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A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

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November 1981
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INTEGRAL EQUATIONS OF THE SECOND KIND WITH KERNELS
OF THE LOGARITHMIC POTENTIAL FORM
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SUMMARY

Two-dimensional Fredholm integral equations with logarithmic potential kernels are numerically solved. The explicit convergence of these solutions to their true solutions is demonstrated. The results are based on a previous work in which numerical solutions were obtained for Fredholm integral equations of the second kind with continuous kernels.

INTRODUCTION

Previously (ref. 1), the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proven. Later on (ref. 2), these results were extended to two dimensions for continuous kernels. However, since a class of physical problems involves kernels of the logarithmic potential form, this study extends the theory to kernels of this type.

MAIN DEVELOPMENT

In a recent report (ref. 2), it was shown that the following system of equations,

\[ x(t_{ij}) - \lambda \sum_{k=1}^{n} \sum_{\ell=1}^{n} h(t_{ij}, t_{k\ell}) x(t_{k\ell}) = y(t_{ij}) \] (1)

converge to the exact solution of the two-dimensional Fredholm integral equation of the second kind,

\[ x(r,s) - \lambda \int_{0}^{1} \int_{0}^{1} h(r,s; t_{1}, t_{2}) x(t_{1}, t_{2}) dt_{1} dt_{2} = y(r,s) \] (2)

when the kernel \( h(r,s; t_{1}, t_{2}) \) is a continuous function over the unit square \([0,1] \times [0,1]\) and \( y(r,s) \) is also a continuous function over the unit square.
Equation (2) will be regarded as a functional equation in the Banach space $X = C^0$ of continuous functions on the unit square $([0,1] \times [0,1])$ and, typically, will be expressed in the following form:

$$Kx \equiv x - \lambda Hx = y$$

(3)

The system (1) is regarded as an approximate functional equation in the space $X = \mathbb{R}^n$ and typically is expressed in the following form:

$$\tilde{K}\tilde{x} \equiv \tilde{x} - \lambda \tilde{H}\tilde{x} = \phi y$$

(4)

Let $\tilde{X}$ be a subspace of $X$. Define the mapping $\tilde{\phi}_0$ in $\tilde{X}$ onto $\tilde{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\tilde{\phi}_0 \tilde{x} = \tilde{x}$.

The differences between this work and that of reference 2 are due to differences arising in proving the following three conditions:

$$\forall \tilde{x} \in \tilde{X}, \quad \|\tilde{\phi}_0 \tilde{\tilde{H}} - \tilde{H}\tilde{\phi}_0 \tilde{x}\| \leq \xi \|\tilde{x}\|$$

(5)

$$\forall \tilde{x} \in \tilde{X}, \quad \exists \tilde{x} \in \tilde{X} : \|\tilde{H}\tilde{x} - \tilde{x}\| \leq \xi_1 \|\tilde{x}\|$$

(6)

$$\exists \tilde{y} \in \tilde{X} : \|\tilde{y} - y\| \leq \xi_2 \|y\|$$

(7)

and in showing that $\xi$, $\xi_1$, and $\xi_2$ go to zero as the mesh size goes to zero.

Conditions (5)-(7) are shown for the case of continuous kernels in reference 2. In this work, for the case of logarithmic kernels,

Let $\tilde{H}$ of equation (4) be defined as:

$$\tilde{H} \equiv \Delta$$

$$\begin{pmatrix}
0 & h(t_{11}, t_{12}) & \ldots & h(t_{11}, t_{1n}) & \ldots & h(t_{11}, t_{nn}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h(t_{nn}, t_{11}) & \ldots & \ldots & 0 & \ldots & h(t_{nn}, t_{nn-1}) \\
\end{pmatrix}$$

(8)
Lemma 1: $H$ maps $C^0 \to C^0$.

Proof of Lemma 1: Since $H$ is a bounded linear operator, this result is immediate.

In $\tilde{X}$ (a subspace of $X$),
\[
\tilde{K}\tilde{x} = \tilde{x} - \lambda H\tilde{x} = Py
\]
which is the approximate equation as in reference 2, since
\[
\phi^{-1}\phi = P
\]
Equation (9) becomes
\[
\lambda H\tilde{x} = \phi^{-1}\phi y
\]
Also,
\[
\phi_0(\tilde{x} - \lambda H\tilde{x}) = \phi y
\]
\[
\tilde{x} - \lambda H\tilde{y} = \phi y
\]
where $\tilde{x} = \phi_0\tilde{x}$.

Lemma 2: Condition (5) is satisfied; that is,
\[
\forall \tilde{x} \in \tilde{X}, \quad ||\phi H\tilde{x} - H\phi_0\tilde{x}|| \leq \varepsilon ||\tilde{x}||
\]

Proof of Lemma 2: For a given $\varepsilon > 0$, there exists an integral operator $H_1$ with a continuous kernel $\phi$ (see the appendix),
\[
||H_1 - H|| < \varepsilon
\]
Since $H_1$ is an integral operator with a continuous kernel, the results of reference 2 are applicable. In particular,
\[
\exists H_1 \quad \forall \tilde{x} \in \tilde{X}, \quad ||\phi H_1\tilde{x} - H\phi_0\tilde{x}|| \leq \varepsilon ||\tilde{x}||
\]
Hence,
\[
||\phi H\tilde{x} - H\phi_0\tilde{x}|| \leq ||\phi H\tilde{x} - \phi H_1\tilde{x}|| + ||H_1\phi_0\tilde{x} - H\phi_0\tilde{x}|| + ||H_1\tilde{x} - H_1\phi_0\tilde{x}||
\]
\[
\leq ||\phi|| ||H_1 - H|| ||\tilde{x}|| + ||H_1 - H|| ||\phi_0|| ||\tilde{x}||
\]
\[
+ ||\phi H_1\tilde{x} - H_1\phi_0\tilde{x}||
\]
\[
\leq \varepsilon ||\tilde{x}||
\]
Lemma 3: Condition (6) is satisfied; that is,
\[ \forall x \in X, \quad \exists \tilde{x} \in \tilde{X} : ||Hx - \tilde{x}|| \leq \zeta_1 ||x|| \] (18)

Proof of Lemma 3: From the proof of Lemma 2, we know that
\[ \forall x \in X, \quad \exists \tilde{x} \in \tilde{X} : ||H_1x - \tilde{x}|| \leq \zeta_1 ||x|| \] (19)

Again, from the proof of Lemma 2, \( \exists H_1 \Theta \) for a given \( \epsilon > 0 \),
\[ ||H_1 - H|| < \epsilon \] (20)

Noting that
\[ Hx - \tilde{x} = (Hx - H_1x) + H_1x - \tilde{x} \] (21)

Now, applying Schwarz's inequality,
\[ ||Hx - \tilde{x}|| \leq ||Hx - H_1x|| + ||H_1x - \tilde{x}|| \] (22)
\[ \leq ||H - H_1|| ||x|| + ||H_1x - \tilde{x}|| \] (23)
\[ \leq \zeta_1 ||x|| \] (24)

Condition (7) follows readily from Lemma 3.

From the proofs of Lemmas 2 and 3, it readily follows that \( \zeta, \zeta_1, \) and \( \zeta_2 \) tend to zero as the mesh size of the partition tends to zero.

CONCLUSION

For a given \( \epsilon > 0 \), the solution \( x^* \) of the Fredholm integral equation
\[ x - \lambda Hx = y \] on \([0,1] \times [0,1]\) with a logarithmic kernel for continuous \( y \)
can be approximated by a function \( \tilde{x}^* \Theta \|x - \Phi^{-1}_0 \tilde{x}^*\| < \epsilon; \tilde{x}^* \) satisfies an
equation of the form \( \tilde{x} - \lambda \tilde{H}x = \tilde{y} \). Hence, the desired accuracy can be
achieved by appropriately restricting the mesh size \( \Delta \).
APPENDIX

CONSTRUCTION OF APPROXIMATE INTEGRAL OPERATOR $H_1$

Let $H$ be an integral operator defined as follows:

$$H[ ] = \int_D \log u[ ] \, dS$$

**Lemma:** Given $\epsilon > 0$, there exists an integral operator $H_1$ with a continuous kernel such that $\|H_1 - H\| < \epsilon$.

**Proof:**

$$H_1[ ] = \int_{D_0} \log r[ ] \, dS + \int_{D_{r_e}} \log r[ ] \, dS$$

In $D_0$, define

$$H_1[ ] = \int_{D_0} \log r[ ] \, dS$$

In $D_{r_e}$,

$$H[ ] = \int_{D_{r_e}} \log r[ ] \, dS$$
In $D_{r,E}$, define

$$H_{1} = \int_{D_{r,E}} \log r \, ds$$

$$= \log r \int_{D_{r,E}} \, ds$$

Therefore,

$$H_{1} - H_{2} = \int_{D_{r,E}} \log r_{E} \, ds_{Q} - \int_{D_{r,E}} \log r_{E} \, ds_{Q}$$

$$= \int_{D_{r,E}} [\log r - \log r_{E}] \, ds_{Q}$$

$$\|H - H_{1}\| \leq \|H\| = \int_{D_{r,E}} \log r \, ds_{Q} - \int_{D_{r,E}} \log r_{E} \, ds_{Q}$$

$$\int_{D_{r,E}} \log r_{E} \, ds = (\log r_{E}) \frac{r_{E}^{2}}{2}$$

$$|\log r - \log r_{E}| \, ds \leq \int \log r_{E} \, ds$$

$$\leq \frac{r_{E}^{2}}{2} \log r_{E}$$

Therefore,

$$\|H - H_{1}\| \leq \frac{r_{E}^{2}}{2} \log r_{E}$$
REFERENCES


