Development of a Simplified Procedure
For Thrust Chamber Life Prediction

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An analytical design procedure for predicting thrust chamber life considering cyclically induced thinning and bulging of the hot-gas-wall is developed. The hot-gas-wall, composed of ligaments connecting adjacent cooling channel ribs and separating the coolant flow from the combustion gas, is subjected to pressure loading and severe thermal cycling. Thermal transients during start-up and shut-down cause plastic straining through the ligaments. The primary bending stress superimposed on the alternate in-plane cyclic straining causes incremental bulging of the ligaments during each firing cycle. This basic mechanism of plastic ratcheting is analyzed and a method developed for determining ligament deformation and strain. The method uses a yield surface for combined bending and membrane loading to determine the incremental permanent deflection and progressive thinning near the center of the ligaments which cause the geometry of the ligaments to change as the incremental strains accumulate. Fatigue and tensile instability are affected by these local geometry changes. Both are analyzed and a failure criterion developed. Results of the simplified analyses are shown to compare favorably with experimental data and finite element analysis results for OFHC (Oxygen Free High Conductivity) copper. They are also in reasonably good agreement with experimental data for NARloy Z, a copper-zirconium-silver alloy developed by the Rocketdyne Division of Rockwell International.
DEVELOPMENT OF A SIMPLIFIED PROCEDURE FOR THRUST CHAMBER LIFE PREDICTION

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DEVELOPMENT OF A SIMPLIFIED PROCEDURE  
FOR THRUST CHAMBER LIFE PREDICTION  

SUMMARY

An analytical design procedure for predicting thrust chamber life considering cyclically induced thinning and bulging of the hot-gas-wall is developed. The hot-gas-wall, composed of ligaments connecting adjacent cooling channel ribs and separating the coolant flow from the combustion gas, is subjected to pressure loading and severe thermal cycling. Thermal transients during start-up and shut-down cause plastic straining through the ligaments. The primary bending stress superimposed on the alternate in-plane cyclic straining causes incremental bulging of the ligaments during each firing cycle. This basic mechanism of plastic ratcheting is analyzed and a method developed for determining ligament deformation and strain. The method uses a yield surface for combined bending and membrane loading to determine the incremental permanent deflection and progressive thinning near the center of the ligaments which cause the geometry of the ligaments to change as the incremental strains accumulate. Fatigue and tensile instability are affected by these local geometry changes. Both are analyzed and a failure criterion developed. Results of the simplified analyses are shown to compare favorably with experimental data and finite element analysis results for OFHC (Oxygen Free High Conductivity) copper. They are also in reasonably good agreement with experimental data for NARloy Z, a copper-zirconium-silver alloy developed by the Rocketdyne Division of Rockwell International.
1.0 INTRODUCTION

Life predictions of regeneratively liquid cooled rocket thrust chambers have been based on low cycle fatigue principles. Tests of thrust chambers, however, [1, 2] revealed that coolant channel walls in the failed areas have exhibited progressive incremental thinning and bulging during the heating and cooling cycle associated with each firing. Failure analyses performed as a part of this work indicate that ductile rupture is a more limiting mode of failure than fatigue failure. Thus, material tensile instability was considered in addition to fatigue damage mechanisms in the present structural evaluation.

The mechanism of hot-gas-wall or ligament incremental distortion can be investigated by inelastic finite element analyses. For the cyclic histories of interest, however, such analyses are difficult to perform and require extensive computer time. Their use for a single characteristic case is feasible and provides valuable information. In fact, such analyses were carried out within NASA-Lewis research programs for simplified histories of thermal loading. Such methods, however, are not suitable for evaluating the effects of changing various design and operating parameters such as geometric configurations, material properties, pressures and temperatures. More general methods which do not require lengthy inelastic finite element solutions are therefore needed.

The present work is aimed at: (1) analyzing the failure mode of the cyclically loaded thrust chamber, (2) developing simplified but conservative methods for evaluating the strains

*Numbers in brackets designate references at end of report.
and deformations, and (3) providing methods for predicting the cyclic life of the thrust chamber.

One or two cycles of elastic and inelastic analysis were analyzed to determine stress and strain resultants for the simplified models. Other available inelastic finite element results and test data were also used to determine the accuracy of the simplified models.

The scope of the work is as follows:

1. Develop a simplified analytical model for determining hot-gas-wall deformation and strain.

2. Determine accuracy of simplified model by comparison with inelastic finite element analysis results.

3. Develop a failure mode evaluation criterion.

4. Provide a simple design procedure for making life predictions for thrust chambers.

Each of these items will be discussed in the sections to follow. Also the assumptions made in the development of the analysis procedure for each item are discussed in their respective sections.

2.0 PLUG NOZZLE THRUST CHAMBER

The plug nozzle thrust chamber shown in Figure 1 is considered. Figure 1 provides dimensions for the thrust chamber tested at NASA Lewis Research Center. The sample solution and computer finite element calculations given in the Appendices were performed for this thrust chamber. However, the resulting simplified method of analysis is applicable for a general case where the geometry, loading conditions and materials can be altered to comply with the design of a specific engine. The
FIGURE 1  PLUG NOZZLE THRUST CHAMBER
plug nozzle assembly consisting of the contoured centerbody and flanged cylinder is shown along with cross sectional details of the cylinder. The inner liner of the cylinder contains axial flow coolant channels of constant cross section and is constructed from either OFHC copper, half-hard Amzirc (American Metal Climax, Inc. copper alloy) or NARloy-Z. The closeout wall is electroformed copper in all cases. Since no failures occurred in the centerbody only the cylindrical part of the nozzle was investigated.

Typical firing cycle history data for the thrust chamber is illustrated in Figure 2. The data refer to the critical section where failures were observed during testing. Only this section was considered in the analysis. The first cycle begins with a cooling period when the liquid hydrogen enters the channels. The uniform temperature 29°K (53°R) is reached before a sudden rise of temperature beginning at the ignition point of 1.5 seconds.

The mechanical properties used to characterize the cylinder wall materials were provided by NASA-Lewis. Reference 3 provides temperature dependent thermal expansion, modulus of elasticity and static stress-strain properties of half-hard Amzirc, NARloy-Z, annealed OFHC and EFCU (electroformed copper).

3.0 DEVELOPMENT OF A SIMPLIFIED ANALYTICAL MODEL FOR CALCULATING HOT-GAS-WALL DEFORMATION AND STRAIN

3.1 Simplified Model

The inner wall shown in Figure 1 separating the coolant flow and the combustion gas is subjected to severe thermal cycling. The temperature difference between the closeout and inner walls that occur during the thermal transient causes plastic straining through the ligament within the
FIGURE 2 THE TIME HISTORIES OF HEAT TRANSFER COEFFICIENTS AND TEMPERATURES FOR A CHILLDOWN AND FIRING CYCLE
range of 1 to 2 percent. The temperature history for the closeout and inner walls at the middle plane of the cooling channel is shown for one cycle in Figure 3. The inner wall is attached to the closeout wall by radial ribs. The outer wall remains elastic within the entire thermal cycle forcing the inner wall ligaments into captive plastic straining. Straining in the hoop direction is accompanied by high axial strains.

The ligaments separating the coolant and the combustion gas are subjected to coolant pressure load and thermally induced through-the-wall bending. The resulting stresses interact during plastic straining causing incremental bulging or ratcheting directed radially inward. The mechanism of pressure induced ratcheting is illustrated in Figure 4. Primary bending stresses persist during alternate in-plane cyclic straining resulting in a small repetitive incremental permanent deflection of the ligaments during each firing cycle. The simple model in Figure 4 does not include interaction of shear stress due to pressure and the thermally induced bending. Their effects are evaluated later in the report.

3.2 Analysis of Basic Mechanism of Plastic Ratcheting

The approach used to determine increments of plastic strain within the load cycles is explained in Figure 5. The axial forces and bending moments acting on the rectangular cross section as shown in Figure 5a are considered. The bar simulates the response of the ligament of a cooling channel between the two ribs. The bar is cyclically stretched and squeezed plastically in the presence of sustained bending. Note that the "X" direction in the beam model corresponds to the hoop direction in the ligament.
FIGURE 3 MIDPLANE TEMPERATURE HISTORY
CLOSE-OUT WALL

FIGURE 4  PLASTIC RATCHETING DUE TO INTERACTION OF PRESSURE AND THERMAL CYCLING

Inner Wall

Cooling Channel

Rib

Ligament

2H

$q_d$

$p$

$\pm \Delta \varepsilon_1$

(Hoop Strain)

$\pm \Delta \varepsilon_2$

(Axial Strain)
X - Hoop Direction
Y - Axial Direction

![Diagram showing plastic ratchet increments in a bar of rectangular cross section subjected to sustained bending moment and cyclic axial strain]

**FIGURE 5** PLASTIC RATCHET INCREMENTS IN A BAR OF RECTANGULAR CROSS SECTION SUBJECTED TO SUSTAINED BENDING MOMENT AND CYCLIC AXIAL STRAIN

a) Ligament Beam Element Loading
b) Interaction Curve
To obtain a conservative bound on the plastic strain increments, strain hardening of the material is ignored.

The yield curve for a beam subjected to bending and axial force is composed of two parabolas as shown in Figure 5b and is defined by the relation, [4]:

\[ F = m + n^2 - 1 = 0 \]  

(1)

where \( m \) and \( n \) are dimensionless variables defined by:

\[ m = \frac{M}{M_o} \]  

(2a)

\[ n = \frac{N}{N_o} \]  

(2b)

In the above, \( M \) and \( N \) denote the bending moment and axial force, while \( M_o \) and \( N_o \) denote the yield bending moment and yield axial force given by:

\[ M_o = H^2 S_Y \]  

(3a)

\[ N_o = 2H S_Y \]  

(3b)

for a rectangular beam of unit width, height \( 2H \) and yield stress \( S_Y \).

Plastic flow vectors are normal to the yield curve. The curve is plotted using generalized stresses \( n \) and \( m \) related to axial and bending tractions, respectively. The associated generalized strain rates \( \dot{\epsilon} \) and \( \dot{\epsilon} \) are defined by the average hoop strain rate \( \dot{\epsilon} \) and the rate of curvature of the bent bar, \( \dot{k} \), respectively, as shown in Figure 5. If hoop increments of plastic strain are known the corresponding increment of curvature is defined by the slope of the yield surface and can be determined (based on the normality
of resulting strain rate vector $\dot{q}$) as shown in Figure 5b.

The generalized strain rates can be obtained by partial differentiation of equation (1) as follows:

$$\dot{\lambda} = A \frac{3F}{3n} = 2nA \quad (4a)$$

$$\dot{\theta} = A \frac{3F}{3m} = A \quad (4b)$$

where $A$ is an arbitrary positive scalar. Thus, for the interaction of hoop and bending loads, the relation between $\dot{\lambda}$ and $\dot{\theta}$ components of the $\dot{q}$ vector is simply:

$$\frac{\dot{\theta}}{\dot{\lambda}} = \left(\frac{\theta}{\lambda}\right) = \frac{1}{2n} \quad (5)$$

For known values of bending $m$ which remain constant within the cycle, $n$ can be obtained from the yield line of equation (1). Hoop increments of plastic strain are being reversed within two halves of the thermal cycle. However, plastic increments of curvature within each half of the thermal cycle are of the same sign. Thus, they accumulate causing incremental bulging of the cooling channel wall.

3.3 Interaction of Shear Stress

For high pressures in the cooling channels the model of the thrust chamber must also include shear stress $\tau$. The method of solution is analogous. However, the solution includes generalized shear stress and the corresponding shear strain.

The yield surface based on Tresca's yield criterion is given by Peterson et al.

$$F = m + \frac{n^2}{k} - k = 0 \quad (6a)$$
where

\[ k = \sqrt{1 - s^2} \quad (6b) \]

and \( s = \frac{2r}{S_Y} \) is the dimensionless shear stress.

The generalized strains \( \lambda, \theta \) and \( \phi \) are related to the hoop strain, curvature and shear strain, respectively. Using the normality law:

\[ \dot{\lambda} = A \frac{\partial F}{\partial n} = 2An \quad (7a) \]

\[ \dot{\theta} = A \frac{\partial F}{\partial m} = A \sqrt{1 - s^2} \quad (7b) \]

\[ \dot{\phi} = A \frac{\partial F}{\partial s} = A \left( 2s - \frac{ms}{\sqrt{1 - s^2}} \right) \quad (7c) \]

The proportions between the strain components are thus:

\[ \frac{\partial F}{\partial m} : \frac{\partial F}{\partial n} : \frac{\partial F}{\partial s} = \left( \sqrt{1 - s^2} \right) : (2n) : \left( 2s - \frac{ms}{\sqrt{1 - s^2}} \right) \quad (8) \]

The generalized variables \( \lambda, \theta \) and \( \phi \) can be related to the hoop strain, curvature and shear strain by considering the rate of dissipation \( D \) given by [4]:

\[ D = M \ddot{\kappa} + N \ddot{\epsilon} + S \ddot{\gamma} \quad (9) \]

where \( M, N \) and \( S \) denote the bending moment, hoop force and shear force, respectively, and \( \ddot{\kappa}, \ddot{\epsilon} \) and \( \ddot{\gamma} \) denote the curvature rate, hoop strain rate and shear strain rate, respectively.
Using equation (2), equation (9) can be written as:

\[ D = m \ddot{M} + N_0 \dot{\varepsilon} + sS_0 \dot{\gamma} \]  
(10a)

where

\[ s = \frac{S}{S_0} = \frac{2\tau}{S_y} = \frac{s}{H} \]  
(10b)

since the cross sectional area for unit width equals 2H.

Then using (3), (10a) and (10b):

\[ D = S_y [m(H^2\ddot{k}) + n(2\dot{H} \dot{\varepsilon}) + s(\dot{H} \dot{\gamma})] \]  
(11a)

Also in terms of the generalized strain rates, the rate of dissipation is given by:

\[ D = S_y [m \dot{\lambda} + n \dot{\lambda} + s \dot{\phi}] \]  
(11b)

Thus the relationship between the generalized strain rates and the hoop strain rate, curvature rate and shear strain rate are, respectively:

\[ \dot{\lambda} = 2H \dot{\varepsilon} \]  
(12a)

\[ \dot{\phi} = H^2 \dot{k} \]  
(12b)

\[ \dot{\phi} = H \dot{\gamma} \]  
(12c)

For known increments of hoop strain \( \lambda \) and known values of \( m \) and \( s \), the remaining components of strain \( \theta \) and \( \phi \) (shear) are determined by the ratios (8). Since the load point must remain on the yield surface, the known values of \( m \) and \( s \) determine \( n \) which is obtained from equation (6a).
The curvature and shear strain as determined from (12) can then be integrated along the length of the ligament to obtain the bending and shear deflections. Once the total deflection is known, the corresponding thinning in the ligament is calculated by assuming that the volume of the material remains constant. Integration of the deflection and determination of ligament thinning is described in Section 3.4.

It can be shown from the generalized yield surfaces derived in Ref. [5] that though the yield surface given by equation (6a) is derived for a uniaxial model, it also holds for biaxial conditions when \( n_2 < n_1 \) (where \( n_1 = n = \) hoop stress variable in equation (6a)) and \( m \) and \( s \) are small, as in the present case. Thus the generalized strain relationships derived above are also applicable for the biaxial case.

### 3.4 Deformation History

From equations (7a), (7b) and (12a), (12b):

\[
\frac{\dot{\theta}}{\lambda} = \frac{\theta}{\lambda} = \frac{\sqrt{1-s^2}}{2n} = \frac{\pi R}{2 \varepsilon} = \frac{H R}{2 \varepsilon} \tag{13}
\]

For the complete cycle, the plastic strain generated by the temperature difference during heat-up in the hoop direction is fully reversed when the temperature becomes uniform at the end of the cycle. Thus:

\[
\varepsilon = \varepsilon_1 = 2 \left( \Delta \varepsilon_{P1} \right) \tag{14}
\]

where \( \Delta \varepsilon_{P1} \) is given by sum of plastic strain range due to differential thermal expansion \( \Delta \varepsilon_{P1} \) and the correction due to thermally induced bending \( \Delta \varepsilon_{B1} \) which are evaluated in Sections 3.5 and 3.6, respectively.

Thus, the curvature is given by:
Similarly from equations (7a), (7c) and (12a), (12c):

$$\frac{\phi}{\lambda} = \frac{\phi}{\lambda} = \left(2s - \frac{ms}{\sqrt{1-s^2}}\right) / 2n = \gamma / \epsilon = \gamma / \epsilon$$  \hspace{1cm} (16)

Making use of equation (14), the shear strain is then:

$$\gamma = 2 \left(2s - \frac{ms}{\sqrt{1-s^2}}\right) \frac{(\Delta \epsilon_p)}{n}$$  \hspace{1cm} (17)

The curvature and shear strain as determined from equations (15) and (17) for each cross section can then be integrated along the length of the ligament to obtain the bending and shear deflections. Once the total deflection is known, the corresponding thinning in the ligament is calculated by assuming that the volume of the material remains constant.

Experimental evidence shows that the deformed shape of the ligament can be approximated by a linear variation in thickness as shown in Figure 6. If \( \delta \) denotes the deflection per cycle, the thinning after \( N \) cycles is then given by:

$$t_N = \frac{N \delta w}{(t + w)}$$  \hspace{1cm} (18)

Note that even though the incremental deflection remains constant, the incremental strain at the center of the ligament increases with each cycle due to ligament thinning.
t(x) = \left( t_{\text{max}} - t_{\text{min}} \right) \frac{2x}{\ell} + t_{\text{min}}

FIGURE 6  LIGAMENT LINEAR THINNING MODEL
3.5 Stress and Strain Resultants

The analysis of incremental plastic deformation involves generalized stresses $m$, $n$ and $s$ and the associated generalized strains. The generalized moment and shear and the generalized strain in the hoop direction are needed as input data for the solution. Their numerical values can be obtained by crude hand calculations using results of the thermal analysis and elastic finite element calculation or by performing the full inelastic analysis of the first two load cycles.

Hand Calculations

The bending moment distribution along the beam can be approximated by the solution for a clamped beam under uniformly distributed load:

$$M = \frac{-pL^2}{2} \left( \frac{1}{6} - \frac{x}{l} + \frac{x^2}{l^2} \right)$$

where $p$ is the unit load, $l$ is the length of the beam and $x$ is the axial coordinate measured from the rib as shown in Figure A-1 of Appendix A.

Shear force is simply:

$$S = \frac{pL}{2} - px$$

The average hoop strain range, $\Delta \varepsilon_1^p$, in the ligament is evaluated based on the differential thermal expansion of the ligament at average temperature $T_1$ and the closeout wall at average temperature $T_0$. Since the thickness of the ligament is small with respect to the thickness of the closeout wall, the elastic deformation of the closeout wall can be disregarded and the range of plastic strain in the hoop direction is:
where \( \alpha_i \) and \( \alpha_o \) are the thermal expansion coefficients of the ligament and closeout, respectively, and \( (T_i \alpha_i - T_o \alpha_o)_{\text{max}} \) and \( (T_i \alpha_i - T_o \alpha_o)_{\text{min}} \) are the maximum and minimum thermal strains, respectively, that occur during the loading cycle. \( S_{y_{\text{max}}} \) and \( S_{y_{\text{min}}} \) are respectively the ligament material absolute yield strengths corresponding to the ligament average temperatures at the times in the cycle when \( (T_i \alpha_i - T_o \alpha_o)_{\text{max}} \) and \( (T_i \alpha_i - T_o \alpha_o)_{\text{min}} \) are calculated.

The average temperature of the ligament can be assessed assuming steady heat transfer conditions between the hot gas at adiabatic temperature and the evaporating hydrogen in the cooling channel.

Using a one dimensional heat flow model and assuming that heat enters the metal through the inner surface of the thrust chamber and is being received by two sides of the cooling channel as shown in Figure 7, the resulting temperature of the sides of the ligament are:

\[
\Delta \varepsilon_i = \left[ (T_i \alpha_i - T_o \alpha_o)_{\text{max}} - (T_i \alpha_i - T_o \alpha_o)_{\text{min}} \right] - \left( S_{y_{\text{max}}} + S_{y_{\text{min}}} \right) / E
\]

\[
T_2 = \left( \frac{4H}{k} \frac{1}{A + A_1 + A_2 + h_1 A_1 + h_2 A_2} \right) \frac{h H_k T_1 + T_4}{1 + h_2 A_2} (22a)
\]

\[
T_3 = T_4 + \frac{h H_k}{h_1 A_1 + h_2 A_2} \left[ T_1 - \left( \frac{4H}{k} \frac{1}{A + A_1 + A_2} \frac{1}{h_1 A_1 + h_2 A_2} \right) \frac{h H_k T_1 + T_4}{1 + h_2 A_2} \right] (22b)
\]

where \( h \) and \( A \) denote heat transfer coefficients and surface areas as shown in Figure 7.
One Dimensional Heat Flow Model:

\[ h_H A_H (T_1 - T_2) = \frac{k}{2H} \left( \frac{A_H + A_1 + A_2}{2} \right) (T_2 - T_3) = (h_1 A_1 + h_2 A_2) (T_3 - T_4) \]

**FIGURE 7 THERMAL MODEL OF LIGAMENT**
Thus the average temperature of the ligament is:

$$T_i = \frac{T_2 + T_3}{2}$$  \hspace{1cm} (23)

A certain part of heat bypasses the cooling channel and heats the closeout wall, thus reducing the differential straining. The temperature of the closeout wall is approximately proportional to the average temperature of the ligament:

$$T_o = \bar{A} T_i$$  \hspace{1cm} (24)

The approximate value of constant $\bar{A}$ for the considered thrust chamber is 0.35. This value for $\bar{A}$ was obtained for a specific geometry and will probably be different for other geometries.

The temperatures $T_2$ and $T_3$ can also be used to determine the thermally induced bending stress through the ligaments which for equibiaxial stress field is:

$$\sigma_b = \frac{\Delta T \mathrm{a}E}{2(1 - \nu)}$$  \hspace{1cm} (25)

where $\Delta T$ is the temperature difference through the ligament.

**Finite Element Thermal and Elastic Analyses**

A more accurate evaluation of stress and strain resultants may be obtained by performing thermal and stress analyses for the extremal conditions using the finite element method. The formulation of the problem and instructions to prepare the input data for the MARC program are given in Appendix B. Selection of thermal steps is optimized by the program. The stress run should be performed for the instants resulting in the maximum range of
stress. The calculation of bending and shear tractions is straightforward. The plastic strain range in the hoop direction should be obtained by averaging the total strains through the thickness and correcting the result to account for elastic unloading as follows:

\[
\Delta \varepsilon_{pl} = \Delta \varepsilon_1 - \frac{(1 - \nu)}{E} \left( S_{y_{max}} + S_{y_{min}} \right)
\]  

(26)

In order to obtain conservative evaluation of plastic strain the correction for unloading derived by the second term in equation (26) is given for the equibiaxial state of stress.

The thermal bending through the wall should be evaluated by stress linearization.

**Inelastic Solution for One or Two Cycles**

If better accuracy of the results is required, the resultants may be obtained by inelastic analysis of one or two cycles. The thermal analysis remains the same. Instructions for performing the inelastic stress and strain analyses using the MARC program are given in Appendix B. The program optimizes the load steps to obtain accurate results within the shortest computer time. As in elastic analysis, the bending and shear stress resultants are determined simply by linearizing the stresses through the thickness of the ligament. The range of plastic strain in the hoop direction is determined by adding absolute values of strains averaged through the thickness of the ligament within the entire cycle and taking half of the sum as a strain range, \( \Delta \varepsilon_{p1} \).
3.6 **Thermally Induced Stresses**

The history of thermally induced stresses for the analyzed ligament is shown in Figure 8. As explained before, the ligaments are subjected to severe thermal straining due to the temperature difference between the outside and inside of the thrust chamber. As can be seen in Figure 3, there is also an essential temperature drop through the wall of the ligament. This causes bending since the ligament is constrained at the ends. The bending stress arises during the initial cooling phase, when the rapid chilldown creates temperature gradients through the ligament. Tensile membrane stresses in the ligament exceed yield. The stresses increase until the temperature difference between the ligament and closeout wall reaches a maximum, after which unloading occurs. The stresses become compressive in the ligament in the residual state when a uniform temperature is attained.

Following ignition, the inner surface of the cylinder is subjected to intensive heat flux through the ligament. The temperature drop through the ligament results in thermally induced bending stresses. Compressive membrane stresses in the ligament exceed yield. Plastic flow in the hoop direction erases the effect of bending and the stresses remain essentially uniform through the ligament. Unloading occurs after the temperature difference between the ligament and closeout wall reaches a maximum and the stresses are relieved. The bending stresses generated in the ligament during unloading are again erased by straining in the hoop direction as the entire nozzle cools to a uniform temperature. The repetitive distribution of stress in subsequent cycles are illustrated in Figures 8a and b.
FIGURE 8 REPETITIVE CYCLE OF THERMALLY INDUCED BENDING STRESS IN LIGAMENT
The thermally induced bending though acting only for a short portion of the cycle, may enhance the ratcheting strain. Its effect may be assessed by computing the elastic energy of the thermally induced bending stresses and correcting the hoop strain accordingly.

The maximum thermal bending stress is given by equation (25), where $\Delta T$ is the temperature drop across the ligament. Thus, referring to Figure 9, the elastic energy is:

$$E_{el} = \frac{2}{E} \int_{0}^{H} \left( a \frac{Z}{b H} \right)^2 dz$$

$$= \frac{E H (a \Delta T)^2}{6 (1 - \nu)^2}$$

(27a)

(27b)

It is conservatively assumed that all of the available elastic energy goes into plastic straining of the ligament.

If $\Delta \varepsilon_p$ denotes the corresponding plastic hoop strain, then the plastic work is:

$$E_{p1} = 2 S_Y \Delta \varepsilon_p^H$$

(28)

Equating (27a) and (28):

$$\Delta \varepsilon_p^H = \frac{E (a \Delta T)^2}{12 (1 - \nu)^2 S_Y}$$

(29)

where $S_Y$ is the average of the absolute values of $S_{Y_{max}}$ and $S_{Y_{min}}$.

This must be added to the plastic strain range due to differential thermal expansion, $\Delta \varepsilon_{p1}^H$, to obtain the plastic strain range in the hoop direction, $\Delta \varepsilon_{p1}^H$. 

25
FIGURE 9 THERMAL BENDING IN LIGAMENT
4.0 DEVELOPMENT OF FAILURE MODE EVALUATION CRITERION

The ligaments are subjected to incremental plastic deformations during each firing cycle of the thrust chamber. The geometry of these ligaments changes as the incremental strains accumulate. They are subjected to incremental inward bending and simultaneously to a progressive thinning near the center of the ligament. Both fatigue damage and the tensile stability of the material are affected by local geometry changes.

Plastic tensile instability occurs when the incremental strain hardening of the deforming metal is less than the incremental increase of true stress due to local thinning. Once conditions of tensile instability are reached, further stretching occurs as minimum ligament section necking. The pressure in the chamber causes ultimate failure of the necked ligament.

4.1 Plastic Instability - Necking

The plastic tensile instability of the ligament in the displacement controlled thermal cyclic strain field is analyzed first by considering the ligament as a biaxially loaded shell subjected to monotonic tensile straining in the hoop and axial directions.

For a biaxially stretched sheet, the Mises yield condition is:

$$
\bar{\sigma}^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2
$$

(30a)

where $\bar{\sigma}$ is the effective stress and $\sigma_1$ and $\sigma_2$ are principal stresses in the hoop and axial directions, respectively.

Denoting the stress ratio $\frac{\sigma_1}{\sigma_2}$ by $\alpha$, equation (30a) can be written as:

$$
\bar{\sigma} = \sigma_1(1 - \alpha + \alpha^2)^{\frac{1}{2}}
$$

(30b)
In the present analysis elastic strains will be neglected and the Levy-Mises flow rule equations assumed to apply. The relation between the strain increments can then be written as [6]:

\[
\frac{d\varepsilon_1}{2\sigma_1 - \sigma_2} = \frac{d\varepsilon_2}{2\sigma_2 - \sigma_1} = \frac{d\varepsilon_3}{-\sigma_1 - \sigma_2} = \frac{d\bar{\varepsilon}}{2\bar{\sigma}} \tag{31a}
\]

where

\[
d\bar{\varepsilon} = \sqrt{\frac{2}{3}} \left( d\varepsilon_1^2 + d\varepsilon_2^2 + d\varepsilon_3^2 \right)^{\frac{1}{2}} \tag{31b}
\]

In the above, \(d\bar{\varepsilon}\) is the increment of effective strain and \(d\varepsilon_1, d\varepsilon_2\) and \(d\varepsilon_3\) are the strain increments in the hoop, axial and radial (thickness) directions, respectively. Equation (31a) can be written in terms of \(\alpha\) as:

\[
\frac{d\varepsilon_1}{2\alpha - 1} = \frac{d\varepsilon_2}{1 + \alpha} = \frac{-d\varepsilon_3}{2\alpha - \alpha^2} = \frac{d\bar{\varepsilon}}{2(1 - \alpha + \alpha^2)^{\frac{1}{2}}} \tag{31c}
\]

For proportional loading the stress ratio remains constant and equation (31c) can be integrated to give the total strains:

\[
\frac{\varepsilon_1}{2 - \alpha} = \frac{\varepsilon_2}{2\alpha - 1} = \frac{-\varepsilon_3}{1 + \alpha} = \frac{\bar{\varepsilon}}{2(1 - \alpha + \alpha^2)^{\frac{1}{2}}} \tag{31d}
\]

Instability occurs when the hoop force \(F\) reaches a maximum, i.e.:

\[
dF = d(2H\alpha_1) = 0 \tag{32}
\]

where \(2H\) denotes the thickness of the ligament and \(\alpha_1\) its
axial length. Equation (32) is differentiated to give:

\[
\frac{d\sigma_1}{\sigma_1} = \frac{da}{a} - \frac{dH}{H}
\]

\[
\frac{d\sigma}{\sigma} = -d\varepsilon_2 - d\varepsilon_3
\]

(33)

From the incompressibility condition of plastically deformed material:

\[
d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 = 0
\]

(34a)

or

\[
d\varepsilon_1 = -d\varepsilon_2 - d\varepsilon_3
\]

(34b)

For proportional loading:

\[
\frac{d\sigma_1}{\sigma_1} = \frac{d\bar{\sigma}}{\bar{\sigma}}
\]

(35)

Thus, equation (33) becomes:

\[
\frac{d\bar{\sigma}}{\bar{\sigma}} = d\varepsilon_1
\]

(36)

Using the flow law equation (31c), the condition for instability is given by:

\[
\frac{d\bar{\sigma}}{d\varepsilon} = \frac{(2-a)\bar{\sigma}}{2(1-a+a^2)\bar{\sigma}}
\]

(37)

If the stress-strain law is given by:

\[
\bar{\sigma} = A\bar{\varepsilon}^n
\]

(38)
where $A$ and $n$ are material constants, then on differentiating:

$$\frac{d\sigma}{d\epsilon} = \frac{n\sigma}{\epsilon},$$

$$\frac{d\sigma}{d\epsilon} = \frac{\sigma}{z}$$  \hspace{1cm} (39)

where $z = \frac{\bar{\epsilon}}{n}$ is the subtangent in Figure 10.

From equations (37) and (39):

$$z = \frac{2(1 - a + a^2)^{\frac{1}{2}}}{(2 - a)}$$  \hspace{1cm} (40)

and the critical effective strain is:

$$\frac{\bar{\epsilon}}{\epsilon_{cr}} = \frac{2n(1 - a + a^2)^{\frac{1}{2}}}{(2 - a)}$$  \hspace{1cm} (41)

The critical strain in the minimum ligament section from equation (31d) is then:

$$\frac{\epsilon}{\epsilon_{cr}} = n$$  \hspace{1cm} (42)

For cyclic loading, the compressive straining of a ligament at the beginning of the loading cycle is followed by tensile strains. In each loading cycle the ligament accumulates plastic strain. The net increments in the hoop and radial directions occur at the end of the cycle when the axial strain is zero. Thus, the limits of material stability can be approximated by the analysis for plane strain conditions. The plastic strains in the
ARC TAN $\frac{\partial \sigma}{\partial \varepsilon}$

$z$

FIGURE 10 - CONSIDERE'S CONSTRUCTION
hoop direction $d\varepsilon_1$, and in the radial direction $d\varepsilon_3$ satisfy the condition of incompressibility. Since $d\varepsilon_2 = 0$:

$$d\varepsilon_1 = -d\varepsilon_3 \quad (43)$$

Now,

$$\varepsilon_3 = \ln\left(\frac{t}{2H}\right) \quad (44)$$

where $2H$ denotes the original ligament thickness and $t$ the ligament variable thickness. Then using equations (42), (43) and (44), the critical thickness at instability is given by:

$$t_{cr} = 2He^{\varepsilon_1_{cr}} \quad (45)$$

$$t_{cr} = 2He^{-n} \quad (46)$$

4.2 Strain Range - Fatigue

For fatigue calculations, the maximum local hoop strain range at the minimum ligament section can be obtained from the average hoop strain range by integration if the geometry of the distorted ligament is known. As mentioned earlier, tests indicate that the deformed shape of the ligament can be approximated by a linear function as shown in Figure 6. Thus, the thickness variation can be written as:

$$t(x) = \left(t_{max} - t_{min}\right)\frac{2x}{h} + t_{min} \quad (47)$$

where $t_{max}$ and $t_{min}$ are as shown in the figure. Now, the average strain in the hoop direction is given by:
From equilibrium of forces in the hoop direction:

\[ \sigma_1 (x) t(x) = \sigma_{\min} t_{\min} \quad (49) \]

Using the yield condition equation (30b), since \( a \) is constant equation (49) becomes:

\[ \bar{\sigma}(x) t(x) = \bar{\sigma}_{\min} t_{\min} \quad (50) \]

Substituting the stress-strain law, equation (38):

\[ \bar{\varepsilon}(x)^n t(x) = \bar{\varepsilon}_{\min}^n t_{\min} \quad (51) \]

Now making use of the flow law, equation (31d) for constant \( a \):

\[ \varepsilon^f(x)^n t(x) = \varepsilon_{\min}^n t_{\min} \quad (52) \]

Substituting for \( \varepsilon_1 (x) \) into equation (48), the relation between the average strain and local strain is given by:

\[ \varepsilon_1^{\text{avg}} = \frac{2 \varepsilon_1_{\min}}{t_{\min}} \int_o^{x/2} \left\{ \frac{t_{\min}}{t(x)} \right\}^{1/n} \, dx \quad (53a) \]

Substituting for \( t(x) \) from equation (47) and integrating:

\[ \varepsilon_1^{\text{avg}} = \varepsilon_1_{\min} \frac{n}{n-1} \frac{t_{\min}}{t_{\max} - t_{\min}} \left[ \left( \frac{t_{\max}}{t_{\min}} \right)^{n-1} - 1 \right] \quad (53b) \]
or

\[
\varepsilon_{1\text{min}} = \varepsilon_{1\text{avg}} \left( \frac{n-1}{n} \right) \left( \frac{t_{\text{max}}}{t_{\text{min}}} - 1 \right) / \left[ \left( \frac{t_{\text{max}}}{t_{\text{min}}} \right)^{n-1} - 1 \right]
\]  

(53c)

As the deformation increases with each cycle, the thicknesses \( t_{\text{max}} \) and \( t_{\text{min}} \) also change per cycle. This, in turn, changes \( \varepsilon_{1\text{min}} \), the local strain in the minimum ligament section which intensifies with each cycle.

The axial strain in the minimum ligament section is given by:

\[
\varepsilon_{2\text{min}} = \alpha (T_i - T_o)
\]

(54a)

Also, from the condition of incompressibility:

\[
\varepsilon_{3\text{min}} = -(\varepsilon_{1\text{min}} + \varepsilon_{2\text{min}})
\]

(54b)

Now the effective strain range in the minimum ligament section for entering the fatigue curve is:

\[
\varepsilon_{\text{min}} = \sqrt[3]{ \frac{2}{3} \sqrt{(\varepsilon_{1\text{min}} - \varepsilon_{2\text{min}})^2 + (\varepsilon_{1\text{min}} - \varepsilon_{3\text{min}})^2 + (\varepsilon_{2\text{min}} - \varepsilon_{3\text{min}})^2} }
\]

(55a)

Substituting equation (54b) into (55a):

\[
\varepsilon_{\text{min}} = \sqrt[3]{ \frac{2}{3} \sqrt{(\varepsilon_{1\text{min}} - \varepsilon_{2\text{min}})^2 + (2\varepsilon_{1\text{min}} + \varepsilon_{2\text{min}})^2 + (2\varepsilon_{2\text{min}} + \varepsilon_{1\text{min}})^2} }
\]

\[
= \frac{2}{\sqrt[3]{3}} \sqrt{\varepsilon_{1\text{min}}^2 + \varepsilon_{1\text{min}} \varepsilon_{2\text{min}} + \varepsilon_{2\text{min}}^2}
\]

(55b)
For the linear variation assumed, the thickness $t_{\text{min}}$ and $t_{\text{max}}$ after the $N$th cycle are given by:

\[
\begin{align*}
    t_{\text{min}} &= \frac{2H(\ell + w) - N \delta w}{(\ell + w)} \\
    t_{\text{max}} &= \frac{2H(\ell + w)^2 + N \delta \ell w}{(\ell + w)^2}
\end{align*}
\]  

where $\delta$ is the deformation per cycle.

Equations (55) and (56) along with the fatigue curve can be used to determine the fatigue life. However, since $\varepsilon_{\text{th}}^{\text{min}}$ changes with each cycle, a numerical procedure is necessary for performing the calculations to determine fatigue life. A FORTRAN program has been written for determining the cycles to failure and is included in Appendix C.

5.0 THRUST CHAMBER LIFE PREDICTIONS

The results obtained using the analyses described can be used to determine the life of the thrust chamber. For small ligament distortion it can be conservatively assumed that the incremental thinning of the ligament remains constant during subsequent cycles. The number of cycles to failure can then be bounded from below by considering the number of cycles needed to reduce the thickness of the ligament below the critical value resulting in tensile instability or by considering the fatigue damage of the gradually thinned ligament, whichever results in smaller number of cycles. For OFHC copper this bounding technique provides a realistic evaluation of the cycles to failure observed in experiments where failure was due to tensile instability.
For NARloy Z the thinning gradually diminishes in consecutive cycles and the fatigue mode of failure prevails. This bounding technique provides a conservative evaluation. However, the efficiency of the bound is significantly lower than for copper.

The tendency of diminishing increments of accumulated strain in consecutive cycles is the result of kinematic hardening. It has been demonstrated [7], [8], that for such material the cyclically loaded structure always achieves the condition of alternating plastic straining if there is no limit on the hardening capacity and the changes of geometry can be ignored. The net increment of plastic ratchet strain vanishes after sufficient hardening. The amount of plastic strain accumulation needed to achieve such plastic shakedown can be uniquely determined. For real materials with limited hardening capacity and less than ideal kinematic behavior, the stable cyclic state occurs at much larger accumulated strains than for an ideal kinematically hardening material or may never be achieved. Materials more closely obeying the theoretical concept of kinematic hardening can be expected to fail due to fatigue since slowed down thinning allows the material to be exposed to more fatigue cycles, as in the case of NARloy Z.

This trend is also in general agreement with the results of [9]. The number of cycles at which thinning stops is related to the strain hardening parameter $n$. Assuming this to be the dominating parameter on which the number of cycles to thinning depend and hypothesizing a power relationship, the following empirical criterion is obtained:

$$N_T = 750n^{1.25}$$  \hspace{1cm} (57)

where $N_T$ denotes the number of cycles at which thinning stops.
Note that the results of [9] have been partly used in determining the constants for the power relationship since these were the only quantifiable numbers available.

If the number of cycles to reach the critical thickness obtained from plastic instability analysis is less than $N_T$ for that material, then tensile stability is the limiting failure mode. The thrust chamber life in this case is given by the number of cycles to instability. If on the other hand, $N_T$ is greater, fatigue is the failure mode and thrust chamber life is given by $N_T$ plus the cycles obtained from the fatigue curve corresponding to the effective strain range which remains constant from that point on.

The numerical calculations for the case of OFHC copper and NARloy Z using the procedure developed herein are presented in Appendix A.

6.0 FINITE ELEMENT ANALYSIS

The MARC program was used for the thermal and mechanical part of the analysis. The MARC program was adapted to analyze the fast thermal and pressure thrust nozzle transient. Optimized selection of the time and load steps was used for the thermal and elastic-plastic analyses.

The finite element inelastic analysis confirmed the fact that the simplified model includes the essence of the physical behavior of the ligament as described herein during the course of the development procedure. Moreover, the results of the finite element analysis were employed in deriving and calibrating the thermal model of the ligament. A description of the model and details of the finite element analysis for the case of OFHC copper are given in Appendix B.
7.0 COMPARISON OF RESULTS

The numerical results obtained for OFHC copper, using the simplified analysis procedure developed herein, are compared with the finite element results in Table 1. The results are seen to be in good agreement. The life prediction of 103 cycles for OFHC copper and 833 cycles for NARloy Z is also in general agreement with experimental results [2] indicating that the simplified design procedure can be used for predicting the life of the thrust chamber without performing detailed inelastic analysis.

TABLE 1

Comparison of Analytical vs. Finite Element Results

<table>
<thead>
<tr>
<th></th>
<th>Analytical Results</th>
<th>Finite Element Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deflection of Pressure Surface/Cycle</td>
<td>.00029&quot; (.00737 mm)</td>
<td>.00027&quot; (.00686 mm)</td>
</tr>
</tbody>
</table>

8.0 CONCLUSIONS

A simplified design procedure for predicting thrust chamber life is developed herein. The method uses a yield surface for combined bending and membrane loading to determine the incremental inward bulging and progressive thinning near the center of the ligaments at the inner liner of the thrust chamber. Failure analyses indicate that plastic tensile instability is the dominant mode of failure for OHFC copper. Both fatigue and plastic tensile instability must be analyzed for NARloy Z to determine the limiting failure mode. Results of the
simplified analyses are shown to compare favorably with those obtained from detailed inelastic finite element analyses for OFHC copper.

Further experiments are needed to provide better characterization of plastic response of the materials used for thrust chambers.

For longer cycle times, the accuracy of the evaluation can be improved by including the time dependent response of the material into the analysis.
REFERENCES


Example I

Consider the ligament for OFHC copper. The geometry is as shown in Figure A-1.

For a unit width of the ligament $\Delta p = 547$ lb/in. ($95794.47$ N/m)

The average yield stress $S_y = 9000$ psi. ($62$ MPa)

Then from elastic solution for a clamped beam (eqs. (19) and (20))

\[
M = \frac{-pL^2}{2} \left( \frac{1}{6} - \frac{x}{l} + \frac{x^2}{l^2} \right)
\]

\[
S = \frac{pL}{2} - px
\]

From eq. (3a):

Yield Moment $M_0 = H^2 S_y$

\[\therefore\] From eq. (2a):

\[
m = \frac{M}{M_0}
\]

\[
m = \frac{-547 \cdot (.0664)^2}{2 \cdot (.035)^2 \cdot 9000} \left[ \frac{1}{6} - \frac{x}{.0664} + \frac{x^2}{(.0664)^2} \right]
\]

\[
m = -0.0729 + 6.5888x - 99.229x^2 \quad (A-1)
\]

From eq. (10b):

\[
s = \frac{S}{H S_y}
\]

\[
s = \frac{1}{(.035)^2 \cdot 9000} \left( \frac{547 \cdot .0664}{2} - 547x \right)
\]

\[
= .1153 - 3.473x \quad (A-2)
\]
FIGURE A-1 GEOMETRY OF OFHC LIGAMENT

\[ \ell = 0.0664" \quad (1.686 \text{ mm}) \]
\[ 2H = 0.035" \quad (0.889 \text{ mm}) \]
\[ w = 0.05" \quad (1.27 \text{ mm}) \]
From eq. (6b):

\[
k = \sqrt{1 - s^2}
\]

\[
= (.9867 + .8009x - 12.062x^2)^{\frac{1}{4}} \quad (A-3)
\]

From eq. (6a) the yield surface:

\[
F = m + \frac{n^2}{k} - k = 0
\]

\[
n^2 = k^2 - km \quad (A-4a)
\]

Substituting eqs. (A-1) and (A-3):

\[
n = \left[ .9867 + .8009x - 12.062x^2 - (.9867 + .8009x - 12.062x^2)^{\frac{1}{4}} \right]^{\frac{1}{4}} \quad (A-4b)
\]

From eq. (21), taking \( a_i = a_o = a \) and \( (S_{Y\text{max}} + S_{Y\text{min}})/2 = S_y \), the plastic strain range in the hoop direction is:

\[
\Delta \varepsilon'_{p1} = \left[ a \left\{ (T_i - T_o)_{\text{max}} - (T_i - T_o)_{\text{min}} \right\} - \frac{2S_y}{E} \right] \quad (A-5a)
\]

From thermal analysis, \( (T_i - T_o)_{\text{max}} - (T_i - T_o)_{\text{min}} \approx 780^\circ \text{F} \approx 433^\circ \text{K} \)

Then for \( a = 9.5 \times 10^{-6} \ \text{in./in./}^\circ \text{F}, \ (17.1 \times 10^{-6} \ \text{oC}^{-1}) \) and \( E = 17 \times 10^6 \ \text{psi} \ (117215 \ \text{MPa}) \):

\[
\Delta \varepsilon'_{p1} = \left[ 9.5 \times 10^{-6} \cdot 780 - \frac{2 \cdot 9000}{17 \cdot 10^6} \right]
\]

\[
\Delta \varepsilon'_{p1} = 0.00635
\]
From eq. (29):

\[
\Delta \varepsilon^* = \frac{E a^2 (\Delta T)^2}{12(1-\nu)^2 s_y}
\]  

(A-5b)

The temperature drop across the ligament \(\Delta T = 200^\circ F\):

\[
\Delta \varepsilon^* = \frac{17 \cdot 10^6 \cdot (9.5)^2(10^{-6})^2(200)^2}{12(1 - 0.3)^2 \cdot 9000}
\]

\[
\Delta \varepsilon^* = 0.0016
\]

\[
\therefore \Delta \varepsilon^* = \Delta \varepsilon_1 + \Delta \varepsilon_2 = 0.0071
\]

(A-6)

From eqs. (6b) and (15):

\[
K = \frac{2k}{\frac{\Delta \varepsilon_1}{H}}
\]

(A-7)

For the present problem, \(m\) and \(s\) are small compared to \(n\). Thus it is found from equation (A-4a) that \(n = k\). Eq. (A-7) then simplifies to:

\[
K = \frac{2(\Delta \varepsilon_1)}{H}
\]

(A-8)

\[
\therefore K = \frac{1}{R} = \frac{2(0.0071)}{0.0175}
\]

where the radius \(R\) is assumed for the length shown in Figure A-2.

Thus \(R = 1.1651"\) (29.59 mm)
FIGURE A-2  DEFORMED LIGAMENT
Deflection $\delta_1 = 2R(1 - \cos \alpha)$

$$= 2(1.1651)\left[1 - \{(1.1651)^2 - (0.0166)^2\} / 1.1651\right]$$

$$\therefore \delta_1 = 0.0002365" \quad (0.006 \text{ mm})$$

From eq. (17), since the contribution of $m$ and $s$ is small, the product term $\frac{ms}{\sqrt{1 - s^2}}$ can be neglected and $n$ taken $= 1$ as seen from Fig. 5b.

Thus:

$$\gamma = 4s(\Delta e_{p1}) \quad (A-9)$$

Substituting from eqs. (A-2) and (A-6):

$$\gamma = 4(0.1153 - 3.473x)(0.00751)$$

$$\gamma = 0.003464 - 0.104328x \quad (A-10)$$

Integrating and determining the deflection at $x = l/2$:

$$\delta_2 = 0.003464x - 0.104328x^2/2$$

$$\delta_2 = 0.0000575" \quad (0.0015 \text{ mm})$$

$$\therefore \text{Total} \quad \delta = \delta_1 + \delta_2 = 0.000294" \quad (0.0075 \text{ mm})$$

From eq. (18), thinning after $N$ cycles:

$$t_N = \frac{0.000294 \cdot 0.05N}{(0.0664 + 0.05)}$$

$$\therefore \quad t_N = 0.0001263N \text{ inches} \quad (A-11)$$

**Material Instability**

The strain hardening parameter $n$ in the stress-strain law is approximately given by LeRC as:
where $S_u$ denotes the ultimate strength and $S_y$ the yield strength.

Towards the end of the cycle, the value of $S_u$ and $S_y$ are approximately 46 ksi (317 MPa) and 9 ksi (62 MPa), respectively. Thus:

$$n = 0.2 \left( \frac{S_u - S_y}{S_y} \right)^{0.6}$$

From Eq. (46), critical thickness for $n = 0.467$

$$t_{cr} = (0.035)e^{-0.467}$$

$$t_{cr} = 0.02194\text{"} (0.557 \text{mm})$$

.: Thinning for Instability $= (0.035 - 0.02194)$

$= 0.01306\text{"} (0.332 \text{mm})$ \hspace{1cm} (A-12)

Equating (A-11) and (A-12), the number of cycles to instability are:

$$N = 103$$

From equation (57)

$$N_p = 750 (0.467)^{1.25}$$

$= 289$ cycles

Thus thinning continues for 289 cycles.

Since $N < N_p$, the failure mode is material instability and thrust chamber life equals 103 cycles.

Fatigue

To demonstrate that fatigue results in a higher number of cycles, the procedure described in Section 4.2 was used to
determine the fatigue life of the OFHC ligament. The fatigue life was based on the fatigue curve for OFHC copper shown in Figure A-3. The calculations were performed using the FORTRAN program listed in Appendix C. The fatigue life was determined to be 132 cycles.
FIGURE A-3  TYPICAL LOW-CYCLE FATIGUE LIFE OF OFHC COPPER ANNEALED CONDITION
**Example II:**

Consider now an example of NARloy 2. The geometry is the same as shown in Figure A-1.

Also, \( \Delta p = 547 \text{ lb/in.} \) \( (95794.47 \text{ N/m}) \)

The average yield stress \( S_y = 30,000 \text{ psi} \) \( (207 \text{ MPa}) \)

As before, from Eq. (21)

\[
\Delta \varepsilon'_{P_1} = a \left\{ \left( T_i - T_o \right)_{\text{max}} - \left( T_i - T_o \right)_{\text{min}} \right\} - \frac{2S_y}{E}
\]  \hspace{1cm} (A-13)

Since the thermal conductivity for NARloy 2 is approximately the same as that of OFHC copper, temperatures \( T_i \) and \( T_o \) are approximately the same.

Then for \( a = 9.5 \times 10^{-6} \text{ in./in./°F}, \) \( (17.1 \times 10^{-6} \text{ °C}^{-1}) \) and \( E = 18 \times 10^6 \text{ psi} \) \( (124110 \text{ MPa}) \):

\[
\Delta \varepsilon'_{P_1} = 9.5 \times 10^{-6} \cdot 780 - \frac{2 \cdot (30000)}{18 \times 10^6}
\]

\[
\therefore \Delta \varepsilon'_{P_1} = 0.004077
\]

From Eq. (29)

\[
\Delta \varepsilon''_{P_1} = \frac{18 \times 10^6 \cdot (9.5)^2 \cdot (10^{-6})^2 \cdot (200)^2}{12(1-0.3)^2 \cdot 30000}
\]

\[
\therefore \Delta \varepsilon''_{P_1} = 0.000368
\]

\[
\therefore \Delta \varepsilon_{P_1} = \Delta \varepsilon'_{P_1} + \Delta \varepsilon''_{P_1} = 0.004445
\]

Substituting in:

\[
K = \frac{2(\Delta \varepsilon_{P_1})}{H}
\]

\[
K = \frac{1}{R} = 2(0.004445)
\]

\[
\therefore R = 1.9685'' \text{ (50 mm)}
\]
\( \text{Deflection } \delta_1 = 2R(1-\cos \alpha) = 0.00014" \) (0.0035 mm)

From Eq. (10b) and (20)

\[
\begin{align*}
S &= \frac{S}{H\sqrt{\gamma}} \\
\therefore \quad s &= \frac{1}{0.035 \cdot 30000} \left( \frac{547 \cdot 0.0664}{2} - 547\gamma \right) \\
&= 0.0346 - 1.042x
\end{align*}
\]

Substituting in:

\[
\gamma = 4s(\Delta \varepsilon_{P1})
\]

\[
\therefore \quad \gamma = 4(0.0346 - 1.042x)(0.004445)
\]

\[
= 0.000615 - 0.018527x
\]

Integrating and determining the deflection at \( x = l/2 \)

\( \delta_2 = 0.00001" \) (0.000254 mm)

\( \therefore \quad \text{Total } \delta = \delta_1 + \delta_2 = 0.00015" \) (0.00381 mm)

From Eq. (18), thinning after \( N \) cycles:

\[
\tau_N = \frac{0.00015 \cdot 0.05N}{(0.0664 + 0.05)}
\]

\[
\therefore \quad \tau_N = 0.0000644 N \text{ inches}
\]

**Material Instability**

For \( S_u = 55 \text{ ksi (379 MPa)} \) and \( S_y = 30 \text{ ksi (207 MPa)} \)

\[
n = 0.2 \left( \frac{55-30}{30} \right)
\]

\[
\therefore \quad n = 0.18
\]

From Eq. (46)

\[
\tau_{cr} = (0.035)e^{0.18}
\]

\[
= 0.0292" \text{ (0.742 mm)}
\]

\( \therefore \quad \text{Thinning for instability } = 0.035 - 0.0292\)

\[
= 0.0058" \text{ (0.147 mm)}
\]

51
Equating (A-16) and (A-17), the number of cycles to instability

\[ N = 90 \]

From Eq. (57)

\[ N_T = 750(0.18)^{1.25} \]
\[ = 88 \]

Thus thinning stops after 88 cycles.

Since \( N > N_T \), material instability is not the failure mode and the thrust chamber life must be determined from fatigue calculations.

**Fatigue**

The procedure of Section 4.2 was used for determining the fatigue life of NARloy Z. The fatigue life was based on the fatigue curve of Figure A-4. The FORTRAN Program listed in Appendix C was used for performing the necessary calculations. The input and output obtained are attached in this Appendix.

The notation for the listed results is as follows:

- **NCYC** - Number of cycles
- **TMIN** - \( t_{\text{min}} \), minimum ligament thickness - in.
- **TMAX** - \( t_{\text{max}} \), maximum ligament thickness - in.
- **EEQVT** - effective strain range %
- **NF** - Number of cycles to failure from fatigue curve
- **USAGE** - Usage factor

Thus if the ligament were to continue thinning, the fatigue life would be 272 cycles. However since thinning stops after 88 cycles, the remaining fatigue life is based on the effective strain range of 2.2416% which now remains constant and equals 747 cycles.

Thus cycles to failure \( = 88 + 747 \)
\[ = 835 \]
Figure A-4 Typical Low-Cycle Fatigue Life of NARloy Z
CONSTANTS INPUT

<table>
<thead>
<tr>
<th>N</th>
<th>18000</th>
<th>n (material strain hardening exponents)</th>
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<tr>
<td>DELTA</td>
<td>15000E-03</td>
<td>δ (Deflection per cycle, in.)</td>
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<tr>
<td>$\varepsilon_{\text{avg}}$</td>
<td>174100%</td>
<td>$\varepsilon_{\text{avg}}$ (Average hoop strain in ligament, $\alpha \Delta T$) = $\varepsilon_{\text{min}}$</td>
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<tr>
<td>H</td>
<td>17500E-01</td>
<td>half ligament thickness, in.</td>
</tr>
<tr>
<td>L</td>
<td>66400E-01</td>
<td>$L$ (width of ligament in hoop direction, in.)</td>
</tr>
<tr>
<td>W</td>
<td>50000E-01</td>
<td>$w$ (width of rib, in.)</td>
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FATIGUE CURVE

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<th>STRAIN RANGE</th>
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<tr>
<td>100.00</td>
<td>5.9000</td>
</tr>
<tr>
<td>4000.0</td>
<td>1.0000</td>
</tr>
<tr>
<td>50000E+06</td>
<td>0.10000</td>
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<td>148934.80</td>
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</table>
APPENDIX B

FINITE ELEMENT ANALYSIS

INTRODUCTION

The MARC general purpose finite element computer program was adopted to study the response of the experimental thrust chamber, during its firing tests. Each test consisted of a chilldown cycle and a number of repetitive firing cycles. Finite element heat transfer and structural analyses were performed on the model. In the heat transfer analysis, the temperature distributions vs. time were obtained for a typical chilldown and firing cycle. In the structural analysis, the resultant temperature distributions and the cyclic pressure loads were used to determine the linear and non-linear response of the structure. Optimized selection of the time and load steps was used for the thermal and elastic-plastic analyses.

THERMAL ANALYSIS

1) Model Description

The finite element mesh used in the analysis is shown in Figure B-1. This model was established based on the geometric symmetry planes and loading conditions. It consists of thirty-five elements and 138 nodal points. All elements are the eight-node planar bi-quadratic quadrilateral type (element 41 in MARC element library). Temperature and position (coordinates) within these elements are interpolated from eight sets of nodal values, the four corner nodes and four mid-side nodes. The interpolation function is such that each edge has a parabolic variation along itself.
FIGURE B-1  FINITE ELEMENT MODEL OF THE NOZZLE
Material Properties

The properties used to characterize the response of the nozzle in the thermal analysis were obtained from Reference 3. They were thermal conductivity and specific heat as a function of temperature of the thrust chamber material as shown in Figures B-2 and B-3, respectively.

Load Cycles

The thermal analysis consisted of a chilldown and a firing cycle which lasted 1.5 and 3.5 seconds, respectively. At the beginning of the chilldown period, the room temperature value of 294°K (530°F) was assumed to be the uniform initial temperature of the entire thrust chamber. The thermal operating conditions (heat transfer coefficient) were defined in terms of thermal boundary conditions on the cooling channel surfaces (cold side) and the combustion gas surface (hot side). These boundary conditions for the chilldown and a firing cycle (totally five seconds) are plotted vs. time in Figures B-4a through B-4D. To improve the convergence process, the temperature of the coolant flow at the beginning of the chilldown period was assumed to be the same as the initial model temperature 294°K (530°F), and then it was suddenly dropped to 29°K (53°F) in 0.1 second. For more accuracy in results of the heat transfer analysis, the time steps were controlled by the program to limit the maximum temperature change at any node to 28°K (50°F).

Results of Heat Transfer Analysis

The heat transfer analysis of the model was completed in 107 iteration cycles, 25 iteration cycles for the chilldown and 82 iteration cycles for a firing cycle. Results show that the temperature of the entire model drops from
FIGURE B-2 MATERIAL CONDUCTIVITY VS. TEMPERATURE

FIGURE B-3 MATERIAL SPECIFIC HEAT VS. TEMPERATURE
a) Hot Side
Heat Trans. Coef.
W/cm\(^2\)/K\(^\circ\)
(BTU/in\(^2\)/S/R\(^\circ\))

b) Cold Side
Heat Trans. Coef.
W/cm\(^2\)/K\(^\circ\)
(BTU/in\(^2\)/S/R\(^\circ\))

C) Hot Side
Adiabatic Temp.
K\(^\circ\)
(R\(^\circ\))

d) Cold Side
Flow Temp.
K\(^\circ\)
(R\(^\circ\))

---

FIGURE B-4 THE TIME HISTORIES OF HEAT TRANSFER COEFFICIENTS
AND TEMPERATURES FOR A CHILDL DOWN AND FIRING CYCLE
$294^\circ K$ ($530^\circ R$) at $t = 0$ to less than $30^\circ K$ ($54^\circ R$) at $t = 0.65$ second and to a uniform temperature of $29^\circ K$ ($53^\circ R$) at $t = 1.13$ seconds. From $t = 1.13$ to $t = 1.5$ seconds the nozzle remains at $294^\circ K$ ($530^\circ R$) temperature. From $t = 1.5$ seconds which is the start of the firing cycle, the temperature of the model begins to rise until the temperature of the hot surface reaches $805^\circ K$ ($1450^\circ R$) at $t = 3.2$ seconds. At this point the temperature of the model begins to drop until the entire model reaches the temperature of $29^\circ K$ ($53^\circ R$) at $t = 5.5$ seconds.

To get a picture of the temperature distribution, plots of continuous isotherms at $t = 3.2$ seconds are reproduced in Figure B-5. The temperature distribution of the centerline of the model is also plotted at different time intervals in a firing cycle in Figure B-6. It appears from the results that the maximum temperature difference between the hot and cold surfaces is $114^\circ K$ ($206^\circ R$) and the maximum temperature difference between the hot surface and the outer surface of the model is $25^\circ K$ ($46^\circ R$) during the time interval of $t = 1.9$ to $t = 3.2$ sec. The temperature histories of a few selected points are also plotted for a chilldown and firing cycle in Figure B-7. Since the temperature of the entire thrust chamber approached $29^\circ K$ ($53^\circ R$) at the end of the first firing cycle, it was concluded that the thermal analysis of this cycle without the chilldown could be used to simulate the remaining cycles.
Figure B-5  Isothermal plots at t=3.2 seconds.
FIGURE B-6  MIDPLANE TEMPERATURE HISTORY
STRUCTURAL ANALYSIS

1) Model Description

The finite element mesh used in the structural analysis is the same as the one used for heat transfer analysis. Eight-node isoparametric distorted quadrilateral generalized plane strain elements are used for the structural analysis. Displacement and position (coordinates) within the element are interpolated from eight sets of nodal values. The four corners and four mid-side nodes. The interpolation function is such that each edge has parabolic variation along itself.

2) Material Properties

The properties used to characterize the response of the thrust chamber in the structural analysis were taken from Ref. 3. They were time-dependent thermal expansion, Modulus of Elasticity, and stress-strain curves (work hardening).

3) Loading Cycles

The temperature distribution from the heat transfer analysis and the corresponding cyclic pressure load were supplied to the structural analysis program. The cyclic pressure load as a function of time is plotted in Figure B-8. The structural analysis was performed for a chill-down and two consecutive firing cycles. Due to the nonlinear response of the thrust chamber material, iteration cycles were controlled to limit the maximum temperature change input to the structural analysis to 13.9°C (25°F) at any node. Both isotropic and kinematic work hardening were used in the analysis.
FIGURE B-8 CYCLIC PRESSURE LOAD VS. TIME
4) **Results of Structural Analysis**

The structural analysis of the model was completed in thirty-four (34) iteration cycles simulating the chilldown and 130 iteration cycles simulating each firing cycle. Some plots of typical stress and strain distributions are reproduced in Figures B-9 through B-14.
FIGURE B-9 STRESS INTENSITY (VON MISES) DISTRIBUTION FOR THE MAXIMUM PRESSURE DIFFERENCE (ELASTIC ANALYSIS)
FIGURE B-10 STRESS INTENSITY (VON MISES) DISTRIBUTION AT TIME = 0.2 SECONDS AFTER THE BEGINNING OF THE FIRST FIRING CYCLE (ELASTIC ANALYSIS)
FIGURE B-11  STRAIN DISTRIBUTION IN THE RADIAL DIRECTION AT TIME = 0.2 SECONDS
AFTER THE BEGINNING OF THE FIRST FIRING CYCLE (ELASTIC ANALYSIS)
FIGURE B-12 STRESS INTENSITY (VON MISES) DISTRIBUTION AT THE END OF THE SECOND FIRING CYCLE

1 psi = 6.895 kPa

TIME = 8.5
FIGURE B-13 TOTAL EQUIVALENT PLASTIC STRAIN DISTRIBUTION AT THE END OF THE SECOND FIRING CYCLE

TIME = 8.5
FIGURE B-14 PLASTIC STRAIN DISTRIBUTION IN HOOP DIRECTION AT THE END OF THE SECOND FIRING CYCLE
APPENDIX C

FORTRAN Program

for

Fatigue Calculation
SUBROUTINE CYCLE

REAL X,Y,Z
DIMENSION CYCLE(100), STRAIN(100)

INTEGER N, M, NCYC

NCYC = 1

DATA A, B, C / 0.1, 0.2, 0.3 /

DO 10 CYCLE = 1, 100

10 X = A * CYCLE + B

IF (X .GT. C) THEN

CALL FATIGUE(X, Y)

ELSE

CALL STRAIN(X, Y)

ENDIF

NCYC = NCYC + 1

IF (NCYC .LE. 100) GOTO 20

END

SUBROUTINE FATIGUE

REAL X, Y

DATA N, M / 0.1, 0.2 /

Y = N * X + M

RETURN

END

SUBROUTINE STRAIN

REAL X, Y

DATA N, M / 0.1, 0.2 /

Y = N * X + M

RETURN

END

END
APPENDIX D

SYMBOLS

a - axial length
A - surface area, also equation arbitrary; positive scalar
D - rate of dissipation
E - modulus of elasticity
\( E_{el} \) - elastic energy
\( E_{pl} \) - plastic work
F - yield surface
h - convective heat transfer coefficient
2H - thickness of ligament
k - \( \sqrt{1-s^2} \); also thermal conductivity
K - curvature
\( \ell \) - width of ligament in hoop direction
m - generalized bending stress variable
M - bending moment in ligament
\( M_0 \) - yield bending moment
n - generalized hoop stress variable; also strain hardening exponent in stress strain law
N - hoop force in ligament; also number of cycles
\( N_0 \) - yield hoop force
p - pressure
s - generalized shear stress variable
S - shear force in ligament
SYMBOLS - continued

$S_o$ - yield shear stress

$S_y$ - average yield strength in tension

$S_{y_{min}}$ - ligament yield strength for minimum $\alpha(T_i - T_o)$

$S_{y_{max}}$ - ligament yield strength for maximum $\alpha(T_i - T_o)$

$t$ - ligament thickness

$t_N$ - thinning after $N$ cycles

$t_{max}$ - maximum ligament thickness

$t_{min}$ - minimum ligament thickness

$T$ - temperature

$T_i$ - average temperature of ligament

$T_o$ - average temperature of closeout wall

$w$ - width of rib

$z$ - $\bar{c}/n$, subtangent in Figure 10

$a$ - coefficient of thermal expansion; also stress ratio $\sigma_2/\sigma_1$

$\gamma$ - shear strain

$\delta$ - deflection per cycle

$\delta_1$ - deflection due to moment

$\delta_2$ - deflection due to shear

$\Delta p$ - pressure difference between coolant pressure and combustion gas pressure

$\Delta \varepsilon_1$ - total strain range in hoop direction

$\Delta \varepsilon_{pl}$ - plastic strain range in hoop direction
SYMBOLS - continued

\( \Delta \varepsilon_P \) - plastic strain range in hoop direction due to differential thermal expansion

\( \Delta \varepsilon_P^{\prime} \) - correction to plastic strain range in hoop direction due to thermally induced bending

\( \varepsilon \) - hoop strain

\( \varepsilon_1 \) - hoop strain

\( \varepsilon_2 \) - axial strain

\( \varepsilon_3 \) - radial strain

\( \varepsilon_{\text{avg}} \) - average hoop strain in ligament

\( \varepsilon_{1\text{min}} \) - hoop strain in minimum ligament section

\( \varepsilon_{2\text{min}} \) - axial strain in minimum ligament section

\( \varepsilon_{3\text{min}} \) - radial strain in minimum ligament section

\( \bar{\varepsilon} \) - effective strain

\( \bar{\varepsilon}_{\text{cr}} \) - critical effective strain

\( \bar{\varepsilon}_{\text{min}} \) - effective strain in minimum ligament section

\( \psi \) - generalized bending strain variable

\( \lambda \) - generalized hoop strain variable

\( \nu \) - Poisson's ratio

\( \sigma_1 \) - hoop stress

\( \sigma_2 \) - axial stress

\( \sigma_b \) - bending stress

\( \bar{\sigma} \) - effective stress

\( \phi \) - generalized shear strain variable

\( \cdot \) - dot above symbol denotes rate