NOTICE

THIS DOCUMENT HAS BEEN REPRODUCED FROM MICROFICHE. ALTHOUGH IT IS RECOGNIZED THAT CERTAIN PORTIONS ARE ILLEGIBLE, IT IS BEING RELEASED IN THE INTEREST OF MAKING AVAILABLE AS MUCH INFORMATION AS POSSIBLE
STUDY OF HIGH SPEED COMPLEX NUMBER ALGORITHMS

Contract No. NAS5-25994

National Aeronautics and Space Administration
Goddard Space Flight Center
Greenbelt, Maryland

Rodney Heisler
School of Engineering
Walla Walla College
College Place, Washington
I. INTRODUCTION

Basic to the design of reflecting antenna systems is the need to compute the far-field radiation pattern for a given antenna geometry, feed placement, and feed radiation characteristic. A simple, yet fast solution to this classic problem has proven to be a formidable challenge. This is especially true for very large reflectors with extensive current distributions and off-focus feeds with their associated asymmetry. These problems are made troublesome because of their large computational requirements.

Recent interest in this problem [1-4] has resulted in the new application of powerful mathematical techniques to produce a more time-efficient means of computing the radiation intensity. Galindo-Israel and Mittra [1] have reformulated the two-dimensional radiation integral to a double Fourier integral "representation." The Fourier integral is evaluated by expanding the integrand in a double summation of circular functions and modified Jacobi polynomials. The result is a rapidly convergent series representation of the radiation intensity, the zeroth term being $J_1(x)/x$, the pattern for a uniform, cophase aperture. Mittra, et al. [2] extend the work of the first paper to offset paraboloid reflectors and introduce a more rigorous corrective technique to compensate for the on-Fourier structure of the radiation integral.

This paper presents a new method which evaluates the radiation integral on the antenna surface. The algorithm is computationally efficient and produces the far-field radiation pattern along planar cuts at any angle.
# CONTENTS

| ABSTRACT                                      | 11 |
| INTRODUCTION                                  | 1  |
| THE THREE-DIMENSIONAL RADIATION INTEGRAL AS A|
| FOURIER TRANSFORM                            | 2  |
| AN EFFICIENT ALGORITHM FOR THE THREE-DIMENSIONAL DISCRETE FOURIER TRANSFORM | 5  |
| AN ANY ANGLE PATTERN CUT                      | 14 |
| CALCULATION OF THE INDUCED ANTENNA CURRENTS  | 16 |
| COMPUTATIONAL RESULTS                         | 17 |
| CONCLUSIONS                                   | 25 |
| ACKNOWLEDGEMENT                               | 25 |
| REFERENCES                                    | 26 |
| APPENDIX                                      | 27 |

## ILLUSTRATIONS

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>8</td>
<td>Radiation pattern for a uniform, cophase aperture by the three-dimensional DFT</td>
</tr>
<tr>
<td>9</td>
<td>Gain and phase patterns for a focal-point feed</td>
</tr>
<tr>
<td>10</td>
<td>Gain and phase patterns for an offset feed</td>
</tr>
<tr>
<td>11</td>
<td>Diagonal axis radiation pattern for a diagonally offset feed</td>
</tr>
<tr>
<td>12</td>
<td>Selected sectional cuts through the radiation pattern</td>
</tr>
<tr>
<td>13</td>
<td>Comparison of $\lambda/8$ and $\lambda/4$ sampling for an offset feed problem</td>
</tr>
</tbody>
</table>
I. INTRODUCTION

Basic to the design of reflecting antenna systems is the need to compute the far-field radiation pattern for a given antenna geometry, feed placement, and feed radiation characteristic. A simple, yet fast solution to this classic problem has proven to be a formidable challenge. This is especially true for very large reflectors with extensive current distributions and off-focus feeds with their associated asymmetry. These problems are made troublesome because of their large computational requirements.

Recent interest in this problem [1-4] has resulted in the new application of powerful mathematical techniques to produce a more time-efficient means of computing the radiation intensity. Galindo-Israel and Mittra [1] have reformulated the two-dimensional radiation integral to a double Fourier integral "representation." The Fourier integral is evaluated by expanding the integrand in a double summation of circular functions and modified Jacobi polynomials. The result is a rapidly convergent series representation of the radiation intensity, the zeroth term being $J_1(x)/x$, the pattern for a uniform, cophase aperture. Mittra, et al. [2] extend the work of the first paper to offset paraboloid reflectors and introduce a more rigorous corrective technique to compensate for the non-Fourier structure of the radiation integral.

This paper presents a new method which evaluates the radiation integral on the antenna surface. The algorithm is computationally efficient and produces the far-field radiation pattern along planar cuts at any angle.
through the three-dimensional pattern. The physical optics approximation is used to compute the induced surface current which is the input to the algorithm. The method is developed for focal-point and translated feeds and is easily extended to offset antennas and arbitrary surfaces.

On the premise that simplification and efficiency may come from first generalizing, the radiation integral is reformulated to three dimensions. The result is shown to be a triple Fourier integral. To evaluate this integral, an extremely fast algorithm is introduced which evaluates, along planes of constant $\phi$, a subset of the total three-dimensional Fourier transform results. No approximations are made other than those normally associated with digitization and the discrete Fourier transform (DFT). To further reduce the computation time, the Winograd Fourier transform algorithm (WFT) is used in place of the standard radix-2 FFT when a DFT is called for in the algorithm. The any $\phi$ angle feature of the program is implemented using a technique similar to one used in computerized tomography (cross-sectional x-rays) [5].

While other methods exist for evaluating the radiation integral, this new theory brings a fast, simple and direct approach. Due to its speed it is especially useful for very large asymmetric antennas.

II. THE THREE-DIMENSIONAL RADIATION INTEGRAL AS A FOURIER TRANSFORM

The far-field radiation intensity of a volume distribution of current may be expressed in terms of the three-dimensional radiation integral

$$E(\theta, \phi) = \iiint \tilde{K}(r', \theta', \phi') e^{-jkr' \cdot \hat{R}} \, dr'$$

(1)
The geometry for (1) is given in Figure 1. In formulating this equation, it is assumed that the phase center is at the origin, the observation point is far from the current distribution and the currents are bounded in a region dimensionally small compared to \( R \).

The \(-\hat{r}' \cdot \hat{R}\) (\( \hat{R} \) a unit vector) term is the distance from the source point to a plane through the origin and normal to \( \hat{R} \). Hence, the exponent \( k(\hat{r}' - \hat{r}' \cdot \hat{R}) \) is the total phase delay modulo \( 2\pi R \).

\[
\hat{r}' \cdot \hat{R} = r'_x w + r'_u + r'_y v
\]
\[
= r' \cos \theta \cos \phi + r' \cos \phi ' \sin \theta ' \cos \phi ' \sin \phi ' + r' \sin \phi ' \sin \phi ' \sin \phi ' \sin \phi
\]
\[
= r' \cos \theta ' \cos \phi + r' \sin \phi ' \sin \theta \cos (\phi - \phi ')
\]

The radiation integral of (1) may now be rewritten as
\[ \tilde{E}(\theta, \phi) = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \tilde{K}(r', \theta', \phi') e^{-jkr'} e^{jkr' \cos \theta' \cos \phi'} \mathrm{d}r' \mathrm{d}\theta' \mathrm{d}\phi' \]  

Equation (2) is a three-dimensional Fourier transform in spherical coordinates as presented by Bracewell [6]. An important corollary to this observation is that the far-field radiation pattern may be computed as the three-dimensional Fourier transform of a volume current distribution. Techniques will be presented for efficiently evaluating equation (2).

If the current exists only on a surface within the volume, the volume current function may be expressed in terms of a surface current as

\[ \tilde{K}(r', \theta', \phi') = \mathcal{J}(\theta', \phi') \delta(r' - \rho) \]

where \( \rho = \rho(\theta', \phi') \) defines the geometry of the surface. Substituting this expression into (2) and integrating with respect to \( r' \) (with the aid of the sifting property of the Dirac function) we arrive at

\[ \tilde{E}(\theta, \phi) = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \mathcal{J}(\theta', \phi') e^{-jkr' \cos \theta' \cos \phi'} r'^2 \sin \theta' \mathrm{d}\theta' \mathrm{d}\phi' \]

\[ e^{jk\rho \sin \theta' \sin \theta \cos (\phi' - \phi)} \]

Equation (3) agrees with equation (5) of Galindo-Israel [1]. To phrase it in exactly the same form requires two modifications. First, the phase center must be moved from the origin to a point defined by the feed position vector \( \mathcal{E} \). Secondly, the integration must be transferred from the reflector surface to the aperture plane. The latter is accomplished by use of the "Jacobian of the transformation" between the reflector surface and the projected aperture. A single component of \( \mathcal{J}(\theta', \phi') \), say \( J_x(\theta', \phi') \), may be transformed to an "equivalent" aperture distribution.
over the circular disk of the aperture. This results in the transformation

\[ J_x(\phi', \phi') \rho^2 \sin \theta ' d \theta ' d \phi ' = J_x(\phi', \phi') \sqrt{1 + (dz/ds)^2} \int ds \int d\phi ' \]

\[ = f(s, \phi') ds d\phi ' \]

\( f(s, \phi') \) is the equivalent aperture distribution and is not to be confused with the physical distribution on the aperture such as might be derived approximately by the ray technique.

Equation (3) now becomes

\[ E(\theta, \phi) = \int_{\phi=0}^{2\pi} \int_{s=0}^{a} f(s, \phi') e^{-jkp e^{j k \cos \theta ' \cos \phi}} \int ds d\phi ' \]

\[ \times e^{j k p [\sin \theta ' \sin \theta \cos (\phi - \phi ')]} ds d\phi ' \]

which is the same as (5) in Galindo-Israel [1] except for a difference in phase center. Equation (4) is not a Fourier transform as \( \exp(j k p \cos \theta ' \cos \phi) \) is a function of both source and observation coordinates.

III. AN EFFICIENT ALGORITHM FOR THE THREE-DIMENSIONAL DISCRETE FOURIER TRANSFORM

The three-dimensional Fourier transform of a volume source is defined by Bracewell [6] as

\[ F(u, v, w) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-j 2\pi (ux + vy + wz)} dx dy dz \]  

(5)

where \( u, v, \) and \( w \) are the direction cosines

\[ u = \sin \theta \cos \phi \]

\[ v = \sin \theta \sin \phi \]

\[ w = \cos \theta \]  

(6)
Note that while there appear to be three independent variables in this transform, i.e., the frequencies \( u, v, \) and \( w \), there are in fact only two geometric variables, \( z \) and \( \phi \). These may be solved for as

\[
\phi = \tan^{-1}v/u
\]

Similarly, the three-dimensional discrete Fourier transform (DFT) is given by the triple summation

\[
F(k_3, k_2, k_1) = \sum_{n_2=0}^{N_2-1} \sum_{n_1=0}^{N_1-1} \sum_{n_3=0}^{N_3-1} f(n_3, n_2, n_1)e^{-j\frac{2\pi}{N_3}n_3k_3}e^{-j\frac{2\pi}{N_1}n_1k_1}e^{-j\frac{2\pi}{N_2}n_2k_2}
\]

for \( k_1 = 0, 1, 2, \ldots, N_1-1 \)

\( k_2 = 0, 1, 2, \ldots, N_2-1 \)

\( k_3 = 0, 1, 2, \ldots, N_3-1 \)

This procedure may be broken into three operations by the use of intermediate results. Consider the intermediate transforms

\[
g(k_3, n_2, n_1) = \sum_{n_3=0}^{N_3-1} f(n_3, n_2, n_1)e^{-j\frac{2\pi}{N_3}n_3k_3}
\]

\[
n_1 = 0, 1, 2, \ldots, N_1-1
\]

\[
n_2 = 0, 1, 2, \ldots, N_2-1
\]

\[
k_3 = 0, 1, 2, \ldots, N_3-1
\]

\[
h(k_3, n_2, k_1) = \sum_{n_1=0}^{N_1-1} g(k_3, n_2, n_1)e^{-j\frac{2\pi}{N_1}n_1k_1}
\]

\[
k_1 = 0, 1, 2, \ldots, N_1-1
\]

\[
n_2 = 0, 1, 2, \ldots, N_2-1
\]

\[
k_3 = 0, 1, 2, \ldots, N_3-1
\]
From these the DFT is found to be

\[ F(k_3, k_2, k_1) = \sum_{n_2=0}^{N_2-1} h(n_3, n_2, k_1) e^{-j\frac{2\pi n_2 k_2}{N_2}} \]

(11)

\[ k_1 = 0, 1, 2, \ldots, N_1-1 \]
\[ k_2 = 0, 1, 2, \ldots, N_2-1 \]
\[ k_3 = 0, 1, 2, \ldots, N_3-1 \]

Comparing the frequency variables of the Fourier integral and the DFT, we note the correspondence

\[ u \sim \frac{k_1}{N_1 T_1}, \quad v \sim \frac{k_2}{N_2 T_2}, \quad w \sim \frac{k_3}{N_3 T_3} \]

(12)

where each \( T \) is the sampling period for the respective axes. Combining (12) and (7), we arrive at expressions for \( \theta \) and \( \phi \) in terms of the frequency indices of the DFT

\[ \theta = \sin^{-1} \left( \frac{k_2}{N_1 T_1} \right)^2 + \left( \frac{k_2}{N_2 T_2} \right)^2 = \cos^{-1} \left( \frac{k_3}{N_3 T_3} \right) \]

(13)

\[ \phi = \tan^{-1} \left( \frac{k_3}{N_2 T_2 \cdot \frac{N_1 T_1}{k_1}} \right) \]

A direct evaluation of the DFT in (8) is impractical, even for modest transform lengths. Nor does use of an FFT algorithm provide sufficient improvement. However, the computation time may be reduced by orders of magnitude in the case where one is interested in a limited amount of the total DFT information. It is possible in this way to compute a two-dimensional planer cut through the far-field radiation pattern with no compromise in accuracy. Once the algorithm is developed, it may be extended to include an any angle planer cut.

The simplification is begun by selecting \( k_1 = 0 \). From equation (13) this requires that \( \phi = 90^\circ \). Next, the value of \( k_2 \) is chosen. From (13)
this determines the polar angle \( \theta = \sin^{-1}(k_2/N_2 T_2) \) and forces \( k_3 \) to a specific value given by

\[
\left( \frac{k_3}{N_3} \right)^2 + \left( \frac{k_2}{N_2} \right)^2 = 1
\]

(14)

Clearly, there are going to be digitization problems in evaluating (14) for \( k_3 \) based on a selected integer value of \( k_2 \). This works to our advantage in decreasing the computation time.

Figure 2. Antenna Geometry for the DFT

The summation in equation (9) may now be reduced. \( k_3 \) is a determined single value and along the \( n_3 \) dimension (see Figure 2) there is only one non-zero data element for each \((n_1, n_2)\) pair. Hence (9) becomes

\[
g(k_3, n_2, n_1) = f(n_3, n_2, n_1) e^{-\frac{j 2 \pi}{N_3} n_3 k_3}
\]

(15)

\( n_1 = 0, 1, 2, \ldots, N_1 - 1 \)

\( n_2 = 0, 1, 2, \ldots, N_2 - 1 \)

\( k_3 \) a constant
Equation (15) describes a coalescing of the data to aplanar array and hence a reduction to two-dimensions as shown in Figure 3. Each point in the $n_1,n_2$ plane corresponds to a $g(k_3,n_2,n_1)$. The $f(n_3,n_2,n_1)$, (i.e. the current density) data will be present in polar form so that the complex multiplications indicated here are simple additions of $\frac{2\pi}{N_3}n_3k_3$ to the phase of $f(n_3,n_2,n_1)$. Furthermore, the phase term $\frac{2\pi}{N_3}n_3k_3$ need be calculated only once for each $n_3$ and recalled as a linear array variable for the individual phase additions.

![Figure 3. The coalesced two-dimensional array](image)

With the selection of $k_1 = 0$ comes the simplification of equation (10) to

$$h(k_3,n_2,0) = \sum_{n_1=0}^{N_1-1} g(k_3,n_2,n_1)$$

(16)

$n_2 = 0, 1, 2, \ldots, N_2-1$

$k_3$ a constant

Equation (16) describes the coalescing of a two-dimensional array to one dimension. The $g(k_3,n_2,n_1)$ data along each row are summed to form the linear array $h(k_3,n_2,0)$ in $n_2$.

The final operation is described by equation (11)
The relationship between $k_2$ and $k_3$ in equation (14) is given by:

$$ F(k_3, k_2, 0) = \sum_{n_2=0}^{N_2-1} h(k_3, n_2, 0) e^{-j \frac{2\pi}{N_2} n_2 k_2} \tag{17} $$

$$ k_2 = N_2 T_2 \sin \theta $$

$$ k_3 = N_3 T_3 \sqrt{1 - \left(\frac{k_2}{N_2 T_2}\right)^2} $$

Note that (17) is not a DFT since the sum is performed for only one value of $k_2$. It is in fact only one of the $N_2$ harmonic constituents of the DFT corresponding to one value of $\theta$ in the spectrum. In practice, one may want to compute the DFT of $h(k_3, n_2, 0)$, as digitization problems in evaluating $k_3$ make (17) applicable for many values of $\theta$ (or $k_2$) based on a single value of $k_3$.

Generally, it is not necessary to execute the above process for more than several values of $k_3$. For $NT = (2520)(1/8)$ in equation (14), $k_3 = 315$ for $k_2$ in the range of zero to 17. This requires that for $\theta$ between zero and $3.1^\circ$, the generation of the $h(k_3, n_2, 0)$ one-dimensional array by coalescing to a plane and thence to a line need be done only once. Radiation data at $\theta$ increments within this bound are found by evaluating (17) for $k_2$ between zero and 17. This may be done by computing the FFT or WFT [7] of $h(k_3, n_2, 0)$ and using only the first 18 transform results. Subsequent integer values of $k_3$ yield $\theta$ bounds of $3.1^\circ$ to $5.5^\circ$, $5.5^\circ$ to $7.1^\circ$, etc.

The three-dimensional DFT produces a spectrum in the three variables $k_1, k_2$, and $k_3$. Such a thing is difficult to represent graphically and fortunately, in this case it is not necessary as $k_1 = 0$. What results, then, is a two-dimensional spectrum or a surface in $k_2$ and $k_3$. Only a portion of this surface is of interest; that section defined by the elliptic relationship between $k_2$ and $k_3$ in equation (14). It is the spectrum
along this elliptic arc (see Figure 4) that describes the far-field radiation pattern for a constant $\phi$ cut.

The DFT described in (8) is a "one-sided" transformation, i.e., it operates on data sequences in positive time or position only. Furthermore, the DFT is cyclic and requires that the input data be periodic. The first condition leads to a necessary format of data input. The second requires a large number of zeros to be embedded in the data to adequately isolate the antenna from distant images regularly spaced threedimensionally about it. The geometric association of data to the $(n_1, n_2, n_3)$ indices is demonstrated in Figure 5. The location of the origin for the $n_1$ and $n_2$ axes is along the center line of the antenna and is a clear extension from two-dimensional DFT theory. There is no obvious point of symmetry along the $n_3$ axis to fix the three-dimensional origin. Consequently, it would be nice to discover that the choice did not affect the DFT results and could be made arbitrarily. It will be shown that this is true for the amplitude results, but not the phase.
Figure 5. Data input format for the three-dimensional DFT
Equation (15) contains the $n_3$ coordinate dependence of the DFT. A shift in the origin along the $n_3$ axis by an integer amount $\gamma$ will alter the phase term to

$$e^{-\frac{2\pi}{N_3}k_3(n_3+\gamma)} = e^{-\frac{2\pi}{N_3}k_3\gamma} e^{-\frac{2\pi}{N_3}k_3n_3} = e^{-J\beta} e^{-\frac{2\pi}{N_3}k_3n_3}$$

When this data is coalesced to a line, the resulting one-dimensional data vector will have a constant phase shift as seen from equation (16)

$$\sum_{n_1=0}^{N_1-1} f(n_3,n_2,n_1)e^{-\frac{2\pi}{N_3}n_3k_3}e^{-J\beta}$$

$$= e^{-J\beta N_1-1} \sum_{n_1=0}^{N_1-1} g(k_3,n_2,n_1)$$

$$= e^{-J\beta} h(k_3,n_2,k_1)$$

Hence, a constant phase shift, $\beta$, will be present in all the DFT terms, but the amplitude results remain unaltered. A constant phase shift is certainly not a problem. However, difficulty does arise when the algorithm is repeated for other values of $k_3$. Recall that repeated application is necessary in order to sweep through a broad range in $\theta$ and that the number of separate values of $k_3$ is determined by this $\theta$ range (three applications for $\theta$ out to 7.1°). Clearly, $\beta$ is a function of $k_3$ and the separate composite ranges will each be shifted by a different constant phase. If phase information is important, it is an easy matter to compute these phase terms and subtract them from the results.

Finally, the direction of the $n_3$ coordinate is important. The Fourier transform may be formulated with either a plus or minus exponential
phase term in the integrand. The \( n_3 \) coordinate orientation shown in Figures 2 and 5 is consistent with the negative phase form and when used with this definition produces correct transform results.

**IV. AN ANY ANGLE PATTERN CUT**

The selection of \( k_1 = 0 \) in the previous section was significant in that it allowed us to simplify the three-dimensional DFT by coalescing the planer data to a linear array. However, it also limited our consideration to the \( \phi = 90^\circ \) plane. It is imperative that we be able to examine the radiation intensity along other planer cuts through the pattern. This may be done using a projection technique similar to that used in computerized tomography [5] and multiangular scanning in gases [8].

Equation (16) indicates that the planer data (see Figure 3) be summed along rows of constant \( n_2 \) to produce a one-dimensional data vector along the \( n_2 (\phi=90^\circ) \) axis. Following the same procedure, the data may be projected to a diagonal line \( n_2' \) (see Figure 6) at an angle \( \phi \) to the \( n_1 \) axis. Conceptually, this is equivalent to a rotation of the coordinate system about the \( n_3 \) axis. The resulting data vector is then operated on by equation (17) to produce a planer cut at the new angle \( \phi \).

From Figure 6, the complex data are summed along grid tubes (of width \( T \)) perpendicular to the diagonal axis. Looking along these tubes, one observes that the data points do not lie in the centers of the new grid squares and that a few squares contain no data points, while others contain two. The first difficulty is related to quantization error which is always present when converting a continuous function to discrete data. The assumption that an interval of a continuous function may be
Figure 6. Coalescing to a diagonal line for an any angle cut
represented by a constant value is a first order type of approximation; that the constant is not the value of the function at the center point is second order.

As for the second difficulty, the concern here is that the sum along each grid tube be essentially the same as if resulting from a regular grid with one and only one value associated with each square. This will be true if the sampling rate is sufficient and if the sums are adjusted to account for any irregularity in the number of data in each tube.

V. CALCULATION OF THE INDUCED ANTENNA CURRENTS

The physical optics approximation was used to compute the antenna surface currents. This comes from applying the magnetic boundary conditions

\[ \mathbf{J} = 2\mathbf{n} \times \mathbf{H} \]  

(18)

Generally, this is considered sufficiently accurate for studies of the main beam and several side lobes. It is only necessary to determine the unit vector (\(\mathbf{n}\)) normal to the surface and the magnetic field (\(\mathbf{H}\)). The currents are derived for a paraboloid reflector with a \((-\cos \theta')^n\) offset feed and a feed displacement \(\mathbf{p}_c = (x_c, y_c, z_c)\). Assuming a unit electric polarization in the y-direction, a \(1/\rho\) space divergence of the field and a \(e^{-jkp'}\) phase delay, we determine

\[ J_x = \frac{-x'y/2f}{\sqrt{1 + (a/2f)^2x^2 + z'^2}} (-\cos \theta')^n e^{-jkp'} \]

\[ J_y = \frac{-z' + x'x/2f}{\sqrt{1 + (a/2f)^2x^2 + z'^2}} (-\cos \theta')^n e^{-jkp'} \]  

(19)

\[ J_z = \frac{y'z'/2f}{\sqrt{1 + (a/2f)^2x^2 + z'^2}} (-\cos \theta')^n e^{-jkp'} \]
The geometric variables for (19) are defined in Figure 7.

![Figure 7. Geometry of focal-point and translated feeds.](image)

VI. COMPUTATIONAL RESULTS

Performance of the new three-dimensional algorithm is verified by comparing computed results with other published work. A driver program was used to generate, on the paraboloid, a surface current of constant amplitude. Furthermore, a phase advance $e^{j2\pi n_3/T_3}$ was assigned to the current at each point so that the field at the aperture would be both uniform and cophase. This provided a test of the new algorithm with classical theory. The computed result was the expected $J_1(x)/x$ (see Figure 8).

Radiation patterns were computed for currents derived by the physical-optics approximation. The reflector is characterized by an $f/D = 0.5$, and the feed pattern as a circularly symmetric $(-\text{cose}^i)^n$. The value of $n$ was chosen to produce a $-10\text{dB}$ taper to the edge of the reflector. Figures 9(a)
and 10(a) present radiation patterns in the $\phi = 90^\circ$ plane for a focal-point feed and an offset feed ($y_c = -1.25\lambda$). The phase results are given in Figures 9(b) and 10(b). These results may be compared with Figures 3 and 4 of Galindo-Israel and Mittra [1] where amplitude and phase patterns were computed for similar antenna parameters using the Jacobi polynomial method. Careful examination reveals strong agreement between the two methods.

A more rigorous test of the algorithm is to translate the feed $1.25\lambda$ along a line bisecting the $-x$ and $-y$ axes and compute a $\phi = 45^\circ$ cut through the pattern. In this case, the feed displacement parameters are $x_c = y_c = -0.883883\lambda$ and $z_c = 0$. The resulting pattern should be essentially the same (for $\theta < 10^\circ$) as computed for $y_c = -1.25\lambda$, $x_c = z_c = 0$, and $\phi = 90^\circ$. Excellent agreement is clearly seen in Figure 11. Due to polarization, the $\phi = 45^\circ$ results fall increasingly further below the $\phi = 90^\circ$ results as $\theta$ becomes larger.

Figure 12 demonstrates the flexibility of this new algorithm. For a feed offset $y_c = -1.25\lambda$, and $x_c = z_c = 0$, pattern cuts are computed for $\phi = 90^\circ$, $60^\circ$, $30^\circ$, and $0^\circ$. Repeated application to other $\phi$ angles can adequately describe the three-dimensional nature of the radiation pattern.

By experience, the new three-dimensional algorithm was found to be computationally very fast. Timing was done on the Goddard Space Flight Center's IBM 360/91 using the interval timer available in the system library. With a very close $\lambda/8$ sampling, the cpu run time per pattern (for the general case) was approximately 25 seconds. A $\lambda/4$ sampling was found to be sufficient to study the main beam and several side lobes (see Figure 13), and ran in a mere seven seconds. Circularly symmetric current sheets may be treated as a special case and consequently, run times reduced by about 75 percent. For the cpu times cited, most of the time is
Figure 9. Gain and phase patterns for a focal-point feed.
$D/\lambda = 50$, $f/D = 0.5$, $T = \lambda/8$ and $N = 2520$. 
Figure 10. Gain and phase patterns for an offset feed.

$y_c = -1.25\lambda$, $x_c = z_c = 0$, $D/\lambda = 50$, $f/D = 0.5$,

$T = \lambda/8$ and $N = 2520$. 
Figure 11. Diagonal axis radiation pattern for a diagonally offset feed.

\( \bigcirc \bigcirc x_c = y_c = -0.883883 \lambda, 
\quad z_c = 0 \text{ and } \phi = 45^\circ. 
\bigtriangleup y_c = -1.25 \lambda, 
\quad x_c = z_c = 0, \text{ and } \phi = 90^\circ. \)
Figure 12. Selected sectional cuts through the radiation pattern. $y_c = -1.25\lambda$, $x_c = z_c = 0$, $D/\lambda = 50$, $f/D = 0.5$, $T = \lambda/8$ and $N = 2520$. 
Figure 13. Comparison of $\lambda/8$ and $\lambda/4$ sampling for an offset feed problem.

$\bigcirc - \bigcirc$ $T = \lambda/8$; $\bigtriangleup$ $T = \lambda/4$. 
consumed in computing the induced currents. The partial three-dimensional DFT results are computed in much less than half of these times.

VII. CONCLUSIONS

A method of evaluating the radiation integral on the curved surface of a reflecting antenna has been presented. The result is a two-dimensional radiation cross-section along a planer cut at any angle $\phi$ through the far-field pattern. This section is produced by evaluating the radiation integral via a three-dimensional Fourier transform. A unique feature of this method is a new algorithm for evaluating a subset of the total three-dimensional DFT results. The algorithm is extremely fast so that the computer time required to produce a radiation pattern is primarily determined by the computation of the antenna currents.

High quality gain and phase results have been computed for a paraboloid reflector with translate feed. However, the method is easily extended to offset antenna systems and reflectors of arbitrary shapes. The new method provides a direct but fast approach to the analysis of large asymmetric reflector antennas.

ACKNOWLEDGMENT

This work was supported by NASA Contract No. NAS5-25994. The author wishes to thank R. F. Schmidt of the Goddard Space Flight Center for many helpful discussions.
REFERENCES


APPENDIX

This program computes the surface currents on a parabolic
reflecting antenna as produced by an offset cosine to the nth
order and then the far-field pattern by means of a 3-D DFT.

The inputs are:
- \( \alpha \) in wavelengths
- \( \beta \) in degrees

\( \alpha = \) Y-feed offset in wavelengths
\( \beta = \) Z-feed offset in wavelengths

\( k_4 \) is a parameter that specifies the theta limit of the pattern.

**Original page is of poor quality.**
C INPUT DATA FOR QUADRANT ONE
DO 20 J=1,10
X=1.0
IF(J,EQ.1) GAMA=0.0*RAD
IF(J.GT.1) GAMA=ATAN(Y/X)
PHI=PHI-GAMA
SIGMA2=X*X+Y*Y
SIGMA=ATAN2(SIGMA2)
PROJAN=SIGMA*COS(SIGMA)
II=1+INT(PROJAN+0.5)
IF(PROJAN.LT.-0.5) II=II+INT(PROJAN-0.5)
CALL CEXP
20 CONTINUE
C INPUT DATA FOR QUADRANT TWO
IF(II.GT.X2.GT.Y) GO TO 200
DO 30 J=1,10
Y=(-(-J+1.0)
IF(J.EQ.1) GAMA=0.0*RAD
IF(J.GT.1) GAMA=ATAN(-X/Y)
PHI=PHI+GAMA
SIGMA2=X*X+Y*Y
SIGMA=ATAN2(SIGMA2)
PROJAN=SIGMA*COS(SIGMA)
II=1+INT(PROJAN+0.5)
IF(PROJAN.LT.-0.5) II=II+INT(PROJAN-0.5)
CALL CEXP
30 CONTINUE
GO TO 200
C INPUT DATA FOR QUADRANT FOUR
DO 40 J=1,10
X=-1.0
IF(J.EQ.1) GAMA=0.0*RAD
IF(J.GT.1) GAMA=ATAN(Y/X)
PHI=PHI+GAMA
SIGMA2=X*X+Y*Y
SIGMA=ATAN2(SIGMA2)
PROJAN=SIGMA*COS(SIGMA)
II=1+INT(PROJAN+0.5)
IF(PROJAN.LT.-0.5) II=II+INT(PROJAN-0.5)
CALL CEXP
40 CONTINUE
C INPUT DATA FOR QUADRANT THREE
IF(II.LT.X2.GT.Y) GO TO 200
DO 70 J=1,4*I*X5,11
X=-(I*-1)*.0)
GAMMA*ATAN(Y/Y)
PSI=AR*GAMMA
SIGMA=SQR(SIGMA2)
PROJAN=SIGMA*GAM(S(PSI)
11=INT(PROJAN+0.5)+1
IF(PROJAN+LT.0.5) 11=INT(-PROJAN+0.5)+1
CALL KCOMP
70 CONTINUE
200 CONTINUE
DO 250 I=1,N
IF (ICOUNT(I).EQ.0) GO TO 250
RATIO=DFLOAT(I)*ITX(I))/DFLOAT(ICOUNT(I))
SIGMA(I)=SIGMA(I)*RATIO
SIGMA(I)=SIGMA(I)*RATIO
250 CONTINUE
CALL ZOODFT(N)
CALL FB.OUT
K3=3-1
IF(K3.63.3) GO TO 1
CALL RKT100(ICPU2)
ICPU3=ICPU-ICPU2
ICPY=ICPY+1
FORMAT(1H1.5+ICPU=110)
STOP
END
SUBROUTINE KCOMP
REAL*8 SIGMA, SUMR
COMMON/XDATA/5040, SUMR(5040)
COMMON/RK411, K1, K2, XE, YE, Y2, SIGMA2, F, KLIMIT, PI
PHASE3(1001), ICOUNT(5040)
COMMON N, T
F2=2.0*F
K=K-LIMIT-INT((SIGMA2/(4.0*F)+0.5)
XPH=X*YE
YP=Y-YE
ZP=F-KLIMIT+X+YE
XP2=X*P*XP
YP2=YP*YP
ZP2=ZP*ZP
RE-=COS(T(XP2+YP2+ZP2)
COT=XP/YP
AP=SIGMA2/(F*F)
DP=SNRT(SIGMA2+XP2+YP2+ZP2)
Q=SNRT((1.0-A*XP2)/2)
XPHASE=-2.0*AP
5SNRT(Q*XP)*1
XPH=XPAG*5N1(XPHASE)
S1M2111=SUMR(I1)+X*E
S1M2111=SUMR(I1)+X*F
ICOUNT(I1)=ICOUNT(I1)+1
RETURN
END
SUBROUTINE ZOODFT(N)
C THE SUBROUTINE ZOODFT COMPUTES A LENGTH N DFT OF THE INPUT DATA WHICH IS 1
C ORDER FACTORS FOR TRANSFORMS OF LENGTH M1
M1 = M1
M2 = M2
M3 = M3
M4 = M4
GO TO 20
10 GO TO (12, 13, 14), MS
C ORDER FACTORS FOR TRANSFORMS OF LENGTH M2
12 M1 = M2
M2 = M1
M3 = M3
M4 = M4
GO TO 20
C ORDER FACTORS FOR TRANSFORMS OF LENGTH M3
13 M1 = M3
M2 = M1
M3 = M3
M4 = M4
GO TO 20
C ORDER FACTORS FOR TRANSFORMS OF LENGTH M4
14 M1 = M4
M2 = M1
M3 = M2
M4 = M4
C INDEXING INITIALIZATION FOR THE TRANSFORMS
20 N2 = 0
M3 = 0
M4 = 0
K1 = M1 * M2 * M3 * M4
K2 = M1 * M2 * M3 * M4
K3 = M1 * M2 * M3 * M4
K4 = M1 * M2 * M3 * M4
I(1) = 1
C INPUT INDEXING ALONG ONE DIMENSION
21 DO 22 J = 2, MM1
I(J) = I(J-1) + K
10 (I(J), J, 1, 22)
22 END
30 DO 31 J=1,MM1
   1.J=1(J)+1
   UR(J)=XR(J,1)
31   U(J)=X(J,1)
C TRANSFROM 1R3:31
   GO TO(50,200,300,400,500,700,800,900,50,50,50,50,50,50,1400),MM
   21
C PLACE RESULTS OF TRANSFORM BACK IN XR AND XI
   40 DO 41 J=1,MM1
      1.J=1(J)+1
      XR(J,1)=UR(J)
   41   X(J)=U(J)
C TESTING FOR COMPLETION OF THIS FACTOR'S TRANSFORMS
   IF(N2.NE.MM2-1) GO TO 51
      N2=0
   IF(N3.NE.MM3-1) GO TO 52
      N3=0
   IF(N4.NE.MM4-1) GO TO 53
      N4=N4-1
   IF(F.EQ.0) GO TO 1000
   GO TO 10
C INPUT INDEXING ALONG OTHER DIMENSIONS
   51 N2=N2+1
   DO 54 J=1,MM1
      1.(J)=1.(J)+1
      IF(1.(J).LT.N) GO TO 54
      1.(J)=1.(J)-N
   54 CONTINUE
   GO TO 30
   52 N3=N3+1
      1.(1)=<3*x;3<4*M4
   IF(1.(1).LT.N) GO TO 21
      1.(1)=1.(1)-N
   GO TO 21
   53 N4=N4+1
      1.(1)=<4*M4
   GO TO 21
C UNSCRAMBING TRANSFORM RESULTS
1000  1=1
   J=1
   GO TO 1001
1001  IF(J.GT.N) GO TO 1003
      1=1+<UNIT
   1004  IF(1.LT.N) GO TO 1001
      1=1-1
   GO TO 1004
1002  A(J)=X*(1)
   B(J)=-X*(1)
   J=J+1
   GO TO 1002
C 2 POINT TRANSFORM
200 U(X)=U*(1)+UR(2)
   UR(1)=U*(1)+U(2)
   UR(2)=UR(1)+UR(2)
   U(2)=U*(2)+U(2)
   U(1)=U*X
   U(1)=U*X
   GO TO 40
C 3 POINT TRANSFORM
300 AR=UR(2)+UR(3)
   A1=U(2)+U(3)
   X=1=-1.5000*A1
\[
\begin{align*}
\text{A}[1] & = \text{A}[2] + \text{A}[4] \\
\text{A}[2] & = \text{A}[3] + \text{A}[4] \\
\text{A}[3] & = \text{A}[4] + \text{A}[5] \\
\text{A}[4] & = \text{A}[5] + \text{A}[6] \\
\text{A}[5] & = \text{A}[6] + \text{A}[7] \\
\text{A}[6] & = \text{A}[7] + \text{A}[8] \\
\text{A}[7] & = \text{A}[8] + \text{A}[9] \\
\text{A}[8] & = \text{A}[9] + \text{A}[10] \\
\text{A}[9] & = \text{A}[10] + \text{A}[11] \\
\text{A}[10] & = \text{A}[11] + \text{A}[12] \\
\text{A}[11] & = \text{A}[12] + \text{A}[13] \\
\text{A}[12] & = \text{A}[13] + \text{A}[14] \\
\end{align*}
\]
ORIGINAL PAGE IS OF POOR QUALITY
1003 READS
END
SUBROUTINE RSTOUT
S=1,2,C,2,A,2,C,2,R,2,S
C=(340)/(878(8340)),S(5040)
C=(340)/(878(8340)),S(5040)
C=340,S,7
I=ST65-1,START,STOP
S(1),6=1(4),(1)x=2
O=57,28578
I+(<3,3,315)GO TO 101
C(126x(11)x=2x(1))x=2
ST4=1
ST5=14
GO TO 200
101 I=(C,3,3,314)GO TO 102
START=14
STOP=3
GO TO 200
102 I=(C,3,3,3,13)GO TO 103
START=39
STOP=40
GO TO 200
103 I=(C,3,3,3,312)GO TO 104
START=61
STOP=47
GO TO 200
104 I=(<3,3,3,3,311)GO TO 105
START=58
STOP=56
GO TO 200
105 I=(<3,3,3,3,310)GO TO 80
START=65
STOP=59
GO TO 200
200 GO TO 80 I=START,STOP
S=(4,1,0)/(35)
S=1,2,5-<A(1)<4,30,F-
I+6(1,1,1,1,1)PHA=PHA+1x8,0
I+6(1,55,1,0)GO TO 60
S=1,0-S=S