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A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

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A NUMERICAL SOLUTION FOR TWO-DIMENSIONAL FREDHOLM
INTEGRAL EQUATIONS OF THE SECOND KIND WITH KERNELS
OF THE LOGARITHMIC POTENTIAL FORM

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SUMMARY

Two-dimensional Fredholm integral equations with logarithmic potential kernels are numerically solved. The explicit convergence of these solutions to their true solutions is demonstrated. The results are based on a previous work in which numerical solutions were obtained for Fredholm integral equations of the second kind with continuous kernels.

INTRODUCTION

Previously (ref. 1), the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proven. Later on (ref. 2), these results were extended to two dimensions for continuous kernels. However, since a class of physical problems involves kernels of the logarithmic potential form, this study extends the theory to kernels of this type.

MAIN DEVELOPMENT

In a recent report (ref. 2), it was shown that the following system of equations,

\[ x(t_{ij}) - \lambda \sum_{k=1}^{n} \sum_{k'=1}^{n} h(t_{ij}; t_{kk'}) x(t_{kk'}) \Delta = y(t_{ij}) \]  (1)

converge to the exact solution of the two-dimensional Fredholm integral equation of the second kind,

\[ x(r,s) - \lambda \int_{0}^{1} \int_{0}^{1} h(r,s; t_1, t_2) x(t_1, t_2) dt_1 dt_2 = y(r,s) \]  (2)

when the kernel \( h(r,s; t_1, t_2) \) is a continuous function over the unit square \([0,1] \times [0,1]\) and \( y(r,s) \) is also a continuous function over the unit square.
Equation (2) will be regarded as a functional equation in the Banach space $X = C^0$ of continuous functions on the unit square $([0,1] \times [0,1])$ and, typically, will be expressed in the following form:

$$Kx \equiv x - \lambda Hx = y$$

(3)

The system (1) is regarded as an approximate functional equation in the space $\tilde{X} = \mathbb{R}^N$ and typically is expressed in the following form:

$$\tilde{K}\tilde{x} = \tilde{x} - \lambda \tilde{H}\tilde{x} = \phi y$$

(4)

Let $\tilde{X}$ be a subspace of $X$. Define the mapping $\phi_0$ in $\tilde{X}$ onto $\tilde{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\phi_0 \tilde{x} = \tilde{x}$.

The differences between this work and that of reference 2 are due to differences arising in proving the following three conditions:

$$\forall \tilde{x} \in \tilde{X}, \quad \| \phi_0 \tilde{x} - \tilde{H}\tilde{x} \| \leq \zeta_1 \| \tilde{x} \|$$

(5)

$$\forall x \in X, \quad \exists \tilde{x} \in \tilde{X} \exists \| Hx - \tilde{x} \| \leq \zeta_2 \| x \|$$

(6)

$$\exists \tilde{y} \in \tilde{X} \exists \| y - \tilde{y} \| \leq \zeta_2 \| y \|$$

(7)

and in showing that $\zeta$, $\zeta_1$, and $\zeta_2$ go to zero as the mesh size goes to zero.

Conditions (5)-(7) are shown for the case of continuous kernels in reference 2. In this work, for the case of logarithmic kernels, let it be defined as:

$$\tilde{H}$$

of equation (4) be defined as:

$$\tilde{H} \equiv \Delta$$

$$\begin{bmatrix}
0 & h(t_{11}, t_{12}) \ldots h(t_{11}, t_{1n}) \ldots h(t_{11}, t_{nn}) \\
& \vdots \\
& h(t_{nn}, t_{11}) \ldots h(t_{nn}, t_{1n}) \ldots h(t_{nn}, t_{nn-1}) \\
\end{bmatrix}$$

(8)
Lemma 1: \( H \) maps \( C^0 \rightarrow C^0 \).

Proof of Lemma 1: Since \( H \) is a bounded linear operator, this result is immediate.

In \( \tilde{X} \) (a subspace of \( X \)),
\[
\tilde{K}\tilde{x} \equiv \tilde{x} - \lambda \tilde{H}\tilde{x} = Py
\]
which is the approximate equation as in reference 2, since
\[
\phi_0^{-1}\phi = P
\]

Equation (9) becomes
\[
- \lambda \tilde{H}\tilde{x} = \phi_0^{-1}\phi y
\]
Also,
\[
\phi_0(\tilde{x} - \lambda \tilde{H}\tilde{x}) = \phi y
\]
\[
\rightarrow \tilde{x} - \lambda \tilde{H}\tilde{y} = \phi y
\]
where \( \tilde{x} = \phi_0\tilde{x} \).

Lemma 2: Condition (5) is satisfied; that is,
\[
\forall \tilde{x} \in \tilde{X}, \quad ||\phi H\tilde{x} - \tilde{H}\phi_0\tilde{x}|| \leq \varepsilon_1 ||\tilde{x}||
\]

Proof of Lemma 2: For a given \( \varepsilon > 0 \), \( \exists \) an integral operator \( H_1 \) with a continuous kernel \( \phi \) (see the appendix),
\[
||H_1 - H|| < \varepsilon
\]
Since \( H_1 \) is an integral operator with a continuous kernel, the results of reference 2 are applicable. In particular,
\[
\exists \tilde{H}_1 \therefore \forall \tilde{x} \in \tilde{X}, \quad ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}|| \leq \varepsilon_1 ||\tilde{x}||
\]
Hence,
\[
||\phi H\tilde{x} - \tilde{H}_1\phi_0\tilde{x}|| \leq ||\phi H\tilde{x} - H_1\phi_0\tilde{x}|| + ||H_1\phi_0\tilde{x} - \tilde{H}_1\phi_0\tilde{x}|| + ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}||
\]
\[
\leq ||\phi|| ||H_1 - H|| ||\tilde{x}|| + ||H_1 - \tilde{H}|| ||\phi_0|| ||\tilde{x}||
\]
\[
+ ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}||
\]
\[
\leq \varepsilon_1 ||\tilde{x}||
\]
Lemma 3: Condition (6) is satisfied; that is,

\[ \forall x \in X, \; \exists \tilde{x} \in X : \|Hx - \tilde{x}\| \leq \zeta_1 \|x\| \]  \hspace{1cm} (18)

Proof of Lemma 3: From the proof of Lemma 2, we know that

\[ \forall x \in X, \; \exists \tilde{x} \in X : \|H_1 x - \tilde{x}\| \leq \zeta_1 \|x\| \]  \hspace{1cm} (19)

Again, from the proof of Lemma 2, \( \exists H_1 \) \ for \( \forall \epsilon > 0 \), \( \|H_1 - H\| < \epsilon \)  \hspace{1cm} (20)

Noting that

\[ Hx - \tilde{x} = (Hx - H_1 x) + H_1 x - \tilde{x} \]  \hspace{1cm} (21)

Now, applying Schwarz's inequality,

\[ \|Hx - \tilde{x}\| \leq \|Hx - H_1 x\| + \|H_1 x - \tilde{x}\| \]  \hspace{1cm} (22)

\[ \leq \|H - H_1\| \|x\| + \|H_1 x - \tilde{x}\| \]  \hspace{1cm} (23)

\[ \leq \zeta_1 \|x\| \]  \hspace{1cm} (24)

Condition (7) follows readily from Lemma 3.

From the proofs of Lemmas 2 and 3, it readily follows that \( \zeta_1 \) and \( \zeta_2 \) tend to zero as the mesh size of the partition tends to zero.

CONCLUSION

For a given \( \epsilon > 0 \), the solution \( x^* \) of the Fredholm integral equation 

\[ x - \lambda Hx = y \]  \hspace{1cm} on \( [0,1] \times [0,1] \) with a logarithmic kernel for continuous \( y \)

can be approximated by a function \( \tilde{x}^* \) \( \|x - \phi^{-1} \tilde{x}^*\| < \epsilon \); \( \tilde{x}^* \) satisfies an 
equation of the form \( \tilde{x} - \lambda \tilde{H} \tilde{x} = \phi y \). Hence, the desired accuracy can be achieved by appropriately restricting the mesh size \( \Delta \).
APPENDIX

CONSTRUCTION OF APPROXIMATE INTEGRAL OPERATOR $H_1$

Let $H$ be an integral operator defined as follows:

$$H[\cdot] = \int_{D} \log u[\cdot]_{PQ} dS_{Q}$$

**Lemma:** Given $\epsilon > 0$, there exists an integral operator $H_1$ with a continuous kernel $\|H_1 - H\| < \epsilon$.

**Proof:**

$$H[\cdot] = \int_{D} \log r[\cdot]_{PQ} dS_{Q}$$

$D = D_0 \cup D_{r_{\epsilon}}$

$r = |P - \bar{Q}|$

$$= \int_{D_0} \log r[\cdot]_{PQ} dS_{Q} + \int_{D_{r_{\epsilon}}} \log r[\cdot]_{PQ} dS_{Q}$$

In $D_0$, define

$$H_1[\cdot] = \int_{D_0} \log r[\cdot] dS$$

In $D_{r_{\epsilon}}$

$$H[\cdot] = \int_{D_{r_{\epsilon}}} \log r[\cdot] dS$$

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In $D_r$, define

$$H_2[ ] = \int_{D_r} \log r \, dS$$

$$= \log r \int_{D_r} [ ] \, dS$$

$$H_1[ ] - H_2[ ] = \int_{D_r} \log r \, dS_Q - \int_{D_r} \log r \, dS$$

$$= \int_{D_r} [ \log r - \log r ] \, dS_Q$$

$$\| (H - H_1)[ ] \| \leq \| \| \int_{D_r} \log r \, dS_Q - \int_{D_r} \log r \, dS$$

$$\int_{D_r} \log r \, dS = (\log r) \frac{r_c^2}{2}$$

$$| \log r - \log r_e | \, dS \leq \int \log r_e \, dS$$

$$\leq \frac{r_c^2}{2} \log r_e$$

$$\Rightarrow \| H - H_1[ ] \| \leq \frac{r_c^2}{2} \log r_e$$
REFERENCES


