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March 1982
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SUMMARY

One of the common ingredients of the inviscid-viscous interacting flow fields about bodies at angle of attack is the predilection of the boundary layers growing around the body to detach from the leeward surface along swept separation lines to form coiled vortex motions. In all cases of three-dimensional flow separation and reattachment, the assumption of continuous vector fields of skin-friction lines and external flow streamlines, coupled with simple laws of topology, provides a flow grammar whose elemental constituents are the singular points: the nodes, spiral nodes (foci), and saddles. The phenomenon of three-dimensional separation may be construed as either a local or a global event, depending on whether the skin-friction line that becomes a line of separation originates at a node or a saddle point. Adopting these notions enables us to create a taxonomy of plausible flow structures, to deduce mean flow characteristics, to expose flow mechanisms, and to aid theory and experiment where lack of resolution in wind-tunnel observations or in the results from numerical computations causes imprecision in diagnosis.

1. INTRODUCTION

The separation of three-dimensional (3-D) turbulent boundary layers from the lee of flight vehicles at high angles of attack results in dominant, large-scale, coiled vortex motions that pass along the body in the general direction of the free stream. Such complex and highly interactive flow fields are an area of fluid mechanics and aerodynamics that is still beyond the reach of definitive theory or numerical computation. If the aerodynamic design of a lifting vehicle with flow separation is to be successful over the full range of flight conditions, the vehicle must be controllable at all times and possess no unpleasant changes in force and moment characteristics. To achieve these aims, the primary lines of separation should remain symmetrically placed and preferably fixed on the body and give rise to symmetric vortices to eliminate the development of potentially uncontrollable side forces and yawing moments. In fact, flow separation in three dimensions is of vital significance to the entire spectrum of aerodynamic design, for the skin-friction-line pattern containing swept lines of attachment, separation, and reattachment, in association with a limited number of singular points, constitutes the skeleton structure around which the elements of the entire flow field can be assembled (Ref. 1).

The obtainment of these skin-friction lines (that is, the loci of the local skin-friction vectors) has usually been attempted with oil-streak techniques on the surfaces of wind-tunnel models (Refs. 2-4), where it has been customarily considered that a necessary condition for the occurrence of flow separation is the convergence of oil-streak lines toward a particular line. Whether this is also a sufficient condition is a matter of current debate. Of the many attempts to make sense of a few of the contending arguments have lent themselves to a precise mathematical formulation. Here, we wish to draw attention to the hypothesis supplied by Legendre (Ref. 5) as being one that provides a mathematical framework of considerable depth. The ensuing discussion follows closely that given already by Tobak and Peake (Ref. 6).

2. LIMITING STREAMLINES AND SKIN-FRiction LINES

Legendre (Ref. 5) proposed that a pattern of streamlines immediately adjacent to the surface (in his notation, "wall streamlines," but more conventionally termed "limiting streamlines") be viewed as trajectories having properties consistent with those of a continuous vector field, the principal one being that through any regular (nonsingular) point there passes one and only one trajectory. On the basis of this postulate, it follows that the elementary singular points of the field, namely the nodes, spiral nodes (foci), and saddles (see Fig. 1) can be categorized mathematically. Hence, the types of singular points, their number, and the rules governing the relations between them, can be said to characterize the pattern. Flow separation in this view has been defined by the convergence of wall streamlines toward a particular wall streamline that originates from a singular point of particular type, the saddle point. This view of flow separation is not universally accepted, however, and situations exist in which a more nuanced description of flow separation appears to be required.

Addressing himself specifically to viscous flows, Lighthill (Ref. 7) tied the postulate of a continuous vector field to the pattern of skin-friction lines rather than to the limiting streamlines just above the surface. Parallel with Legendre's definition, the convergence of skin-friction lines toward a particular skin-friction line originating from a saddle point was defined as the necessary condition for flow separation [Note that in the above, the separation line is the asymptote of the adjacent skin-friction lines and not the envelope, as Haskell (Ref. 1) had proposed.]. More recently, Kays (Ref. 8) have shown that the notions of elementary singular points and the simple rules that they collectively obey can be extended to the flow above the surface in planes of symmetry, in projections of actual flows (Ref. 9), in crossflow planes, and so on (see also Ref. 10). Further applications and extensions can be found in the various contributions of Legendre (Ref. 11-13), Osawatitch (Ref. 14), and in the review article by Peake and Tobak (Ref. 4).

The question of an adequate, yet convincing, description of 3-D separated flow arises with especial poignancy when one asks how 3-D separated flow patterns originate and how they succeed one another as the relevant parameters of the problem (e.g., angle of attack, Mach number, and Reynolds number) are varied. In a recent essay (Ref. 6), we suggested that we might answer this question by placing Legendre's hypothesis (utilizing skin-friction lines) within a framework broad enough to include the notions of topological structure and structural stability (see Refs. 15, 16) coupled with arguments from bifurcation theory (see Refs. 17, 18, 19). In the following, we shall try to show that the emergence of a description of 3-D
separated flow about configurations at angle of attack will, in fact, be facilitated by this broader framework. In so doing, we limit our attention to 3-D viscous flows that are steady in the mean.

3. HYPOTHESIS

The postulate that the skin-friction lines on the surface of the body are the trajectories of a continuous vector field can be interpreted mathematically as follows. Let \((\xi, \eta, \zeta)\) be general curvilinear coordinates with \((\xi, \eta)\) set as orthogonal axes in the surface and \(\zeta\) normal to them. Let the length parameters be \(h_1(\xi, \eta)\) and \(h_2(\xi, \eta)\). Except at singular points, it follows from the adherence condition that very close to the surface, the components of the velocity vector \((u_1, u_2)\) parallel to the surface must grow from zero linearly with \(\zeta\). Hence, a particle on a streamline near the surface will have velocity components

\[
\begin{align*}
 h_1(\xi, \eta) \frac{d\xi}{dt} &= \zeta \frac{au_1}{\zeta} (\xi, \eta, 0) = -c u_2(\xi, \eta) = cP(\xi, \eta) \\
 h_2(\xi, \eta) \frac{d\eta}{dt} &= \zeta \frac{au_2}{\zeta} (\xi, \eta, 0) = c u_1(\xi, \eta) = cQ(\xi, \eta)
\end{align*}
\]

(1)

where \((u_1, u_2)\) are the local orthogonal components of the surface vorticity vector. (Note that the surface vortex lines that exist everywhere at right angles to the skin-friction lines are also trajectories of a continuous vector field.) The specification of a steady flow allows \((u_1, u_2)\) to be independent of time. With \(\zeta\) treated as a parameter and \(P\) and \(Q\) functions only of the coordinates, Eq. (1) is composed of a pair of autonomous ordinary differential equations. Their form places them in the same category as the equations studied by Poincaré (Refs. 20 and 21; an English translation of his complete works is given in Ref. 22) in his classical investigation of the curves defined by ordinary differential equations.

Letting

\[
\begin{align*}
 T_{W_1} &= \mu \frac{au_1}{\zeta} (\xi, \eta, 0) \\
 T_{W_2} &= \mu \frac{au_2}{\zeta} (\xi, \eta, 0)
\end{align*}
\]

(2)

be components of the skin-friction parallel to \(\xi\) and \(\eta\), respectively, we have for the equation governing the trajectories of the surface shear-stress vector, from Eqs. (1),

\[
\frac{h_1 d\xi}{T_{W_1}} = \frac{h_2 d\eta}{T_{W_2}}
\]

(3)

Alternatively, for the trajectories of the surface vorticity vector, the governing equation is

\[
\frac{h_1 d\xi}{u_1} = \frac{h_2 d\eta}{u_2}
\]

(4)

4. SINGULAR POINTS

Singular points in the pattern of skin-friction lines occur at isolated points on the surface where the skin-friction \((T_{W_1}, T_{W_2})\) in Eq. (3) or, alternatively, the surface vorticity \((u_1, u_2)\) in Eq. (4), becomes identically zero. Singular points are classifiable into two main types: nodes and saddle points. Nodes may be further subdivided into two subclasses: nodal points and spiral nodes (often called foci of attachment or separation).

A nodal point (Fig. 1a) is the point common to an infinite number of skin-friction lines. At the point, all of the skin-friction lines except one (labeled AA in Fig. 1a) are tangential to a single line BB. At a nodal point of attachment, all of the skin-friction lines are directed outward away from the node. At a nodal point of separation, all of the skin-friction lines are directed inward toward the node. In the presence of axisymmetry, the node degenerates into a “star-like” or “source-like” form.

A spiral node or focus (Fig. 1b) differs from a nodal point in Fig. 1a in that it has no common tangent line. An infinite number of skin-friction lines spiral around the singular point, either away from it (at attachment) or into it (at separation). Spiral nodes of attachment occur generally in the presence of rotation, either of the flow or of the surface, and will not be included in this study. In the exceptional case, the trajectories of the spiral node form closed paths around the singular point. The spiral node is then called a center.

At a saddle point (Fig. 1c), there are only two particular lines, CC and DD, that pass through the singular point. The directions on either side of the singular point are inward on one particular line and outward on the other particular line. The remainder of the skin-friction lines take directions consistent with the direction of the adjacent particular lines. As can be determined from Fig. 1c, the particular lines act as barriers in the field of skin-friction lines, making one set of skin-friction lines inaccessible to an adjacent set.

For each of the patterns in Figs. 1a-1c, the surface vortex lines form a system of curves orthogonal at every point to the system of skin-friction lines. Of all the possible patterns of skin-friction lines
5. TOPOGRAPHY OF SKIN-FRICTION LINES

The singular points, acting either in isolation or in combination, fulfill certain characteristic functions that largely determine the distribution of skin-friction lines on the surface. The nodal point of attachment is typically a stagnation point on a forward-facing surface, such as the nose of a body, where the external flow from far upstream attaches itself to the surface. The nodal point of attachment thereby acts as a source of skin-friction lines that emerge from the point and spread out over the surface. Conversely, the nodal point of separation is typically a point on a rearward-facing surface; it acts as a sink where the skin-friction lines that have circumscribed the body surface may vanish.

The saddle point acts typically to separate the skin-friction lines issuing from or entering into adjacent nodes; for example, adjacent nodal points of attachment. An example of this function is illustrated in Fig. 2a (Ref. 7) and in the skin-friction line pattern on the cockpit windows of a Space Shuttle model (Fig. 2b, courtesy of L. Seegmiller, Ames Research Center). Skin-friction lines emerging from the nodal points of attachment are prevented from crossing by the presence of a particular skin-friction line emerging from the saddle point. Lighthill (Ref. 7) called this particular line a line of separation, and identified the existence of a saddle point from which the particular line emerges as the necessary condition for flow separation. As the patterns in Fig. 2 illustrate, skin-friction lines from either side tend to converge on the particular line issuing from the saddle point. However, the convergence of skin-friction lines on either side of a particular line occurs in situations in which a saddle point can neither be seen nor can be rationally argued to exist. It can happen, for example, that a skin-friction line, one of the infinite set of lines emanating from a nodal point of attachment, may become a line toward which others of the set converge.

In the following, we shall attempt to construct an appropriate physical description of flow separation by utilizing the notions already advanced and by appealing to the theory of structural stability and bifurcation. Adopting a terminology that is suggested by the theoretical framework, we say that a skin-friction line emerging from a saddle point is a global line of separation and leads to global flow separation. In the alternative case, in which the skin-friction line on which other lines converge does not originate or enter a nodal point, we shall identify the line as being a local line of separation, leading to local separation. (When no modifier is used, what is said will apply to either case.)

The notion of local separation may be clarified by taking the example of the flow over a smooth slender body of revolution that is inclined at a small angle of attack to a uniform oncoming stream. A streamline in the oncoming flow attaches itself to the nose at the stagnation point and nodal singular point of attachment. This is the source of the continuous pattern of skin-friction lines that emerge from this point and envelop the body, all of which disappear into a nodal point of separation at the rear. Because in Fig. 2a (Ref. 7) and in the skin-friction line pattern on the circumferential direction, all the way from the windward ray to the leeward ray, the skin-friction lines emanating from the nodal point of attachment sweep around the sides of the body and converge on either side of the particular skin-friction line running along the leeward ray. This particular leeward skin-friction line, beginning at the node of attachment and finishing at the node of separation, is hence a local separation line. It follows that a body of revolution experiences flow separation at all angles of attack other than zero.

The converse of the line of separation is the line of attachment, from which adjacent skin-friction lines diverge. Two lines of attachment are illustrated in Fig. 2a, emanating from each of the nodal points of attachment.

The limiting streamlines, that is, the ones that pass very close to the surface, leave the proximity of the surface very rapidly in the vicinity of a separation line. A simple argument due to Lighthill (Ref. 7) illustrates the flow mechanism. Referring to Eq. (3), let us align \( c_n \) with the external streamline coordinates so that \( \tau_w \) and \( \tau_w' \) are the streamwise and crossflow skin-friction components respectively. If \( n \) is the distance between two adjacent limiting streamlines (see Fig. 3) and \( h \) is the height of a rectangular streamtube (being assumed small so that the local resultant velocity vectors are coplanar and form a linear profile), then the mass flux through the streamtube is

\[
\dot{m} = \rho h n u
\]

where \( \rho \) is the density and \( u \) is the mean velocity of the cross section. But the resultant skin friction at the wall is the resultant of \( \tau_w \) and \( \tau'_w \) or

\[
\tau_w = \frac{u}{h/2}
\]

so that

\[
u = \frac{\tau_w h}{2\rho}
\]

Hence,

\[
h^2 n \frac{\rho}{2u} = \text{constant}
\]

yielding

\[
h = C \left( \frac{\rho}{\tau_w} \right)^{1/4} ; \quad \nu = \frac{u}{\rho}
\]
Thus, as the line of separation is approached, h, the height of the limiting streamline above the surface, increases rapidly. There are two reasons for this increase in h: first, whether the line of separation is global or local, the distance n between adjacent limiting streamlines falls rapidly as the limiting streamlines converge toward the line of separation; second, the resultant skin-friction $\tau_w$ drops toward a minimum as the line of separation is approached and, in the case of the global line of separation, actually approaches zero as the saddle point is approached.

Limiting streamlines rising on either side of the line of separation are prevented from crossing by the presence of a stream surface stemming from the line of separation itself. The existence of such a stream surface is characteristic of flow separation; how it originates determines whether the separation is of global or local form. In the former case, the presence of a saddle point as the origin of the global line of separation provides a mechanism for the creation of a new stream surface that originates at the wall. Emanating from a saddle point and terminating at nodal points of separation (either nodes or spiral nodes), the global line of separation traces a smooth curve on the wall which forms the base of the stream surface, the streamlines of which have all entered the fluid through the saddle point. We shall call this new stream surface a dividing surface. The dividing surface extends the function of the global line of separation into the flow, acting as a barrier separating the set of limiting streamlines that have arisen from the surface on one side of the global line of separation from the set arisen from the other side of the stream surface.

Now we consider the origin of the stream surface characteristic of local flow separation. We note that if a skin-friction line emanating from a nodal point of attachment ultimately becomes a local line of separation, then there will be a point on the line beyond which each of the orthogonal surface vortex lines crossing the line is shaped concave facing downstream. At this point, the skin-friction along the line has become locally minimum. A surface, starting from this point and stemming from the skin-friction line downstream, can be constructed that will be the locus of a set of limiting streamlines originating from far upstream; this surface may also roll up on its development downstream.

This section concludes with a discussion of the remaining type of singular point, the spiral node (also called focus). The spiral node invariably appears on the surface in company with a saddle point. Together they allow a particular form of global flow separation. One leg of the (global) line of separation emanating from the saddle point winds into the spiral node to form the continuous curve on the surface from which the dividing surface stems. The spiral node on the wall extends into the fluid as a new stream surface rolls up with the same sense of rotation as the vortex filament. When the dividing surface extends downstream, it quickly draws the vortex filament into its core. In effect, then, the extension into the fluid of the spiral node on the wall serves as the vortical core about which the dividing surface coils. This flow behavior was first hypothesized by Legendre (Ref. 11), who also noted (Ref. 12) that experimental confirmation existed in the results of earlier experiments carried out by Verlé (Ref. 24). Figure 4a shows Legendre's original sketch of the skin-friction lines; Fig. 4b is a photograph illustrating the experimental confirmation. The dividing surface that coils around the extension of the spiral node (Fig. 4c) will be termed here a "horn-type dividing surface." On the other hand, it can happen that the dividing surface to which a spiral node is connected does not extend downstream. In this case the vortex filament emanating from the spiral node remains distinct and is seen as a separate entity on crossflow planes downstream of its origin on the surface.

6. CHARACTER OF DIVIDING SURFACES

We have seen how the combination of a spiral node and a saddle point in the pattern of skin-friction lines allows a particular form of global flow separation characterized by a horn-type dividing surface. The nodal points of attachment and separation may also combine with saddle points to allow additional forms of global flow separation, again characterized by their particular dividing surfaces. The characteristic dividing surface formed from the combination of a nodal point of attachment and a saddle point is illustrated in Fig. 5a. This form of dividing surface typically occurs in the flow before an obstacle (see Fig. 34 in Peake and Tobak, Ref. 4), or near the nose of a blunt body at a very high angle of attack (see Fig. 81 in Peake and Tobak, Ref. 4). In the example illustrated in Fig. 5, it will be noted that the dividing surface admits of a point in the external flow at which the fluid velocity is identically zero. This is a three-dimensional singular point, which in Fig. 5a acts as the origin of the streamline running through the vortical core of the rolled-up dividing surface.

The characteristic dividing surface formed from the combination of a nodal point of separation and a saddle point is illustrated in Fig. 5b, again a form of global separation. This form of dividing surface often occurs in nominally two-dimensional separated flows such as in the separated flow behind a backward-facing step (see Fig. 24 in Tobak and Peake, Ref. 25) and the separated flow at a cylinder-flare junction (both two and three dimensional; compare Figs. 47 and 48 in Peake and Tobak, Ref. 4). We note in both Figs. 5a and 5b that the streamlines on the dividing surface have all entered the fluid through the saddle point in the pattern of skin-friction lines.

Finally, in Fig. 5c the form of dividing surface materializing at a typical local separation is sketched. Here the eruption of the fluid from the surface, and hence the commencement of the dividing surface, occurs not from a nodal point of separation nor from a saddle point of separation, but from a regular (nonsingular) point on the surface where the surface vortex lines crossing a particular skin-friction line change curvature from concave facing upstream to concave facing downstream. At this point the skin-friction lines adjacent to the one particular skin-friction line begin to converge rapidly toward this particular line, the local separation line. The departure of the fluid from the vicinity of the surface again results in a tight coiling of the dividing surface and the adjacent external flow streamlines.
7. TOPOLOGY OF STREAMLINES IN TWO-DIMENSIONAL SECTIONS OF THREE-DIMENSIONAL FLOWS

Results reported by Smith (Refs. 9, 26), Perry and Fairlie (Ref. 10), and Hunt et al. (Ref. 8), have made it clear that the rules governing the behavior of skin-friction lines may be adapted and extended to yield similar rules governing the behavior of the flow field itself. This is possible when we construct two-dimensional sections of the three-dimensional flow, for example, crossflow planes and streamwise planes of symmetry, which are especially useful for flows around bodies at angle of attack. In particular, Hunt et al. (Ref. 8) have noted that if

\[ y = [u(x,y,z_0), v(x,y,z_0), w(x,y,z_0)] \]

is the mean velocity vector, whose \( u, v \) components are measured in a plane \( z = z_0 = \text{constant} \), above a surface situated at \( y = Y(x; z_0) \) (see Fig. 6), then the mean streamlines in the plane are the solutions of

\[ \frac{dx}{u} = \frac{dy}{v} \]

which are a direct counterpart of Eq. (3) for skin-friction lines on the surface. For a streamwise plane of symmetry \( w(x,y,z_0) = 0 \), then the streamlines defined by Eq. (5) are identifiable with particle path lines in the plane when the flow is steady, or with instantaneous streamlines when the flow is unsteady. Note, however, that if an arbitrary, two-dimensional section of the flow is chosen, Eq. (5) will not necessarily represent the projections of the three-dimensional streamlines on to that plane \( z = z_0 \).

In any case, since \([u(x,y), v(x,y)]\) is a continuous vector field \( \mathbf{V}(x,y) \), with only a finite number of singular points in the interior of the flow at which \( \mathbf{V} = 0 \), it follows that nodes and saddles can be defined in the plane just as they were for skin friction lines on the surface. Nodes and saddles within the flow, excluding the boundary \( y = Y(x; z_0) \), are labeled \( N \) and \( S \), respectively, and are shown in their typical form in Fig. 6. The only new feature of the analysis that is required is the treatment of singular points on the boundary, \( y = Y(x; z_0) \). Since, for a viscous flow, \( \mathbf{V} \) is zero everywhere on the boundary, the boundary is itself a singular line in the plane \( z = z_0 \). Singular points on the line occur where the component of the surface vorticity vector normal to the plane \( z = z_0 \) is zero. Thus, for example, it is ensured that a singular point will occur on the boundary wherever it passes through a singular point in the pattern of skin-friction lines, since the surface vorticity is identically zero there. As introduced by Hunt et al. (Ref. 8), singular points on the boundary are defined as half-nodes \( N' \) and half-saddles \( S' \) (Fig. 6). With this simple amendment to the types of singular points allowable, all of the previous notions and descriptions relevant to the analysis of skin-friction lines carry over to the analysis of the flow within the plane.

In a parallel vein, Hunt et al. (Ref. 8) have recognized that just as the singular points in the pattern of skin-friction lines on the surface obey a topological rule, so must the singular points in any of the sectional views of three-dimensional flows obey topological rules. Although a very general rule applying to general bodies can be derived (Ref. 8), we list here only those special rules that will be useful in subsequent studies of the flow past wings and bodies at angle of attack. In the five topological rules listed below, we assume that the body is simply connected and immersed in a flow that is uniform far upstream.

1. Skin-friction lines on a three-dimensional body (Refs. 7, 23):

\[ \Sigma_N - \Sigma_S = 2 \quad (6) \]

2. Skin-friction lines on a three-dimensional body B connected simply (without gaps) to a plane wall P that either extends to infinity both upstream and downstream or is the surface of a torus:

\[ (\Sigma_N - \Sigma_S)_{P+B} = 0 \quad (7) \]

3. Streamlines on a two-dimensional plane cutting a three-dimensional body:

\[ (\Sigma_N + \frac{1}{2} \Sigma_N') - (\Sigma_S + \frac{1}{2} \Sigma_S') = -1 \quad (8) \]

4. Streamlines on a vertical plane cutting a surface that extends to infinity both upstream and downstream:

\[ (\Sigma_N + \frac{1}{2} \Sigma_N') - (\Sigma_S + \frac{1}{2} \Sigma_S') = 0 \quad (9) \]

5. Streamlines on the projection onto a spherical surface of a conical flow past a three-dimensional body (Ref. 9):

\[ (\Sigma_N + \frac{1}{2} \Sigma_N') - (\Sigma_S + \frac{1}{2} \Sigma_S') = 0 \quad (10) \]
8. TOPOLOGICAL STRUCTURE AND STRUCTURAL STABILITY

Now, then, do 3-D separated flow patterns originate and how do they succeed one another as the relevant parameters (e.g., angle of attack, Reynolds number, and Mach number) are varied? Our approach to dealing with this question will be offered in physical terms, although our definitions should be compatible with whatever set of partial differential equations is assumed to govern the fluid motion. Specifically, we shall apply definitions of topological structure and structural stability to the properties of skin-friction-line patterns, since in so doing, we shall be able to utilize experimental oil-streak patterns directly.

A pattern of skin-friction lines on a given part of the body surface is a map (called a "phase portrait" by Andronov et al., Ref. 16) of the surface shear-stress vector. Two maps have the same topological structure if the paths in the first map are preserved in the second. This may be visualized by imagining a map of skin-friction lines on a deformed sheet of rubber. Disallowing folding or tearing of the sheet, every deformation is a path-preserving mapping. A topological property is then defined as any characteristic of the map that stays invariant under all deformations. The number and types of singular points and the existence of paths connecting the singular points are examples of topological properties. The set of all topological properties of the map describes the topological structure.

Let us also define the structural stability of a map relative to a parameter $\alpha$; for instance, $\alpha$ may be the angle of attack. The map is said to be structurally stable at a given value of $\alpha$ if the map resulting from a very small change in $\alpha$ has the same topological structure as the initial one. Structurally stable maps of the surface shear-stress vector then have two properties in common: first, the singular points in the map are all elementary singular points (i.e., simple nodes or saddles); and second, there are no saddle-point-to-saddle-point connections in the map.

In speaking of the stability of the viscous/inviscid flow external to the surface, we shall find it necessary to distinguish between structural stability and asymptotic stability of the flow. The distinction in structural stability follows from that introduced in reference to the map of the surface shear-stress vector. An external flow is called structurally stable relative to $\alpha$ if a small change in that parameter does not alter the topological structure (e.g., the number and types of 3-D singular points) of the external 3-D velocity vector field. Asymptotic stability is defined as follows: Suppose that the fluid motions evolve according to time-dependent equations of the general form

$$u_t = G(u,a)$$

(11)

where $a$ again is a parameter. Solutions of $G(u,a) = 0$ represent steady mean flow of the kind we have been considering. A mean flow $u_0$ is an asymptotically stable flow if small perturbations from it (at fixed $a$) decay to zero as time $t \to \infty$. When the parameter $\alpha$ is varied, one mean flow may persist (in the mathematical sense) that it remains a valid solution of $G(u,a) = 0$ but becomes unstable to small disturbances as $\alpha$ crosses a critical value. At such a transition point, a new mean flow may bifurcate from the known flow. A characteristic property of the bifurcation flow (such as, e.g., a transverse velocity component) that was zero in the known flow takes on increasing values as the parameter increases beyond the critical point. Finally, we shall find it convenient to distinguish between local and global characteristics of the instabilities. We shall call an instability global if it permanently alters the topological structure of either the external 3-D velocity vector field or the map of the surface shear-stress vector. We shall call an instability local if it does not result in an alteration of the topological structure of either vector field.

This distinction between local and global events suggests why we distinguish between local and global lines of separation in the pattern of skin-friction lines. If an (asymptotic) instability in the flow field alters the topological structure of the map of surface skin-friction lines, then the convergence of skin-friction lines on to one (or several) particular skin-friction line(s) can only be a local event. Accordingly, we label the particular lines local lines of separation, and these will usually all stem from a node of attachment (the stagnation point) on a forward-facing part of the body. If, on the other hand, a global instability (asymptotic structural) of the flow field does change the topological structure of the skin-friction-line map, then resulting in the emergence of a saddle point in this pattern, then this is construed as a global event insofar as the skin-friction-line map is concerned. Accordingly, we label the particular skin-friction line emanating from the saddle point a global line of separation.

9. BIFURCATION

The bifurcation phenomenon alluded to in the discussion of Eq. (11) is conveniently displayed on a bifurcation diagram, two examples of which are shown in Fig. 7. Flows that bifurcate from the known flow are represented by the ordinate $\psi$, which may be any quantity that characterizes the bifurcation flow alone. Stable flows are indicated by solid lines, unstable flows by dashed lines. Thus, over the range of $\alpha$ where the known flow is stable, $\psi$ is zero, and the stable known flow is represented along the abscissa by a solid line. The known flow becomes unstable for all values of $\alpha$ larger than $\alpha_c$, the dashed line along the abscissa indicates. New mean flows bifurcate from $\alpha = \alpha_c$ either supercritically or subcritically.

At a supercritical bifurcation (Fig. 7a), as the parameter $\alpha$ is increased just beyond the critical point $\alpha_c$, the bifurcation flow that replaces the unstable known flow can differ only infinitesimally from it. The bifurcation flow breaks the symmetry of the known flow, adopting a form of lesser symmetry in which dissipative structures arise to absorb just the amount of excess available energy that the more symmetrical known flow no longer was able to absorb. Because the bifurcation flow initially departs only infinitesimally from the unstable known flow, the structural stability of the surface shear-stress initially is unaffected. However, as $\alpha$ continues to increase beyond $\alpha_c$, the bifurcation flow departs significantly from the unstable known flow and begins to affect the structural stability of the surface shear stress. Ultimately a value of $\alpha$ is reached at which the surface shear stress becomes structurally unstable, evidenced either by one of the elementary singular points of its map becoming a singular point of
we find that the instability occurs at the critical point $\alpha_c$ where a supercritical bifurcation (Fig. 7a) leads to a new stable mean flow.

At a subcritical bifurcation (Fig. 7b), when the parameter is increased just beyond the critical point $\alpha_c$, there are no adjacent bifurcation flows that differ only infinitesimally from the unstable known flow. Here, there must be a finite jump to a new branch of flows that may represent a radical change in the topological structure of the external flow and perhaps in the map of the surface shear-stress vector and a new external flow from which additional flows ultimately will bifurcate with further increases of the parameter.

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This completes a framework of terms and notions that should suffice to describe how the structural forms of three-dimensional separated flows originate and succeed each other. The following section is devoted to illustrations of the use of this framework in two examples involving supercritical and subcritical bifurcations.

10. SUPERCRITICAL AND SUBCRITICAL BIFURCATIONS

10.1 Blunt Body of Revolution at Angle of Attack

Let us first consider how a separated flow may originate on a slender round-nosed body of revolution, as one of the main parameters of the problem, angle of attack, is increased from zero in increments. We adopt this example to illustrate a sequence of events in which supercritical bifurcation is the agent leading to the formation of large-scale dissipative structures.

At zero angle of attack (Fig. 8a) the flow is everywhere attached. All skin-friction lines originate at the nodal point of attachment at the nose and, for a sufficiently smooth slender body, disappear into a nodal point of separation at the tail. The relevant topological rule, Eq. (6), is satisfied in the simplest possible way ($N = 2, S = 0$).

At a very small angle of attack (Fig. 8b) the topological structure of the pattern of skin-friction lines remains unaltered. All skin-friction lines again originate at a nodal point of separation. However, the favorable circumferential pressure gradient drives the skin-friction lines leeward where they tend to converge on the skin-friction line running along the leeward ray. Emanating from a node rather than a saddle point and being a line onto which other skin-friction lines converge, this particular line qualifies as a local line of separation according to our definition. The flow in the vicinity of the local line of separation provides a rather innocuous form of local flow separation, typical of the flows leaving surfaces near the symmetry planes of wakes.

As the angle of attack is increased further, a critical angle $\alpha_c$ is reached just beyond which the external flow becomes locally unstable. Coming into play here is the well-known susceptibility of inflexional boundary-layer velocity profiles to instability (Refs. 27-29). The inflexional profiles develop on crossflow planes that are slightly inclined from the plane normal to the external inviscid flow direction. Called a crossflow instability, the event is often a precursor of boundary-layer transition, typically occurring at Reynolds numbers just entering the transitional range (Refs. 30, 31).

Referring to the bifurcation diagrams of Fig. 7 and identifying the parameter $\alpha$ with angle of attack, we find that the instability occurs at the critical point $\alpha_c$ where a supercritical bifurcation (Fig. 7a) leads to a new stable mean flow.

Within the local space influenced by the instability, the new mean flow contains an array of dissipative structures. The structures, illustrated schematically in Fig. 8c, are initially of very small scale, with spacing of the order of the boundary-layer thickness. Because they resemble an array of streamwise vortices having axes slightly skewed from the direction of the external flow, the structures will be called vortical structures. The representation of the structures on a crossflow plane in Fig. 8c is intended to be merely schematic; nevertheless, the sketch satisfies the topological rule for streamlines in a crossflow plane, Eq. (6). As illustrated in the side view of Fig. 8c, the array of vortical structures is reflected in the pattern of skin-friction lines by the appearance of a corresponding array of alternating lines of attachment and (local) separation. Because the bifurcation is supercritical, however, the vortical structures initially are of infinitesimal size and cannot affect the topological structure of the pattern of skin-friction lines. Therefore, once again, these are local lines of separation, each of which leads to a locally separated flow that is initially of very small scale.

At Reynolds numbers typical of those at which boundary-layer transition occurs, the production of longitudinal vortices within the rapidly skewing three-dimensional boundary layer appears on not only blunt body shapes but on pointed configurations also. Figure 9a, for instance, shows evidence of structures on the surface of a hemisphere-cylinder at an angle of attack of $15^\circ$; the structures are formed by the scarring effect of the longitudinal vortices on the sublimation material. Fig. 9b shows evidence of the longitudinal vortices on a circular cone at an angle of attack of $5^\circ$ as evidenced in an oil-flow pattern (see also Figs. 180-182 in Peake and Tobak, Ref. 4).

Although the vortical structures are initially all very small, they are not of equal strength, being immersed in a nonuniform crossflow. Viewed in a crossflow plane, the strength of the structures increases from zero starting from the windward ray, reaches a maximum near halfway around, and diminishes and dies zero on the leeward ray. Recalling that the parameter $\psi$ in Fig. 7 was supposed to characterize the bifurcation flow, we find it convenient to let $\psi$ designate the maximum crossflow velocity induced by the largest of the vortical structures. Thus, with further increase in angle of attack, $\psi$ increases accordingly, as Fig. 7a indicates. Physically, $\psi$ increases because the dominant vortical structure captures the greater part of the oncoming flow feeding the structures, thereby growing while the nearby structures
structures diminish and are drawn into the orbit of the dominant structure. Thus, as the angle of attack increases, the number of vortical structures near the dominant structure diminishes while the dominant structure grows rapidly. Meanwhile, with the incursion in angle of attack, the flow in a region closer to the nose becomes subject to the crossflow instability and develops an array of small vortical structures similar to those that had developed farther downstream at a lower angle of attack. The situation is illustrated on Fig. 8d. We believe that this description is a true representation of the type of flow that Wang (Refs. 32, 33) has characterized as an "open long separation." We note that a dominant vortical structure now appears to represent a full-fledged case of flow separation, the surface shear-stress vector has remained structurally stable so that, in our terms, this is still a case of a local flow separation.

With further increase in the angle of attack, the crossflow instability in the region upstream of the dominant vortical structure prepares the way for the forward movement of the structure and its associated local line of separation. Eventually an angle of attack is reached at which the influence of the vortical structures is great enough to alter the structural stability of the surface shear-stress vector in the immediate vicinity of the nose. A new (unstable) singular point of second order appears at the origin of each of the local lines of separation. With a slight further increase in angle of attack, the unstable singular point splits into a pair of elementary singular points—a spiral node of separation and a saddle point. This combination produces the horn-type dividing surface described earlier (Fig. 4) and Fig. 5, which is flow separation. A new stable mean flow has emerged from which additional flows ultimately will bifurcate with further increase of the angle of attack.

10.2 Asymmetric Vortex Breakdown on Slender Wing

In contrast to supercritical bifurcations, which are normally benign events, beginning as they must with the appearance of only infinitesimal dissipative structures, subcritical bifurcations may be drastic events and thus occur suddenly in a flow structure. Although we are only beginning to appreciate the role of bifurcations in the study of separated flows, we can anticipate that sudden large-scale events, such as those involved in aircraft buffet and stall, will be describable in terms of subcritical bifurcations. Here we cite one example where it is already evident that a fluid dynamical phenomenon involving a subcritical bifurcation can significantly influence the aircraft's dynamical behavior. This is the case of asymmetric vortex breakdown which occurs with slender swept wings at high angles of attack.

We leave aside the vexing question of the mechanisms underlying vortex breakdown itself (see Ref. 35), as well as its topological structure, to focus on events subsequent to the breakdown of the wing's primary vortices. Lawson (Ref. 36) noted that when a slender delta wing was slowly pitched to a sufficiently large angle of attack with sideslip angle held fixed at zero, the breakdown of the pair of leading-edge vortices, which at lower angles had occurred symmetrically (i.e., side by side), became asymmetric. As the angle of attack increased, the vortex breakdown moved closer to the wing apex than the other. Which of the two possible asymmetric patterns was observed after any single pitch-up was probabilistic. Once established, however, the relative positions of the two vortex breakdowns would persist over the wing, even as the angle of attack was reduced to values at which the breakdowns had occurred initially downstream of the wing leading edge. After identifying terms, we show that these observations are perfectly compatible with our previous description of a subcritical bifurcation (Fig. 7b).

Let us denote by \( \alpha_c \) the difference between the chordwise positions of the left-hand and right-hand vortex breakdowns and let \( \alpha_c \) be positive when the left-hand breakdown position is the closer of the two to the wing apex. Referring now to the subcritical bifurcation diagram in Fig. 7b, we identify the bifurcation parameter \( \psi \) with \( \alpha_c \) and the parameter \( \alpha \) with angle of attack. We see that, in accordance with observations, there is a range of \( \alpha, \alpha < \alpha_c \), in which the vortex breakdown positions can coexist side by side, a stable state represented by \( |\alpha| < \alpha_c \). At the critical angle of attack \( \alpha_c \), the breakdowns can no longer sustain themselves side by side, so that for \( \alpha > \alpha_c \) the previously defined stable state \( |\alpha| = 0 \) is no longer stable. Immediately beyond \( \alpha = \alpha_c \) there are no adjacent bifurcation flows, and \( |\alpha| \) must jump to a distant branch of stable flows; this represents the sudden shift forward of one of the vortex breakdown positions. Further, with \( |\alpha| > \alpha_c \) on the new branch, as the angle of attack is reduced \( |\alpha| \) does not return to zero at \( \alpha_c \) but only after \( \alpha \) has passed a smaller value \( \alpha_0 \). All of this is in accordance with observations (Ref. 36). At any angle of attack at which \( |\alpha| \) can be nonzero under symmetric boundary conditions, the variation of \( \alpha \) with sideslip or roll angle must necessarily be hysteretic. This also has been demonstrated experimentally (Ref. 32). Further, since \( \alpha \) must be directly proportional to the rolling moment, the consequent hysteretic behavior of the rolling moment with sideslip or roll angle makes the aircraft susceptible to the dynamical phenomenon of wing-rock (Ref. 38).

11. CONCLUSION

Holding strictly to the notion that patterns of skin-friction lines and external streamlines above bodies at angle of attack reflect the properties of continuous vector fields enables us to characterize the patterns on the surface and on particular projections of the flow (the crossflow plane, for example) by a restricted number of singular points (nodes, saddle points, and spiral nodes). It is useful to consider the restricted number of singular points, and the topological rules that they obey, as components of an organizing principle: a flow grammar whose finite number of elements can be combined in myriad ways to describe, understand, and connect the properties common to all steady, three-dimensional, viscous, separated flows. Introducing a distinction between local and global properties of the flow resolves an ambiguity in the proper definition of a three-dimensional separated flow. Adopting the notion of a topological structure, structural stability and bifurcation gives us a framework in which to describe how three-dimensional separated flows originate on bodies and how they succeed each other as the relevant parameters of the problem, for example, angle of attack, are varied.
12. REFERENCES
21. Poincaré, H., "Les courbes définies par une équation différentielle." Journal de Mathématiques 3 to 7, 1881, p. 375 (see also Oeuvres Complètes, Tome 1).


Figure 1. Singular points.
Figure 1. Saddle point and separation line.

Figure 2. Example of adjacent nodes and saddle point.

Figure 3. Limiting streamlines near 3-D separation line.

Figure 4. Spiral node of separation.
(a) Global dividing surface formed from nodal point of attachment and saddle point.

(b) Global dividing surface formed from nodal point of separation and saddle point.

(c) Local dividing surface.

Figure 5. Global and local dividing surfaces.

(a) Supercritical bifurcation.

(b) Subcritical bifurcation.

Figure 7. Examples of bifurcation.
Figure 8. Sequence of flows leading to global three-dimensional flow separation on round-nose body of revolution as angle of attack is increased.
(a) Sublimation material on hemisphere-cylinder at $\alpha = 19^\circ$, $M = 1.2$, $R_{\infty} = 4.9 \times 10^6$, $L = 7.5$, $D = 2.6$ in.

(b) Oil-film study on a $10^\circ$ semi-angle cone at $\alpha = 5^\circ$, $M = 7.4$, $R_{\infty} = 3 \times 10^6$, $T_r = 1050^\circ K$ (Ref. 30).

Figure 9. Evidence of streamwise vortices on blunt and sharp configurations at angle of attack.
**Abstract**

One of the common ingredients of the inviscid-viscous interacting flow fields about bodies at angle of attack is the predilection of the boundary layers growing around the body to detach from the leeward surface along swept separation lines to form coiled vortex motions. In all cases of three-dimensional flow separation and reattachment, the assumption of continuous vector fields of skin-friction lines and external flow streamlines, coupled with simple laws of topology, provides a flow grammar whose elemental constituents are the singular points: the nodes, spiral nodes (foci), and saddles. The phenomenon of three-dimensional separation may be construed as either a local or a global event, depending on whether the skin-friction line that becomes a line of separation originates at a node or a saddle point. Adopting these notions enables us to create sequences of plausible flow structures, to deduce mean flow characteristics, to expose flow mechanisms, and to aid theory and experiment where lack of resolution in wind-tunnel operations or in the results from numerical computations causes imprecision in diagnosis.