

EVALUATION OF AN IMPROVED FINITE-ELEMENT THERMAL STRESS CALCULATION TECHNIQUE

Charles J. Camarda
Aerospace Engineer, Langley Research Center
Hampton, Virginia

INTRODUCTION

The accurate calculation of thermal stresses in complicated airframe structures often requires a refined finite-element grid and a corresponding large computational time. In looking toward combined thermal-structural design and optimization calculations, it is essential to avoid having to solve an excessively large system of equations since such calculations are performed many times during optimization procedures.

A procedure for generating accurate thermal stresses with coarse finite-element grids was developed and described by Ojalvo (ref. 1). The procedure is based on the observation that for linear thermoelastic problems, the thermal stresses may be envisioned as being composed of two contributions--the first due to the strains in the structure which depend on the integral of the temperature distribution over the finite element and the second due to the local variation of the temperature in the element. Ojalvo's key idea was that the first contribution could be accurately predicted with a coarse finite-element mesh. The resulting strain distribution could then be combined via the constitutive relations with detailed temperatures from a separate thermal analysis. The result would be accurate thermal stresses from coarse finite-element structural models even where the temperature distributions have sharp variations.

Although this intriguing idea was proposed in 1974, its use has not been documented in the open literature except for the original AIAA Technical Note. It has recently received attention by this author because of the current interest at Langley in rapid analysis and design-oriented analysis techniques. The range of applicability of the method for various classes of thermostructural problems such as in-plane or bending type problems and the effect of the nature of the temperature distribution and edge constraints was not fully documented. These questions are addressed in this paper. Ojalvo's method is used in conjunction with the SPAR finite-element program (ref. 2) and extensive calculations are carried out and are described. Results are obtained for rods, membranes, a box beam and a stiffened panel.

SUMMARY OF OJALVO'S METHOD

The following is a brief summary of "Ojalvo's" method (ref. 1) for thermal stress calculations using a finite-element (F.E.) analysis procedure. For a detailed explanation of the following subject topics see references 1 and 3. The equations shown in figure 1 represent the system of matrix equations involved in a typical structural F.E. analysis. The system stiffness matrix $[K]$ relates the vector of element node point displacements $\{\delta\}$ and mechanical and thermal load vectors ($\{P_{mech}\}$ and $\{P_T\}$ respectively) by the first equation. The stiffness matrix can be calculated from the strain shape function matrix $[B]$ and the constitutive relations $[D]$ as shown. If a structure, idealized by finite elements, has sufficient nodal constraints so the stiffness matrix $[K]$ is nonsingular, the node-point displacements $\{\delta\}$ can be obtained from the first equation. These displacements are then used to calculate corresponding strains $\{\epsilon\}$ and stresses $\{\sigma\}$.

As explained in reference 1, the dependence of $\{\epsilon\}$ upon the temperature T for a linear elastic analysis is through $\{P_T\}$ and $\{P_T\}$ is a function of the spatial integral of temperature throughout the element. Hence it appears $\{\epsilon\}$ is related to integrals of T . The stresses $\{\sigma\}$, however, are not only related to integrals of T through $\{\epsilon\}$ but are also directly related to the local temperature. Thus, one can expect a greater accuracy in the numerical calculation of strain than stress in thermomechanical problems since some of the errors of approximations in T are self-cancelling in the integrals which determine strain. In essence one should calculate accurate strains with as coarse a F.E. grid as possible to approximate the thermal load vector. Theoretically one could then use a coarser structural F.E. grid than a corresponding thermal F.E. grid to obtain the same desired accuracies in each analysis. One could improve the coarse-grid F.E. stress results by using the coarse-grid structural results for $\{\epsilon\}$ and coupling them with the fine-grid thermal results for T . This last statement and the last equation of the figure summarize Ojalvo's method. By comparison, in the conventional F.E. procedure there is no separation in the strain and stress calculation; the same F.E. grid is used to calculate consistent nodal thermal load, and nodal strains and stresses. Also, in most finite elements the temperature field in the element is similar to the displacement field; hence, sharp variations in temperature are not accounted for.

FOR FINITE ELEMENT ANALYSIS:

$$\begin{aligned}
 [K] \{\delta\} &= \{P_{mech}\} + \{P_T\} & \text{WHERE } \{P_T\} &= \int T(x, y, z) [B]^T [D] \{\alpha\} dV \\
 & & [K] &= \int [B]^T [D] [B] dV \\
 \{\epsilon\} &= [B] \{\delta\} = [B] \left\{ [K]^{-1} \left(\{P_{mech}\} + \int T(x, y, z) [B]^T [D] \{\alpha\} dV \right) \right\} \\
 \{\sigma\} &= [D] \left(\{\epsilon\} - T \{\alpha\} \right) \\
 & \bullet \{\epsilon\} \text{ IS RELATED TO INTEGRAL OF } T \\
 & \bullet \{\sigma\} \text{ IS RELATED TO THE LOCAL TEMPERATURE} \\
 & \bullet \text{ THEREFORE } \{\epsilon\} \text{ SHOULD BE NUMERICALLY MORE ACCURATE THAN } \{\sigma\} \\
 \text{HENCE CALCULATE STRESS AS: } \{\sigma\} &= [D] \left(\underbrace{\{\epsilon\}}_{\substack{\text{COARSE-GRID} \\ \text{STRUCT. ANALYSIS}}} - T \underbrace{\{\alpha\}}_{\substack{\text{FINE-GRID} \\ \text{THERMAL ANALYSIS}}} \right)
 \end{aligned}$$

Figure 1

SHUTTLE ORBITER VERTICAL FIN STRUCTURAL IDEALIZATION ANALYZED BY OJALVO

To demonstrate the usefulness of Ojalvo's method, reference 1 presents results for a hot structure concept for the vertical fin of the Space Shuttle Orbiter (ref. 4). Temperatures were calculated by a detailed lumped parameter analysis and used as input to a coarse-grid F.E. structural analysis. As shown in figure 2, only a section of the fin main structural box between ribs 9 and 10 was idealized by finite elements using rod, membrane and shear elements. Rene' 41 was as the structural material at locations where the temperature exceeded 923K (1200°F) and Inconel 718 was used for locations below that temperature. The spars were corrugated to partially alleviate excessive thermal stresses.

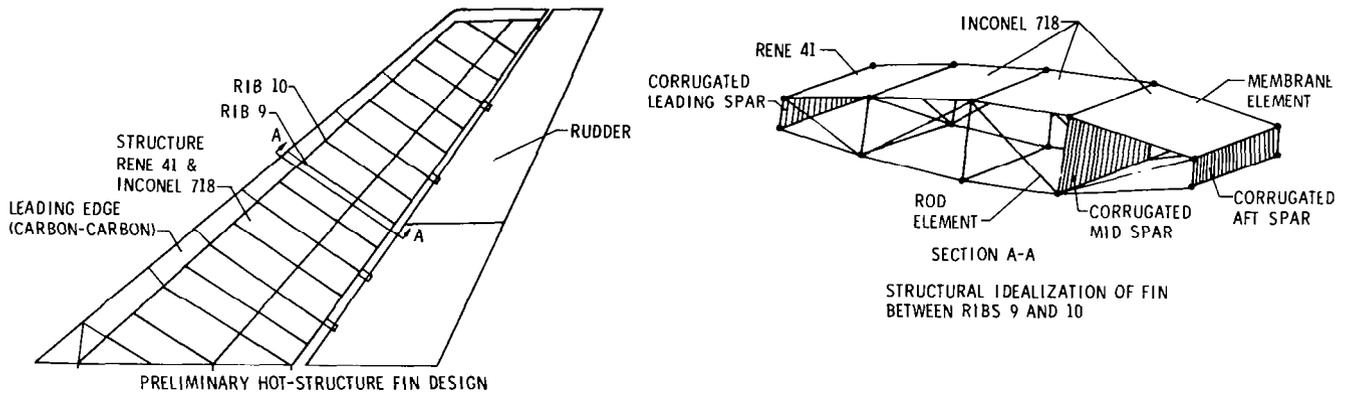


Figure 2

RESULTS OF OJALVO'S METHOD FOR THE SHUTTLE ORBITER VERTICAL FIN

A chordwise temperature distribution from a detailed lumped-parameter transient thermal analysis is shown as a solid curve in the upper left-hand part of figure 3. The massive spars and spar caps act as heat sinks causing sharp temperature drops in their vicinity during transient heating. The dashed lines are the integrated average values of temperature which were used as input for the cover panels in the structural F.E. model. Since the fin is long and slender and since the section analyzed is far enough away from the ends beam theory may be used. If z is the coordinate direction through the depth of the beam (along the chord in this example), and assuming $T = T(z)$ and beam symmetry in two directions, the following equation can be used to predict strains (ref. 1):

$$\epsilon = \frac{\int E\alpha T dA}{\int E dA} + \frac{z \int E\alpha T z dA}{\int E z^2 dA}$$

Because of the above relationship, the sharp drops in temperature near the spars have little effect on the strains. Hence, as shown in figure 3, the strain distribution in the covers predicted by coarse-grid F.E. is linear and very close to the predicted beam theory solution regardless of the nature of the temperature distribution. Thus a coarse grid is sufficient to obtain accurate strains for a thermal stress problem which would require a fine grid to predict the temperature distribution. Also, mesh refinement of the structural grid may not be necessary for performing quasi-static thermal/structural analysis.

A comparison of stress distributions using beam theory, conventional F.E., and Ojalvo's method is shown in figure 3 at the right. The conventional F.E. procedure, using constant-strain constant-temperature quadrilateral membrane elements and integrated average temperature values miss large peak stresses in the covers. However, stress results using the strains from the coarse-grid structural F.E. analysis and the existing detailed temperature distribution (Ojalvo's method) agree closely with beam theory results. Ojalvo's method works well for this problem because the strain distribution is linear and the thermal load vector can be closely approximated by a coarse grid (4 elements).

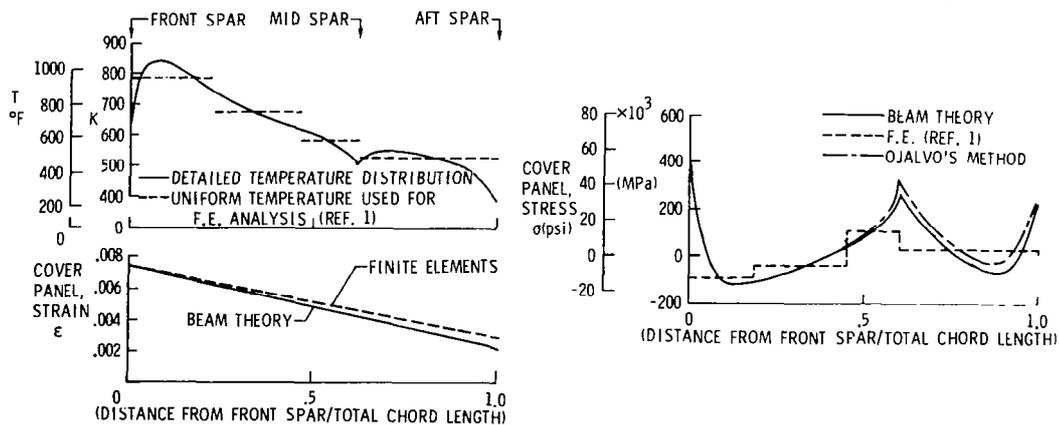


Figure 3

QUALIFICATION OF OJALVO'S METHOD

With regard to earlier statements in reference 1, it is true that the node point displacements $\{\delta\}$ are directly related to the spatial integral of the temperature distribution as shown by the first equation in figure 4. However, the strains are directly related to the strain shape function matrix $[B]$ which relates strains to node point displacements $\{\delta\}$. As shown in the figure $[B]$ is related to the strain operator matrix $[L]$ which performs the operation of differentiation and the shape function matrix $[N]$ which relates the displacement vector $\{u\}$ to the vector of node point displacements $\{\delta\}$. The $[B]$ matrix operates on the $\{\delta\}$ and transforms it into $\{\epsilon\}$; embedded in this operation is the differentiation of the shape function matrix $[N]$. The direction of differentiation (implied in $[L]$) is dependent upon the type of element. Hence, depending on the arguments of integration in $\{P_T\}$ and the direction of differentiation in $[L]$ the strains may be directly related to the local temperatures and not their integrals as stated in reference 1.

As stated in reference 5, deflections caused by heating were "acceptable" but "special attention" was necessary for the "interpolation of the stresses which could deviate considerably from the true stress state." Here, reference 5 indicates that deflections were less sensitive to temperature distributions than their corresponding stresses, as stated in reference 1; however, no mention is made of the sensitivity of strains to temperature distributions. It appears that for bending problems, beams, or plates, there is a direct relationship between ϵ and integrals of T . The reason for this relationship is that for beam-type problems the argument of integration for $\{P_T\}$ is through the depth of the beam while the desired stresses are in the longitudinal direction. This means the direction of differentiation implied by $[B]$ is along the length and hence does not affect the integration. The direction of differentiation for membrane-type problems, however, is the same as the arguments of integration in $\{P_T\}$ and hence the strain variation should be similar to the temperature variation. This reasoning suggests that while there appears to be an advantage in using Ojalvo's method for bending problems, it does not appear to be appropriate for membrane or plane-stress problems.

$$\{\delta\} = [K]^{-1} \left(\{P_{\text{mech}}\} + \int T(x, y, z) [B]^T [D] \{a\} dv \right)$$

$$\{\epsilon\} = [B] \{\delta\} = [L][N] \{\delta\}$$

WHERE FOR EXAMPLE

$$[L] \{u\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}; \quad [L] = \text{STRAIN OPERATOR (DIFFERENTIATION) MATRIX}$$

$$[N] \{\delta\} = \{u\}; \quad N = \text{SHAPE FUNCTION MATRIX}$$

HENCE $[B]$ PERFORMS DIFFERENTIATION

- BENDING PROBLEM $\{\epsilon\} \sim \int T dv$
- MEMBRANE OR PLANE STRESS PROBLEM $\{\epsilon\} \sim T$

Figure 4

COMPARISON OF STRESS CALCULATION METHODS FOR CLAMPED ROD PROBLEM

To qualify the usefulness of Ojalvo's method, analytical and numerical results for several classes of thermostructural problems are presented beginning with the simplest one-dimensional example, a rod. As shown in figure 5, an aluminum rod of length ℓ is clamped at both ends and subjected to a sinusoidal temperature variation along its length; the governing equations are as follows:

$$(1) \quad \frac{d^2u}{dx^2} - \alpha \frac{dT}{dx} = 0 \quad u(0) = u(\ell) = 0$$

Integrating and substituting $T = T_0 \sin \frac{\pi x}{\ell}$

$$(2) \quad u(x) = -\frac{\alpha T_0 \ell}{\pi} \cos \frac{\pi x}{\ell} - \frac{2\alpha T_0 x}{\pi} + \frac{\alpha T_0 \ell}{\pi}$$

and

$$(3) \quad \epsilon_x(x) = \alpha T_0 \sin \frac{\pi x}{\ell} - \frac{2\alpha T_0}{\pi} = \alpha T - \frac{2\alpha T_0}{\pi}$$

$$\sigma = -2E\alpha T_0/\pi$$

From eq. (3) and figure 5 the strain distribution is directly related to the local temperature T rather than the integral of T . Thus, to obtain accurate results for strains and stresses one would need as fine a structural F.E. grid as the thermal F.E. grid. Analytical results for stress from eq. 4 give $\sigma = -179.8$ MPa (-26.07 ksi); conventional F.E. results using SPAR for 2, 4, 8, and 16 elements per length converge from -141.18 MPa (-20.475 ksi) for the 2-element grid to -184.38 MPa (-26.741 ksi) for the 16-element grid. A comparison of analytical, conventional F.E., and Ojalvo's method for stresses is shown in figure 5 at the right. Ojalvo's method was employed using strain results from two different F.E. model, the 4-element and the 8-element models. From the figure note that conventional F.E. results (dashed line) are comparable to results using Ojalvo's method (shown by symbols). In some instances Ojalvo's method is more accurate than conventional results and in some instances (where the coarse strains are not close to exact values) Ojalvo's method is less accurate. For other boundary conditions similar results were obtained; the strains varied as the local temperature and no benefit was realized using Ojalvo's method.

COMPARISON OF STRESS CALCULATION METHODS FOR CLAMPED ROD PROBLEM

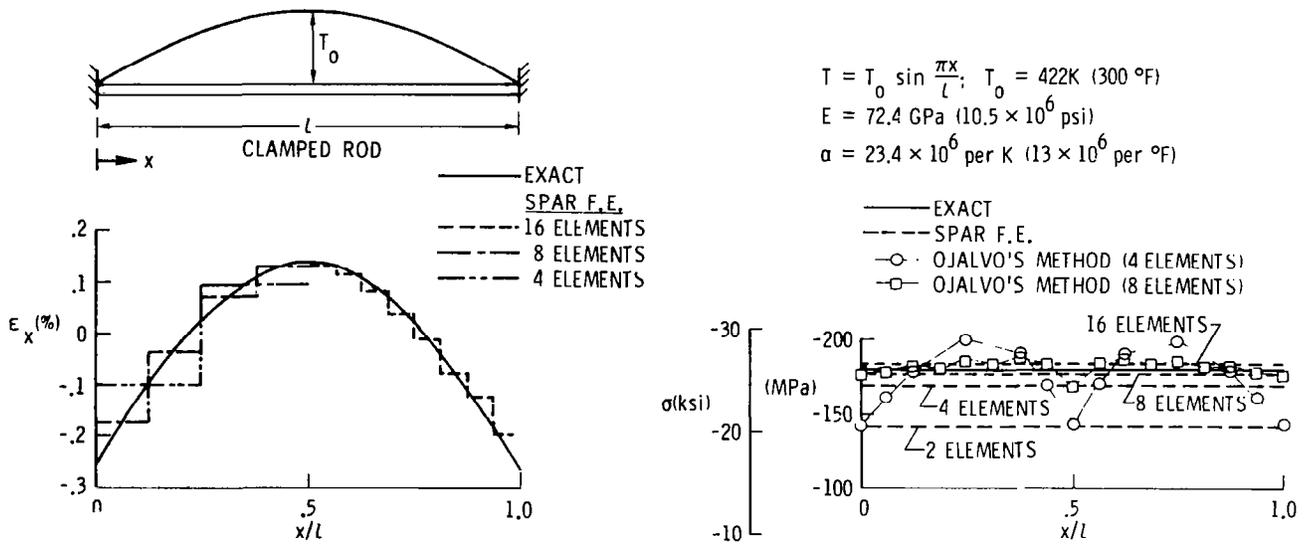


Figure 5

THIN FREE MEMBRANE SUBJECT TO A PARABOLIC
TEMPERATURE DISTRIBUTION

The problem shown in figure 6 is that of a thin rectangular titanium plate of uniform thickness, free to expand, in which the temperature is an even function of y :

$$T = T_0(1 - y^2/c^2)$$

From reference 6

$$\sigma_x = \frac{1}{2c} \int_{-c}^c \alpha T E dy - \alpha T E \quad \text{and} \quad \sigma_y = 0$$

From the thermoelastic constitutive relations and the above equations

$$\sigma_x = \frac{E}{1-\nu^2} [\epsilon_x + \nu\epsilon_y - (1 + \nu)\alpha T] = \frac{1}{2c} \int_{-c}^c \alpha T E dy - \alpha T E$$

$$\sigma_y = \frac{E}{1-\nu^2} [\epsilon_y + \nu\epsilon_x - (1 + \nu)\alpha T] = 0$$

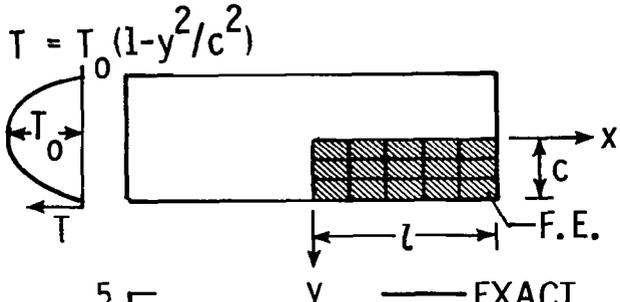
solving for ϵ_x and ϵ_y gives

$$\epsilon_x = \frac{2\alpha T_0}{3} = \text{Constant}$$

$$\epsilon_y = (1+\nu)\alpha T - \frac{2}{3}\nu\alpha T_0$$

From the above equations the strain in the x -direction is a constant, but the strain in the y -direction is directly related to the local temperature T . Hence for accurate results for σ_x accurate values of ϵ_x , ϵ_y , and T are needed; since the ϵ_y varies as the local temperature in the y -direction as fine a discretization as that used in the thermal analysis is needed. If accurate or converged values of ϵ_y are not used, Ojalvo's method will produce less accurate results for stresses as seen in the previous problem. Hence if the linear strain distribution results of the 8×1 grid (triangular symbol) are used with the fine or exact temperature distribution (Ojalvo's method) the results for stresses are not as accurate as the conventional F.E. method in regions where strains are not converged as shown in the right side of the figure. Also, conventional F.E. results for the 8×2 grid are close to the exact solution; therefore if Ojalvo's method is used with this grid, it will give little improvement.

THIN FREE MEMBRANE SUBJECT TO A PARABOLIC TEMPERATURE DISTRIBUTION (REF. 6)



$T_0 = 644 \text{ K (} 700^\circ\text{F)}$
 $E = 110 \text{ GPa (} 16 \times 10^6 \text{ psi)}$
 $\alpha = 10.8 \times 10^{-6} \text{ per K (} 4.8 \times 10^{-6} \text{ per } ^\circ\text{F)}$
 $\nu = 0.29$

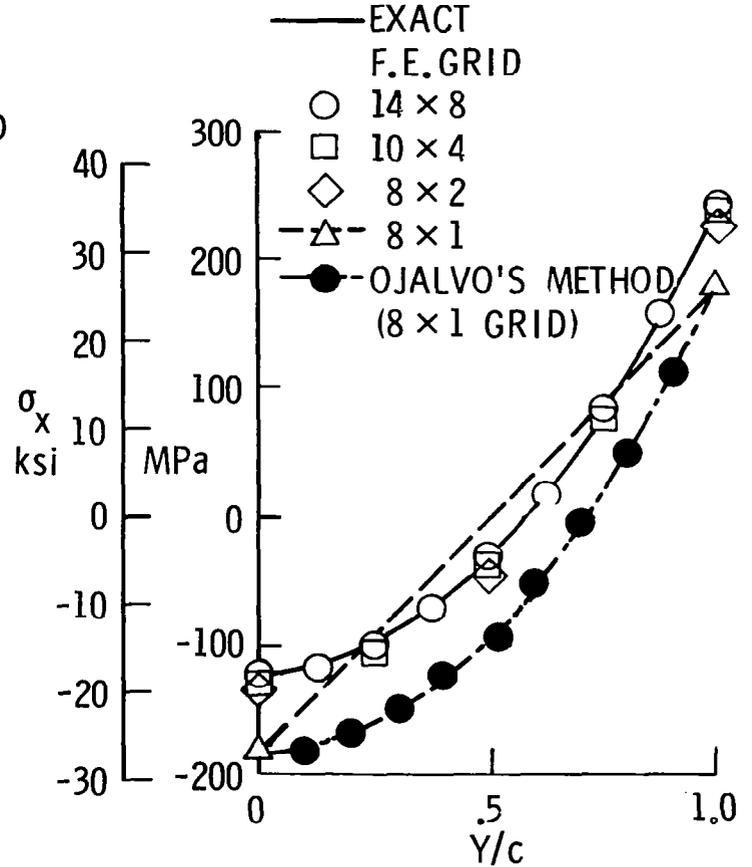
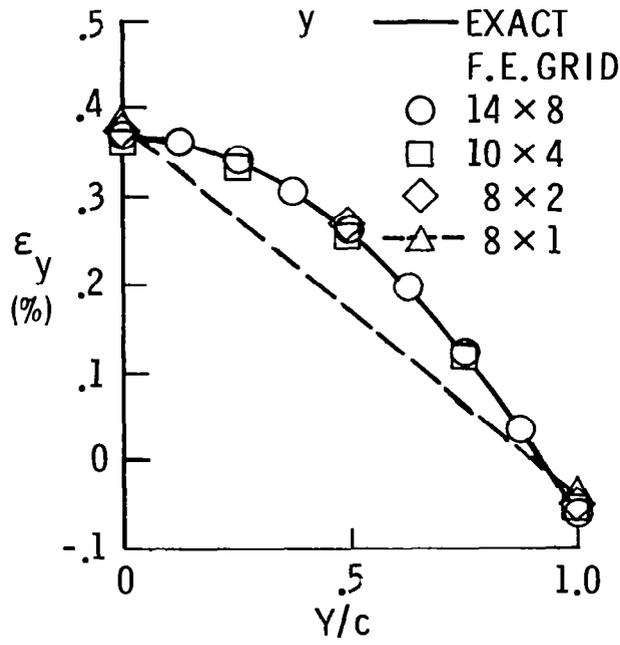


Figure 6

CANTILEVERED TITANIUM BOX BEAM WITH TEMPERATURE VARYING THROUGH THE DEPTH

A titanium box-beam was idealized as a built-up structure consisting of rod and membrane elements. The fine-grid F.E. model consists of 1555 nodes, 1512 membrane elements and 216 rod elements representing one-half of the entire box beam. The load applied to the structure is the temperature which varies in the y -direction only as shown in figure 7. This temperature distribution (solid curve) is a least-squares fit of a fifth-degree polynomial curve to the temperature distribution of reference 1. The dashed lines represent integrated average values of temperature which were used in an approximate beam theory solution to compare with F.E. results from several different models.

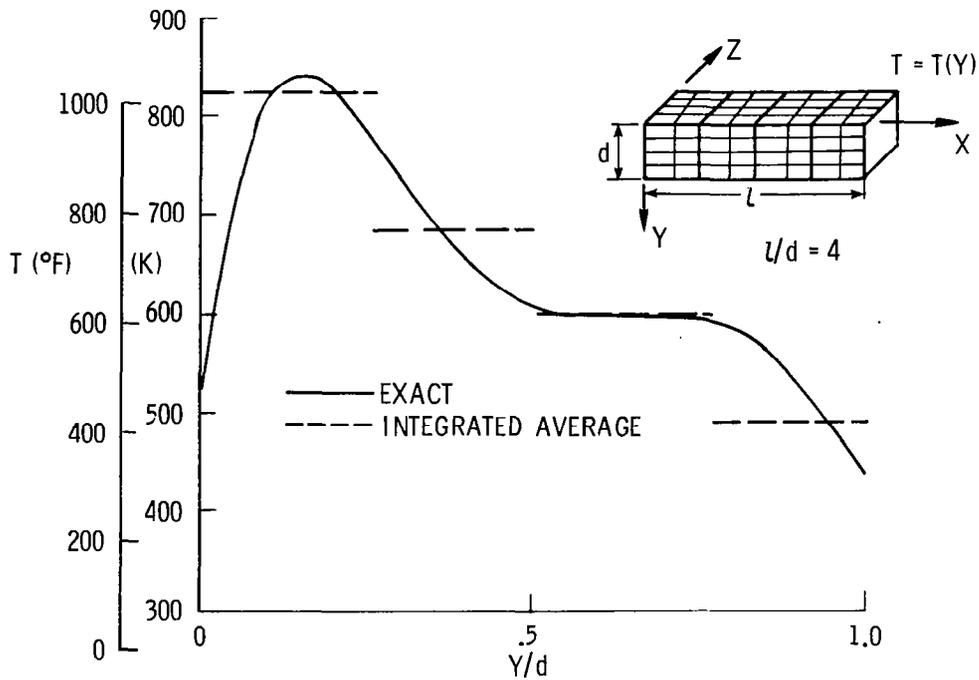


Figure 7

COMPARISON OF OJALVO'S METHOD WITH CONVENTIONAL F.E.
FOR A CANTILEVERED BOX BEAM

The box beam is cantilevered as shown in figure 8. Finite-element results for strain in the membranes at $Z = 0$ at various stations along the X-axis using the fine grid (48 x 12 grid) are shown as the dashed curves. Beam theory results are shown as a solid line, and results of the coarser grids are shown by the symbols. Far from the ends of the beam, plane sections remain plane; the strains are very nearly linear and results are close to predicted beam theory results. At the free end ($x/l = 1.0$), however, the strain distribution is similar to the temperature distribution; this is necessary to insure the stress-free end condition.

Nondimensionalized stress results at the center of the beam ($x/l = 0.5$) are shown in figure 8 at the right. Notice that results from the coarse structural grid (diamond symbol and dashed line) miss critical (peak) stresses located within the element predicted by beam theory (solid curve) and the fine-grid F.E. results (circular symbol). The use of Ojalvo's method, shown by the single-dashed curve, gives a good representation of the actual stress distribution consistent with results of reference 1.

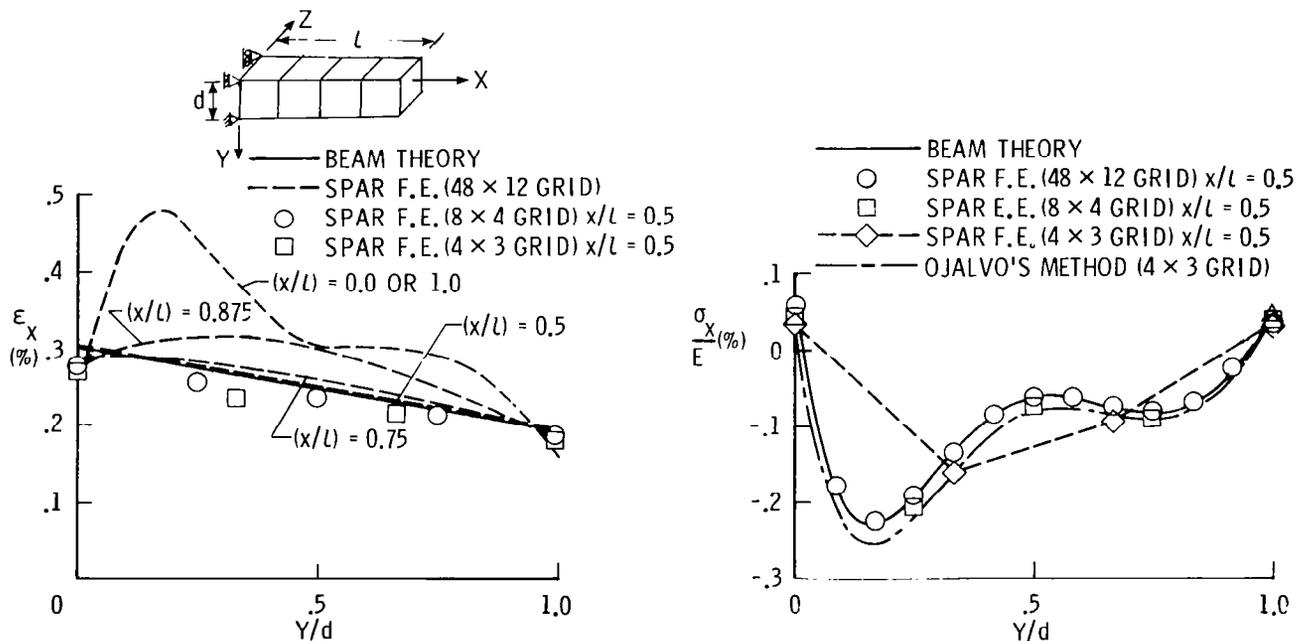


Figure 8

COMPARISON OF OJALVO'S METHOD WITH CONVENTIONAL F.E.
FOR A CANTILEVERED BOX-BEAM (FREE END)

A comparison of the exact solution ($\sigma_x=0$) with conventional F.E. and Ojalvo's method for the free edge of the beam ($x/l=1.0$) is shown in figure 9. Since the strain distribution at $x/l=1.0$ is similar to the temperature distribution, it is not surprising that the coarse-grid F.E. results for ϵ are inaccurate. For this particular case the results from Ojalvo's method are worse than the results using the Conventional F.E. with a coarse grid.

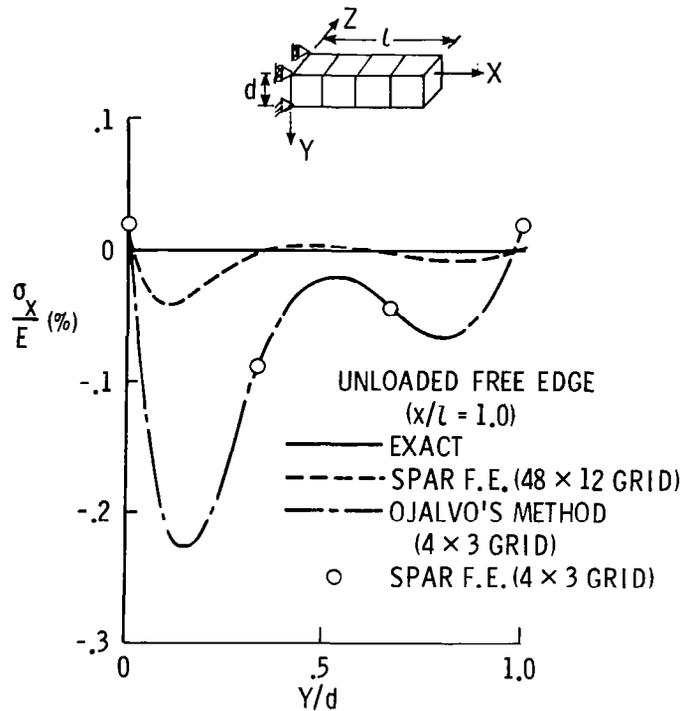


Figure 9

TEMPERATURE DISTRIBUTION IN SKIN AND STIFFENER OF A HEATED STIFFENED PANEL

Another application for which Ojalvo's method might be useful, other than for wing and fin-type structures, would be in the stiffeners of a heated stiffened panel (ref. 7). As indicated in reference 8, stresses in stiffeners can be critical during transient heating. Shown in figure 10 are temperature distributions in the skin and stiffener of a stiffened panel subjected to a hypersonic heating simulation (ref. 7). The specimen was cooled prior to heating to simulate a cold soak condition which explains the sub-ambient temperatures over a portion of the stinger. These temperatures were used as input for a F.E. structural analysis.

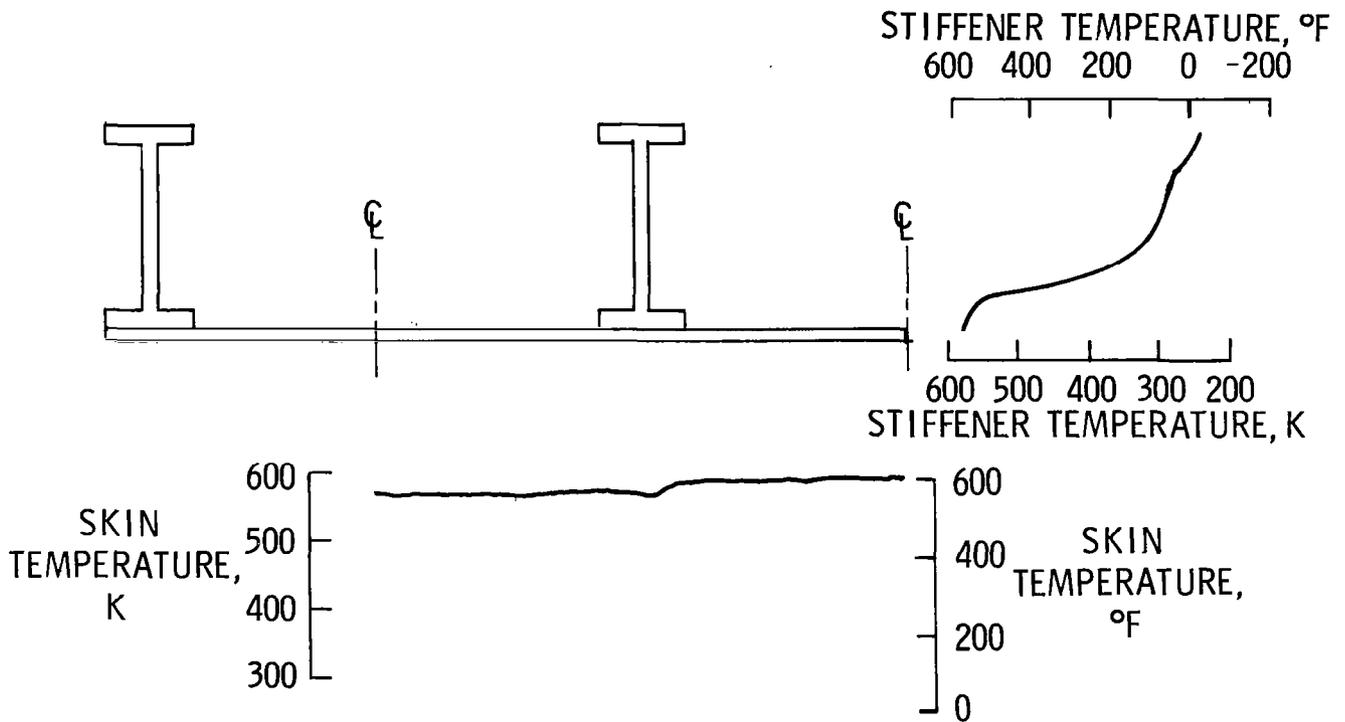


Figure 10

FINITE-ELEMENT IDEALIZATIONS OF A SQUARE STIFFENED TITANIUM PANEL

One-quarter of a stringer-stiffened panel was idealized by two different F.E. grids (fig. 11). The fine-grid model has ten membrane elements through the stringer depth and the coarse-grid model has one membrane element through the depth. Rod elements were used to represent the upper and lower flanges of the stiffener. Symmetry boundary conditions were imposed along the x- and y- axes and the point $x=y=z=0$ was fully constrained.

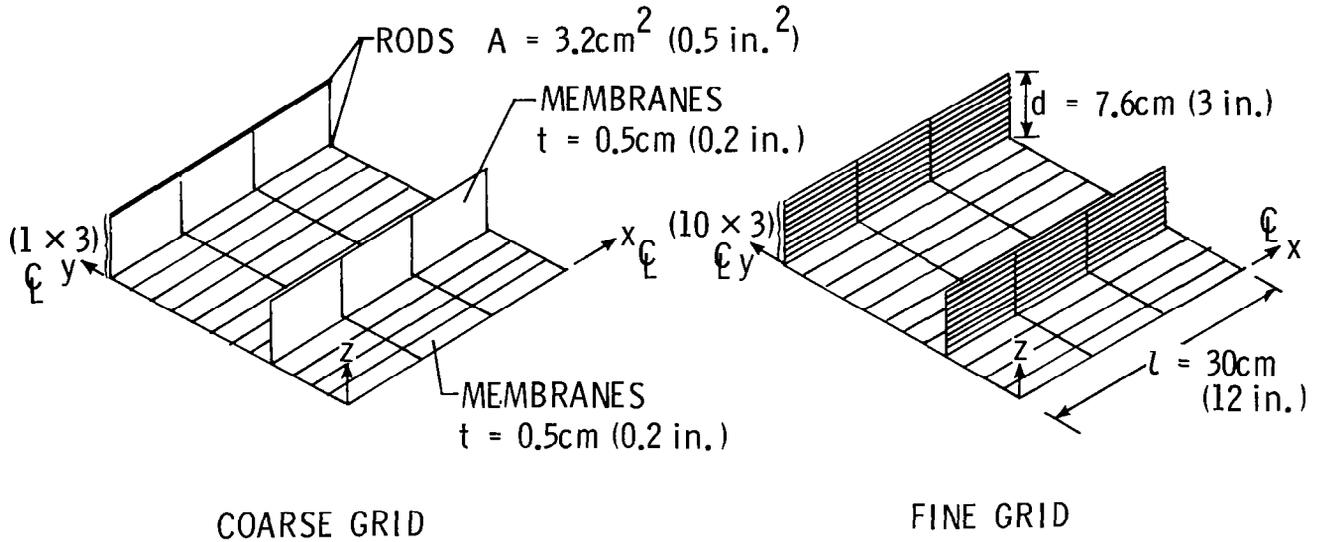


Figure 11

COMPARISON OF OJALVO'S METHOD WITH CONVENTIONAL F.E.
FOR A STIFFENED PANEL

A comparison of strains of the fine- and coarse-grid models indicates that the coarse grid is sufficient in obtaining a good approximation to the linear strain distribution. The coarse-grid results for stresses (solid symbol and dashed curve) badly miss the peak stresses which occur in the center of the element as shown in figure 12 by the solid line. However, when the coarse grid strains are coupled with the detailed temperatures (Ojalvo's method) a good representation of the actual stress distribution (the square symbols) is obtained. This indicates that stiffeners in a heated panel are appropriate applications of Ojalvo's method in thermal/structural analysis and design.

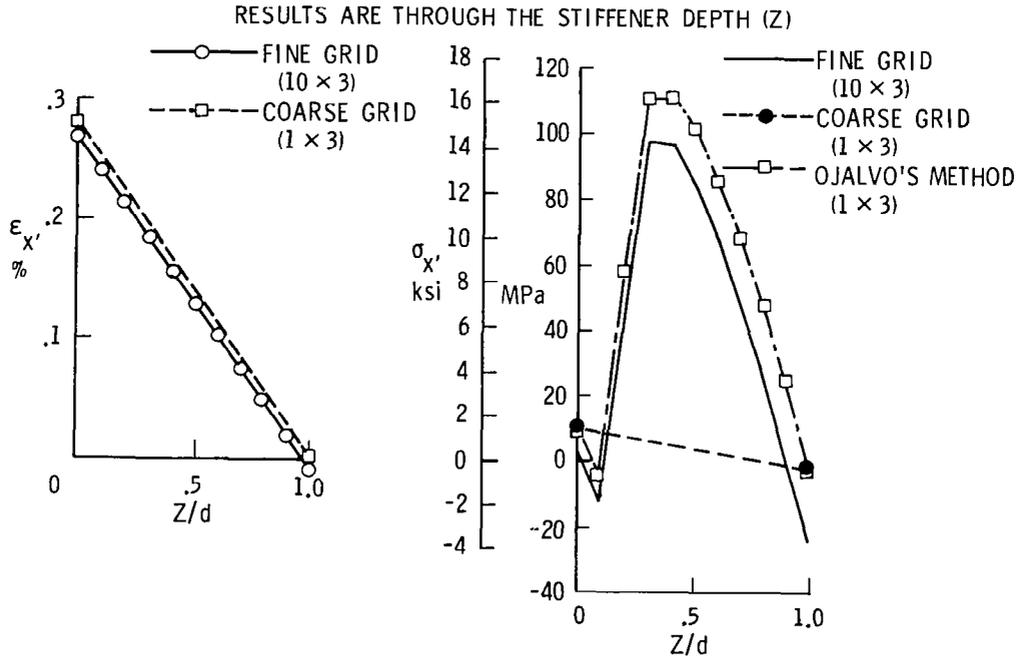


Figure 12

CONCLUDING REMARKS

The usefulness of a thermal/structural analysis technique for improving thermal stress calculations termed Ojalvo's method was investigated by numerical examples of several classes of thermostructural problems (fig. 13). The problems investigated include a rod, a thin membrane, a box beam, and a stiffened panel. The basis of Ojalvo's method is an observation that "strains in heated structures idealized by conventional components are generally less sensitive to spatially distributed temperature variations than are their corresponding stresses." Results of most bending-type problems indicate that Ojalvo's method is useful since the strains are related to the integrals of temperature and hence are less sensitive to local temperature variations. This means that for those problems where Ojalvo's method is appropriate the structural F.E. idealization may be coarser than the thermal idealization and also that the same coarse structural representation can be used for many different time slices in a quasi-static thermostructural analysis. For plane stress or membrane type problems the strain distributions are similar to the temperature distribution and a finite-element grid fine enough to calculate accurate temperatures would be necessary to calculate accurate stresses. This negates the usefulness of Ojalvo's method for this class of problems. Several useful areas for application of Ojalvo's method include built-up structures which can be idealized as bending elements (beams or plates) and stiffeners in a stiffened panel.

- EVALUATE OJALVO'S METHOD FOR THERMAL STRESS ANALYSIS
- KEY FEATURE - STRAINS ARE LESS SENSITIVE TO TEMPERATURE VARIATIONS THAN STRESSES
- PROBLEMS ANALYZED
 - ROD
 - MEMBRANE
 - BOX BEAM
 - STIFFENED PANEL
- FOR MEMBRANE AND PLANE STRESS PROBLEMS STRAINS ARE RELATED TO LOCAL TEMPERATURES AND OJALVO'S METHOD IS GENERALLY NOT USEFUL
- FOR BENDING PROBLEMS STRAINS ARE RELATED TO THE INTEGRALS OF TEMPERATURE AND OJALVO'S METHOD IS BENEFICIAL

Figure 13

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