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A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

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A Numerical Solution for Two-Dimensional Fredholm Integral Equations of the Second Kind With Kernels of the Logarithmic Potential Form

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A NUMERICAL SOLUTION FOR TWO-DIMENSIONAL FREDHOLM
INTEGRAL EQUATIONS OF THE SECOND KIND WITH KERNELS
OF THE LOGARITHMIC POTENTIAL FORM

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SUMMARY

Two-dimensional Fredholm integral equations with logarithmic potential kernels are numerically solved. The explicit convergence of these solutions to their true solutions is demonstrated. The results are based on a previous work in which numerical solutions were obtained for Fredholm integral equations of the second kind with continuous kernels.

INTRODUCTION

Previously (ref. 1), the convergence of a numerical scheme for solving one-dimensional Fredholm integral equations of the second kind was proven. Later on (ref. 2), these results were extended to two dimensions for continuous kernels. However, since a class of physical problems involves kernels of the logarithmic potential form, this study extends the theory to kernels of this type.

MAIN DEVELOPMENT

In a recent report (ref. 2), it was shown that the following system of equations,

\[ x(t_{ij}) = \sum_{k=1}^{n} \sum_{\tilde{k}=1}^{n} h(t_{ij}; t_{\tilde{k}\tilde{k}}) x(t_{\tilde{k}\tilde{k}}) + \Delta = y(t_{ij}) \]  

converge to the exact solution of the two-dimensional Fredholm integral equation of the second kind,

\[ x(r,s) = \int_{0}^{1} \int_{0}^{1} h(r,s;t_1,t_2) x(t_1,t_2) dt_1 dt_2 = y(r,s) \]

when the kernel \( h(r,s;t_1,t_2) \) is a continuous function over the unit square \([0,1] \times [0,1]\) and \( y(r,s) \) is also a continuous function over the unit square.
Equation (2) will be regarded as a functional equation in the Banach space $X = C^0$ of continuous functions on the unit square $([0,1] \times [0,1])$ and, typically, will be expressed in the following form:

$$Kx \equiv x - \lambda Hx = y \quad (3)$$

The system (1) is regarded as an approximate functional equation in the space $\tilde{X} = \mathbb{R}^n$ and typically is expressed in the following form:

$$\tilde{K}\tilde{x} \equiv \tilde{x} - \tilde{\lambda} H\tilde{x} = \phi y \quad (4)$$

Let $\tilde{X}$ be a subspace of $X$. Define the mapping $\phi_0$ in $\tilde{X}$ onto $\tilde{X}$ as follows: if $\tilde{x} \in \tilde{X}$, then $\phi_0\tilde{x} = \tilde{x}$.

The differences between this work and that of reference 2 are due to differences arising in proving the following three conditions:

$$\forall \tilde{x} \in \tilde{X}, \quad \|\phi_0\tilde{x} - \tilde{H}\phi_0 \tilde{x}\| \leq \zeta \|\tilde{x}\| \quad (5)$$

$$\forall x \in X, \quad \exists \tilde{x} \in \tilde{X} \exists \|Hx - \tilde{x}\| \leq \zeta_1 \|x\| \quad (6)$$

$$\exists \tilde{y} \in \tilde{X} \exists \|y - \tilde{y}\| \leq \zeta_2 \|y\| \quad (7)$$

and in showing that $\zeta$, $\zeta_1$, and $\zeta_2$ go to zero as the mesh size goes to zero.

Conditions (5)-(7) are shown for the case of continuous kernels in reference 2. In this work, for the case of logarithmic kernels,

Let $\tilde{H}$ of equation (4) be defined as:

$$\tilde{H} \equiv \Delta \begin{bmatrix}
0 & h(t_{11}, t_{12}) & \ldots & h(t_{11}, t_{1n}) & \ldots & h(t_{11}, t_{nn}) \\
h(t_{12}, t_{11}) & 0 & \ldots & h(t_{12}, t_{1n}) & \ldots & h(t_{12}, t_{nn}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h(t_{1n}, t_{11}) & \ldots & \ldots & 0 & \ldots & h(t_{1n}, t_{nn}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
h(t_{nn}, t_{11}) & \ldots & h(t_{nn}, t_{12}) & \ldots & h(t_{nn}, t_{nn-1}) & 0
\end{bmatrix} \quad (8)$$
Lemma 1: \( H \) maps \( C^0 \to C^0 \).

Proof of Lemma 1: Since \( H \) is a bounded linear operator, this result is immediate.

In \( \tilde{X} \) (a subspace of \( X \)),

\[
\tilde{K}x = \tilde{x} - \lambda \tilde{H}x = Py
\]

which is the approximate equation as in reference 2, since

\[
\phi_0^{-1} \phi = P
\]

Equation (9) becomes

\[
\lambda \tilde{H}x = \phi_0^{-1} \phi y
\]

Also,

\[
\phi_0 (\tilde{x} - \lambda \tilde{H}x) = \phi y
\]

\[
\tilde{x} - \lambda \tilde{H}y = \phi y
\]

where \( \tilde{x} = \phi_0 \tilde{x} \).

Lemma 2: Condition (5) is satisfied; that is,

\[
\forall \tilde{x} \in \tilde{X} , \quad \| \phi Hx - \tilde{H} \phi_0 \tilde{x} \| \leq \tau \| \tilde{x} \|
\]

Proof of Lemma 2: For a given \( \epsilon > 0 \), \( \exists \) an integral operator \( H_1 \) with a continuous kernel \( \beta \) (see the appendix),

\[
\| H_1 - H \| < \epsilon
\]

Since \( H_1 \) is an integral operator with a continuous kernel, the results of reference 2 are applicable. In particular,

\[
\exists H_1 \beta \forall \tilde{x} \in \tilde{X} , \quad \| \phi H_1 \tilde{x} - \tilde{H}_1 \phi_0 \tilde{x} \| \leq \tau_1 \| \tilde{x} \|
\]

Hence,

\[
\| \phi Hx - \tilde{H}_1 \phi_0 \tilde{x} \| \leq \| \phi Hx - \phi H_1 \tilde{x} \| + \| \tilde{H}_1 \phi_0 \tilde{x} - \tilde{H} \phi_0 \tilde{x} \| + \| \phi H_1 \tilde{x} - \tilde{H}_1 \phi_0 \tilde{x} \|
\]

\[
\leq \| \phi \| \| \tilde{H}_1 - \tilde{H} \| \| \tilde{x} \| + \| \tilde{H}_1 - \tilde{H} \| \| \phi_0 \| \| \tilde{x} \|
\]

\[
+ \| \phi H_1 \tilde{x} - \tilde{H}_1 \phi_0 \tilde{x} \|
\]

\[
\leq \tau \| \tilde{x} \|
\]
Lemma 3: Condition (6) is satisfied; that is,

$$
\forall x \in X, \; \exists \tilde{x} \in X \; \exists \|Hx - \tilde{x}\| \leq \zeta_1 \|x\| \tag{18}
$$

Proof of Lemma 3: From the proof of Lemma 2, we know that

$$
\forall x \in X, \; \exists \tilde{x} \in X \; \exists \|H_1x - \tilde{x}\| \leq \zeta_1 \|x\| \tag{19}
$$

Again, from the proof of Lemma 2, $\exists \zeta_1 \; \text{for a given } \epsilon > 0,$

$$
\|H_1 - H\| < \epsilon \tag{20}
$$

Noting that

$$
Hx - \tilde{x} = (Hx - H_1x) + H_1x - \tilde{x} \tag{21}
$$

Now, applying Schwarz's inequality,

$$
\|Hx - \tilde{x}\| \leq \|Hx - H_1x\| + \|H_1x - \tilde{x}\| \tag{22}
$$

$$
\leq \|H - H_1\| \|x\| + \|H_1x - \tilde{x}\| \tag{23}
$$

$$
\leq \zeta_1 \|x\| \tag{24}
$$

Condition (7) follows readily from Lemma 3.

From the proofs of Lemmas 2 and 3, it readily follows that $\zeta, \zeta_1, \text{ and } \zeta_2$ tend to zero as the mesh size of the partition tends to zero.

CONCLUSION

For a given $\epsilon > 0,$ the solution $x^*$ of the Fredholm integral equation $x - \lambda Hx = y$ on $[0,1] \times [0,1]$ with a logarithmic kernel for continuous $y$ can be approximated by a function $\tilde{x}^* \in \|x - \phi^{-1}\tilde{x}^*\| < \epsilon ; \tilde{x}^*$ satisfies an equation of the form $\tilde{x} - \lambda H\tilde{x} = \tilde{y}.$ Hence, the desired accuracy can be achieved by appropriately restricting the mesh size $\Delta.$
CONSTRUCTION OF APPROXIMATE INTEGRAL OPERATOR $H_1$

Let $H$ be an integral operator defined as follows:

$$H[f] = \int_D \log u[ ]_Q dS_Q$$

**Lemma:** Given $\epsilon > 0$, $\exists$ an integral operator $H_1$ with a continuous kernel $\|H_1 - H\| < \epsilon$.

**Proof:**

$$H_1[f] = \int_{D_0} \log r[ ]_Q dS_Q + \int_{D_{r_\epsilon}} \log r[ ]_Q dS_Q$$

In $D_0$, define

$$H_1[f] = \int_{D_0} \log r[ ] dS$$

In $D_{r_\epsilon}$,

$$H[f] = \int_{D_{r_\epsilon}} \log r[ ] dS$$
In $D_{r_e}$, define

$$H_1[ ] = \int_{D_{r_e}} \log r[ ] \, ds$$

$$= \log r_e \int_{D_{r_e}} [ ] \, ds$$

$$= \int_{D_{r_e}} [ \log r - \log r_e ] [ ] \, ds$$

$$\therefore \|H - H_1\| \leq \|H\| \leq \int_{D_{r_e}} [ \log r - \log r_e ] \, ds$$

$$\leq \int_{D_{r_e}} \log r_e \, ds = (\log r_e) \frac{r_e^2}{2}$$

$$|\log r - \log r_e| \, ds \leq \int \log r_e \, ds$$

$$\leq \frac{r_e^2}{2} \log r_e$$

$$\therefore \|H - H_1\| \leq \frac{r_e^2}{2} \log r_e$$
REFERENCES


