NOTICE

THIS DOCUMENT HAS BEEN REPRODUCED FROM MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED IN THE INTEREST OF MAKING AVAILABLE AS MUCH INFORMATION AS POSSIBLE
A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

Ralph E. Gabrielsen and Aynur Ünal

November 1981
A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

Ralph E. Gabrielsen, Aeromechanics Laboratory
AVRACOM Research and Technology Laboratories
Ames Research Center, Moffett Field, California

Aynur Ünal, NASA Stanford Joint Institute for Aeronautics and Astronautics
Stanford, California
A NUMERICAL SOLUTION FOR TWO-DIMENSIONAL FREDHOLM
INTEGRAL EQUATIONS OF THE SECOND KIND WITH KERNELS
OF THE LOGARITHMIC POTENTIAL FORM
Ralph E. Gabrielsen and Aynur Ünal
Ames Research Center

SUMMARY

Two-dimensional Fredholm integral equations with logarithmic potential kernels are numerically solved. The explicit convergence of these solutions to their true solutions is demonstrated. The results are based on a previous work in which numerical solutions were obtained for Fredholm integral equations of the second kind with continuous kernels.

INTRODUCTION

Previously (ref. 1), the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proven. Later on (ref. 2), these results were extended to two dimensions for continuous kernels. However, since a class of physical problems involves kernels of the logarithmic potential form, this study extends the theory to kernels of this type.

MAIN DEVELOPMENT

In a recent report (ref. 2), it was shown that the following system of equations,

\[ x(t_{ij}) - \lambda \sum_{k=1}^{n} \sum_{\ell=m}^{n} h(t_{ij};t_{k\ell})x(t_{k\ell}) = y(t_{ij}) \]  

converge to the exact solution of the two-dimensional Fredholm integral equation of the second kind,

\[ x(r,s) - \lambda \int_{0}^{1} \int_{0}^{1} h(r,s;t_{1},t_{2})x(t_{1},t_{2})dt_{1} dt_{2} = y(r,s) \]

when the kernel \( h(r,s;t_{1},t_{2}) \) is a continuous function over the unit square \([0,1] \times [0,1]\) and \( y(r,s) \) is also a continuous function over the unit square.
Equation (2) will be regarded as a functional equation in the Banach space \( X = C^0 \) of continuous functions on the unit square \([0,1] \times [0,1]\) and, typically, will be expressed in the following form:

\[ Kx = x - \lambda Hx = y \quad (3) \]

The system (1) is regarded as an approximate functional equation in the space \( \tilde{X} = R^n \) and typically is expressed in the following form:

\[ \tilde{K}\tilde{x} = \tilde{x} - \lambda \tilde{H}\tilde{x} = \phi \quad (4) \]

Let \( \tilde{X} \) be a subspace of \( X \). Define the mapping \( \phi_0 \) in \( \tilde{X} \) onto \( \tilde{X} \) as follows: if \( \tilde{x} \in \tilde{X} \), then \( \phi_0 \tilde{x} = \tilde{x} \).

The differences between this work and that of reference 2 are due to differences arising in proving the following three conditions:

\[ \forall \tilde{x} \in \tilde{X}, \quad \| \phi\tilde{x} - \tilde{H}\phi_0 \tilde{x} \| \leq \zeta_1 \| \tilde{x} \| \quad (5) \]

\[ \forall x \in X, \quad \exists \tilde{x} \in \tilde{X} \exists \tilde{y} \in \tilde{X} \exists \| Hx - \tilde{x} \| \leq \zeta_1 \| x \| \quad (6) \]

\[ \exists \tilde{y} \in \tilde{X} \exists \| y - \tilde{y} \| \leq \zeta_2 \| y \| \quad (7) \]

and in showing that \( \zeta_1, \zeta_2 \), and \( \zeta_2 \) go to zero as the mesh size goes to zero.

Conditions (5)-(7) are shown for the case of continuous kernels in reference 2. In this work, for the case of logarithmic kernels,

Let \( \tilde{H} \) of equation (4) be defined as:

\[
\tilde{H} \equiv \Delta \left[ \begin{array}{cccc}
0 & h(t_{11}, t_{12}) & \ldots & h(t_{11}, t_{1n}) & \ldots & h(t_{11}, t_{nn}) \\
h(t_{12}, t_{11}) & 0 & \ldots & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
h(t_{in}, t_{i1}) & \ldots & \ldots & 0 & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
h(t_{nn}, t_{nn}) & h(t_{nn}, t_{nn}) & \ldots & h(t_{nn}, t_{nn}) & 0 & \ldots \\
h(t_{nn}, t_{nn}) & \ldots & \ldots & \ldots & 0 & \ldots \\
\end{array} \right] \quad (8)
\]
Lemma 1: \( H \) maps \( C^0 \to C^0 \).

Proof of Lemma 1: Since \( H \) is a bounded linear operator, this result is immediate.

In \( \tilde{X} \) (a subspace of \( X \)),

\[
\tilde{K}\tilde{x} = \tilde{x} - \lambda \tilde{H}\tilde{x} = \tilde{P}y
\]

which is the approximate equation as in reference 2, since

\[
\phi_0^{-1}\phi = P
\]

Equation (9) becomes

\[
\lambda \tilde{H}\tilde{x} = \phi_0^{-1}\phi y
\]

Also,

\[
\phi_0(\tilde{x} - \lambda \tilde{H}\tilde{x}) = \phi y
\]

\[
\tilde{x} - \lambda \tilde{H}\tilde{y} = \phi y
\]

where \( \tilde{x} = \phi_0\tilde{x} \).

Lemma 2: Condition (5) is satisfied; that is,

\[
\forall \tilde{x} \in \tilde{X}, \quad ||\phi\tilde{H}\tilde{x} - \tilde{H}\phi_0\tilde{x}|| \leq \epsilon ||\tilde{x}||
\]

Proof of Lemma 2: For a given \( \epsilon > 0 \), \( \exists \) an integral operator \( H_1 \) with a continuous kernel \( \phi \) (see the appendix),

\[
||H_1 - H|| < \epsilon
\]

Since \( H_1 \) is an integral operator with a continuous kernel, the results of reference 2 are applicable. In particular,

\[
\exists H_1 \land \forall \tilde{x} \in \tilde{X}, \quad ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}|| \leq \epsilon \||\tilde{x}\||
\]

Hence,

\[
||\phi\tilde{H}\tilde{x} - \tilde{H}_1\phi_0\tilde{x}|| \leq ||\phi\tilde{H}\tilde{x} - \phi H_1\tilde{x}|| + ||\tilde{H}_1\phi_0\tilde{x} - \tilde{H}\phi_0\tilde{x}|| + ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}||
\]

\[
\leq ||\phi|| \||\tilde{H}_1 - H|| \||\tilde{x}|| + \||\tilde{H}_1 - H|| \||\phi_0|| \||\tilde{x}||
\]

\[
+ ||\phi H_1\tilde{x} - \tilde{H}_1\phi_0\tilde{x}||
\]

\[
\leq \epsilon \||\tilde{x}||
\]
Lemma 3: Condition (6) is satisfied; that is,

\[ \forall x \in X, \ \exists \tilde{x} \in X : \|Hx - \tilde{x}\| \leq \zeta_1\|x\| \]  

(18)

Proof of Lemma 3: From the proof of Lemma 2, we know that

\[ \forall x \in X, \ \exists \tilde{x} \in X : \|H_1x - \tilde{x}\| \leq \zeta_1\|x\| \]  

(19)

Again, from the proof of Lemma 2, \( \exists H_1 \) for a given \( \epsilon > 0 \),

\[ \|H_1 - H\| < \epsilon \]  

(20)

Noting that

\[ Hx - \tilde{x} = (Hx - H_1x) + H_1x - \tilde{x} \]  

(21)

Now, applying Schwarz's inequality,

\[ \|Hx - \tilde{x}\| \leq \|Hx - H_1x\| + \|H_1x - \tilde{x}\| \]  

(22)

\[ \leq \|H - H_1\| \||x\| + \|H_1x - \tilde{x}\| \]  

(23)

\[ \leq \zeta_1\|x\| \]  

(24)

Condition (7) follows readily from Lemma 3.

From the proofs of Lemmas 2 and 3, it readily follows that \( \zeta, \zeta_1, \) and \( \zeta_2 \) tend to zero as the mesh size of the partition tends to zero.

CONCLUSION

For a given \( \epsilon > 0 \), the solution \( x^* \) of the Fredholm integral equation

\[ x - \lambda Hx = y \text{ on } [0,1] \times [0,1] \]  

with a logarithmic kernel for continuous \( y \)
can be approximated by a function \( \tilde{x}^* \) \( \|x - \phi_0^{-1}\tilde{x}^*\| < \epsilon; \tilde{x}^* \) satisfies an equation of the form \( \tilde{x} - \lambda H\tilde{x} = \phi y \). Hence, the desired accuracy can be achieved by appropriately restricting the mesh size \( \Delta \).
APPENDIX

CONSTRUCTION OF APPROXIMATE INTEGRAL OPERATOR $H_1$

Let $H$ be an integral operator defined as follows:

$$H[f] = \int_{D} \log u [ ]_{Q} dS_Q$$

**Lemma:** Given $\epsilon > 0$, there exists an integral operator $H_1$ with a continuous kernel such that $\|H_1 - H\| < \epsilon$.

**Proof:**

$$H[f] = \int_{D} \log r [ ]_{Q} dS_Q$$

Let $D = D_0 \cup D_{r_{c}}$

$$r = |P - Q|$$

$$= \int_{D_0} \log r [ ]_{Q} dS_Q + \int_{D_{r_{c}}} \log r [ ]_{Q} dS_Q$$

In $D_0$, define

$$H_1[f] = \int_{D_0} \log r [ ] dS$$

In $D_{r_{c}}$:

$$H[f] = \int_{D_{r_{c}}} \log r [ ] dS$$
In $D_{r_{\epsilon}}$, define

$$H_1[ ] = \int_{D_{r_{\epsilon}}} \log r_{\epsilon} \, ds$$

$$= \log r_{\epsilon} \int_{D_{r_{\epsilon}}} [ ] \, ds$$

$$\therefore H_1 - H_2 = \int_{D_{r_{\epsilon}}} \log r \, ds_Q - \int_{D_{r_{\epsilon}}} \log r_{\epsilon} \, ds_Q$$

$$= \int_{D_{r_{\epsilon}}} [ \log r - \log r_{\epsilon} ] \, ds_Q$$

$$\| (H - H_1) \| \leq \| H \|$$

$$\int_{D_{r_{\epsilon}}} \log r \, ds_Q - \int_{D_{r_{\epsilon}}} \log r_{\epsilon} \, ds_Q$$

$$\int_{D_{r_{\epsilon}}} \log r_{\epsilon} \, ds = (\log r_{\epsilon}) \frac{r_{\epsilon}^2}{2}$$

$$\left| \log r - \log r_{\epsilon} \right| \leq \int \log r_{\epsilon} \, ds$$

$$\leq \frac{r_{\epsilon}^2}{2} \log r_{\epsilon}$$

$$\therefore \| H - H_1 \| \leq \frac{r_{\epsilon}^2}{2} \log r_{\epsilon}$$
REFERENCES


