	-9.57248
4.15	12.78480	23.39911	30.04059	31.68292	27.76260	19.22999	7.43230	-5.62710	-10.42110
4.20	13.59295	24.82307	31.73856	33.43699	28.77509	19.41236	6.67480	-7.22317	-11.30539
4.25	14.44964	26.32903	33.52319	35.35329	29.79909	19.54421	5.81126	-8.95554	-12.22882
4.30	15.35879	27.92118	35.40033	36.43378	30.83400	19.62006	4.83418	-10.83187	-13.19576
4.35	16.32199	29.60462	37.37210	38.17967	31.87475	19.63404	3.73539	-12.85889	-14.20550
4.40	17.34398	31.38809	39.44560	39.99246	32.92155	19.57805	2.50627	-15.04334	-15.25753
4.45	18.42697	33.26452	41.62271	41.87442	33.96871	19.44793	1.13825	-17.39317	-16.35594
4.50	19.57496	35.25221	43.91034	43.82618	35.01549	19.23339	-0.37847	-19.91473	-17.48622
4.55	20.79078	37.35197	46.31171	45.84948	36.05376	18.92768	-2.05329	-22.61604	-18.65768
4.60	22.08080	39.57086	48.83440	47.94560	37.08947	18.52211	-3.89591	-25.50371	-19.90953
4.65	23.44685	41.91487	51.48143	50.11620	38.10728	18.00736	-5.91688	-28.58517	-21.24286
4.70	24.89493	44.39080	54.25994	52.36241	39.18882	17.37461	-8.12772	-31.86775	-22.66952
4.75	26.42868	47.00577	57.17411	54.68461	40.08625	16.61284	-10.53934	-35.35806	-24.19470
4.80	28.05318	49.76587	60.23129	57.08382	41.03426	15.71119	-13.10333	-39.06232	-25.81309
4.85	29.77446	52.68266	63.43774	59.56233	41.94801	14.65923	-16.01128	-42.98903	-27.52091
4.90	31.59865	55.76074	66.80009	62.11968	42.81995	13.44368	-19.09696	-47.14317	-29.40949
4.95	33.53372	59.01029	70.32348	64.75653	43.64205	12.05221	-22.43158	-51.53051	-31.47549
5.00	35.57524	62.43977	74.01696	67.47298	44.40839	10.47156	-26.02900	-56.15699	-33.71007
5.05	37.74028	66.05954	77.88556	70.26885	45.10814	8.68697	-29.90282	-61.02880	-36.11670
5.10	40.03462	69.87976	81.93084	73.14531	45.73417	6.68364	-34.06773	-66.14922	-38.69449
5.15	42.46241	73.91028	86.18268	76.10094	46.27475	4.44600	-38.53714	-71.52304	-41.45241
5.20	45.03390	78.16221	90.62792	79.13470	46.72150	1.95677	-43.32546	-77.15346	-44.40623
5.25	47.75459	82.64822	95.27899	82.24709	47.06995	-0.80131	-48.49397	-83.04459	-47.57098
5.30	50.63687	87.39073	100.14788	85.43551	47.28131	-3.84637	-53.91831	-89.19574	-50.99063
5.35	53.68075	92.37019	105.23799	88.69833	47.36868	-7.19733	-59.75210	-95.61030	-54.74623

v = p β	$\lambda = \frac{v}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5.40	56.91579	07.63381	110.59670	92.03456	47.31137	-10.87637	-05.90891	-102.28768	-109.49763
5.45	60.33213	103.18328	116.13663	95.44072	47.00076	-14.90206	-72.57731	-109.22556	-114.22396
5.50	63.94794	109.03412	121.90098	98.91508	46.69378	-19.29975	-80.82406	-116.42247	-118.90065
5.55	67.51612	115.20209	128.04446	102.45231	46.40013	-24.08865	-87.04658	-123.87475	-123.51331
5.60	71.82292	121.70374	134.40549	106.04897	45.29145	-29.29760	-94.93865	-131.57687	-128.01946
5.65	76.10530	128.55643	141.04662	109.69961	44.33370	-34.94752	-103.28465	-139.52141	-132.38963
5.70	80.63900	135.77843	147.98702	113.40098	42.93872	-41.06691	-112.10839	-147.70011	-136.60478
5.75	85.27302	143.38889	155.22747	117.14637	41.38177	-47.68423	-121.41775	-156.10248	-140.65859
5.80	90.50455	151.40793	162.78995	120.92652	39.51172	-54.82757	-131.19874	-164.71558	-144.32472
5.85	95.36888	159.85070	170.67644	124.73597	37.31002	-62.52424	-141.56281	-173.52547	-147.77131
5.90	101.54536	168.75738	178.90717	128.56705	34.75496	-70.82586	-152.42937	-182.51188	-150.88345
5.95	107.54014	178.13134	187.48910	132.40744	31.81677	-79.70887	-163.83854	-191.65855	-153.60007
6.00	113.85901	188.00473	196.43836	136.24917	28.46576	-89.26211	-175.80916	-200.93858	-155.87586
6.05	120.60860	198.40322	205.76154	140.07952	24.66628	-99.50061	-188.34331	-210.33133	-157.64618
6.10	127.70862	209.35145	215.47974	143.88649	20.38897	-110.46070	-201.46650	-219.80454	-158.85600
6.15	135.09780	220.87072	225.59969	147.65234	15.59415	-122.17766	-215.17546	-229.32597	-159.43350
6.20	143.15286	233.01514	236.14162	151.36449	10.24421	-134.68993	-229.48707	-238.85912	-159.34743
6.25	151.54150	245.79099	247.11192	155.00699	4.27100	-148.03721	-245.40282	-248.36551	-158.42845
6.30	160.41400	259.23985	258.53191	158.55974	-7.73562	-162.20010	-258.84282	-257.80219	-156.69459
6.35	170.00452	273.73487	270.74273	162.20604	-9.57255	-177.61097	-276.41461	-267.45539	-154.22127
6.40	179.71043	288.28900	282.76107	165.31433	-17.56489	-193.49264	-292.83290	-276.26717	-150.35251
6.45	190.42762	304.34211	296.06237	168.47668	-26.38810	-210.85140	-310.59151	-285.54008	-145.75378
6.50	201.26807	320.45121	308.94428	171.43987	-35.96408	-228.73171	-328.19344	-293.80062	-139.59961
6.55	213.24162	338.21950	323.20549	174.42014	-46.56128	-248.27198	-347.21600	-302.45053	-132.49713
6.60	225.35036	356.05004	337.20224	176.72390	-57.98952	-268.33130	-365.98851	-309.91188	-123.67586

Appendix 3

VLASOV FUNCTIONS $\phi_3 = \text{sh } y \cdot \cos ky$

$\nu = p \beta$	$\lambda = \frac{q}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.05000	0.05000	0.04999	0.04999	0.04998	0.04998	0.04997	0.04996	0.04995
0.10	0.10019	0.10018	0.10015	0.10012	0.10007	0.10002	0.09995	0.09988	0.09979
0.15	0.15058	0.15053	0.15045	0.15033	0.15017	0.14999	0.14977	0.14952	0.14923
0.20	0.20130	0.20118	0.20098	0.20070	0.20033	0.19989	0.19937	0.19877	0.19806
0.25	0.25251	0.25229	0.25190	0.25135	0.25063	0.24977	0.24875	0.24757	0.24624
0.30	0.30438	0.30397	0.30329	0.30233	0.30110	0.29960	0.29783	0.29579	0.29349
0.35	0.35697	0.35631	0.35522	0.35370	0.35173	0.34934	0.34652	0.34328	0.33961
0.40	0.41042	0.40943	0.40780	0.40550	0.40256	0.39898	0.39475	0.38990	0.38443
0.45	0.46486	0.46345	0.46110	0.45782	0.45360	0.44848	0.44243	0.43551	0.42796
0.50	0.52045	0.51849	0.51525	0.51071	0.50490	0.49782	0.48950	0.47996	0.46922
0.55	0.57727	0.57466	0.57029	0.56422	0.55642	0.54695	0.53582	0.52308	0.50875
0.60	0.63550	0.63207	0.62636	0.61840	0.60822	0.59584	0.58132	0.56470	0.54606
0.65	0.69527	0.69087	0.68354	0.67333	0.66027	0.64443	0.62586	0.60465	0.58088
0.70	0.75672	0.75116	0.74191	0.72941	0.71259	0.69265	0.66932	0.64271	0.61295
0.75	0.82000	0.81308	0.80158	0.78559	0.76517	0.74046	0.71157	0.67869	0.64198
0.80	0.88527	0.87677	0.87154	0.84333	0.81800	0.78774	0.75246	0.71235	0.66769
0.85	0.95266	0.94231	0.92519	0.89182	0.87105	0.83444	0.79180	0.74345	0.68972
0.90	1.02230	1.00993	0.98933	0.96072	0.92433	0.88046	0.82946	0.77175	0.70778
0.95	1.09451	1.07969	1.05512	1.02104	0.97774	0.92565	0.86518	0.79695	0.72149
1.00	1.16933	1.15153	1.12271	1.08243	1.03133	0.96994	0.89884	0.81877	0.73052
1.05	1.24694	1.22631	1.19214	1.14489	1.08812	1.01190	0.93014	0.83360	0.73145
1.10	1.32758	1.30346	1.26358	1.20843	1.13867	1.05515	0.95888	0.85101	0.73326
1.15	1.41139	1.38337	1.33704	1.27309	1.19229	1.09578	0.98474	0.86074	0.72534
1.20	1.49861	1.46620	1.41270	1.33887	1.24582	1.13483	1.00750	0.86570	0.71145
1.25	1.58941	1.55212	1.49059	1.40581	1.29009	1.17211	1.02601	0.86552	0.69070
1.30	1.68404	1.64131	1.57085	1.47389	1.35205	1.20739	1.04238	0.85075	0.66262
1.35	1.78270	1.73391	1.65538	1.54310	1.40455	1.24047	1.05380	0.84796	0.62669
1.40	1.88548	1.83015	1.73880	1.61344	1.45648	1.27104	1.06073	0.82866	0.58237
1.45	1.99310	1.93015	1.82666	1.68186	1.50769	1.29880	1.06277	0.80438	0.52911
1.50	2.10537	2.03349	1.91731	1.75738	1.55795	1.32358	1.05947	0.77157	0.46633
1.55	2.22261	2.09738	2.01072	1.83091	1.60714	1.34488	1.05036	0.73067	0.39343
1.60	2.34721	2.25688	2.10888	1.90705	1.65648	1.36358	1.03586	0.68170	0.31008
1.65	2.47338	2.37216	2.20646	1.98087	1.70444	1.37582	1.01281	0.62230	0.21486
1.70	2.60748	2.49417	2.30095	2.05716	1.74666	1.38464	0.98327	0.55357	-0.10791
1.75	2.74774	2.61423	2.40064	2.13422	1.78862	1.38834	0.94578	0.47429	-0.01172
1.80	2.89286	2.73358	2.50353	2.21195	1.82888	1.38673	0.89977	0.38372	-0.14470
1.85	3.04332	2.89142	2.63575	2.29021	1.86642	1.37962	0.84451	0.28149	-0.29171
1.90	3.20933	3.08301	2.75146	2.35889	1.90102	1.36475	0.77939	0.16592	-0.45346
1.95	3.37793	3.18466	2.87060	2.44781	1.93249	1.34337	0.70353	0.03719	-0.63966
2.00	3.55458	3.34055	2.99339	2.52687	1.95959	1.31423	0.61646	-0.10590	-0.82402
2.05	3.73956	3.50301	3.10058	2.60579	1.98274	1.27666	0.51709	-0.26412	-1.03427
2.10	3.93350	3.67232	3.24978	2.68443	2.00116	1.22996	0.40472	-0.43834	-1.26210
2.15	4.13668	3.84875	3.38350	2.76255	2.01429	1.17338	0.27840	-0.62941	-1.50822
2.20	4.34969	4.03257	3.52107	2.83985	2.02174	1.10616	0.13723	-0.83829	-1.77339
2.25	4.57290	4.22416	3.66240	2.91607	2.02269	1.02741	-0.01970	-1.06583	-2.05820
2.30	4.80697	4.42376	3.80763	2.99091	2.01670	0.93624	-0.19348	-1.31303	-2.36347
2.35	5.05016	4.63179	3.95659	3.06402	2.00292	0.83173	-0.38516	-1.58087	-2.68966
2.40	5.30957	4.84849	4.10957	3.13499	1.98974	0.71290	-0.59576	-1.87027	-3.03758
2.45	5.57919	5.07431	4.26623	3.20344	1.94925	0.57873	-0.82642	-2.18223	-3.40766
2.50	5.86210	5.30953	4.42687	3.26892	1.90775	0.42799	-1.07845	-2.51779	-3.80055
2.55	6.15862	5.55456	4.59123	3.33099	1.85525	0.25901	-1.35284	-2.87790	-4.21674
2.60	6.46972	5.80982	4.75935	3.38901	1.79084	0.07230	-1.65112	-3.26355	-4.65659

Appendix 3 continued

v = p β	$\lambda = \frac{q}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.65	6.79584	6.07564	4.93115	3.44247	1.71345	-0.13520	-1.97442	-3.67583	-5.12050
2.70	7.13793	6.35242	5.10082	3.49079	1.62204	-0.36424	-2.32416	-4.11596	-5.60891
2.75	7.49655	6.64058	5.28546	3.53325	1.51534	-0.61629	-2.70189	-4.58403	-6.12189
2.80	7.87293	6.94009	5.46778	3.56906	1.39238	-0.81092	-3.10842	-5.08194	-6.65962
2.85	8.26730	7.25289	5.65319	3.59750	1.25157	-1.10532	-3.54581	-5.61029	-7.22196
2.90	8.68123	7.57796	5.84188	3.61784	1.09168	-1.52563	-4.01547	-6.17010	-7.80934
2.95	9.11516	7.91621	6.03314	3.62895	0.91124	-1.88529	-4.51876	-6.76203	-8.42004
3.00	9.57047	8.26815	6.22721	3.63007	0.70866	-2.27608	-5.05752	-7.38707	-9.05686
3.05	10.04774	8.63418	6.42342	3.62000	0.48225	-2.69997	-5.63294	-8.04607	-9.71624
3.10	10.54843	9.01490	6.62183	3.59763	0.23028	-3.15889	-6.24701	-8.73943	-10.39890
3.15	11.07336	9.41081	6.81606	3.56177	-0.04892	-3.65482	-6.90090	-9.46811	-11.10341
3.20	11.62428	9.82242	7.02326	3.51114	-0.35758	-4.18993	-7.59885	-10.23253	-11.82927
3.25	12.20100	10.25015	7.22537	3.44427	-0.69761	-4.76636	-8.33604	-11.03313	-12.57552
3.30	12.80738	10.69479	7.42810	3.35983	-1.07112	-5.38665	-9.12047	-11.87928	-13.33901
3.35	13.44242	11.15675	7.56047	3.25589	-1.48060	-6.05250	-9.95143	-12.74411	-14.11966
3.40	14.10859	11.63661	7.83242	3.13133	-1.92814	-6.76704	-10.83103	-13.65499	-14.91552
3.45	14.80615	12.13509	8.03268	2.98386	-2.41663	-7.53249	-11.76030	-14.60255	-15.72362
3.50	15.53965	12.65246	8.23112	2.81175	-2.94872	-8.35155	-12.74163	-15.58680	-16.54197
3.55	16.30772	13.18962	8.42657	2.61284	-3.52681	-9.22679	-13.77574	-16.60721	-17.36725
3.60	17.11336	13.74719	8.61848	2.38479	-4.15446	-10.16123	-14.86516	-17.66339	-18.19696
3.65	17.95784	14.32540	8.80552	2.12563	-4.83434	-11.15761	-16.01041	-18.75430	-19.02671
3.70	18.84359	14.92543	8.98715	1.83256	-5.57003	-12.21873	-17.21375	-19.87802	-19.85355
3.75	19.77199	15.54752	9.16184	1.50314	-6.36465	-13.34785	-18.47582	-21.03608	-20.67251
3.80	20.74572	16.19250	9.32871	1.13417	-7.22211	-14.54809	-19.79852	-22.22414	-21.47912
3.85	21.76625	16.86075	9.48736	0.72313	-8.14585	-15.82244	-21.18216	-23.41778	-22.26815
3.90	22.83704	17.55315	9.63323	0.26665	-9.14015	-17.17414	-22.62890	-24.68542	-23.03433
3.95	23.95877	18.27058	9.76776	-0.23881	-10.20878	-18.60974	-24.13866	-25.95365	-23.77083
4.00	25.13565	19.01316	9.88877	-0.79686	-11.25670	-20.12331	-25.71311	-27.24325	-24.47251
4.05	26.38305	19.78177	9.99392	-1.41098	-12.58750	-21.72760	-27.35201	-28.55125	-25.13082
4.10	27.66204	20.57702	10.08130	-2.08569	-13.90673	-23.42279	-29.05613	-29.87351	-25.73892
4.15	29.01701	21.39985	10.14882	-2.82466	-15.31867	-25.21255	-30.82539	-31.20590	-26.12927
4.20	30.43847	22.25023	10.19471	-3.63325	-16.82951	-27.10023	-32.66062	-32.54361	-26.77021
4.25	31.92756	23.12937	10.21578	-4.51527	-18.44307	-29.08957	-34.55084	-33.88206	-27.17503
4.30	33.48928	24.03792	10.20996	-5.47673	-20.16644	-31.18438	-36.52442	-35.21501	-27.49380
4.35	35.12521	24.97666	10.17433	-6.52260	-22.00409	-33.38766	-38.55151	-36.53584	-27.71408
4.40	36.84079	25.94430	10.10571	-7.65848	-23.96331	-35.70350	-40.64193	-37.83841	-27.82675
4.45	38.63817	26.94355	10.00072	-8.89029	-26.04930	-38.13518	-42.79222	-39.11504	-27.81812
4.50	40.52255	27.97432	9.85611	-10.22468	-28.26954	-40.68587	-45.00121	-40.35690	-27.67685
4.55	42.49703	29.03708	9.66751	-11.66829	-30.63993	-43.35951	-47.26550	-41.55555	-27.38829
4.60	44.56635	30.13175	9.43215	-13.22809	-33.13885	-46.15954	-49.58444	-42.70082	-26.93962
4.65	46.73528	31.25916	9.14460	-14.91193	-35.80139	-49.08927	-51.95008	-43.78207	-26.31484
4.70	49.00875	32.42019	8.80054	-16.72708	-38.62729	-52.15078	-54.46493	-44.78877	-25.50014
4.75	51.38955	33.61402	8.39541	-18.68222	-41.62283	-55.34860	-56.81814	-45.70843	-24.47719
4.80	53.88572	34.84139	7.92303	-20.78564	-44.79654	-58.68340	-59.30666	-46.52672	-23.23084
4.85	56.49994	36.10361	7.37911	-23.04742	-48.15772	-62.16105	-61.82639	-47.23167	-21.74264
4.90	59.24067	37.39897	6.75641	-25.47672	-51.71414	-65.78156	-64.36824	-47.80653	-19.99464
4.95	62.11104	38.72870	6.04834	-28.08394	-55.47333	-69.54706	-66.92416	-48.23565	-17.96785
5.00	65.11925	40.09199	5.24913	-30.87966	-59.44716	-73.46043	-69.48834	-48.50219	-15.64304
5.05	68.26951	41.48935	4.34973	-33.87499	-63.64207	-77.52282	-72.04822	-48.58808	-12.99692
5.10	71.57158	42.92048	3.34510	-37.08234	-68.07066	-81.73482	-74.59685	-48.47241	-10.01152
5.15	75.02932	44.38410	2.22343	-40.51314	-72.13951	-86.09819	-77.11912	-48.13608	-6.66598
5.20	78.65344	45.88042	0.97884	-44.18195	-77.06282	-90.61251	-79.60781	-47.55714	-2.93471
5.25	82.44813	47.40872	-0.40018	-48.10236	-82.84640	-95.27670	-82.04319	-46.71222	1.20149
5.30	86.42415	48.96811	-1.92318	-52.28761	-88.30426	-100.09179	-84.41582	-45.57649	5.76755

Appendix 3 continued

y-p β	$h = \frac{q}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
5.35	90.58638	50.55720	-3.60130	-56.75312	-91.04447	-105.05470	-86.70498	-44.12540	10.78600
5.40	94.94931	52.17668	-5.44427	-61.51652	-100.08141	-110.16516	-88.89840	-42.33204	16.28189
5.45	99.51624	53.82317	-7.45074	-66.59204	-100.42205	-115.41915	-90.96186	-40.10633	22.27687
5.50	104.30004	55.49520	-9.67085	-71.99940	-113.08248	-120.82217	-92.90797	-37.50990	28.79731
5.55	109.31016	57.19076	-12.74078	-77.75530	-120.06890	-126.34031	-94.68256	-34.60170	35.86609
5.60	114.55017	58.90888	-14.73790	-83.87071	-127.33884	-131.99873	-96.27454	-31.13917	43.50831
5.65	120.05105	60.56286	-17.60880	-90.39210	-135.07799	-137.85514	-97.69570	-27.17793	52.45681
5.70	125.80683	62.40132	-20.73360	-97.31462	-143.12450	-143.67740	-98.80147	-22.63378	60.60664
5.75	131.82992	64.47122	-24.12803	-104.68842	-151.54265	-149.68205	-99.67912	-17.61806	70.10779
5.80	138.13992	65.95031	-27.81097	-112.47588	-160.35230	-155.78106	-100.25986	-11.94517	80.27692
5.85	144.74345	67.73618	-31.79946	-120.76017	-169.55661	-161.99613	-100.50616	-5.62168	92.02674
5.90	151.66008	69.52451	-36.11836	-129.54000	-179.17729	-168.22260	-100.38820	-1.38890	102.69704
5.95	158.91886	71.31240	-40.78503	-138.86788	-189.21598	-174.53559	-99.86154	9.13135	114.98707
6.00	166.48194	73.09278	-45.82923	-148.74127	-199.69451	-180.88820	-98.89189	17.64990	128.02532
6.05	174.41551	74.85978	-51.27074	-159.20055	-210.61336	-187.26183	-97.43005	26.98616	142.88289
6.10	182.72274	76.60912	-57.13861	-170.27068	-221.99369	-193.63728	-95.43537	37.71910	159.49166
6.15	191.41393	78.33169	-63.45936	-181.99510	-233.83500	-199.99143	-92.85484	48.30579	171.65394
6.20	200.51847	80.02212	-70.23326	-194.39116	-246.15164	-206.29593	-89.64054	60.38368	187.93125
6.25	210.04306	81.66960	-77.57390	-207.49963	-258.96660	-212.63320	-85.73662	73.46949	204.89921
6.30	220.01447	83.27021	-85.44577	-221.85412	-272.27414	-218.65849	-81.08648	87.61488	222.68014
6.35	230.73290	84.91484	-94.01103	-236.28727	-286.45433	-224.02918	-75.72347	103.09238	241.56937
6.40	241.36929	86.28027	-102.96036	-251.44715	-300.40710	-230.46689	-69.30226	119.29138	260.66739
6.45	253.10847	87.77062	-112.82488	-268.09088	-315.64004	-236.36844	-62.11387	137.09078	281.21333
6.50	264.75237	88.99249	-123.11078	-284.97506	-330.61787	-241.42258	-53.76993	155.81572	301.85056
6.55	277.61038	90.20258	-134.44346	-303.51120	-346.94296	-246.78423	-44.47143	176.24394	324.00713
6.60	290.35838	91.21779	-146.23956	-322.27248	-362.94522	-251.17421	-33.90988	197.69057	346.14525

Appendix 4

VLASOV FUNCTIONS $\phi_4 = \text{ch } y \cdot \sin ky$

$y = p \beta$	$k = \frac{y}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	0	0	0	0	0	0	0	0	0
0.05	0.00501	0.01000	0.01502	0.02002	0.02503	0.03004	0.03504	0.04004	0.04503
0.10	0.01001	0.02002	0.03004	0.04003	0.05004	0.06003	0.07002	0.08000	0.09000
0.15	0.01517	0.03034	0.04549	0.06064	0.07577	0.09090	0.10600	0.12106	0.13610
0.20	0.02040	0.04080	0.06116	0.08151	0.10183	0.12211	0.14234	0.16252	0.18260
0.25	0.02578	0.05155	0.07727	0.10139	0.12859	0.15413	0.17958	0.20491	0.22912
0.30	0.03130	0.06260	0.09395	0.12514	0.15621	0.18715	0.21791	0.24848	0.27883
0.35	0.03713	0.07421	0.11118	0.14805	0.18471	0.22116	0.25732	0.29319	0.32869
0.40	0.04323	0.08639	0.12941	0.17224	0.21478	0.25697	0.29876	0.34007	0.38083
0.45	0.04962	0.09913	0.14844	0.19746	0.24607	0.29419	0.34172	0.38854	0.43459
0.50	0.05630	0.11257	0.16851	0.22403	0.27807	0.33323	0.38666	0.43912	0.49048
0.55	0.06350	0.12680	0.18972	0.25208	0.31423	0.37430	0.43381	0.49200	0.54870
0.60	0.07108	0.14191	0.21223	0.28179	0.35033	0.41760	0.48330	0.54895	0.60950
0.65	0.07916	0.15799	0.23615	0.31333	0.38916	0.46337	0.52561	0.60559	0.67300
0.70	0.08779	0.17515	0.26165	0.34688	0.43040	0.51181	0.59072	0.66673	0.73947
0.75	0.09701	0.19348	0.28884	0.38260	0.48067	0.56315	0.64915	0.73103	0.80101
0.80	0.10687	0.21308	0.31791	0.42071	0.52082	0.61760	0.71043	0.79871	0.88187
0.85	0.11744	0.23406	0.34899	0.46139	0.57046	0.67541	0.77547	0.86995	0.95814
0.90	0.12881	0.25657	0.38225	0.50483	0.62435	0.73681	0.83570	0.92495	1.00797
0.95	0.14098	0.28069	0.41787	0.55127	0.67971	0.80201	0.91709	1.02389	1.12143
1.00	0.15405	0.30656	0.45602	0.60091	0.73980	0.87128	0.99408	1.10694	1.20874
1.05	0.16808	0.33433	0.49689	0.65396	0.80382	0.93485	1.07546	1.19424	1.29985
1.10	0.18317	0.36412	0.54067	0.71069	0.87212	1.02300	1.16152	1.28600	1.39103
1.15	0.19938	0.39557	0.58759	0.77132	0.94484	1.10593	1.25236	1.38228	1.49391
1.20	0.21675	0.43039	0.63781	0.83659	1.02237	1.19931	1.34829	1.48327	1.59693
1.25	0.23543	0.46719	0.69167	0.90536	1.10489	1.28722	1.44944	1.58905	1.70384
1.30	0.25549	0.50668	0.74932	0.97930	1.19277	1.38610	1.55603	1.69971	1.81472
1.35	0.27703	0.54902	0.81102	1.05827	1.28623	1.49083	1.66825	1.81536	1.92940
1.40	0.30014	0.59442	0.87075	1.14254	1.38565	1.60165	1.78632	1.93602	2.04785
1.45	0.32496	0.64306	0.94766	1.23241	1.49125	1.71886	1.91032	2.06172	2.16984
1.50	0.35194	0.69518	1.02323	1.32826	1.60350	1.84271	2.04053	2.19254	2.29536
1.55	0.38008	0.75101	1.10335	1.43044	1.72259	1.97348	2.17702	2.32838	2.42390
1.60	0.41064	0.81079	1.19022	1.53926	1.84897	2.11143	2.31998	2.46926	2.55545
1.65	0.44339	0.87475	1.28235	1.65512	1.98399	2.25687	2.46946	2.61507	2.68909
1.70	0.47849	0.94322	1.38073	1.77852	2.12486	2.41004	2.62573	2.76570	2.82597
1.75	0.51609	1.01642	1.48565	1.90959	2.27513	2.57120	2.78871	2.92116	2.96413
1.80	0.55633	1.09167	1.59767	2.04900	2.43417	2.74066	2.95859	3.08993	3.10471
1.85	0.59941	1.17835	1.71705	2.19710	2.60229	2.91866	3.13536	3.24510	3.24403
1.90	0.64547	1.26770	1.84431	2.35454	2.78005	3.10545	3.21913	3.41332	3.38468
1.95	0.69476	1.36316	1.97987	2.52160	2.96770	3.30133	3.50975	3.58526	3.52478
2.00	0.74744	1.46508	2.12429	2.69885	3.16578	3.50652	3.70746	3.76058	3.66382
2.05	0.80376	1.57334	2.27892	2.88681	3.37463	3.72125	3.91193	3.93888	3.80057
2.10	0.86302	1.68988	2.44158	3.08602	3.59486	3.94576	4.12326	4.11961	3.93498
2.15	0.92822	1.81366	2.61562	3.29711	3.82672	4.18030	4.34118	4.30233	4.06527
2.20	0.99985	1.94566	2.80068	3.52067	4.07133	4.25502	4.56570	4.48637	4.18941
2.25	1.07016	2.08636	2.99733	3.75730	4.32771	4.68011	4.79647	4.67114	4.31020
2.30	1.14838	2.23627	3.20639	4.00761	4.59777	4.94570	5.03334	4.85578	4.42248
2.35	1.23183	2.39600	3.42833	4.27237	4.88141	5.22222	5.27587	5.03960	4.52616
2.40	1.32089	2.56609	3.66414	4.55220	5.17930	5.50949	5.52383	5.22459	4.61994
2.45	1.41534	2.74702	3.91439	4.84719	5.49176	5.80767	5.77667	5.38499	4.70219
2.50	1.51713	2.94000	4.18001	5.16013	5.81941	6.11689	6.03410	5.57608	4.77235
2.55	1.62508	3.14514	4.46169	5.48910	6.16272	6.43724	6.29731	5.74634	4.82563
2.60	1.74018	3.36339	4.76051	5.83759	6.52235	6.76860	6.55992	5.91023	4.86326

Appendix 4 continued

v = p β	$k = \frac{v}{p}$								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
2.65	1.86272	3.50500	5.07722	6.20432	6.80854	7.11106	6.82609	6.06640	4.88219
2.70	1.99340	3.84241	5.41296	6.59130	7.20201	7.46443	7.09596	6.21329	4.88047
2.75	2.13248	4.10483	5.76855	6.99892	7.70313	7.82870	7.36575	6.34938	4.85566
2.80	2.29072	4.38377	6.14531	7.42828	8.13265	8.20354	7.63560	6.42419	4.80581
2.85	2.43845	4.68011	6.54416	7.88038	8.58070	8.58894	7.90705	6.58162	4.72807
2.90	2.60631	4.99497	7.01212	8.35625	9.04813	8.98442	8.17040	6.67944	4.62027
2.95	2.78504	5.32946	7.41330	8.85698	9.53509	9.38067	8.43291	6.74765	4.47931
3.00	2.97519	5.68460	7.88630	9.38346	10.04239	9.80459	8.69050	6.80030	4.30279
3.05	3.17747	6.06174	8.40741	9.93695	10.57013	10.22730	8.94124	6.82963	4.08726
3.10	3.39272	6.46204	8.91522	10.51840	11.11905	10.65062	9.18402	6.83272	3.83013
3.15	3.62280	6.88673	9.47419	11.12942	11.68023	11.09903	9.41633	6.80712	3.52800
3.20	3.86501	7.33760	10.06510	11.77086	12.28137	11.54515	9.63667	6.74979	3.17757
3.25	4.12362	7.81576	10.68938	12.44395	12.89545	11.99711	9.84244	6.65746	2.76630
3.30	4.39876	8.32206	11.34891	13.15014	13.53227	12.45403	10.03188	6.52702	2.31789
3.35	4.69105	8.86085	12.04510	13.89050	14.19129	12.91464	10.20170	6.35494	1.80145
3.40	5.01693	9.43106	12.78058	14.66666	14.87365	13.37783	10.35018	6.13763	1.22240
3.45	5.33208	10.03573	13.55646	15.47992	15.57878	13.84198	10.47372	5.87128	0.57672
3.50	5.68282	10.67654	14.37560	16.33169	16.30749	14.30582	10.56948	5.55173	0.13938
3.55	6.05520	11.35554	15.23956	17.22329	17.05884	14.76745	10.63396	5.17474	0.92993
3.60	6.45104	12.07508	16.14631	18.15030	17.83390	15.22488	10.66408	4.73605	1.79923
3.65	6.87121	12.83747	17.11200	19.13232	18.63143	15.67631	10.65544	4.23084	2.75126
3.70	7.31775	13.64494	18.12539	20.15203	19.45247	16.11940	10.60448	3.65462	3.79000
3.75	7.79141	14.52132	19.19315	21.21891	20.29548	16.55134	10.50681	3.00195	4.92028
3.80	8.29413	15.46548	20.31858	22.33293	21.15715	16.96991	10.35775	2.26793	6.14546
3.85	8.82835	16.36428	21.50389	23.40209	22.04784	17.37203	10.15251	1.46687	7.47058
3.90	9.39500	17.37900	22.75298	24.70987	22.95586	17.75412	9.88627	0.53352	8.89930
3.95	9.99607	18.45298	24.06772	25.97627	23.88354	18.11340	9.55316	-0.47824	10.43586
4.00	10.62437	19.58083	25.45236	27.29619	24.83137	18.44562	9.14798	-1.50396	12.08444
4.05	11.31131	20.79251	26.90921	28.67270	25.79651	18.74711	8.66419	-2.82051	13.84876
4.10	12.02942	22.06496	28.44287	30.10630	26.77913	19.01332	8.09597	-4.16372	15.73292
4.15	12.79100	23.41074	30.05552	31.59861	27.77640	19.23955	7.43599	-5.62999	17.73992
4.20	13.59906	24.83423	31.75283	33.15189	28.78862	19.42109	6.67780	-7.22642	19.87432
4.25	14.45965	26.34727	33.54641	34.77736	29.82023	19.55775	5.81528	-8.96175	22.14415
4.30	15.36444	27.93147	35.41337	36.44720	30.84535	19.62729	4.83596	-10.83586	24.53475
4.35	16.32743	29.61448	37.38454	38.19239	31.88537	19.64163	3.73663	-12.86318	27.06657
4.40	17.34920	31.39345	39.45749	40.00451	32.93147	19.58455	2.50703	-15.04787	29.73649
4.45	18.43200	33.27360	41.63407	41.88585	33.97798	19.45324	1.13856	-17.39791	32.54482
4.50	19.57979	35.26091	43.92118	43.83700	35.02414	19.23813	-0.37857	-19.91965	35.49498
4.55	20.79590	37.36032	46.32206	45.85973	36.06181	18.93191	-2.05375	-22.62109	38.58500
4.60	22.08526	39.57886	48.84427	47.95529	37.09697	18.52585	-3.89670	-25.50886	41.81798
4.65	23.45114	41.92254	51.49085	50.12536	38.11425	18.10657	-5.91796	-28.59039	45.19112
4.70	24.89904	44.39814	54.26891	52.37107	39.11519	17.37749	-8.12907	-31.87302	48.70458
4.75	26.43264	47.01260	57.18267	54.69280	40.09226	16.61533	-10.54092	-35.36335	52.35493
4.80	28.05744	49.77343	60.24044	57.09250	41.04049	15.71358	-13.16533	-39.06824	56.14161
4.85	29.77811	52.68911	63.44551	59.56963	41.95315	14.66402	-16.01324	-42.99430	60.05827
4.90	31.60168	55.76609	66.80650	62.12564	42.82406	13.44497	-19.10081	-47.14769	64.06683
4.95	33.53285	59.01621	70.33054	64.76395	43.64643	12.05342	-22.43383	-51.53568	68.26434
5.00	35.57848	62.44545	74.02368	67.47911	44.41243	10.47251	-26.03137	-56.16209	72.54245
5.05	37.74339	66.06497	77.89192	70.27463	45.11185	8.68769	-29.90528	-61.03382	76.92302
5.10	40.03759	69.88495	81.94593	73.15075	45.73757	6.68414	-34.07026	-66.15413	81.40054
5.15	42.46527	73.91524	86.18846	76.10606	46.27788	4.44630	-38.53973	-71.52785	85.95998
5.20	45.03665	78.16697	90.63344	79.13952	46.72434	1.95689	-43.32810	-77.15816	90.59175
5.25	47.75722	82.65277	95.28385	82.25162	47.06354	-0.80135	-48.49064	-83.04916	95.27623
5.30	50.63939	87.38509	100.15287	85.43977	47.28367	-3.81656	-53.92099	-89.20018	100.00462

ORIGINAL PAGE IS
OF POOR QUALITY

Appendix 4 continued

v = p β	$h = \frac{v}{p}$								
	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
5.35	53,68948	92,37488	105,24334	88,70284	47,37108	-7,19770	-59,75513	-95,61516	-104,75155
5.40	56,91811	97,63780	110,57152	92,03831	47,31330	-10,87681	-65,97160	-102,29185	-109,50210
5.45	60,33435	103,18709	116,14032	95,44425	47,09250	-14,00262	-72,57999	-109,22960	-114,22818
5.50	63,95008	109,03777	121,06500	98,91837	46,89534	-19,30040	-79,60328	-116,42637	-118,91003
5.55	67,76486	115,18706	128,02845	102,43947	46,09436	-24,21423	-87,03568	-123,85921	-123,49784
5.60	71,82488	121,70706	134,40917	106,05187	45,29569	-29,29840	-94,94125	-131,58047	-128,02296
5.65	76,10719	128,55961	141,05011	109,70232	44,53431	-34,94838	-103,28721	-139,52487	-132,39191
5.70	80,63981	135,78146	147,99038	113,40351	42,95968	-41,06783	-112,11090	-147,70341	-136,59289
5.75	85,43245	143,39178	155,23061	117,14874	41,38561	-47,68520	-121,42011	-156,10564	-140,66144
5.80	90,50621	151,41070	162,79293	120,92873	39,51245	-54,82855	-131,20694	-164,65584	-144,32736
5.85	95,87047	159,85935	170,67927	124,73804	37,31064	-62,52528	-141,56516	-173,52835	-147,77376
5.90	101,54688	168,75991	178,90985	128,56898	34,75548	-70,80867	-152,43166	-182,51462	-150,88572
5.95	107,54760	178,13375	187,49164	132,40023	31,81720	-79,70995	-163,84076	-191,66148	-153,60215
6.00	113,86041	188,00704	196,44078	136,25085	28,46611	-89,26320	-175,81132	-200,94105	-155,87778
6.05	120,60993	198,40542	205,76382	140,08107	24,86655	-99,50171	-188,34540	-210,33366	-157,64792
6.10	127,70991	209,35357	215,70485	143,88794	20,38918	-110,46182	-201,46854	-219,80676	-158,85069
6.15	135,09903	220,88173	225,60174	147,65368	15,59430	-122,17877	-215,17741	-229,32805	-159,43494
6.20	143,15404	233,01706	236,14356	151,36574	10,24429	-134,69104	-229,48896	-238,86109	-159,31874
6.25	151,54263	245,79282	247,11376	155,00815	4,29693	-148,03825	-244,40454	-248,36736	-158,42968
6.30	160,41509	259,24161	258,53366	158,56081	-2,28993	-162,26119	-259,93321	-257,80393	-156,69565
6.35	170,01148	273,74609	270,75383	162,21269	-9,57294	-177,61825	-276,42594	-264,60020	-154,22759
6.40	179,72337	289,30975	282,78142	165,32624	-17,56606	-193,50657	-292,85399	-276,28706	-150,36333
6.45	190,43458	304,35324	296,07320	168,68285	-26,38905	-210,85911	-310,60296	-285,55051	-145,75911
6.50	201,26899	320,45266	308,94568	171,44064	-35,98424	-228,73275	-328,19493	-292,81035	-139,60024
6.55	213,24250	338,22080	323,20682	174,42085	-46,56147	-248,27300	-347,21743	-302,45527	-132,49976
6.60	225,35119	356,05136	337,20348	176,72456	-57,98074	-268,33229	-365,08986	-309,91302	-123,67632

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