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ABSTRACT

In this paper the elasticity problem for a laminated thick plate which consists of two bonded dissimilar layers and which contains a circular hole is considered. The problem is formulated for arbitrary axisymmetric tractions on the hole surface by using the Love strain function. Through the expansion of the boundary conditions into Fourier series the problem is reduced to an infinite system of algebraic equations which is solved by the method of reduction. Of particular interest in the problem are the stresses along the interface as they relate to the question of delamination failure of the composite plate. These stresses are calculated and are observed to become unbounded at the hole boundary. An approximate treatment of the singular behavior of the stress state is presented and the stress intensity factors are calculated.

1. INTRODUCTION

In this paper the elasticity problem for a thick plate which consists of two bonded dissimilar homogeneous layers is considered. It is assumed that the plate is infinite, contains a circular hole, and is subjected to axisymmetric external loads. Even though the problem as stated may have some applications, from a practical viewpoint the important problem is that of a laminated plate containing a circular hole and subjected to uniaxial membrane or bending loads away from the hole region. The latter problem has important applications in the analysis of delamination failure of perforated multilayered plate and shell structures. In such structures the interface stresses are known to have a power singularity which greatly enhances the possibility of

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delamination failure [1,2,3]. In a laminated plate under general loading conditions one may always separate a homogeneous solution and reduce the problem to a perturbation problem in which the self-equilibrating tractions on the hole surface are the only external loads. By expanding these tractions into Fourier series in θ one may further separate the problem into its simpler components in the independent variables r and z only. Thus, the axisymmetric problem treated in this paper may also be considered as the first component of the general three-dimensional plate problem.

The three-dimensional elasticity problem for laminated plates containing a hole does not seem to have been considered before (see, for example, [4] for a recent review). The existing solutions are mostly based on numerical techniques and are generally highly approximate [5-7]. The circular hole problem for a homogeneous thick plate was considered in [8] and [9]. The technique developed in [9] will be used to solve the laminated plate problem considered in this paper. The related problem of a layered semi-infinite medium (i.e., the limiting case of the hole problem in which the hole radius is infinite) was considered in [10] where the method of singular integral equations was used to solve the problem.

2. THE HOMOGENEOUS SOLUTION

In this section we give the results of some elementary solutions for a laminated plate without a hole which is subjected to certain uniform loading conditions at infinity. First we assume that the composite plate which consists of two layers with the elastic constants E_1, ν_1 and E_2, ν_2, and the thicknesses h_1 and h_2 is subjected to an average radial membrane stress σ_0 at $r=\infty$ and is constrained to remain flat upon deformations. Thus, defining

$$\sigma_{rr1}(r,\theta,z) = \sigma_1 \quad , \quad \sigma_{rr2}(r,\theta,z) = \sigma_2 \quad , \tag{1}$$

we have

$$\sigma_{\theta\theta 1} = \sigma_1 \quad , \quad \sigma_{\theta\theta 2} = \sigma_2 \quad , \tag{2}$$

and all the remaining stress components are zero. From

$$\sigma_1 h_1 + \sigma_2 h_2 = \sigma_o(h_1+h_2) \quad , \quad \varepsilon_{rr1} = \varepsilon_{rr2} \tag{3}$$

it then follows that

$$\sigma_1 = \sigma_o \frac{(1-\nu_2)m(1+k)}{1-\nu_1+mk(1-\nu_2)} \quad ,$$

$$\sigma_2 = \sigma_o \frac{(1-\nu_1)(1+k)}{1-\nu_1+mk(1-\nu_2)} \quad , \tag{4a,b}$$

where

$$m = E_1/E_2 \quad , \quad k = h_1/h_2 \quad . \tag{5}$$

Next we consider the composite plate which is uniformly loaded by an average stress σ_o in x-direction only.

Defining now

$$\sigma_{xx1}(x,y,z) = \sigma_1 \quad , \quad \sigma_{xx2}(x,y,z) = \sigma_2 \quad , \tag{6}$$

and again assuming that the plate is constrained to remain flat, the nonzero stress components σ_1, σ_2, σ_{yy1}, σ_{yy2} may be determined from

$$\sigma_1 h_1 + \sigma_2 h_2 = \sigma_o(h_1+h_2) \quad , \quad \varepsilon_{xx1} = \varepsilon_{xx2} \quad ,$$

$$\sigma_{yy1} h_1 + \sigma_{yy2} h_2 = 0 \quad , \quad \varepsilon_{yy1} = \varepsilon_{yy2} \quad . \tag{7a-d}$$

Solving (7) we have

$$\sigma_1 = \frac{\sigma_0(1+k)[m(1-\nu_1\nu_2) + m^2k(1-\nu_2^2)]}{1-\nu_1^2 + 2mk(1-\nu_1\nu_2) + m^2k^2(1-\nu_2^2)} \quad ,$$

$$\sigma_2 = \sigma_0(1+k) - \sigma_1 k \quad ,$$

$$\sigma_{yy1} = \frac{\sigma_1 - m\sigma_2}{\nu_1 + mk\nu_2} \quad , \quad \sigma_{yy2} = -k\sigma_{yy1} \quad . \qquad (8a\text{-}d)$$

Referring to the cylindrical coordinates the stress states in the layers 1 and 2 may be expressed as

$$\sigma_{rrj} = \frac{\sigma_j+\sigma_{yyj}}{2} + \frac{\sigma_j-\sigma_{yyj}}{2}\cos 2\theta \quad ,$$

$$\sigma_{\theta\theta j} = \frac{\sigma_j+\sigma_{yyj}}{2} - \frac{\sigma_j-\sigma_{yyj}}{2}\cos 2\theta \quad ,$$

$$\sigma_{r\theta j} = -\frac{\sigma_j-\sigma_{yyj}}{2}\sin 2\theta \quad ,$$

$$\sigma_{zzj} = \sigma_{rzj} = \sigma_{\theta zj} = 0 \quad , \quad (j=1,2) \quad . \qquad (9a\text{-}f)$$

Solutions similar to that given by (4) and (8) may also be obtained for other types of uniform external loads such as bending and thermally induced loading.

3. SOLUTION OF THE AXISYMMETRIC PROBLEM

The problem of a laminated plate containing a circular hole with a radius a may now be solved by superimposing on the homogeneous solutions found in the previous section a perturbation solution in which the tractions $-\sigma_{r\alpha j}(a,\theta,z)$, $(\alpha = r,\theta,z;\ j=1,2;\ 0\le\theta<2\pi,\ -h_2<z<h_1)$ acting on the hole boundary are the only external loads, where the stress state $\sigma_{\beta\alpha j}(r,\theta,z)$, $(\beta,\alpha = r,\theta,z;\ j=1,2)$ is given by the homogeneous solution. From, for example, (1), (4), (8), and (9) it may be seen that the simplest such perturbation problem is an axisymmetric problem corresponding to the

axisymmetrically loaded plate or to the first part of the unidirectionally loaded plate (i.e., to the θ-independent part of the solution given by (9)). Thus, in this section we will consider the axisymmetric problem for the composite plate subjected to the following boundary conditions (Figure 1):

$$\sigma_{rrj}(a,z) = \sigma_j(z) \quad , \quad \sigma_{rzj}(a,z) = \tau_j(z) \quad , \quad (j=1,2) \quad (10a,b)$$

where σ_j and τ_j are known tractions acting on the layers 1 and 2, respectively. Note that the solution is independent of θ and the shear stresses $\sigma_{\theta\beta j} = 0$, ($\beta = r,z$; $j=1,2$) everywhere.

To formulate the problem the technique described in [9] will be used. Because of axisymmetry, it is sufficient to use the z-component of the Galerkin vector only which is nothing but the Love strain function $Z(r,z)$ [11,12]. In addition to the surface tractions given by (10) the composite plate is subjected to the following homogeneous boundary, continuity, and regularity conditions (Figure 1):

$$\sigma_{rz1}(r,h_1) = 0 \quad , \quad \sigma_{zz1}(r,h_1) = 0 \quad , \quad (a<r<\infty) \quad , \tag{11}$$

$$\sigma_{rz2}(r,-h_2) = 0 \quad , \quad \sigma_{zz2}(r,-h_2) = 0 \quad , \quad (a<r<\infty) \quad , \tag{12}$$

$$\sigma_{rz1}(r,0) = \sigma_{rz2}(r,0) \quad , \quad \sigma_{zz1}(r,0) = \sigma_{zz2}(r,0) \quad , \quad (a<r<\infty) \quad , \tag{13}$$

$$u_{r1}(r,0) = u_{r2}(r,0) \quad , \quad u_{z1}(r,0) = u_{z2}(r,0) \quad , \quad (a<r<\infty) \quad , \tag{14}$$

$$\sigma_{rrj}(\infty,z) = 0 \quad , \quad \sigma_{rzj}(\infty,z) = 0 \quad , \quad (j=1,2) \quad . \tag{15}$$

Let Z_1 and Z_2 be the Love strain functions for layers 1 and 2, respectively (Figure 1). In the absence of body forces Z_1 and Z_2 satisfy

$$\nabla^2\nabla^2 Z_j(r,z) = 0 \quad , \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad , \quad (j=1,2) \quad , \tag{16}$$

and the displacements and stresses are given by

$$2\mu u_r = -\frac{\partial^2 Z}{\partial z^2}\ , \quad 2\mu u_z = [2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]Z \ ; \tag{17}$$

$$\sigma_{rr} = \frac{\partial}{\partial z}(\nu\nabla^2 - \frac{\partial^2}{\partial r^2})Z \ , \quad \sigma_{\theta\theta} = \frac{\partial}{\partial z}(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r})Z \ ,$$

$$\sigma_{zz} = \frac{\partial}{\partial z}[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]Z \ , \quad \sigma_{rz} = \frac{\partial}{\partial r}[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}]Z \ . \tag{18}$$

Looking for a series type solution in z and taking into account the regularity conditions (15), the solution of (16) may be expressed as

$$Z_1 = \sum_{n=1}^{\infty}[A_{1n}K_o(\alpha_n r) + A_{3n}\alpha_n r K_1(\alpha_n r) + A_{5n} z\alpha_n K_o(\alpha_n r)]\sin\alpha_n z$$

$$+ \sum_{n=1}^{\infty}[B_{1n}K_o(\alpha_n r) + B_{3n}\alpha_n r K_1(\alpha_n r) + B_{5n} z\alpha_n K_o(\alpha_n r)]\cos\alpha_n z \ , \tag{19}$$

$$Z_2 = \sum_{n=1}^{\infty}[C_{1n}K_o(\alpha_n r) + C_{3n}\alpha_n r K_1(\alpha_n r) + C_{5n} z\alpha_n K_o(\alpha_n r)]\sin\alpha_n z$$

$$+ \sum_{n=1}^{\infty}[D_{1n}K_o(\alpha_n r) + D_{3n}\alpha_n r K_1(\alpha_n r) + D_{5n} z\alpha_n K_o(\alpha_n r)]\cos\alpha_n z \ . \tag{20}$$

Substituting from (17)-(20) into the boundary and continuity conditions (11)-(14) and observing that these conditions must be satisfied for all values of r in $a<r<\infty$, after some manipulations, it may be shown that part of the resulting algebraic system which contains only the unknowns A_{3n}, B_{3n}, C_{3n}, and D_{3n} is separated and is overdetermined. Hence for the conditions (11)-(14) to be satisfied it is necessary that

$$A_{3n} = B_{3n} = C_{3n} = D_{3n} = 0 \ . \tag{21}$$

Considering (21) from (17)-(20) the displacements and stresses in the composite plate may now be expressed as follows:

$$u_{r1} = \frac{1}{2\mu_1}\{\sum_{n=1}^{\infty} \alpha_n^2 K_1(\alpha_n r)[A_{1n} + A_{5n} z\alpha_n + B_{5n}]\cos\alpha_n z$$

$$- \sum_{n=1}^{\infty} \alpha_n^2 K_1(\alpha_n r)[B_{1n} + B_{5n} z\alpha_n - A_{5n}]\sin\alpha_n z\} \quad , \qquad (22)$$

$$u_{z1} = \frac{1}{2\mu_1}\{\sum_{n=1}^{\infty} \alpha_n^2 \sin\alpha_n z K_o(\alpha_n r)[(-2+4\nu_1)B_{5n} + A_{1n}$$

$$+ A_{5n} z\alpha_n] + \sum_{n=1}^{\infty} \alpha_n^2 \cos\alpha_n z K_o(\alpha_n r)[(2-4\nu_1)A_{5n}$$

$$+ B_{1n} + B_{5n} z\alpha_n]\} \quad , \qquad (23)$$

$$\sigma_{rr1} = \sum_{n=1}^{\infty} \alpha_n^3\{B_{1n}[K_o(\alpha_n r) + \frac{1}{\alpha_n r} K_1(\alpha_n r)]$$

$$+ B_{5n} z[\alpha_n K_o(\alpha_n r) + \frac{1}{r} K_1(\alpha_n r)]$$

$$- A_{5n}[\frac{1}{\alpha_n r} K_1(\alpha_n r) + (1+2\nu_1)K_o(\alpha_n r)]\}\sin\alpha_n z$$

$$- \sum_{n=1}^{\infty} \alpha_n^3\{A_{1n}[K_o(\alpha_n r) + \frac{1}{\alpha_n r} K_1(\alpha_n r)] + A_{5n} z[\alpha_n K_o(\alpha_n r)$$

$$+ \frac{1}{r} K_1(\alpha_n r)] + B_{5n}[(1+2\nu_1)K_o(\alpha_n r)$$

$$+ \frac{1}{\alpha_n r} K_1(\alpha_n r)]\} \cos\alpha_n z \quad , \qquad (24)$$

$$\sigma_{\theta\theta 1} = - \sum_{n=1}^{\infty} \alpha_n^2 \sin\alpha_n z\{\frac{K_1(\alpha_n r)}{r} [B_{1n} + B_{5n} z\alpha_n - A_{5n}]$$

$$+ 2\nu_1 \alpha_n A_{5n} K_o(\alpha_n r)\} + \sum_{n=1}^{\infty} \alpha_n^2 \cos\alpha_n z\{ \frac{K_1(\alpha_n r)}{r} [A_{1n}$$

$$+ A_{5n} z\alpha_n + B_{5n}] - 2\nu_1 \alpha_n B_{5n} K_o(\alpha_n r)\} \quad , \qquad (25)$$

$$\sigma_{zz1} = - \sum_{n=1}^{\infty} \alpha_n^3 K_o(\alpha_n r)[B_{1n} + B_{5n} z\alpha_n + (1-2\nu_1)A_{5n}]\sin\alpha_n z$$

$$+ \sum_{n=1}^{\infty} \alpha_n^3 K_o(\alpha_n r)[A_{1n} + A_{5n} z\alpha_n + (2\nu_1 - 1)B_{5n}]\cos\alpha_n z \quad , \qquad (26)$$

$$\sigma_{rz1} = - \sum_{n=1}^{\infty} \alpha_n^3 K_1(\alpha_n r)\sin\alpha_n z[2\nu_1 B_{5n} + A_{1n} + A_{5n} z\alpha_n]$$

$$- \sum_{n=1}^{\infty} \alpha_n^3 K_1(\alpha_n r)\cos\alpha_n z[-2\nu_1 A_{5n} + B_{1n} + B_{5n} z\alpha_n] \quad , \qquad (27)$$

$$u_{r2} = \frac{1}{2\mu_2}\{\sum_{n=1}^{\infty} \alpha_n^2 K_1(\alpha_n r)[C_{1n} + C_{5n} z\alpha_n + D_{5n}]\cos\alpha_n z$$

$$- \sum_{n=1}^{\infty} \alpha_n^2 K_1(\alpha_n r)[D_{1n} + D_{5n} z\alpha_n - C_{5n}]\sin\alpha_n z\} \quad , \qquad (28)$$

$$u_{z2} = \frac{1}{2\mu_2}\{\sum_{n=1}^{\infty} \alpha_n^2 \sin\alpha_n z K_o(\alpha_n r)[(-2+4\nu_2)D_{5n}$$

$$+ C_{1n} + C_{5n} z\alpha_n] + \sum_{n=1}^{\infty} \alpha_n^2 \cos\alpha_n z K_o(\alpha_n r)[(2-4\nu_2)C_{5n}$$

$$+ D_{1n} + D_{5n} z\alpha_n]\} \quad , \qquad (29)$$

$$\sigma_{rr2} = \sum_{n=1}^{\infty} \alpha_n^3\{D_{1n}[K_o(\alpha_n r) + \frac{1}{\alpha_n r} K_1(\alpha_n r)]$$

$$+ D_{5n} z[\alpha_n K_o(\alpha_n r) + \frac{1}{r} K_1(\alpha_n r)]$$

$$- C_{5n}[\frac{1}{\alpha_n r} K_1(\alpha_n r) + (1+2\nu_2)K_o(\alpha_n r)]\}\sin\alpha_n z$$

$$- \sum_{n=1}^{\infty} \alpha_n^3\{C_{1n}[K_o(\alpha_n r) + \frac{1}{\alpha_n r} K_1(\alpha_n r)] + C_{5n} z[\alpha_n K_o(\alpha_n r)$$

$$+ \frac{1}{r} K_1(\alpha_n r)] + D_{5n}[(1+2\nu_2)K_o(\alpha_n r) + \frac{1}{\alpha_n r} K_1(\alpha_n r)]\}\cos\alpha_n z, \quad (30)$$

$$\sigma_{\theta\theta 2} = - \sum_{n=1}^{\infty} \alpha_n^2 \sin\alpha_n z\{\frac{K_1(\alpha_n r)}{r}[D_{1n} + D_{5n} z\alpha_n$$

$$- C_{5n}] + 2\nu_2\alpha_n C_{5n}K_o(\alpha_n r)\} + \sum_{n=1}^{\infty} \alpha_n^2 \cos\alpha_n z\{$$

$$\frac{K_1(\alpha_n r)}{r}[C_{1n} + C_{5n}z\alpha_n + D_{5n}] - 2\nu_2\alpha_n D_{5n}K_o(\alpha_n r)\} \quad , \quad (31)$$

$$\sigma_{zz2} = - \sum_{n=1}^{\infty} \alpha_n^3 K_o(\alpha_n r)[D_{1n} + D_{5n}z\alpha_n + (1-2\nu_2)C_{5n}]\sin\alpha_n z$$

$$+ \sum_{n=1}^{\infty} \alpha_n^3 K_o(\alpha_n r)[C_{1n} + C_{5n}z\alpha_n + (2\nu_2-1)D_{5n}]\cos\alpha_n z \quad , \quad (32)$$

$$\sigma_{rz2} = - \sum_{n=1}^{\infty} \alpha_n^3 K_1(\alpha_n r)\sin\alpha_n z[2\nu_2 D_{5n} + C_{1n} + C_{5n}z\alpha_n]$$

$$- \sum_{n=1}^{\infty} \alpha_n^3 K_1(\alpha_n r)\cos\alpha_n z[-2\nu_2 C_{5n} + D_{1n} + D_{5n}z\alpha_n] \quad . \quad (33)$$

Substituting from (22)-(33) into the boundary and continuity conditions (11)-(14) we obtain the following system of homogeneous algebraic equations:

$$-A_{1n}\sin\alpha_n h_1 + [2\nu_1\cos\alpha_n h_1 - \alpha_n h_1\sin\alpha_n h_1]A_{5n}$$

$$-B_{1n}\cos\alpha_n h_1 - [2\nu_1\sin\alpha_n h_1 + \alpha_n h_1\cos\alpha_n h_1]B_{5n} = 0 \quad , \quad (34)$$

$$A_{1n}\cos\alpha_n h_1 + [(2\nu_1-1)\sin\alpha_n h_1 + \alpha_n h_1\cos\alpha_n h_1]A_{5n} - B_{1n}\sin\alpha_n h_1$$

$$+ [(2\nu_1-1)\cos\alpha_n h_1 - \alpha_n h_1\sin\alpha_n h_1]B_{5n} = 0 \quad , \quad (35)$$

$$C_{1n}\sin\alpha_n h_2 + [2\nu_2\cos\alpha_n h_2 - \alpha_n h_2\sin\alpha_n h_2]C_{5n}$$

$$- D_{1n}\cos\alpha_n h_2 + [2\nu_2\sin\alpha_n h_2 + \alpha_n h_2\cos\alpha_n h_2]D_{5n} = 0 \quad , \quad (36)$$

$$C_{1n}\cos\alpha_n h_2 + [(1-2\nu_2)\sin\alpha_n h_2 - \alpha_n h_2\cos\alpha_n h_2]C_{5n}$$

$$+ D_{1n}\sin\alpha_n h_2 + [(2\nu_2-1)\cos\alpha_n h_2 - \alpha_n h_2 \sin\alpha_n h_2]D_{5n} = 0 \quad , \tag{37}$$

$$- 2\nu_1 A_{5n} + B_{1n} + 2\nu_2 C_{5n} - D_{1n} = 0 \quad , \tag{38}$$

$$A_{1n} + (2\nu_1-1)B_{5n} - C_{1n} - (2\nu_2-1)D_{5n} = 0 \quad , \tag{39}$$

$$\beta A_{1n} + \beta B_{5n} - C_{1n} - D_{5n} = 0 \quad , \tag{40}$$

$$\beta(2-4\nu_1)A_{5n} + \beta B_{1n} - (2-4\nu_2)C_{5n} - D_{1n} = 0 \quad , \tag{41}$$

where

$$\beta = \mu_2/\mu_1 \quad . \tag{42}$$

To obtain a non-trivial solution for the system of algebraic equations (34)-(42), the determinant of coefficients must vanish, giving

$$\Delta(\alpha_n) = 0 \tag{43}$$

The characteristic equation (43) gives the eigenvalues α_n, $(n=1,2,\ldots)$ of the problem. A close examination of the roots of (43) shows that if α_n is a root so are $-\alpha_n$, $\bar{\alpha}_n$, and $-\bar{\alpha}_n$. Therefore in solving the problem it is sufficient to use the roots in the first quadrant and consider the real part of the solution only. Further examination of the roots indicates that for dissimilar materials generally there is only one positive real root and the remaining roots are all complex. Furthermore, the complex roots form two distinct sequences in the first quadrant which greatly facilitates their numerical evaluation. When the elastic constants of the layers 1 and 2 become equal (i.e., for a homogeneous layer), the real root disappears and the two sequences of complex eigenvalues become the roots of the following characteristic equations:

$$\sin 2\lambda_n + 2\lambda_n = 0 \quad , \quad \sin 2\lambda_n - 2\lambda_n = 0 \quad , \quad \lambda_n = \alpha_n(h_1+h_2) \quad . \qquad (44a,b)$$

Equations (44) are known to correspond to the extension and bending problems for a homogeneous thick plate containing a circular hole [8,9].

It is found that $\alpha_0 = 0$ is also a root of (43). Therefore, a particular solution must be added to that given by (19) and (20) to account for the zero eigenvalue. Considered as single "plates" the layers are subjected to stretching, bending, and transverse shear. The particular solutions Z_1^0, and Z_2^0 must therefore exhibit the characteristics of all three modes of loading. Thus

$$Z_1^0 = N_1 z \ln r + M_1 z^2 \ln r + P_1 r^2 \ln r + Q_1(z^2 - r^2/2) \quad ,$$

$$Z_2^0 = N_2 z \ln r + M_2 z^2 \ln r + P_2 r^2 \ln r \quad . \qquad (45a,b)$$

where the first terms in each expression correspond to stretching, and the next two terms to combined bending and transverse shear in the individual plates.* The term $Q_1(z^2-r^2/2)$ corresponds to a rigid body translation in z-direction and is added to (45a) to insure continuity of displacements at the interface. The constants M_1, N_1, P_1, M_2, N_2, and P_2 which appear in (45) are not independent. By using expressions (17) and (18) which relate the displacements and the stresses to the Love strain function, all field quantities can be written in terms of these constants. Then, by applying the boundary and continuity conditions (11)-(14), after some lengthy algebra and after redefining the constants we obtain:

$$Z_1^0 = D_0 z \ln r + A_0 z^2 \ln r + \frac{\nu_1}{2(1-\nu_1)} A_0 r^2 \ln r + Q_1(z^2 - r^2/2)$$

$$Z_2^0 = \beta D_0 z \ln r + \beta A_0 z^2 \ln r + \beta \frac{\nu_2}{2(1-\nu_2)} A_0 r^2 \ln r \qquad (46a,b)$$

*Note that Z_1^0 and Z_2^0 are of the form $f(r) + g(r)h(z) + m(z)$ suggested in [9].

The stress field generated by (46) can then be expressed as:

$$\sigma^o_{rr1} = \frac{1}{r^2}(D_o+2A_oz) \quad , \quad \sigma^o_{\theta\theta 1} = -\frac{1}{r^2}(D_o+2A_oz) \quad ,$$

$$\sigma^o_{zz1} = 0 \quad , \quad \sigma^o_{rz1} = 0 \quad , \tag{47a-d}$$

$$\sigma^o_{rr2} = \frac{\beta}{r^2}(D_o+2A_oz) \quad , \quad \sigma^o_{\theta\theta 2} = -\frac{\beta}{r^2}(D_o+2A_oz) \quad ,$$

$$\sigma^o_{zz2} = 0 \quad , \quad \sigma^o_{rz2} = 0 \quad . \tag{48a-d}$$

In the perturbation problem the stress states in layers 1 and 2 are obtained by adding the respective stress components given by (24)-(27), (30)-(33), (47) and (48). Thus, the problem is reduced to one of determining the unknown constants A_o, D_o, A_{1n}, A_{5n}, B_{1n}, B_{5n}, C_{1n}, C_{5n}, D_{1n}, and D_{5n}, $(n=1,2,\ldots)$. However, from (34)-(42) it is clear that the homogeneous system contains only one arbitrary constant for each eigenvalue α_n, $(n=1,2,\ldots)$. For example, one may assume that A_{1n}, $(n=1,2..)$ is the only unknown in (34)-(42) and the remaining seven unknowns $A_{5n},\ldots,D_{5n}$ may be expressed in terms of A_{1n} after solving the related eigenvalue problem. The unknown constants A_o, D_o, and A_{1n}, $(n=1,2,..)$ are then determined from the boundary conditions (10). To do this, we first substitute from the expressions (24), (47a), (30), (48a) and (27), (47d), (33), (48d) into (10a) and (10b), respectively. In the resulting equations by expanding both sides into a series of an appropriate system of orthogonal functions in $-h_2<z<h_1$ and by matching the coefficients we obtain a linear system of algebraic equations to determine the unknown coefficients A_o, D_o, and A_{1n}. The algebraic system is infinite and may be solved by the method of reduction.

If we use the first N+1 functions of a real orthogonal system, the (real parts of) conditions (10) would give 2N+2 equations. On the other hand, since A_o, D_o, A_{11} (corresponding to the real eigenvalue α_1) are real and $A_{12},A_{13},\ldots$ are complex, truncating the series (24), (27), (30) and (33) at the Nth term we would have 2N+1 real unknowns. However, it can be shown that this discrepancy disappears if one selects an orthogonal system in which the first function is a constant. Thus, if we substitute from (27), (47d), (33), and (48d) into

$$\int_{-h_2}^{h_1} \sigma_{rz} dz = \int_{-h_2}^{0} \sigma_{rz2} dz + \int_{0}^{h_1} \sigma_{rz1} dz \quad , \tag{49}$$

corresponding to the coefficient of the first coordinate function in expanding the lefthand side of (10b), it can be shown that the expression becomes identically zero. On the other hand, the static equilibrium of the composite plate requires that

$$\int_{-h_2}^{0} \tau_2 dz + \int_{0}^{h_1} \tau_1 dz = 0 \quad . \tag{50}$$

Thus, the first equation obtained from the series expansion of (10b) becomes an identity, $0 = 0$, and may therefore be ignored.

Now let us assume that the tractions are

$$\sigma_1(z) = -\sigma_1 \quad , \quad \sigma_2(z) = -\sigma_2 \quad , \quad \tau_1(z) = 0 \quad , \quad \tau_2(z) = 0 \quad , \tag{51}$$

where σ_1 and σ_2 are constant, and $\cos[\pi k(z+h_2)/(h_1+h_2)]$, $(k=0,1,2,\ldots)$ is selected as the orthogonal system. By expanding (10) into cosine series and considering the first N+1 terms we then obtain

$$\begin{aligned} \mathrm{Re} \sum_{n=1}^{N} \alpha_n^3 K_1(\alpha_n a) \{ & (2\nu_2 D_{5n} + C_{1n}) a_{nk} + \alpha_n C_{5n} b_{nk} \\ & + (-2\nu_2 C_{5n} + D_{1n}) c_{nk} + \alpha_n D_{5n} d_{nk} + (2\nu_1 B_{5n} + A_{1n}) e_{nk} \\ & + \alpha_n A_{5n} f_{nk} + (-2\nu_1 A_{5n} + B_{1n}) g_{nk} + \alpha_n B_{5n} h_{nk} \} = 0 \quad , \\ & (k=1,\ldots,N) , \end{aligned} \tag{52}$$

$$\mathrm{Re}\sum_{n=1}^{N} \alpha_n^3\{(X_n D_{1n} - Y_n C_{5n})a_{nk} + D_{5n}\alpha_n X_n b_{nk}$$

$$- (X_n C_{1n} + Y_n D_{5n})c_{nk} - \alpha_n C_{5n} X_n d_{nk}$$

$$+ (X_n B_{1n} - Z_n A_{5n})e_{nk} + \alpha_n B_{5n} X_n f_{nk}$$

$$- (X_n A_{1n} + Z_n B_{5n})g_{nk} - \alpha_n A_{5n} X_n h_{nk}\}$$

$$+ \frac{D_o}{a^2}(m_k + \beta n_k) + 2\,\frac{A_o}{a^2\gamma_k^2}\,[(-1)^k - \cos\gamma_k h_2 + \beta(\cos\gamma_k h_2 - 1)]$$

$$= -\sigma_1 m_k - \sigma_2 n_k \quad , \quad (k=1,\ldots,N), \tag{53}$$

$$\mathrm{Re}\sum_{n=1}^{N} \alpha_n^3\{(X_n D_{1n} - Y_n C_{5n})a_{no} + D_{5n}\alpha_n X_n b_{no}$$

$$- (X_n C_{1n} + Y_n D_{5n})c_{no} - \alpha_n C_{5n} X_n d_{no}$$

$$+ (X_n B_{1n} - Z_n A_{5n})e_{no} + \alpha_n B_{5n} X_n f_{no}$$

$$- (X_n A_{1n} + Z_n B_{5n})g_{no} - \alpha_n A_{5n} X_n h_{no}\}$$

$$+ \frac{D_o}{a^2}(h_1 + \beta h_2) + \frac{A_o}{a^2}(h_1^2 - \beta h_2^2) = -\sigma_1 h_1 - \sigma_2 h_2 \quad , \tag{54}$$

where

$$X_n = K_o(\alpha_n a) + \frac{1}{\alpha_n a} K_1(\alpha_n a) \quad ,$$

$$Y_n = (1+2\nu_2)K_o(\alpha_n a) + \frac{1}{\alpha_n a} K_1(\alpha_n a) \quad ,$$

$$Z_n = (1+2\nu_1)K_o(\alpha_n a) + \frac{1}{\alpha_n a} K_1(\alpha_n a) \quad , \quad (n=1,2\ldots,N) \quad ,$$

$$m_k = -\frac{\sin\gamma_k h_2}{\gamma_k} \quad , \quad n_k = -m_k \quad , \quad \gamma_k = \frac{k\pi}{h_1+h_2} \quad , \quad (k=1,2..,N) \quad ,$$

$$a_{nk} = \frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}[\cos(\gamma_k+\alpha_n)h_2 - 1]$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}\sin(\gamma_k+\alpha_n)h_2 - \frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}[\cos(\gamma_k-\alpha_n)h_2 - 1]$$

$$-\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}\sin(\gamma_k-\alpha_n)h_2 \quad , \quad (n=1,\ldots,N;\ k=1,\ldots,N) \quad ,$$

$$b_{nk} = -\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}h_2\cos(\gamma_k+\alpha_n)h_2 + \frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k+\alpha_n)^2}\sin(\gamma_k+\alpha_n)h_2$$

$$-\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}h_2\sin(\gamma_k+\alpha_n)h_2$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k+\alpha_n)^2}[1-\cos(\gamma_k+\alpha_n)h_2] + \frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}h_2\cos(\gamma_k-\alpha_n)h_2$$

$$-\frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k-\alpha_n)^2}\sin(\gamma_k-\alpha_n)h_2 + \frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}h_2\sin(\gamma_k-\alpha_n)h_2$$

$$-\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k-\alpha_n)^2}[1 - \cos(\gamma_k-\alpha_n)h_2] \quad , \quad (n=1,\ldots,N;\ k=1,\ldots,N) \quad ,$$

$$c_{nk} = \frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}\sin(\gamma_k+\alpha_n)h_2 + \frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}[1 - \cos(\gamma_k+\alpha_n)h_2]$$

$$+\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}\sin(\gamma_k-\alpha_n)h_2 + \frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}[1-\cos(\gamma_k-\alpha_n)h_2] \quad ,$$

$$(n = 1,\ldots,N;\ k = 1,\ldots,N) \quad ,$$

$$d_{nk} = -\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}h_2\sin(\gamma_k+\alpha_n)h_2 + \frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k+\alpha_n)^2}[1-\cos(\gamma_k+\alpha_n)h_2]$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}h_2\cos(\gamma_k+\alpha_n)h_2 - \frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k+\alpha_n)^2}\sin(\gamma_k+\alpha_n)h_2$$

$$-\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}h_2\sin(\gamma_k-\alpha_n)h_2 + \frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k-\alpha_n)^2}[1 - \cos(\gamma_k-\alpha_n)h_2]$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}h_2\cos(\gamma_k-\alpha_n)h_2-\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k-\alpha_n)^2}\sin(\gamma_k-\alpha_n)h_2 \quad ,$$

$$(n=1,\ldots,N;\ k=1,\ldots,N) \quad ,$$

$$e_{nk}=-\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}[\cos(\gamma_k+\alpha_n)h_1-1]+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}\sin(\gamma_k+\alpha_n)h_1$$

$$+\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}[\cos(\gamma_k-\alpha_n)h_1-1]-\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}\sin(\gamma_k-\alpha_n)h_1 \quad ,$$

$$(k=1,2,\ldots,N;\ n=1,\ldots,N)] \ ,$$

$$f_{nk}=-\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}h_1\cos(\gamma_k+\alpha_n)h_1+\frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k+\alpha_n)^2}\sin(\gamma_k+\alpha_n)h_1$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}h_1\sin(\gamma_k+\alpha_n)h_1+\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k+\alpha_n)^2}[\cos(\gamma_k+\alpha_n)h_1-1]$$

$$+\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}h_1\cos(\gamma_k-\alpha_n)h_1-\frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k-\alpha_n)^2}\sin(\gamma_k-\alpha_n)h_1$$

$$-\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}h_1\sin(\gamma_k-\alpha_n)h_1-\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k-\alpha_n)^2}[\cos(\gamma_k-\alpha_n)h_1-1] \quad ,$$

$$(n=1,\ldots N;\ k=1,\ldots,N) \quad ,$$

$$g_{nk}=\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}\sin(\gamma_k+\alpha_n)h_1+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}[\cos(\gamma_k+\alpha_n)h_1-1]$$

$$+\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n}\sin\ (\gamma_k-\alpha_n)h_1+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n}[\cos(\gamma_k-\alpha_n)h_1-1] \ ,$$

$$(n=1,\ldots,N;\ k=1,\ldots,N) \quad ,$$

$$h_{nk}=\frac{1}{2}\frac{\cos\gamma_k h_2}{\gamma_k+\alpha_n}h_1\sin\ (\gamma_k+\alpha_n)h_1+\frac{1}{2}\frac{\cos\gamma_k h_2}{(\gamma_k+\alpha_n)^2}[\cos(\gamma_k+\alpha_n)h_1-1]$$

$$+\frac{1}{2}\frac{\sin\gamma_k h_2}{\gamma_k+\alpha_n}h_1\ \cos(\gamma_k+\alpha_n)h_1-\frac{1}{2}\frac{\sin\gamma_k h_2}{(\gamma_k+\alpha_n)^2}\sin(\gamma_k+\alpha_n)h_1$$

$$+ \frac{1}{2} \frac{\cos\gamma_k h_2}{\gamma_k-\alpha_n} h_1 \sin(\gamma_k-\alpha_n)h_1 + \frac{1}{2} \frac{\cos\gamma_k h_2}{(\gamma_k-\alpha_n)^2} [\cos(\gamma_k-\alpha_n)h_1-1]$$

$$+ \frac{1}{2} \frac{\sin\gamma_k h_2}{\gamma_k-\alpha_n} h_1 \cos(\gamma_k-\alpha_n)h_1 - \frac{1}{2} \frac{\sin\gamma_k h_2}{(\gamma_k-\alpha_n)^2} \sin(\gamma_k-\alpha_n)h_1 \quad ,$$

$$(n=1,..,N;\ k=1,..,N) \quad ;$$

$$a_{no} = \frac{\cos\alpha_n h_2 - 1}{\alpha_n}$$

$$b_{no} = -\frac{h_2\cos\alpha_n h_2}{\alpha_n} + \frac{\sin\alpha_n h_2}{\alpha_n^2} \quad ,$$

$$c_{no} = \frac{\sin\alpha_n h_2}{\alpha_n} \quad ,$$

$$d_{no} = -\frac{h_2\sin\alpha_n h_2}{\alpha_n} + \frac{(1-\cos\alpha_n h_2)}{\alpha_n^2} \quad ,$$

$$e_{no} = \frac{1-\cos\alpha_n h_1}{\alpha_n} \quad ,$$

$$f_{no} = -\frac{h_1\cos\alpha_n h_1}{\alpha_n} + \frac{\sin\alpha_n h_1}{\alpha_n^2} \quad ,$$

$$g_{no} = \frac{\sin\alpha_n h_1}{\alpha_n} \quad ,$$

$$h_{no} = \frac{h_1\sin\alpha_n h_1}{\alpha_n} - \frac{(1-\cos\alpha_n h_1)}{\alpha_n^2} \quad , \quad (n=1,2..,N) \quad . \tag{55}$$

and β is given by (42). Even though it is difficult to investigate the regularity of the algebraic system (52)-(54), the numerical results show very good convergence with increasing N.

4. NUMERICAL RESULTS AND DISCUSSION

In the numerical example considered the following material properties and dimensions are used (Figure 1)

$$\frac{E_1}{E_2} = 2 \quad , \quad \nu_1 = 0.3 \quad , \quad \nu_2 = 0.35$$

$$\frac{a}{h_1} = 3 \quad , \quad \frac{a}{h_2} = 1.5 \quad . \tag{56}$$

The first 60 roots of the characteristic equation (43) obtained from the algebraic system (34)-(41) are given in Table 1.

Two separate loading conditions are used to calculate the stresses. In the first it is assumed that*

$$\sigma_{rr1}(a,z) = -\sigma_1 \quad , \quad \sigma_{rr2}(a,z) = -\sigma_2$$

$$\sigma_1/\sigma_2 = 1.964 \quad , \quad \sigma_{rz1}(a,z) = \sigma_{rz2}(a,z) = 0 \quad . \tag{57}$$

The second loading consists of a uniform pressure on the entire hole surface, namely

$$\sigma_{rr1}(a,z) = \sigma_{rr2}(a,z) = -\sigma_2 \quad ,$$

$$\sigma_{rz1}(a,z) = \sigma_{rz2}(a,z) = 0 \quad . \tag{58}$$

Tables 2-5 show the calculated results which are partially displayed also in Figures 2-5. Based on the calculated results one could make the following observations:

(a) Away from the hole boundary generally the convergence is quite good. It becomes slower when the hole boundary is approached. For the loading given by (57) the discontinuity in traction σ_{rr} may be partially responsible for this. However, the main reason for the lack of convergence of the calculated results near the hole boundary appears to be

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*The stress ratio 1.964 corresponds to (8a) and (8b) for the material pair under consideration.

the singular nature of the stress state at the intersection of the interface and the boundary [1-3]. Thus, near the hole boundary more terms in the infinite series were needed to obtain convergence comparable to that found in computing the stresses away from the boundary.*

(b) As $r \to \infty$ all stress components go to zero. However, the decay in $\sigma_{\theta\theta j}$ is much slower than that in σ_{zzj} and σ_{rzj}, $(j=1,2)$.

(c) From Figures 3 and 5 it may be seen that the relative magnitidues of the interface stresses σ_{rz} and σ_{zz} are rather small. Also, the stresses corresponding to $\sigma_1 = \sigma_2$ (Figure 5) are an order of magnitude greater than those obtained for $\sigma_1 = 1.964\ \sigma_2$ (Figure 3). A partial explanation for these results may be found if one considers the homogeneous plates separately. In a homogeneous plate axisymmetrically loaded by $\sigma_{rr1}(a,z) = -\sigma_1$, $\sigma_{rz1}(a,z) = 0$, $(0<z<h_1)$ the stress state is given by

$$\sigma_{rr1} = -\frac{\sigma_1}{(r/a)^2} = -\sigma_{\theta\theta 1} \quad , \quad \sigma_{rz1} = 0 \quad , \quad \sigma_{zz1} = 0 \quad , \tag{59}$$

from which it follows that

$$\varepsilon_{\theta\theta 1} = \frac{1}{2\mu_1}\frac{\sigma_1}{(r/a)^2} = -\varepsilon_{rr1} \quad , \quad \varepsilon_{rz1} = 0 \quad , \varepsilon_{zz1} = 0 \quad . \tag{60}$$

Thus, it is seen that if the second plate is axisymmetrically loaded by $\sigma_{rr2}(a,z) = -\sigma_2$, $\sigma_{rz2}(a,z) = 0$, $(-h_2<z<0)$ and if $\sigma_1/\mu_1 = \sigma_2/\mu_2$, then in the two plates the displacements would be identical along the interface and the stresses σ_{zz} and σ_{rz} would be zero everywhere. In the example under consideration $\mu_1/\mu_2 = 2.077$. Therefore, for $\sigma_1/\sigma_2 = 1.964$ one would expect the magnitude of the interface stresses to be rather small. Similar observations may be made with regard to the comparison of $\sigma_{\theta\theta}$ and σ_{rr} in bonded and unbonded plates. On the other hand, for $\sigma_1 = \sigma_2$

*The numerical results given in the tables are obtained by using 20 to 30 terms in the series for locations away from the hole and up to 60 terms near the hold boundary.

one would expect higher interface stresses because of the greater mismatch in the displacements along the interface.

(d) For the loading $\sigma_1/\sigma_2 = 1.964$ since the solution is close to that of a homogeneous plate, the thickness effect should not be significant. Indeed, by varying a/h_1 it is observed that the results do not change significantly. Also, in this case from Table 3 it may be seen that the variation of $\sigma_{\theta\theta}$ with z is negligible, whereas for $\sigma_1/\sigma_2=1$ Table 5 shows a significant variation in $\sigma_{\theta\theta}$. Again, note that in the homogeneous plate $\sigma_{\theta\theta}$ is independent of z.

(e) The calculated results indicate that on the interface z=0 the stresses become unbounded as r approaches a, the hole radius. Theoretically, this is indeed known to be the case [1-3]. The solution given in this paper is in terms of infinite series, meaning that for z=0 and r=a certain series should be divergent. In problems such as this one would have to determine the eigenvalues γ_n in closed form for large values of n by examining the asymptotic behavior of the characteristic equation, determine the related eigenfunctions again in closed form, and try to separate and sum the divergent part of the infinite series. Such a procedure seems to be quite impossible for the problem under consideration. However, if one has a reasonably good solution for sufficiently small values of r-a, one may then follow an indirect method to establish the singular behavior of the stresses in an approximate manner. To do this we note that from the plane strain solution of two bonded elastic quarter planes one may express the asymptotic behavior of the stresses for z=0 and for small values of r-a as follows [3]

$$\sigma_{ij} \cong \frac{A_{ij}}{(r-a)^\alpha} + B_{ij}(r-a)^{1-\alpha} + \cdots \quad , \quad (i,j=r,z) \tag{61}$$

where α is the root of the characteristic equation in the strip $0<Re(\alpha)<1$

$$(\cos\pi\alpha + c_{11} + \alpha c_{12} + c_{13}\alpha(\alpha+1)/2)(\cos\pi\alpha + c_{21} + c_{22}\alpha$$

$$+ c_{23}\alpha(\alpha+1)/2) - (d_{11}+d_{12}\alpha)(d_{21}+d_{22}\alpha) = 0 \quad , \tag{62}$$

where

$$c_{11} = \frac{1}{2} - \frac{m_1(1+\kappa_1)}{2(m_1+\kappa_2)} - \frac{3(1-m_1)}{2(1+m_1\kappa_1)} ,$$

$$c_{12} = \frac{6(1-m_1)}{1+m_1\kappa_1} , \quad c_{13} = -\frac{4(1-m_1)}{1+m_1\kappa_1} ,$$

$$c_{21} = \frac{1}{2} - \frac{m_2(1+\kappa_2)}{2(m_2+\kappa_1)} - \frac{3(1-m_2)}{2(1+m_2\kappa_2)} ,$$

$$c_{22} = \frac{6(1-m_2)}{1+m_2\kappa_2} , \quad c_{23} = -\frac{4(1-m_2)}{1+m_2\kappa_2} ,$$

$$d_{11} = \frac{3(1+\kappa_2)}{2(m_2+\kappa_1)} - \frac{1+\kappa_1}{2(1+m_2\kappa_2)} , \quad d_{12} = \frac{1+\kappa_1}{1+m_2\kappa_2} - \frac{1+\kappa_1}{m_2+\kappa_1} ,$$

$$d_{21} = \frac{3(1+\kappa_2)}{2(m_1+\kappa_2)} - \frac{1+\kappa_2}{2(1+m_1\kappa_1)} , \quad d_{22} = \frac{1+\kappa_2}{1+m_1\kappa_1} - \frac{1+\kappa_2}{m_1+\kappa_2} ,$$

$$m_1 = \mu_2/\mu_1 \quad m_2 = \mu_1/\mu_2 , \quad \kappa_i = 3-4\nu_i , \quad (i=1,2) .$$

For real material combinations it turns out that in $0<\mathrm{Re}(\alpha)<1$ (62) has only one root which is always real, and $\alpha=0$ is not a root (meaning that there is no need to investigate the possible existence of a logarithmic singularity). For the material constants given by (56) and used in this paper α is found to be

$$\alpha = 0.048940 . \tag{63}$$

Thus, the approximate asymptotic behavior of the stresses around ($z=0$, $r=a$) may be established by assuming that α is known and by using the last two calculated points for σ_{ij} in the expressions (61) to determine the corresponding constants A_{ij} and B_{ij}. The constants A_{ij} are usually referred to as the stress intensity factors. For the present problem they are found to be

σ_1/σ_2	$A_{\theta\theta 1}/\sigma_2 a^\alpha$	$A_{\theta\theta 2}/\sigma_2 a^\alpha$	$A_{zz}/10^{-2}\sigma_2 a^\alpha$	$A_{rz}/10^{-2}\sigma_2 a^\alpha$
1.964	1.608	0.776	0.082	-2.224
1.000	1.144	0.562	0.990	-20.918

This is essentially a curve-fitting process to a smooth data. Consequently, for example, it was observed that the next point calculated from (61) is rather in good agreement with the stresses given by series solution.

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REFERENCES

1. D. B. Bogy, "Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading," J. Applied Mechanics, Vol. 35, Trans. ASME, pp. 460-466, 1968.

2. V. L. Hein and F. Erdogan, "Stress singularities in a two-material wedge," Int. J. of Fracture Mechanics, Vol. 7, pp. 317-330, 1971.

3. F. Erdogan and V. Biricikoglu, "Two bonded half planes with a crack going through the interface," Int. J. Engng. Sci., Vol. 11, pp. 745-766, 1973.

4. N. J. Salamon, "An assesment of the interlaminar stress problem in laminated composites," J. of Composite Materials supplement, Vol. 14, pp. 177-194, 1980.

5. J. R. Dana and R. M. Barker, "Three-dimensional analysis for the stress distribution near circular holes in laminated composites," Report VPI-E-74-18, VPI&SU, Blacksburg, Virginia, 1974.

6. R. M. Barker, J. R. Dana and C. W. Pryor, Jr., "Three-dimensional analysis of stress concentration near holes in laminated composites," Report VPI-E-72-27, VPI&SU, Blacksburg, Virginia, 1972.

7. E. F. Rybicki and D. W. Schmuser, "Effect of stacking sequence and lay-up angle on free edge stresses around a hole in a laminated plate under tension," J. Composite Materials, Vol. 12, pp. 300-313, 1978.

8. J. B. Alblas, "Theorie van de driedimensionale spanningstoestand in een doorboorde plaat," Ph.D. Thesis, H. J. Paris-Amsterdam, 1957.

9. F. Delale and F. Erdogan, "Three-dimensional elasticity solution of an infinite plate with a hole under extension," NASA, Technical Report, Lehigh University, June 1981.

10. F. Erdogan and M. Bakioglu, "Stress-free end problem in layered materials," International J. of Fracture, Vol. 13, pp. 739-749, 1977.

11. A. E. H. Love, A Treatise on the Mathematical Theory of Elasticity, Dover Publications, Inc. New York, 1944.

12. R. W. Little, Elasticity, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1973.

Table 1. The first 60 roots of the characteristic equation (43)

n	$Re(\alpha_n h_1)$	$I_m(\alpha_n h_1)$	
1	2.576640149	0	
2	1.293810176		0.8298514402
3	2.885866996	0.7551679918	
4	4.384613298		1.587138202
5	4.508247667	0.9781794212	
6	6.169990276	1.228997641	
7	7.582619850		2.113409611
8	7.752063359	1.340085172	
9	9.342477506	1.451922791	
10	10.77125258		2.453236163
11	10.92034586	1.528341711	
12	12.50012425	1.602698086	
13	13.94523374		2.705188038
14	14.07574872	1.661164351	
15	15.65192294	1.717452823	
16	17.11030656		2.905787940
17	17.22622651	1.764831059	
18	18.80069246	1.810326620	
19	20.26975009		3.072577814
20	20.37415766	1.850159284	
21	21.94765603	1.888423334	
22	23.42541951		3.215377334
23	23.52055621	1.922785421	
24	25.0934437	1.955843539	
25	26.57843058		3.340252251
26	26.66594450	1.986056608	
27	28.23841688	2.0515177619	
28	29.72949401		3.451217584
29	29.81062486	2.042135663	
30	31.38280031	2.068169865	
31	32.87908457		3.551071015
32	32.95478667	2.092505940	
33	34.52674116	2.116052348	
34	36.02753210		3.641841058
35	36.09855559	2.138231174	
36	37.67034054	2.159728360	
37	39.17507330		3.725045967
38	39.24201876	2.180101156	
39	40.81367039	2.199880230	
40	42.3218827		3.801851958
41	42.38523874	2.218718904	
42	43.95678339	2.237036178	
43	45.46809202		3.873174424

Table 1 (Cont.)

n	$Re(\alpha_n h_1)$	$I_m(\alpha_n h_1)$	
44	45.52826175	2.254555535	
45	47.09971910	2.271613599	
46	48.61380260		3.939745157
47	48.67112277	2.287986286	
48	50.24250782	2.303948118	
49	51.7590938		4.002158438
50	51.81384877	2.319314977	
51	53.38517321	2.334313681	
52	54.90402880		4.060903496
53	54.95646096	2.348791111	
54	56.52773403	2.362936830	
55	58.04865835		4.116387922
56	58.09897623	2.376622132	
57	59.67020534	2.390007063	
58	61.19302391		4.16895489
59	61.24140822	2.402982400	
60	62.81259940	2.415684503	

Table 2. Variation of the stresses at the interface z=0, with r/a for $\sigma_1/\sigma_2 = 1.964$

r/a	$\sigma_{\theta\theta 1}/2\sigma_2$	$\sigma_{\theta\theta 2}/2\sigma_2$	$\sigma_{zz}/(2\sigma_2 x10^{-3})$	$\sigma_{rz}/(2\sigma_2 x10^{-3})$
1.025	0.955	0.460	-	-11.0
1.037	0.933	0.449	-1.0	-9.7
1.050	0.910	0.438	-1.5	-8.9
1.075	0.868	0.417	-1.8	-7.7
1.100	0.829	0.399	-1.8	-6.7
1.150	0.758	0.365	-1.5	-5.1
1.200	0.697	0.335	-1.1	-3.9
1.300	0.594	0.286	-0.3	-2.3
1.500	0.446	0.215	0.3	-0.8
1.750	0.328	0.158	0.3	-0.2
2.0	0.251	0.121	0.1	-0.05
3.0	0.111	0.054	0.	0.
4.0	0.063	0.030	0.	0.
5.0	0.040	0.019	0.	0.

Table 3. Distribution of stresses in z- direction for r/a=1.2 and $\sigma_1/\sigma_2 = 1.964$

	z/h_i	$\sigma_{zzi}/(2\sigma_2 x10^{-3})$	$\sigma_{rzi}/(2\sigma_2 x10^{-2}$	$\sigma_{\theta\theta i}/2\sigma_2$
i=1	0.	-1.1	-3.9	0.697
	0.25	-2.1	-2.1	0.692
	0.50	-1.5	0.4	0.687
	0.75	-0.5	1.2	0.682
	0.90	-0.1	0.7	0.679
	1.0	0.	0.	0.677
i=2	0.	-1.1	-3.9	0.335
	-0.25	0.1	-0.5	0.339
	-0.50	-0.4	1.7	0.344
	-0.75	-0.3	1.6	0.349
	-0.90	-0.08	0.9	0.353
	-1.0	0.	0.	0.355

Table 4. Variation of the stresses at the interface z=0, with r/a for $\sigma_1/\sigma_2 = 1.0$

r/a	$\sigma_{\theta\theta 1}/2\sigma_2$	$\sigma_{\theta\theta 2}/2\sigma_2$	$\sigma_{zz}/(2\sigma_2 x 10^{-2})$	$\sigma_{rz}/(2\sigma_2 x 10^{-2})$
1.025	0.672	0.324	-	-10.4
1.037	0.653	0.312	-0.9	-9.2
1.050	0.635	0.301	-1.4	-8.5
1.075	0.605	0.286	-1.7	-7.3
1.100	0.578	0.273	-1.7	-6.4
1.150	0.529	0.250	-1.4	-4.9
1.200	0.487	0.231	-1.0	-3.7
1.300	0.415	0.199	-0.3	-2.2
1.500	0.313	0.151	0.3	-0.8
1.750	0.230	0.111	0.3	-0.2
2.0	0.176	0.085	0.1	-0.05
3.0	0.078	0.038	0.	0.
4.0	0.044	0.021	0.	0.
5.0	0.028	0.014	0.	0.

Table 5. Distribution of stresses in z- direction for r/a=1.2 and $\sigma_1/\sigma_2 = 1.0$

	z/h_i	$\sigma_{zzi}/(2\sigma_2 x 10^{-2})$	$\sigma_{rzi}(2\sigma_2 x 10^{-2})$	$\sigma_{\theta\theta i}/2\sigma_2$
i=1	0.	-1.0	-3.7	0.487
	0.25	-2.0	-2.0	0.444
	0.50	-1.4	0.4	0.399
	0.75	-0.5	1.1	0.350
	0.90	-0.1	0.7	0.321
	1.0	0.	0.	0.302
i=2	0.	-1.0	-3.7	0.231
	-0.25	0.1	-0.5	0.270
	-0.50	-0.4	1.6	0.318
	-0.75	-0.3	1.6	0.369
	-0.90	-0.07	0.8	0.400
	-1.0	0.	0.	0.423

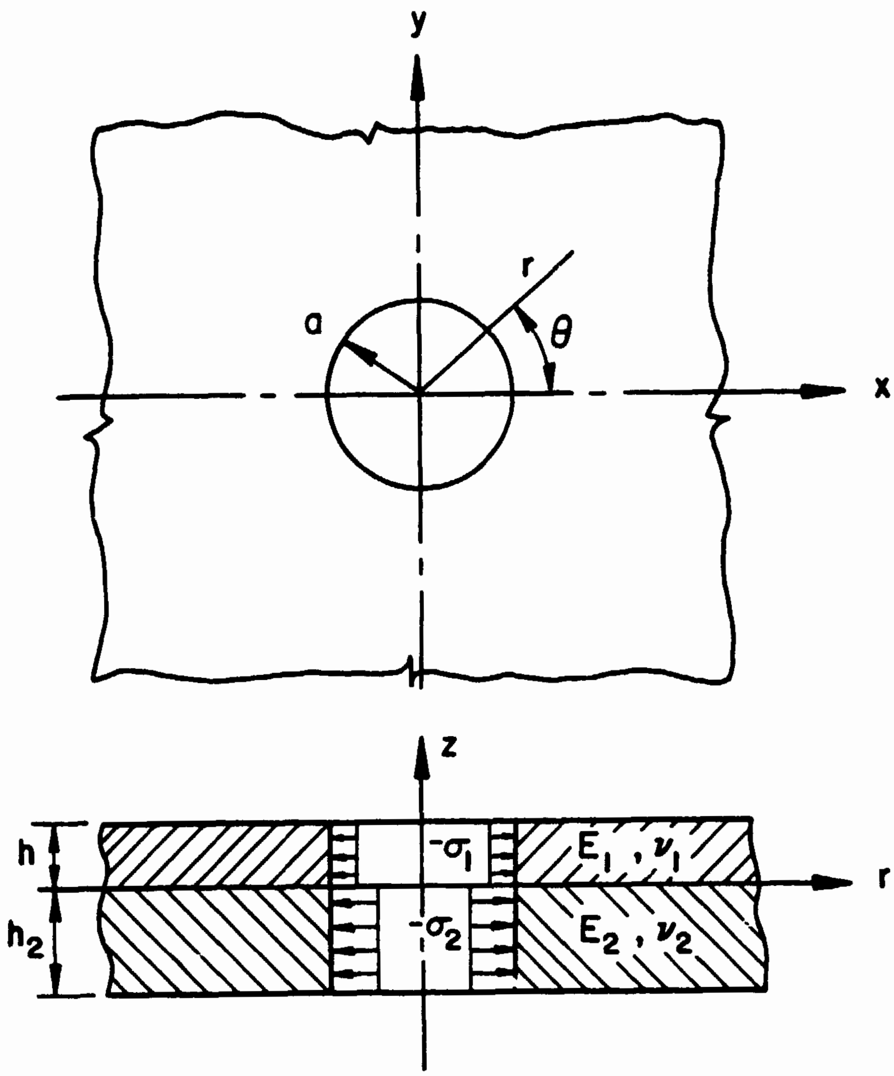

Figure 1. Geometry of the composite plate

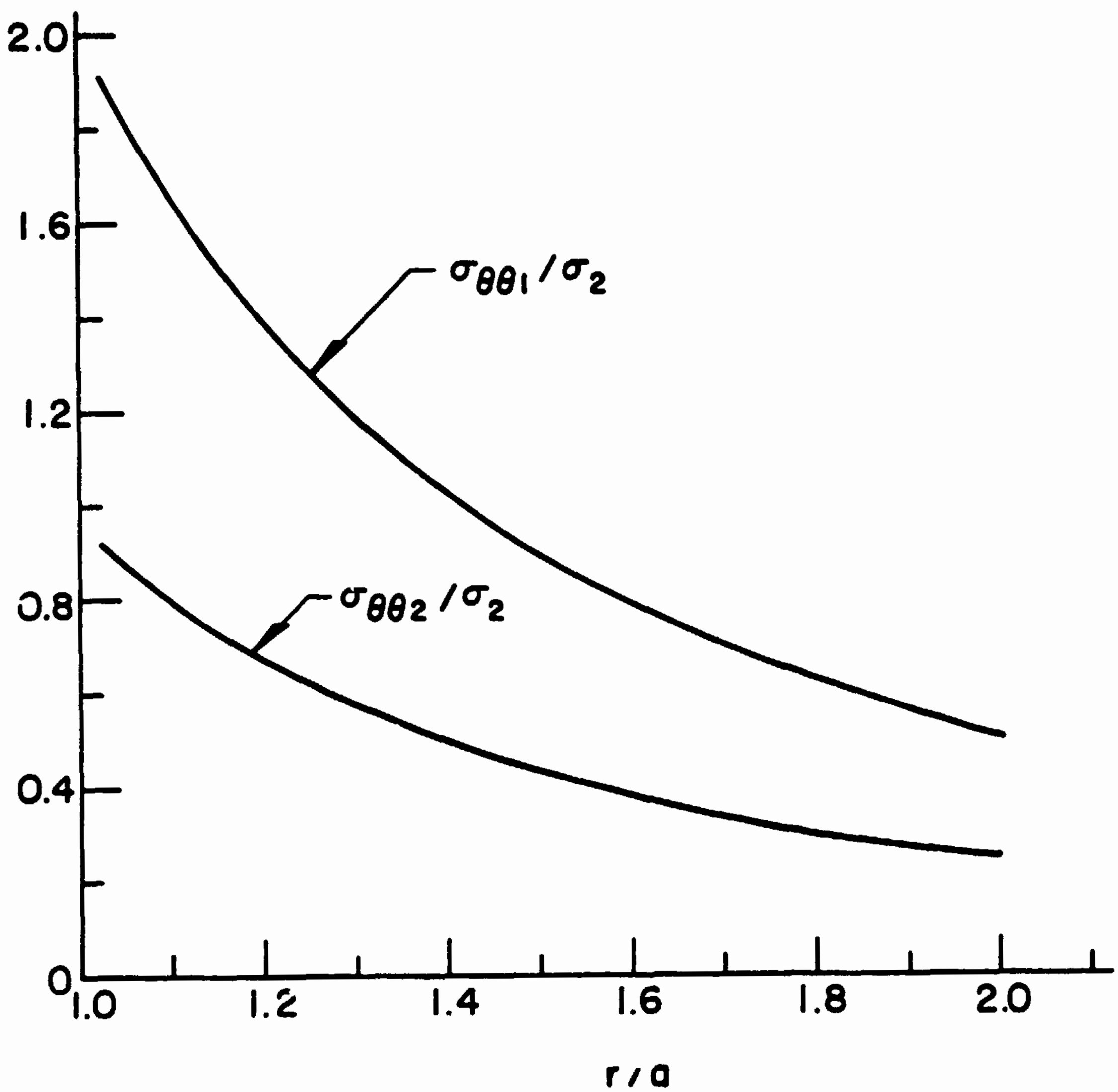

Figure 2. Variation of the hoop stresses $\sigma_{\theta\theta 1}(r,0)$ and $\sigma_{\theta\theta 2}(r,0)$ with r/a for $\sigma_1/\sigma_2 = 1.964$ and $z = 0$

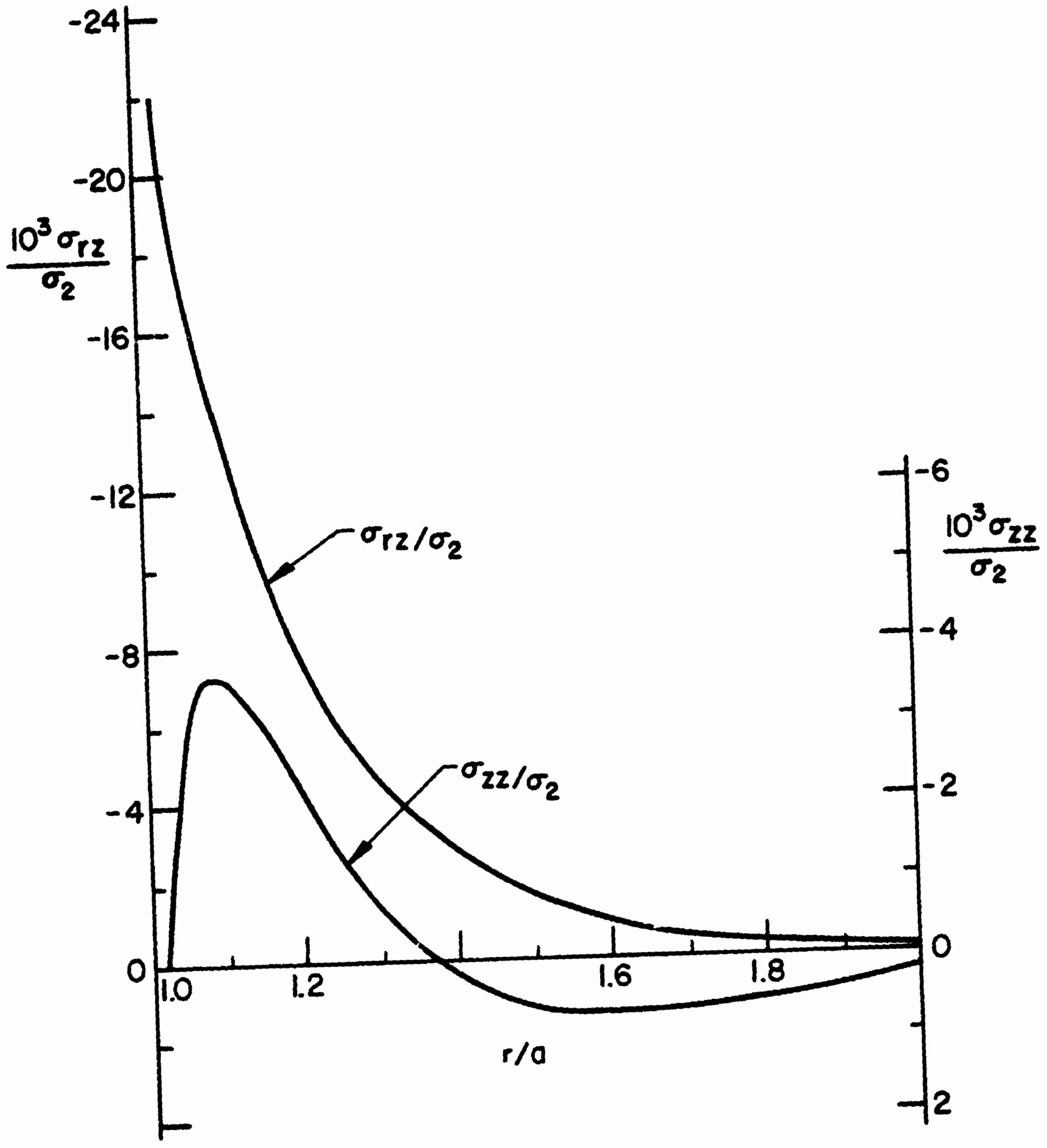

Figure 3. Variation of the interface stresses $\sigma_{zz}(r,0)$ and $\sigma_{rz}(r,0)$ with r/a for $\sigma_1/\sigma_2 = 1.964$

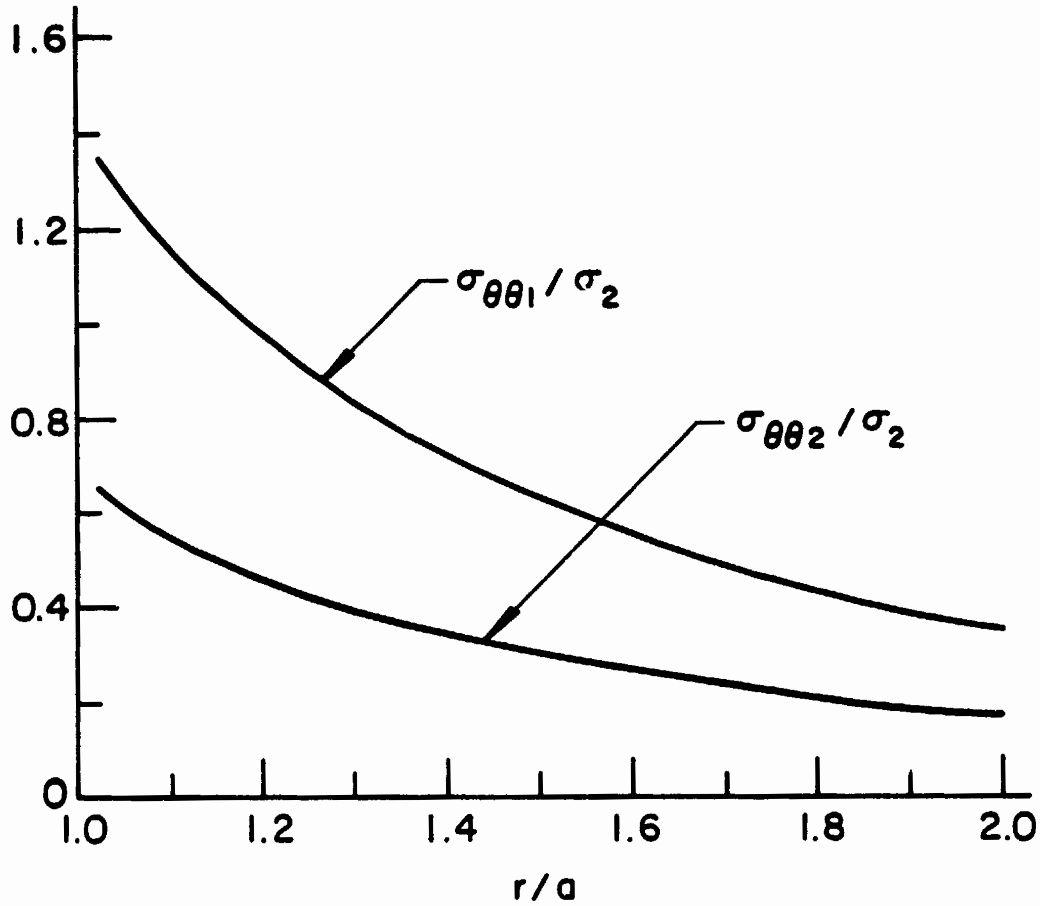

Figure 4. Variation of the hoop stresses $\sigma_{\theta\theta 1}(r,0)$ and $\sigma_{\theta\theta 2}(r,0)$ with r/a for $\sigma_1/\sigma_2 = 1$

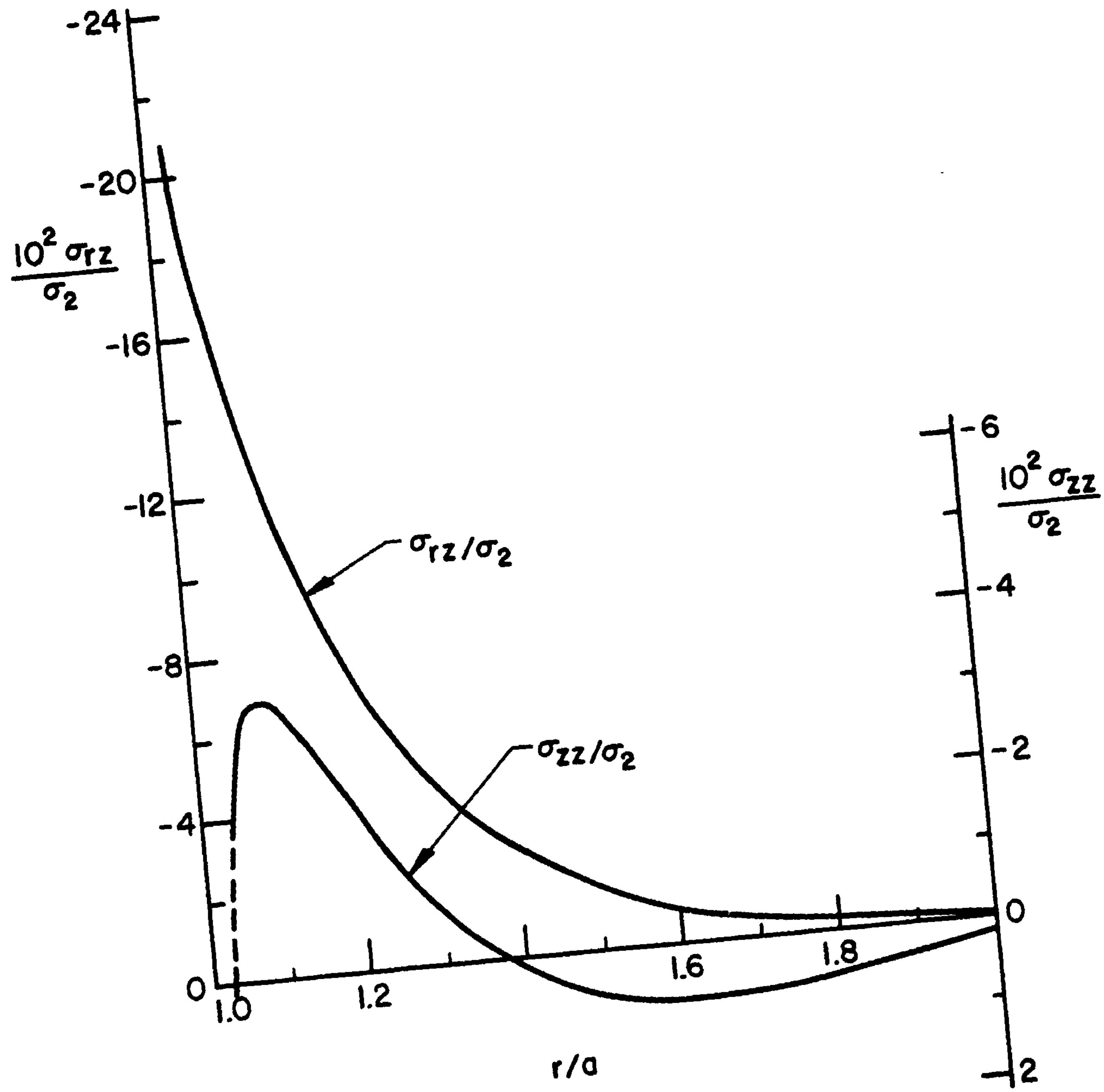

Figure 5. Variation of the interface stresses $\sigma_{zz}(r,0)$ and $\sigma_{rz}(r,0)$ with r/a for $\sigma_1/\sigma_2 = 1$