Mathematical Models for the Synthesis and Optimization of Spiral Bevel Gear Tooth Surfaces

F. L. Litvin, Pernez Rahman, and Robert N. Goldrich

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F. L. Litvin, Pernez Rahman,
and Robert N. Goldrich
University of Illinois at Chicago Circle
Chicago, Illinois

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<table>
<thead>
<tr>
<th>TABLE OF CONTENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>SUMMARY.............</td>
</tr>
<tr>
<td>1. BASIC METHODS OF INVESTIGATION..............................</td>
</tr>
<tr>
<td>1.1 General Kinematic Relations..................................</td>
</tr>
<tr>
<td>1.2 Transfer Velocity...........................................</td>
</tr>
<tr>
<td>1.3 Relative Velocity of Contact Points..........................</td>
</tr>
<tr>
<td>1.4 The General Law of Gearings..................................</td>
</tr>
<tr>
<td>1.5 Contact Lines, Surface of Action, The Enveloped Surface....</td>
</tr>
<tr>
<td>1.6 Relations Between Principal Curvatures and Directions of Two Surfaces Being in Meshing.</td>
</tr>
<tr>
<td>1.7 Contact Ellipse.............................................</td>
</tr>
<tr>
<td>2. GEOMETRY OF SPIRAL BEVEL GEARS..................................</td>
</tr>
<tr>
<td>2.1 Introduction................................................</td>
</tr>
<tr>
<td>2.2 Geometry I: The Line of Action................................</td>
</tr>
<tr>
<td>2.3 Geometry I: Contact Point Path on Surface Σ_i (i=1,2)........</td>
</tr>
<tr>
<td>2.4 Geometry I: The Instantaneous Contact Ellipse...............</td>
</tr>
<tr>
<td>2.5 Geometry II: Generating Surfaces.............................</td>
</tr>
<tr>
<td>2.6 Geometry II: The Line of Action.............................</td>
</tr>
<tr>
<td>2.7 Geometry II: The Instantaneous Contact Ellipse...............</td>
</tr>
<tr>
<td>3. METHODS TO CALCULATE GEAR-DRIVE KINEMATICAL ERRORS............</td>
</tr>
<tr>
<td>3.1 Introduction................................................</td>
</tr>
<tr>
<td>3.2 The Computer Method..........................................</td>
</tr>
<tr>
<td>3.3 Approximate Method...........................................</td>
</tr>
<tr>
<td>3.4 Kinematical Errors of Spiral Bevel Gears Induced by Their Eccentricity.</td>
</tr>
<tr>
<td>3.5 Kinematical Errors Induced by Misalignment...................</td>
</tr>
<tr>
<td>4. CONCLUSION...................................................</td>
</tr>
<tr>
<td>REFERENCES.......................................................</td>
</tr>
</tbody>
</table>
Spiral bevel gears have widespread applications in the transmission systems of helicopters, airplanes, trucks, automobiles, tanks and many other machines. Major requirements in the field of helicopter transmissions are: (a) improved life and reliability, (b) reduction in overall weight (i.e., a large power to weight ratio) without compromising the strength and efficiency during the service life, (c) reduction in the transmission noise.

Spiral bevel gears which used in practice are normally generated with approximately conjugate tooth surfaces by using special machine and tool settings. Therefore, designers and researchers cannot solve the Hertzian contact stress problem and define the dynamic capacity and contact fatigue life until these settings are calculated. The geometry of gear tooth surfaces is very complicated and the determination of principal curvatures and principal directions of tooth surfaces for Hertzian problem is a very hard problem.

The first two parts of this report deal with tooth contact geometry. In this report, a novel approach to the study of the geometry of spiral bevel gears and to their rational design is proposed. The nonconjugate tooth surfaces of spiral bevel gears are, in theory, replaced (or approximated) by conjugated tooth surfaces. These surfaces can be generated: (a) by two conical surfaces which are rigidly connected with each other and are in linear tangency along a common generatrix of tool cones and (b) by a conical surface and a surface of revolution which are in linear tangency along a circle.
We can imagine that four surfaces are in mesh: two of them are tool surfaces $\Sigma_1$ and $\Sigma_2$, $G_1$ and $G_2$ are gear tooth surfaces. Surfaces $\Sigma_1$ and $G_1$ are in linear contact and the contact line moves along the surfaces $\Sigma_1$ and $G_1$ in the process of meshing. Surfaces $\Sigma_1$ and $\Sigma_2$ are rigidly connected and move in the process of meshing as a whole body. Surfaces $G_1$ and $G_2$ are in point contact and the point of their contact moves along the surfaces in the process of meshing. Surfaces $G_1$ and $G_2$ are hypothetical conjugate tooth surfaces which approximate the actual nonconjugate tooth surfaces to within manufacturing tolerances in the neighborhood of any path contact point. It is important to note that these conjugate tooth surfaces are not practical to use and, due to a constant tooth depth, may be undercut partly. However, the dynamic design of the gears is primarily dependent upon the nature of tooth surfaces in the neighborhood of the path of contact, and we propose to use these hypothetical conjugate surfaces for this purpose.

Although these hypothetical conjugate surfaces are simpler than the actual ones, the determination of their principal curvatures and directions is still a complicated problem. Therefore, a new approach to the solution of these is proposed in this report. In this approach, direct relationships between the principal curvatures and directions of the tool surface and those of the generated gear surface are obtained. Therefore, the principal curvatures and directions of gear tooth surface are obtained without using the complicated equations of these surfaces.

The proposed report utilizes effective methods of kinematic and analytic geometry (e.g., matrices for coordinate transformation, kinematic relations between motions of contact point and unit normal vector of two surfaces, etc.). With the aid of these analytical tools, the Hertzian
contact problem for conjugate tooth surfaces can be solved. These results are eventually useful in determining compressive load capacity and surface fatigue life of spiral bevel gears.

In the third part of this report, a general theory of kinematical errors exerted by manufacturing and assembly errors is developed. This theory is used to determine the analytical relationship between gear misalignments and kinematical errors. In the past, the influence of manufacturing errors and assembly errors on two surfaces in contact could be determined only by using numerical methods.
1. BASIC METHODS OF INVESTIGATION

1.1 General Kinematic Relations

Three coordinate systems rigidly connected with mechanism links are considered. One of these - \( S_f(x_f, y_f, z_f) \) - is rigidly connected with the frame. The other two - \( S_i(x_i, y_i, z_i) \) (i=1,2) are rigidly connected with the driving and driven gears.

The tooth surface is represented by vector-function

\[
\mathbf{r}_i(u_i, \theta_i) \in C^1 \quad (u, \theta) \in G
\]

where \((u_i, \theta_i)\) are surface coordinates. The symbol \( C^1 \) means that function (1.1.1) has continuous partial derivatives of first order with respect to all its arguments. The designation \( \epsilon G \) means that surface coordinates belong to the area \( G \).

The normal vector \( \mathbf{N}_i \) and unit normal vector \( \mathbf{n}_i \) are represented by the following equations:

\[
\mathbf{N}_i = \frac{\partial \mathbf{r}_i}{\partial u_i} \times \frac{\partial \mathbf{r}_i}{\partial \theta_i} \quad (1.1.2)
\]

\[
\mathbf{n}_i = \frac{\mathbf{N}_i}{|\mathbf{N}_i|} \quad (1.1.3)
\]

It is assumed that surface \( \Sigma_i \) is a regular one and \( \mathbf{N}_i \neq 0 \).

Surface \( \Sigma_i \) and its unit normal vector may be represented in coordinate system \( S_f \) by equations

\[
\mathbf{r}_i^{(i)} = \mathbf{r}_f^{(i)}(u_i, 0_i, \phi_i) \quad (i=1,2) \quad (u_i, 0_i) \in G, \phi_i^{(1)} < \phi_i < \phi_i^{(2)} \quad (1.1.4)
\]

\[
\mathbf{n}_i^{(i)} = \mathbf{n}_f^{(i)}(u_i, \theta_i, \phi_i) \quad (i=1,2) \quad (1.1.5)
\]

Equations (1.1.4) and (1.1.5) can be obtained with the matrix equations

\[
[r_i^{(i)}] = [M_{fi}] [r_i] \quad (1.1.6)
\]

\[
[n_i^{(i)}] = [L_{fi}] [n_i] \quad (1.1.7)
\]

Matrix \([M_{fi}]\) is represented by
\[
[M_{f1}] = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
  a_{31} & a_{32} & a_{33} & a_{34} \\
  0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
  \cos(x_f^\wedge x_i) & \cos(x_f^\wedge y_i) & \cos(x_f^\wedge z_i) & x_f^{(0_i)} \\
  \cos(y_f^\wedge x_i) & \cos(y_f^\wedge y_i) & \cos(y_f^\wedge z_i) & y_f^{(0_i)} \\
  \cos(z_f^\wedge x_i) & \cos(z_f^\wedge y_i) & \cos(z_f^\wedge z_i) & z_f^{(0_i)} \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

(1.1.8)

where \(x_f^{(0_i)}, y_f^{(0_i)}\) and \(z_f^{(0_i)}\) are "new" coordinates of the "old" origin--the coordinates of origin \(O_i\) of the coordinate system \(S_i\) as defined in coordinate system \(S_f\).

The column matrix \([r_i]\) is represented by

\[
[r_i] = \begin{bmatrix}
  x_i \\
  y_i \\
  z_i \\
  1
\end{bmatrix}
\]

(1.1.9)

Here the coordinates of a point \(M\) are homogeneous coordinates: \(M(x_i, y_i, z_i, 1)\)

Matrix \([L_{f1}]\) is a sub-matrix of \([M_{f1}]\)

\[
[L_{f1}] = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
  \cos(x_f^\wedge x_i) & \cos(x_f^\wedge y_i) & \cos(x_f^\wedge z_i) \\
  \cos(y_f^\wedge x_i) & \cos(y_f^\wedge y_i) & \cos(y_f^\wedge z_i) \\
  \cos(z_f^\wedge x_i) & \cos(z_f^\wedge y_i) & \cos(z_f^\wedge z_i)
\end{bmatrix}
\]

(1.1.10)

The column matrix \([n_i]\) is represented by
In the process of motion tooth surfaces $\Sigma_1$ and $\Sigma_2$ must be in continuous tangency. Therefore, the following equations are to be observed

$$\begin{align*}
\mathbf{r}_f(1)(u_1, \theta_1, \phi_1) &= \mathbf{r}_f(2)(u_2, \theta_2, \phi_2) \\
\mathbf{n}_f(1)(u_1, \theta_1, \phi_1) &= \mathbf{n}_f(2)(u_2, \theta_2, \phi_2),
\end{align*}$$

(1.1.12) \hspace{1cm} (1.1.13)

where $\phi_1$ and $\phi_2$ are the angles of rotation of the driving and driven gears, respectively. Equation (1.1.12) expresses that surfaces $\Sigma_1$ and $\Sigma_2$ have common points. Equation (1.1.13) expresses that surfaces $\Sigma_1$ and $\Sigma_2$ have common unit normals at their common points. Together, equation systems (1.1.12) and (1.1.13) express that surfaces $\Sigma_1$ and $\Sigma_2$ are in tangency.

Figure 1.1.1 shows surfaces $\Sigma_1$ and $\Sigma_2$ which are in tangency at point $M$. Plane $T$ is tangent to these surfaces at their point of tangency, point $M$. Position vectors $\mathbf{x}_f(1)$ and $\mathbf{x}_f(2)$ drawn from the origin $O_f$ of coordinate system $S_f(x_f, y_f, z_f)$ coincide with each other at point $M$. At this point the unit normal vectors $\mathbf{n}_f(1)$ and $\mathbf{n}_f(2)$ coincide, too.

Vector equations (1.1.12) and (1.1.13) yield the following six scalar equations

$$\begin{align*}
\mathbf{x}_f(1)(u_1, \theta_1, \phi_1) &= \mathbf{x}_f(2)(u_2, \theta_2, \phi_2) \\
\mathbf{y}_f(1)(u_1, \theta_1, \phi_1) &= \mathbf{y}_f(2)(u_2, \theta_2, \phi_2) \\
\mathbf{z}_f(1)(u_1, \theta_1, \phi_1) &= \mathbf{z}_f(2)(u_2, \theta_2, \phi_2) \\
\mathbf{n}_x(1)(u_1, \theta_1, \phi_1) &= \mathbf{n}_x(2)(u_2, \theta_2, \phi_2) \\
\mathbf{n}_y(1)(u_1, \theta_1, \phi_1) &= \mathbf{n}_y(2)(u_2, \theta_2, \phi_2) \\
\mathbf{n}_z(1)(u_1, \theta_1, \phi_1) &= \mathbf{n}_z(2)(u_2, \theta_2, \phi_2),
\end{align*}$$

(1.1.14) \hspace{1cm} (1.1.19)
FIG. 1.1.1

Contacting Tooth Surfaces
Scalar equations (1.1.14) - (1.1.19) can be represented as
\[ f_k(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \quad (k=1,2,\ldots,6) \tag{1.1.20} \]
But three equations (1.1.17) - (1.1.19) of the system (1.1.14) - (1.1.19) can provide only two independent equations because \( n_f^{(1)} \) and \( n_f^{(2)} \) are unit vectors. Therefore, \( |n_f^{(1)}| = |n_f^{(2)}| \) and if two projections of each unit vector are equal then the third projections must be equal, too. Consequently, vector equations (1.1.12) and (1.1.13) yield a system of only five independent equations:
\[ f_i(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \quad (i=1,2,\ldots,5) \tag{1.1.21} \]
It is assumed that
\[ \{f_1, f_2, f_3, f_4, f_5\} \in C^1 \tag{1.1.22} \]
In other words, it is assumed that functions \( f_i \) (\( i=1,\ldots,5 \)) have with respect to all arguments continuous partial derivatives of first order at least.

It is known that the instantaneous contact of tooth surfaces can be a linear contact (along a spatial curve, in general) or a point contact. Let us suppose that the system of equations (1.1.20) is satisfied at a point \( M_0 \) by a set of parameters
\[ \mathbf{P} = (u_1^0, \theta_1^0, \phi_1^0, u_2^0, \theta_2^0, \phi_2^0) \tag{1.1.23} \]
If link 1 is the input and tooth surfaces are in contact point in the neighborhood of \( M_0 \) a system of functions
\[ \{\phi_2(\phi_1^0), u_1(\phi_1^0), \theta_1(\phi_1^0), u_2(\phi_1^0), \theta_2(\phi_1^0)\} \in C^1 \]
must exist in the neighborhood of \( M_0 \). This requirement will be satisfied if at point \( M_0 \) the following inequality is observed
\[ \frac{D(f_1, f_2, f_3, f_4, f_5)}{D(u_1^0, \theta_1^0, u_2^0, \theta_2^0, \phi_2^0)} \neq 0 \tag{1.1.24} \]
Here:
\[ \frac{D(f_1, f_2, f_3, f_4, f_5)}{D(u_1^0, \theta_1^0, u_2^0, \theta_2^0, \phi_2^0)} = \begin{vmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \phi_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \phi_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \phi_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \phi_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \phi_2} \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \phi_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \phi_2} \\ \frac{\partial f_5}{\partial u_1} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \phi_1} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \phi_2} \end{vmatrix} \tag{1.1.25} \]
is the Jacobian of system (1.1.20)
Inequality (1.1.24) indicates that the tooth surfaces are in contact at a point. If the inequality (1.1.24) becomes an equality this indicates that surfaces contact each other along a line.

It results from the continuity of surface contact that

\[ \frac{dr_f}{dt}^{(1)}(u_1, \theta_1, \phi_1) = \frac{dr_f}{dt}^{(2)}(u_2, \theta_2, \phi_2) \quad (1.1.26) \]

\[ \frac{dn_f}{dt}^{(1)}(u_1, \theta_1, \phi_1) = \frac{dn_f}{dt}^{(2)}(u_2, \theta_2, \phi_2) \quad (1.1.27) \]

or that

\[ \frac{dr_f}{dt}^{(1)}(u_1, \theta_1, \phi_1) = \frac{dr_f}{dt}^{(2)}(u_2, \theta_2, \phi_2) \quad (1.1.28) \]

\[ \frac{dn_f}{dt}^{(1)}(u_1, \theta_1, \phi_1) = \frac{dn_f}{dt}^{(2)}(u_2, \theta_2, \phi_2) \quad (1.1.29) \]

Let us designate \( \frac{dr_f}{dt}^{(i)} \) by \( v_{\text{abs}}^{(i)} \) and \( \frac{dn_f}{dt}^{(i)} \) by \( \hat{n}_{\text{abs}}^{(i)} \) \((i=1,2)\). Here: \( v_{\text{abs}}^{(i)} \) is the velocity of contact point in the absolute motion (with respect to the frame); \( \hat{n}_{\text{abs}}^{(i)} \) is the velocity of the end of unit normal in absolute motion (with respect to the frame).

The velocity of absolute motion can be represented as a sum of two components: (a) velocity of transfer motion - together with the surface; and (b) velocity of a relative motion - relative to the surface. Consequently,

\[ v_{\text{abs}}^{(1)} = v_{\text{abs}}^{(1)} + v_{\text{abs}}^{(2)}, \quad v_{\text{abs}}^{(2)} = v_{\text{abs}}^{(2)} + v_{\text{abs}}^{(2)} \quad (1.1.30) \]

\[ \hat{n}_{\text{abs}}^{(1)} = \hat{n}_{\text{abs}}^{(1)} + \hat{n}_{\text{abs}}^{(1)}, \quad \hat{n}_{\text{abs}}^{(2)} = \hat{n}_{\text{abs}}^{(2)} + \hat{n}_{\text{abs}}^{(2)} \quad (1.1.31) \]

Equations (1.1.12), (1.1.13), (1.1.30) and (1.1.31) yield

\[ v_{\text{abs}}^{(i)} = \frac{\partial x^{(i)}}{\partial \phi_1} \frac{d\phi_1}{dt}, \quad v_{\text{abs}}^{(1)} = \frac{\partial x^{(1)}}{\partial u_1} \frac{du_1}{dt} + \frac{\partial x^{(1)}}{\partial \theta_1} \frac{d\theta_1}{dt} \quad (1.1.32) \]
Due to continuity of tangency
\[ \begin{align*}
\bar{v}^{(1)} &= v^{(2)} \\
\bar{n}^{(1)} &= n^{(2)}
\end{align*} \] (1.1.34)

Equations (1.1.30), (1.1.31) and (1.1.34) yield
\[ \begin{align*}
\bar{v}_{tr}^{(1)} + v_{r}^{(1)} &= v_{tr}^{(2)} + v_{r}^{(2)} \\
\bar{n}_{tr}^{(1)} + n_{r}^{(1)} &= n_{tr}^{(2)} + n_{r}^{(2)}
\end{align*} \] (1.1.35) (1.1.36)

Equations (1.1.34) and (1.1.35) were proposed by F. Litvin. On the basis of these equations important problems in the theory of gearings, such as problem of tooth-nonundercutting, relations between curvatures of two surfaces in mesh, and the problem of kinematical errors of gear drives caused by errors of manufacturing and assemblage, were solved.

1.2 Transfer Velocity

In addition to equation (1.1.32), transfer velocity may be defined in a kinematical way, too.

Figure 1.2.1 shows a tooth surface \( \Sigma_1 \) of gear 1. The gear rotates with angular velocity \( \omega_1^{(i)} \) about axis \( j-j \). Generally, the axis of rotation does not pass through the origin \( O_f \) of coordinate system \( S_f \).

The sliding vector \( \omega_f^{(i)} \) directed along \( j-j \) may be substituted by the same vector which passes through \( O_f \) and a vector-moment \( \bar{R}_f^{(i)} \times \omega_f^{(i)} \), where \( \bar{R}_f^{(i)} \) is a position vector drawn from \( O_f \) to an arbitrary point on the line of action of \( \omega_f^{(i)} \) (of axis \( j-j \)). Figure 2.1 shows vector \( \bar{R}_f^{(i)} = \bar{O}_f \bar{N}^{(i)} \).

The reduction of the sliding vector \( \omega_f^{(i)} \) passing through point \( N^{(i)} \) by the same vector \( \omega_f^{(i)} \) passing through \( O_f \) and vector-moment \( \bar{R}_f^{(i)} \times \omega_f^{(i)} \) is based on the opportunity to represent the transfer velocity by the following two equations:
\[ \begin{align*}
\bar{v}_{tr}^{(i)} &= \omega_f^{(i)} \times \bar{R}_f^{(i)} \\
\bar{n}_{tr}^{(i)} &= \omega_f^{(i)} \times \bar{R}_f^{(i)}
\end{align*} \] (1.2.1)
FIG 1.2.1

Rotation of Tooth Surface About Axis Not Passing Through Origin
\[
\mathbf{v}_{tr}^{(i)} = \omega_{\mathbf{F}}^{(i)} \times \mathbf{r}_{\mathbf{F}}^{(i)} + \mathbf{R}_{\mathbf{F}}^{(i)} \times \omega_{\mathbf{F}}^{(i)}
\]

(1.2.2)

It is easy to verify that

\[
\omega_{\mathbf{F}}^{(i)} \times \mathbf{r}_{\mathbf{F}}^{(i)} + \mathbf{R}_{\mathbf{F}}^{(i)} \times \omega_{\mathbf{F}}^{(i)} = \omega_{\mathbf{F}}^{(i)} \times \mathbf{r}_{\mathbf{F}}^{(i)},
\]

(1.2.3)

taking into account that

\[
\mathbf{r}_{\mathbf{F}}^{(i)} = \mathbf{O}_{\mathbf{F}}^{(i)} - \mathbf{M}_{\mathbf{F}}^{(i)} = \mathbf{O}_{\mathbf{F}}^{(i)} - \mathbf{Q}_{\mathbf{F}}^{(i)} = \mathbf{r}_{\mathbf{F}}^{(i)} - \mathbf{R}_{\mathbf{F}}^{(i)}
\]

(1.2.4)

Consequently,

\[
\omega_{\mathbf{F}}^{(i)} \times \mathbf{r}_{\mathbf{F}}^{(i)} + \mathbf{R}_{\mathbf{F}}^{(i)} \times \omega_{\mathbf{F}}^{(i)} = \omega_{\mathbf{F}}^{(i)} \times (\mathbf{r}_{\mathbf{F}}^{(i)} - \mathbf{R}_{\mathbf{F}}^{(i)}) = \omega_{\mathbf{F}}^{(i)} \times \mathbf{r}_{\mathbf{F}}^{(i)} - \omega_{\mathbf{F}}^{(i)} \times \mathbf{R}_{\mathbf{F}}^{(i)}
\]

The velocity of transfer motion represented by equation (1.2.2) can be considered as a resultant velocity of two motions: (a) translation with the velocity \( \mathbf{r}_{\mathbf{F}}^{(i)} \), and (b) rotation with angular velocity \( \omega_{\mathbf{F}}^{(i)} \) about axis \( j'-j' \) drawn through \( \mathbf{O}_{\mathbf{F}}^{(i)} \) parallel to axis \( j-j \).

Now, let us define the transfer velocity of the unit normal vector. Fig.1.2.2 shows point \( M^{(i)} \) of the tooth surface \( \Sigma_{\mathbf{i}} \) (i=1,2), the unit normal \( \mathbf{n}_{\mathbf{F}}^{(i)} \), and the tangent plane \( T \) to the surface at point \( M^{(i)} \). The surface rotates about axis \( j-j \) with angular velocity \( \omega_{\mathbf{F}}^{(i)} \).

Unlike the previous case, shown in Fig. 1.2.1, let us move the sliding vector \( \omega_{\mathbf{F}}^{(i)} \) not to point \( \mathbf{O}_{\mathbf{F}}^{(i)} \) but to point \( M^{(i)} \). Then, the transfer motion may be represented as a resultant motion with two components: (a) of translation with velocity \( \mathbf{M}^{(i)} \times \omega_{\mathbf{F}}^{(i)} \); and (b) of rotation about axis \( j'-j' \) with angular velocity \( \omega_{\mathbf{F}}^{(i)} \). Axis \( j'-j' \) is drawn through point \( M^{(i)} \) parallel to \( j-j \) (Fig. 1.2.2). Point \( \mathbf{N}_{\mathbf{F}}^{(i)} \) is an arbitrarily chosen point on axis \( j-j \).

By translation the unit normal vector \( \mathbf{n}_{\mathbf{F}}^{(i)} \) will be moved with the surface point \( M^{(i)} \) parallel to its original direction. So, when surface \( \Sigma_{\mathbf{i}} \) with point \( M^{(i)} \) and unit normal \( \mathbf{n}_{\mathbf{F}}^{(i)} \) is translated with velocity \( \mathbf{M}^{(i)} \times \omega_{\mathbf{F}}^{(i)} \) vector \( \mathbf{n}_{\mathbf{F}}^{(i)} \) does not change its original direction. But the direction of \( \mathbf{n}_{\mathbf{F}}^{(i)} \) will be changed by rotation about axis \( j'-j' \).
FIG. 1.2.2

Transfer Velocity of Unit Normal Vector
Fig. 1.2.2 shows two positions of the unit vector: $\mathbf{n}^{(i)}$ is the initial position and $\mathbf{n}^{(i)}_f$ is the changed position after rotation about axis $j'-j'$ by the angle $d\phi_f^{(i)} = \omega_f^{(i)} dt$. The difference
\[ \mathbf{n}^{(i)} - \mathbf{n}^{(i)}_f = d\mathbf{n}_{tr}^{(i)} \]
represents the displacement of unit normal by rotation about axis $j'-j'$.

Vector $d\mathbf{n}_{tr}$ is represented by the equation
\[ d\mathbf{n}_{tr} = d\phi_f^{(i)} \times \mathbf{n}^{(i)} = (\omega_f^{(i)} \times \mathbf{n}^{(i)}) dt \]

Accordingly, the velocity $\dot{\mathbf{n}}_{tr}^{(i)}$ of transfer motion may be represented by
\[ \dot{\mathbf{n}}_{tr}^{(i)} = \frac{d\mathbf{n}_{tr}}{dt} = \omega_f^{(i)} \times \mathbf{n}^{(i)} \]

### 1.3 Relative Velocity of Contact Points

Consider tooth surfaces $\Sigma_1$ and $\Sigma_2$ which are in mesh. Points $M^{(1)}$ and $M^{(2)}$ are rigidly connected with their respective surfaces and coincide with each other at the point of surface contact.

Let us designate by $\dot{\mathbf{v}}^{(1)}_1$ and $\dot{\mathbf{v}}^{(2)}_2$ the transfer velocities of points $M^{(1)}$ and $M^{(2)}$; the subscript "1" means that $\dot{\mathbf{v}}^{(1)}_1$ and $\dot{\mathbf{v}}^{(2)}_2$ are represented in terms of components of coordinate system $S_1$ rigidly connected with surface $\Sigma_1$. The relative velocity
\[ \dot{\mathbf{v}}_1^{(21)} = \dot{\mathbf{v}}_2^{(2)} - \dot{\mathbf{v}}_1^{(1)} \]
expresses the velocity of point $M^{(2)}$ with respect to point $M^{(1)}$ defined by an observer located at the system $S_1$ at point $M^{(1)}$.

Sample problem 1.3.1

Gears 1 and 2 rotate about crossed axes $z_1$ and $z_2$ with angular velocities $\omega^{(1)}_1$ and $\omega^{(2)}_2$ (Fig. 1.3.1); axes $z_1$ and $z_2$ make an angle $\gamma$; the shortest distance between $z_1$ and $z_2$ is $C$. Points $M^{(1)}$ and $M^{(2)}$ of surfaces $\Sigma_1$ and $\Sigma_2$ coincide with each other at the point of contact $M$. 

14
FIG. 1.3.1

Vectors for Computing Relative Velocity at Point M
The transfer velocities of points $M^{(1)}$ and $M^{(2)}$ are represented by the equations

$$
y^{(1)}_1 = \omega^{(1)}_1 \times \overrightarrow{0_1M} = \begin{pmatrix} \omega^{(1)}_1 x_1 \\ \omega^{(1)}_1 y_1 \\ \omega^{(1)}_1 z_1 \end{pmatrix}
$$

(1.3.2)

$$
v^{(2)}_1 = \omega^{(2)}_1 \times \overrightarrow{0_1M} + \overrightarrow{0_1O_2} \times \omega^{(2)}_1 = 
\begin{pmatrix}
\omega^{(2)}_1 x_1 \\
\omega^{(2)}_1 y_1 \\
\omega^{(2)}_1 z_1
\end{pmatrix}
+ 
\begin{pmatrix}
(0_2 x_1) \\
(0_2 y_1) \\
(0_2 z_1)
\end{pmatrix}
$$

(1.3.3)

Here: $(x_1, y_1, z_1)$ are coordinates of point $M^{(1)} = M$, $\omega^{(1)}, \omega^{(2)}, \omega^{(1)}$ are projections of angular velocity $\omega^{(i)}$ ($i=1, 2$); $x_1, y_1, z_1$ are coordinates of point $O_2$ in terms of coordinate system $S_1$.

Surface $S_1$ rotates about $z_1$ and

$$
\omega^{(1)}_{x_1} = \omega^{(1)}_{y_1} = 0, \quad \omega^{(1)}_{z_1} = \omega^{(1)}
$$

(1.3.4)

It is easy to express $\omega^{(2)}_1$ in terms of components of coordinate system $S_1$ rigidly connected with the frame.

$$
\begin{bmatrix}
\omega^{(2)}_1 \\
\omega^{(2)}_1 \\
\omega^{(2)}_1
\end{bmatrix} =
\begin{bmatrix}
0 \\
-\omega^{(2)} \sin \gamma \\
-\omega^{(2)} \cos \gamma
\end{bmatrix}
$$

(1.3.5)

The angular velocity $\omega^{(2)}_1$ can be expressed in terms of components of coordinate system $S_1$ with the aid of the matrix equation

$$
[\omega^{(2)}_1] = [L_{1f}] [\omega^{(2)}_f]
$$

(1.3.6)

Here: matrix
The matrix $L_{1f}$ describes transformation of vector projections by transition from $S_f$ to $S_1$.

It results from expressions (1.3.5)-(1.3.7) that

$$L_{1f} = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -\omega^{(2)} \sin \gamma \\ -\omega^{(2)} \cos \gamma \end{bmatrix} \quad (1.3.7)$$

Transformation of coordinates of some point given in system $S_f$ to $S_1$ is represented by matrix equation

$$[r_1] = [M_{1f}] [r_f], \quad (1.3.9)$$

where

$$[M_{1f}] = \begin{bmatrix} \cos \phi_1 & \sin \phi_1 & 0 & 0 \\ -\sin \phi_1 & \cos \phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.3.10)$$

For point $O_2$ the column matrix is given by

$$[r_f] = [R_f] = \begin{bmatrix} -C \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (R_f = \vec{O_1 O_2}) \quad (1.3.11)$$

Expressions (1.3.9)-(1.3.14) yield

$$[R_1] = \begin{bmatrix} x_1^{(0)}_2 \\ y_1^{(0)}_2 \\ z_1^{(0)}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} -C \cos \phi_1 \\ C \sin \phi_1 \\ 0 \\ 1 \end{bmatrix} \quad (R_1 = \vec{O_1 O_2}) \quad (1.3.12)$$
The subscripts "f" and "1" for \( [R_f] \) and \( [R_1] \) denote that the same vector 
\( R = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \) is expressed in terms of components of two coordinate systems: 
\( S_f \) and \( S_1 \).

Equations (1.3.2)-(1.3.4), (1.3.8) and (1.3.12) yield

\[
[v_1^{(21)}] = [v_1^{(2)}] - [v_1^{(1)}] = \\
\begin{bmatrix}
y_1(\omega^{(2)} \cos \psi + \omega^{(1)}) - z_1 \omega^{(2)} \sin \psi \cos \phi_1 - C \omega^{(2)} \cos \psi \sin \phi_1 \\
-x_1(\omega^{(2)} \cos \psi + \omega^{(1)}) + z_1 \omega^{(2)} \sin \psi \sin \phi_1 - C \omega^{(2)} \cos \psi \cos \phi_1 \\
\omega^{(2)} \sin \psi (x_1 \cos \phi_1 - y_1 \sin \phi_1 + C)
\end{bmatrix}
\]

(1.3.13)

To express the relative velocity \( v^{(21)} \) in terms of components of coordinate 
\( S_f \) it is sufficient to put in matrix (1.3.13) \( \phi_1 = 0 \) and 
\( x_1 = x_f, \ y_1 = y_f, \ z_1 = z_f \) because with \( \phi_1 = 0 \) the coordinate system \( S_1 \) coincides with \( S_f \).

\[
[v_f^{(21)}] = \\
\begin{bmatrix}
y_f(\omega^{(2)} \cos \psi + \omega^{(1)}) - z_f \omega^{(2)} \sin \psi \\
-x_f(\omega^{(2)} \cos \psi + \omega^{(1)}) - z_f \omega^{(2)} \cos \psi \\
\omega^{(2)} \sin \psi (x_f + C)
\end{bmatrix}
\]

(1.3.14)

For the case when motion is transformed between parallel axes the crossing 
angle \( \gamma \) must be put equal to zero in matrices (1.3.13) and (1.3.14). For 
gear drives with intersecting axes, such as bevel gears, the shortest distance 
\( C \) must be put equal to zero in the same matrices; the angle \( \gamma \) is made by 
intersected axes.

1.4. The General Law of Gearings

Let us suppose that tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \) which are in linear or 
point contact must transform motion with prescribed angular velocity ratio 
\( R_{21} = \omega^{(2)}:\omega^{(1)} \) with prescribed location of the axes of rotation. Because 
the contact of surfaces must be a continuous one the surfaces should not 
interfere each other or lose their contact. Therefore, at a point of contact
the relative velocity \( v_{1}^{(21)} \) must belong to the common tangent plane \( T \) to the surfaces at their contact point \( M \) (Fig. 1.4.1). Consequently, at a point of contact the following equation

\[
N_{1} \cdot v_{1}^{(21)} = 0
\]  

must be observed. Here: \( N_{1} \) is the common surface normal at the contact point \( M \), \( v_{1}^{(21)} \) is the relative velocity represented by equations (1.3.13).

For a surface \( \Sigma_{1} \) represented by vector-function

\[
\mathbf{r}_{1}(u, \theta) \in \mathbb{C}^{1}, \ (u, \theta) \in \mathbb{G}
\]  

the surface normal is defined by equation

\[
N_{1} = \frac{\partial \mathbf{r}_{1}}{\partial u} \times \frac{\partial \mathbf{r}_{1}}{\partial \theta}
\]  

Equations (1.4.1) and (1.4.3) yield that the scalar triple product

\[
\left[ \frac{\partial \mathbf{r}_{1}}{\partial u} - \frac{\partial \mathbf{r}_{1}}{\partial \theta} \right] v_{1}^{(21)} = 0
\]  

provides an equation of meshing

\[
f(u, \theta, \phi_{1}) = 0
\]  

because \( \frac{\partial \mathbf{r}_{1}}{\partial u} \) and \( \frac{\partial \mathbf{r}_{1}}{\partial \theta} \) are functions of surface coordinates \((u, \theta)\) and \( \chi_{1}^{(12)}(x_{1}, y_{1}, z_{1}, \phi_{1}) \) is a function of \((u, \theta, \phi_{1})\).

Surface \( \Sigma_{1} \) can be represented in coordinate system \( \Sigma_{F} \) by the vector-function

\[
\mathbf{r}_{F}(u, \theta, \phi_{1}) \in \mathbb{C}^{1}, \ (u, \theta) \in \mathbb{G}, \ \phi_{1}^{(1)} < \phi_{1} < \phi_{1}^{(2)}
\]  

Different values of \( \phi_{1} \) correspond to different positions of \( L_{1} \) in coordinate system \( S_{F} \). For a definite position of \( \Sigma_{1} \) the motion parameter \( \phi_{1} \) must be considered as a fixed one.
FIG 1.4.1

Contacting Tooth Surfaces and Common Tangent Plane
The equation of meshing (1.4.5) can be obtained by

\[
\begin{bmatrix}
\frac{\partial r_f}{\partial u} & \frac{\partial r_f}{\partial \phi}
\end{bmatrix}
\begin{bmatrix}
\nu_f^{(21)}
\end{bmatrix}
= f(u, \phi) = 0
\] (1.4.7)

Here:

\[
\frac{\partial r_f}{\partial u} \times \frac{\partial r_f}{\partial \phi} = N_f
\] (1.4.8)

is the surface normal; the relative velocity \( \nu_f^{(21)} \) is represented by equations (1.3.14).

For gearings with parallel and intersecting axes the law of meshing can be expressed in another form.

For gears with parallel axes the relative motion can be represented as a rotation about the instantaneous axis of rotation I-I (Fig. 1.4.2). By a given ratio

\[
R_{21} = \frac{\omega^{(2)}}{\omega^{(1)}}
\] (1.4.9)

the relative motion is rolling of two cylinders with operating radii \( r_2' \) and \( r_1' \) defined by equations

\[
\frac{r_1'}{r_2'} = \frac{\omega^{(2)}}{\omega^{(1)}} = R_{21}, \quad r_1' + r_2' = C,
\] (1.4.10)

where \( C = 0_10_2 \) is the distance between the axes of rotation.

With cylinder 1 fixed, cylinder 2 rotates about axis I-I with angular velocity \( \omega^{(21)} = \omega^{(2)} - \omega^{(1)} \). The relative velocity \( \nu_1^{(21)} \) is represented by equation

\[
\nu_1^{(21)} = \omega^{(21)} \times MM,
\] (1.4.11)

where \( M \) is the point of contact of surfaces \( \Sigma_1 \) and \( \Sigma_2; \overrightarrow{MM} \) is a perpendicular to axis I-I drawn from point \( M \).

Equations (1.4.1) and (1.4.11) yield
FIG 1.4.2
Pitch Cylinders and Instantaneous Axis of Rotation

22
Because the scalar triple product is equal to zero, all three vectors must belong to the same plane and the surface normal \( \mathbf{N}_1 \) must intersect the instantaneous axis of rotation I-I (Fig. 1.4.2,B). This fact results in the following theorem:

The contact line of tooth surfaces of gears with parallel axes of rotation must be such that common normal to tooth surfaces at any point of contact intersects the instantaneous axis I-I of rotation (the line of tangency of operating pitch cylinders).

According to this theorem the law of meshing may be defined with the following equations:

\[
\frac{x_1 - x_1(u, \theta)}{N_{x_1}} = \frac{y_1 - y_1(u, \theta)}{N_{y_1}} = \frac{z_1 - z_1(u, \theta)}{N_{z_1}}
\]  

(1.4.13)

Here: \( x_1(u, \theta) \), \( y_1(u, \theta) \), \( z_1(u, \theta) \) are coordinates of a point of surface \( S_1 \); \( x_1(\phi_1) \), \( y_1(\phi_1) \), \( z_1(\theta_1) \) are coordinates of a point which belongs to axis I-I (Fig. 1.4.2). It is assumed that axis \( Z_1 \) is the rotation axis of gear 1 and \( l_1 \) is a coordinate of a point of this axis.

The first equation (1.4.13)

\[
\frac{x_1(\phi_1) - x_1(u, \theta)}{N_{x_1(u, \theta)}} = \frac{y_1(\phi_1) - y_1(u, \theta)}{N_{y_1(u, \theta)}}
\]  

(1.4.14)

yields the equation of meshing (1.4.5).

Equations (1.4.13) can be applied for bevel gears, too.

The equation of meshing can also be defined another way, if instead of (1.4.14) the following equation is used

\[
\frac{x_1(u, \theta, \phi_1)}{N_{x_1(u, \theta, \phi_1)}} = \frac{y_1(u, \theta, \phi_1)}{N_{y_1(u, \theta, \phi_1)}}
\]  

(1.4.15)

Subscript "\( f \)" denotes that all vectors are represented in terms of components of coordinate system \( S_f(x_f, y_f, z_f) \) rigidly connected with the
frame; \( X_f, Y_f, Z_f \) are coordinates of a point which belongs to the axis of instantaneous rotation; \( x_f, y_f, z_f \) are coordinates of a point of surface \( \Sigma_f; N_{xf}, N_{yf}, N_{zf} \) are projection of surface normal.

1.5 Contact Lines, Surface of Action, The Enveloped Surface

The same three coordinate systems mentioned in item 1.1 are considered. The problem to be solved can be formulated as follows: The surface \( \Sigma_1 \) of gear 1 teeth is given; surface \( \Sigma_2 \) of gear 2 teeth, the surface of action \( \Sigma_f \) and lines of contact of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) must be defined. Let us take \( \Sigma_1 \) as the generating surface and \( \Sigma_2 \) as the surface generated by \( \Sigma_1 \).

Let us suppose that surface \( \Sigma_1 \) is represented by vector-function

\[
T_1(u, \theta) \in C^1, \quad (u, \theta) \in \mathcal{U}
\]  

(1.5.1)

Then, contact lines on surface \( \Sigma_1 \) can be represented by the following equations

\[
\begin{align*}
x_1 &= x_1(u, \theta) \\
y_1 &= y_1(u, \theta) \\
z_1 &= z_1(u, \theta) \\
N_1 \cdot \gamma^{(2)}_1 &= f(u, \theta, \phi_1) = 0
\end{align*}
\]  

(1.5.2)

The first three equations represent surface \( \Sigma_1 \), the fourth one represents the equation of meshing; \( \phi_1 \) is a fixed value for every contact line.

Fig. 1.5.1 shows surface \( \Sigma_1 \) covered with contact lines \( CL(\phi_1^{(i)}) \) \((i=1,2,3,...)\), where \( \phi_1^{(i)} \) are fixed values. By a definite value of \( \phi_1^{(i)} \) line \( CL(\phi_1^{(i)}) \) will become the line of instantaneous tangency of \( \Sigma_1 \) and \( \Sigma_2 \).

The to-be-defined surface \( \Sigma_2 \) can be represented as the locus of contact lines in coordinate system \( S_2(x_2, y_2, z_2) \). Consequently, surface \( \Sigma_2 \) can be represented by equations

\[
\begin{align*}
x_2 &= x_2(u, \theta, \phi_1), \quad y_2 = y_2(u, \theta, \phi_1), \\
z_2 &= z_2(u, \theta, \phi_1), \quad f(u, \theta, \phi_1) = 0
\end{align*}
\]  

(1.5.3)

24
FIG. 1.5.1

Surface Covered with Contact Lines
The first of these three equations can be obtained through the matrix equation

\[
\begin{bmatrix}
    r_2 \\
    1
\end{bmatrix} = \begin{bmatrix}
    M_{21}(\phi_1)
\end{bmatrix} \begin{bmatrix}
    r_1(u,\theta) \\
    1
\end{bmatrix},
\]

(1.5.4)

where

\[
\begin{bmatrix}
    x_2 \\
    y_2 \\
    z_2
\end{bmatrix} = \begin{bmatrix}
    x_1(u,0) \\
    y_1(u,\theta) \\
    z_1(u,\theta)
\end{bmatrix},
\]

matrix \( M_{21} \) describes coordinate transformation by transition from \( S_1 \) to \( S_2 \).

The surface of action is a locus of contact lines represented in the coordinate system \( S_f \) by equations

\[
x_f = x_f(u,\theta,\phi_1), \quad y_f(u,\theta,\phi_1), \quad z_f = z_f(u,\theta,\phi_1),
\]

\( f(u,\theta,\phi_1) = 0 \) \hspace{1cm} (1.5.5)

The first three equations are obtained by using the matrix equation

\[
\begin{bmatrix}
    r_f \\
    1
\end{bmatrix} = \begin{bmatrix}
    M_{f1}(\phi_1)
\end{bmatrix} \begin{bmatrix}
    r_1(u,\theta) \\
    1
\end{bmatrix}
\]

(1.5.6)

Sample problem 1.5.1.

The generating process of spiral bevel gears is shown in Fig. 1.5.2. The tool is a head-cutter with blades mounted in it. Both shapes of a blade are straight lines. By rotation about head-cutter axis \( C \) the straight-lined side of the blade describes a cone surface with vertex angle \( 2\psi_c \) (Fig. 1.5.3,a). The angular velocity of the head-cutter rotation is not related to the kinematics of tooth generation.

The head cutter is mounted on the cradle of the cutting machine (Fig. 1.5.2). In the process of cutting the cradle and the to-be-generated gear rotate about intersecting axes \( 0-0 \) and \( a-a \) with angular velocities \( \omega^{(1)} \) and \( \omega^{(2)} \), respectively. The generating surface \( \Sigma_1 \) and the generating gear are shown in Fig. 1.5.3.

The conic surface \( \Sigma_1 \) is represented in coordinate system \( S_c \).
FIG. 1.5.2
Schematic of Cutting Process for Spiral Bevel Gears
FIG. 15.3
Generating Surface and Generating Gear
28
(Fig. 1.5.4,a) by the equation
\[ x_C = r_C \cot \psi_C - u \cos \psi_C, \]
\[ y_C = u \sin \psi_C \sin \theta, \]  \hfill (1.5.7)
\[ z_C = u \sin \psi_C \cos \theta. \]

Here: \( u = |0'N| \) and \( \theta \) are surface coordinates. \( \psi_C \) is the angle made by the cone generatrix and cone axis and \( r_C \) is the mean radius of the head cutter measured in plane \( x_C = 0 \).

Coordinate systems \( S_C \) and \( S_1 \) are rigidly connected with the generating gear. Axis \( x_1 \) is the axis of rotation of the generating gear by cutting. The location of the head cutter (or of system \( S_C \)) is defined by the distance \( O_1O_C = b \) and by the angle \( \varphi \) (Fig. 1.5.4,b and Fig. 1.5.4,c); \( \beta \) is the mean spiral angle; \( M \) is the point of intersection of the cone surface and axis \( z_1 \).

The coordinate transformation from system \( S_C \) to \( S_1 \) is represented by matrix equation
\[ [x_1] = [M_{1C}] [r_C], \]  \hfill (1.5.8)
where (Fig. 1.5.4)
\[ [M_{1C}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & -b \sin \varphi \\ 0 & \sin \varphi & \cos \varphi & b \cos \varphi \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  \hfill (1.5.9)

Equations (1.5.7)-(1.5.9) yield
\[ x_1 = r_C \cot \psi_C - u \cos \psi_C \]
\[ y_1 = u \sin \psi_C \sin (\theta - q) - b \sin q \]  \hfill (1.5.10)
\[ z_1 = u \sin \psi_C \cos (\theta - q) + b \cos q. \]

Equations (1.5.10) represent the generating surface in coordinate system \( S_1 \), represent the generating gear.

The surface normal is represented by the equation
\[ N_1 = \frac{\partial r_1}{\partial \theta} \times \frac{\partial r_1}{\partial u}. \]
The surface unit normal is represented by the equations (it is assumed that $u \sin \psi_c \neq 0$):

$$
\vec{n}_1 = \sin \psi_c \vec{i}_1 + \cos \psi_c \sin(\theta-q) \vec{j}_1 + \cos \psi_c \cos(\theta-q) \vec{k}_1 \tag{1.5.12}
$$

In the process of cutting the generating gear 1 rotates about axis $x_f$ (of coordinate system $S_f$) rigidly connected with the frame, while the generated gear 2 rotates about axis $z_p$ of the auxiliary coordinate system $S_p$ which is rigidly connected with $S_f(x_f,y_f,z_f)$ (Fig. 1.5.5). The angular velocities $\omega^{(1)}$ and $\omega^{(2)}$ are related such that $O_M^{(p)}$ is the instantaneous axis of rotation ($O_M^{(p)}$ is the generatrix of the pitch cone of gear 2). A coordinate system $S_2$ (see below) is rigidly connected with gear 2.

The coordinate transformation is represented by matrix equations

$$
[\vec{r}_f] = [M_{f1}] [\vec{r}_1] \tag{1.5.12}
$$

$$
[\vec{r}_p] = [M_{pf}] [\vec{r}_f] \tag{1.5.13}
$$

$$
[\vec{r}_2] = [M_{2p}] [\vec{r}_2] \tag{1.5.14}
$$

According to the drawings of Figs. 1.5.5-7, the mentioned matrices are given by

$$
[M_{f1}] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi_1 & \sin \phi_1 & 0 \\
0 & -\sin \phi_1 & \cos \phi_1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \tag{1.5.15}
$$
FIG 1.5.5

Positioning of Generating Gear and Generated Gear During Cutting
FIG. 1.5.6

Applied Coordinate Systems (Top View)
FIG. 1.5.7
Coordinate Systems $S_2$ and $S_f$
\[ \mathbf{M}_{\frac{\pi}{2}} = \begin{bmatrix} \cos(\gamma_2 - \Delta_2) & 0 & \sin(\gamma_2 - \Delta_2) & h \cos(\gamma_2 - \Delta_2) \\ 0 & 1 & 0 & 0 \\ -\sin(\gamma_2 - \Delta_2) & 0 & \cos(\gamma_2 - \Delta_2) & -h \sin(\gamma_2 - \Delta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

\[ \mathbf{M}_{\frac{\pi}{2}} = \begin{bmatrix} \cos\phi_2 & \sin\phi_2 & 0 & 0 \\ -\sin\phi_2 & \cos\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]  

Let us define the equation of meshing (1.4.7). The relative velocity \( \mathbf{v}_f^{(21)} \) is represented by equation

\[ \mathbf{v}_f^{(21)} = \mathbf{v}_f^{(2)} - \mathbf{v}_f^{(1)} = \mathbf{\omega}_f^{(2)} \times \mathbf{r}_f + \mathbf{0}_{1,p} \times \mathbf{\omega}_f^{(2)} - \mathbf{\omega}_f^{(1)} \times \mathbf{r}_f = \\
\mathbf{0}_f \times \mathbf{r}_f + \mathbf{0}_{1,p} \times \mathbf{\omega}_f^{(2)} = \\
\begin{bmatrix} \mathbf{i}_f & \mathbf{j}_f & \mathbf{k}_f \\ -\mathbf{\omega}_f^{(2)} \sin(\gamma_2 - \Delta_2) + \mathbf{\omega}_f^{(1)} & 0 & \mathbf{\omega}_f^{(2)} \cos(\gamma_2 - \Delta_2) \\ \mathbf{x}_f & \mathbf{y}_f & \mathbf{z}_f \\ \mathbf{i}_f & \mathbf{j}_f & \mathbf{k}_f \end{bmatrix} + \\
\mathbf{0}_f \times \mathbf{r}_f = \\
\begin{bmatrix} \mathbf{i}_f & \mathbf{j}_f & \mathbf{k}_f \\ -\mathbf{\omega}_f^{(2)} \sin(\gamma_2 - \Delta_2) & 0 & \mathbf{\omega}_f^{(2)} \cos(\gamma_2 - \Delta_2) \end{bmatrix} \]  

\[ (1.5.17) \]

Vectors \( \mathbf{\omega}_f^{(2)} \) and \( \mathbf{\omega}_f^{(1)} \) are related such that \( \mathbf{\omega}_f^{(2)} - \mathbf{\omega}_f^{(1)} = \mathbf{\omega}_f^{(21)} \) coincides with the generatrix of the pitch cone. Consequently (Fig. 1.5.5),

\[ \mathbf{\omega}_f^{(2)} \sin \gamma_2 = \mathbf{\omega}_f^{(1)} \cos \Delta_2 \]  

\[ (1.5.18) \]

Equations (1.5.17) and (1.5.18) yield

\[ \mathbf{v}_f^{(21)} = \mathbf{\omega}_f^{(2)} \cos(\gamma_2 - \Delta_2) \left[ \mathbf{y}_f \mathbf{i}_f + (\mathbf{x}_f + \mathbf{h}) \mathbf{j}_f \right] \]  

\[ (1.5.19) \]

It results from equation of meshing (1.4.7) that \( \mathbf{N}_f \cdot \mathbf{v}_f^{(21)} = 0 \) and from (1.5.19) that

\[-\mathbf{y}_f \mathbf{N}_f \mathbf{x}_f + (\mathbf{x}_f + \mathbf{L} \sin \Delta_2) \mathbf{N}_f \mathbf{y} = 0 \]  

\[ (1.5.20) \]
Here $\sin\Delta_2 = h$ (Fig. 1.5.5), where \( l = O_M(P) \) is the mean length of the generatrix of pitch cone.

Equation (1.5.20) can be obtained another way, on the basis of equation (1.4.14), which was represented above by

\[
\frac{X_f - x}{N_{fx}} - \frac{Y_f - y}{N_{fy}} = 0
\]

(1.5.21)

Here $X_f$ and $Y_f$ are coordinates of an arbitrary point on instantaneous axis of rotation - generatrix $O_M(P)$. In the discussed case putting into equation (1.5.21) coordinates $X_f = h$, $Y_f = 0$ of point $O_p$ (Fig. 1.5.7), equation (1.5.20) will be found.

Equation (1.5.10) and matrix equation (1.5.12) with matrix (1.5.15) yield that

\[
X_f = r_c \cot \psi_c - u \cos \psi_c
\]

\[
y_f = u \sin \psi_c \sin(\theta - q + \phi_1) - b \sin(\phi_1)
\]

\[
z_f = u \sin \psi_c \cos(\theta - q + \phi_1) + b \cos(\phi_1)
\]

(1.5.22)

Equation (1.5.11) and matrix equation

\[
[n_f] = [L_f1] [n_1]
\]

(1.5.23)

does not apply.

Equations (1.5.20), (1.5.22) and (1.5.24) yield

\[
(r_c \cot \psi_c - u \cos \psi_c + L \sin \Delta_2) \cos \psi_c \sin(\theta - q + \phi_1) -
[u \sin \psi_c \sin(\theta - q + \phi_1) - b \sin(\phi_1)] \sin \psi_c =
\]

\[
[(r_c \cot \psi_c + L \sin \Delta_2) \cos \psi_c - u] \sin(\theta - q + \phi_1) +
\]

\[
b \sin \psi_c \sin(\phi_1) - f(u, \theta, \phi_1) = 0
\]

(1.5.25)
Equation (1.5.25) is the equation of meshing.

Equations (1.5.10) and (1.5.25) represent the set of contact lines covering surface $\Sigma_1$. Each contact line of the set is defined by fixed value of $\phi$. Surface $\Sigma_2$ is represented by equations

$$x_2 = x_2(u, \theta, \phi), \quad y_2 = y_2(u, \theta, \phi), \quad z_2 = z_2(u, \theta, \phi_1),$$

$$f(u, \theta, \phi_1) = 0 \quad (1.5.26)$$

The first three equations are defined by equations (1.5.10) and matrix equality

$$[r_2] = [M_{2p}] [M_{pf}] [M_{f1}] [r_1] \quad (1.5.27)$$

### 1.6 Relations Between Principal Curvatures and Directions of Two Surfaces

**Being in Meshing**

Generally, equations of the enveloped surface are considerably more complicated than of the enveloping one. Therefore a direct way to obtain the principal curvatures and directions of the enveloped surface is a very hard problem. The solution of this problem can be significantly simplified if relations between the principal curvatures and directions of two surfaces which are in mesh are known. Such relations were worked out first by F. L. Litvin. It is necessary to emphasize that the principal curvatures and directions of two contacting surfaces are necessary to define the size and direction of contact ellipse at the contact point.

Let us suppose that surfaces $\Sigma_1$ and $\Sigma_2$ contact each other at point $M$ given in the coordinate system $S_f$ rigidly connected with the frame. Principal directions of surface $\Sigma_1$ are represented by unit vectors $i_1$ and $i_{II}$ and principal curvatures $\kappa_1$ and $\kappa_{II}$ of $\Sigma_1$ are known. At the point of contact the equation of meshing

$$n_f^{(1)}v_f^{(12)} = n_f^{(1)} \left[ (\omega_f^{(1)} - \omega_f^{(2)}) \nabla f^{(1)} - (R \times \omega_f^{(2)}) \right] = 0 \quad (1.6.1)$$

is satisfied
Here: \( n_f^{(1)} \) is the surface \( S_{1} \) unit normal; \( v_f^{(12)} \) is the relative velocity \( (v_f^{(12)} = v_f^{(1)} - v_f^{(2)} ; v_f^{(1)}(i=1,2) \) is the transfer velocity of a point rigidly connected with surface \( S_{i} ; R_f \) is a vector-radius drawn from the origin of coordinate system \( S_f \) to an arbitrary point of the line of action of angular velocity \( \omega_f^{(2)} \); vector \( v_f^{(12)} = -v_f^{(21)} = (v_f^{(2)} - v_f^{(1)}) \) where \( v_f^{(21)} \) is the vector represented by equations (1.3.14).

Equation of meshing (1.6.1) must be observed not only at the point of contact \( M \), but in the neighborhood of \( M \), too. Therefore, equation (1.6.1) can be differentiated which yields:

\[
\frac{\partial}{\partial t} \vec{v}_f^{(1)}(1,2) - \vec{n}_f^{(1)}(1,2) \times \vec{\omega}_f^{(1,2)} = 0
\]  

(1.6.2)

It is assumed that \( \omega_f^{(1)} = \text{const}; R_f(21) = \frac{\omega_f^{(2)}}{\omega_f^{(1)}} = \text{const}; R = \text{const} \). Lower subscript "f" is eliminated for simplification.

According to results demonstrated in items (1.1) and (1.2) by equations (1.1.31) and (1.2.7) it yields that

\[
\vec{n}_f^{(1)} = \vec{\omega}_f^{(1)} \times \vec{n}_f^{(1)}, \quad \vec{n}_f^{(1)}
\]  

(1.6.3)

Equation (1.1.30) yields

\[
\vec{\omega}_f^{(1)}(1,2) = \vec{\omega}_f^{(1)}(1) + \vec{\omega}_f^{(1)}(2)
\]  

(1.6.4)

It results from equations (1.6.2), (1.6.3) and (1.6.4) that

\[
\begin{bmatrix}
\vec{\omega}_f^{(1)}(1,2) \\
\vec{n}_f^{(1)}(1,2)
\end{bmatrix}
+ \vec{\omega}_f^{(1)}(1,2) \\
\vec{n}_f^{(1)}(1,2)
= 0
\]  

(1.6.5)

where

\[
\vec{\omega}_f^{(12)} = \vec{\omega}_f^{(1)}(1) - \vec{\omega}_f^{(2)}
\]  

(1.6.6)

\[
\vec{\omega}_f^{(12)} = \vec{\omega}_f^{(1)} - \vec{\omega}_f^{(2)}
\]  

(1.6.7)

Equations (1.6.5)-(1.6.7) yield

\[
\begin{aligned}
\hat{n}_f^{(1)}(1,2) \cdot \vec{v}_f^{(1)}(1,2) - \vec{n}_f^{(1)}(1,2) \times \vec{\omega}_f^{(1,2)} \\
+ \vec{n}_f^{(1)}(1,2) \times \vec{v}_f^{(1,2)} - \vec{n}_f^{(1)}(1,2) \times \vec{v}_f^{(1,2)}
= 0
\end{aligned}
\]  

(1.6.8)

38
Two other equations

\[ v_r(2) = v_r(1) + v(12) \]  
\[ n_r(2) = n_r(1) + \omega(12) \times n(1) \]  

were represented before in item (1.1) by equations (1.1.35) and (1.1.36).

Relations between the principal curvatures and principal directions of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) will be composed on the basis of equations (1.6.8) - (1.6.10). Before this, let us recall the following equations from differential geometry. The normal curvature of a surface is represented by equation

\[ \kappa = - \frac{\dot{n}_r \times r}{n_r \times r} \]  

Along the principal direction, vectors \( \ddot{n}_r \) and \( v_r \) are co-linear and the principal curvature is represented by equation

\[ \dot{n}_r \times i = - \kappa_1 (v_r \times i), \]  

where \( i \) is the unit vector directed along the principal direction.

Now, let us place two right trihedrons at the contact point \( M \) (Fig. 1.6.1): \( S_a(i_I, i_{II}, n) \) and \( S_b(i_{III}, i_{IV}, n) \). Here: \( i_I, i_{II} \) are unit vectors directed along principal directions of surface \( \Sigma_1 \); \( i_{III} \) and \( i_{IV} \) are unit vectors directed along principal directions of surface \( \Sigma_2 \); \( n \) is the common unit normal of surfaces \( \Sigma_1 \) and \( \Sigma_2 \). It is assumed that unit vectors \( i_I \) and \( i_{III} \) make an angle \( \varphi \) (Fig. 1.6.1). Vectors \( v_r(1), \dot{n}_r(1) \) and \( v_r(2), \dot{n}_r(2) \) can be expressed in terms of components of coordinate systems \( S_a \) and \( S_n \) by following equations

\[ v_r(1) = v_{rI}(1) i_I + v_{rII}(1) i_{II} \]  
\[ \dot{n}_r(1) = \dot{n}_{rI}(1) i_I + \dot{n}_{rII}(1) i_{II} \]  
\[ v_r(2) = v_{rIII}(2) i_{III} + v_{rIV}(2) i_{IV} \]  
\[ \dot{n}_r(2) = \dot{n}_{rIII}(2) i_{III} + \dot{n}_{rIV}(2) i_{IV} \]
Fig. 1.6.1

Principal Directions of Surfaces $\Sigma_1$ and $\Sigma_2$
Vectors $v_r^{(2)}$ and $\hat{n}_r^{(2)}$ can be expressed in terms of components of coordinate system $S, (i_I, i_{II}, i_{III})$ by the following equations

$$v_{rI}^{(2)} = v_{rII}^{(2)} i_{II} + v_{riv}^{(2)} i_{IV}, i_{II}$$  \hspace{1cm} (1.6.17)

$$v_{rII}^{(2)} = v_{rIII}^{(2)} i_{III} + v_{riv}^{(2)} i_{IV}, i_{II}$$  \hspace{1cm} (1.6.18)

$$\hat{n}_r^{(2)} = \hat{n}_r^{(2)} i_{IV} = \hat{n}_r^{(2)} i_{III, i_{IV}}, i_{III} + \hat{n}_r^{(2)} i_{IV}, i_{IV}$$  \hspace{1cm} (1.6.19)

$$\hat{n}_r^{(2)} = \hat{n}_r^{(2)} i_{IV} = \hat{n}_r^{(2)} i_{III} + \hat{n}_r^{(2)} i_{IV}, i_{IV}$$  \hspace{1cm} (1.6.20)

Here (Fig. 1.6.1):

$$i_{III, i_{IV}} = \cos \sigma, \ i_{IV, i_{II}} = -\sin \sigma, \ i_{III, i_{III}} = \sin \sigma, \ i_{IV, i_{IV}} = \cos \sigma$$  \hspace{1cm} (1.6.21)

Equations (1.6.17)-(1.6.21) yield

$$v_{rI}^{(2)} = v_{rIII}^{(2)} \cos \sigma - v_{riv}^{(2)} \sin \sigma$$  \hspace{1cm} (1.6.22)

$$v_{rII}^{(2)} = v_{rIII}^{(2)} \sin \sigma + v_{riv}^{(2)} \cos \sigma$$  \hspace{1cm} (1.6.22, a)

$$\hat{n}_r^{(2)} = \hat{n}_r^{(2)} i_{IV} = \hat{n}_r^{(2)} i_{III, i_{IV}}, i_{III} + \hat{n}_r^{(2)} i_{IV}, i_{IV}$$  \hspace{1cm} (1.6.23)

$$\hat{n}_r^{(2)} = \hat{n}_r^{(2)} i_{IV} = \hat{n}_r^{(2)} i_{III} + \hat{n}_r^{(2)} i_{IV}, i_{IV}$$  \hspace{1cm} (1.6.24)

Equations (1.6.8)-(1.6.10), (1.6.12) and (1.6.22)-(1.6.24) yield the following system of 9 linear equations in 8 unknowns $v_{rI}^{(1)}, v_{rII}^{(1)}, v_{rIII}^{(1)}, v_{riv}^{(1)}, \hat{n}_r^{(1)}, \hat{n}_r^{(1)}, \hat{n}_r^{(1)}, \hat{n}_r^{(1)}$:

$$\begin{bmatrix}
\hat{n}_r^{(1),}} & v_{rI}^{(12)} + \hat{n}_r^{(1),}} v_{rII}^{(12)} - v_{rI}^{(12)} - v_{rII}^{(12)} & v_{rIII}^{(12)} + \hat{n}_r^{(1),}} v_{riv}^{(12)} - v_{rIII}^{(12)} - v_{riv}^{(12)} \\
\hat{n}_r^{(1),}} & v_{rI}^{(12)} + \hat{n}_r^{(1),}} v_{rII}^{(12)} - v_{rI}^{(12)} - v_{rII}^{(12)} & v_{rIII}^{(12)} + \hat{n}_r^{(1),}} v_{riv}^{(12)} - v_{rIII}^{(12)} - v_{riv}^{(12)} \\
\end{bmatrix} = 0$$  \hspace{1cm} (1.6.25)

$$v_{rII}^{(2)} \cos \sigma - v_{riv}^{(2)} \sin \sigma - v_{rI}^{(1)} = v_{I}^{(12)}$$  \hspace{1cm} (1.6.26)

$$v_{rIII}^{(2)} \sin \sigma + v_{riv}^{(2)} \cos \sigma - v_{rII}^{(1)} = v_{II}^{(12)}$$  \hspace{1cm} (1.6.27)

$$\hat{n}_r^{(2)} \cos \sigma - \hat{n}_r^{(2)} \sin \sigma - \hat{n}_r^{(1),}} = \hat{n}_r^{(1),}}$$  \hspace{1cm} (1.6.28)

$$\hat{n}_r^{(2)} \sin \sigma + \hat{n}_r^{(2)} \cos \sigma - \hat{n}_r^{(1),}} = \hat{n}_r^{(1),}}$$  \hspace{1cm} (1.6.29)

$$\hat{n}_r^{(2)} \sin \sigma + \hat{n}_r^{(2)} \cos \sigma - \hat{n}_r^{(1),}} = \hat{n}_r^{(1),}}$$  \hspace{1cm} (1.6.29)
Here: \( \kappa_I \) and \( \kappa_{II} \), \( \kappa_{III} \) and \( \kappa_{IV} \) are principal curvatures of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) at contact point \( M \).

After eliminating 6 unknowns a system of 3 linear equations in two unknowns \( x_1 = v_{III}^{(1)} \), \( x_2 = v_{III}^{(1)} \) can be got:

\[
\begin{align*}
   a_{11}x_1 + a_{12}x_2 &= b_1 \\
   a_{21}x_1 + a_{22}x_2 &= b_2 \\
   a_{31}x_1 + a_{32}x_2 &= b_3
\end{align*}
\]

Here:

\[
\begin{align*}
   a_{11} &= -\kappa_I + 1/2 \left[ (\kappa_{III} + \kappa_{IV}) + (\kappa_{III} - \kappa_{IV})\cos 2\sigma \right] \\
   a_{12} &= a_{21} = 1/2 \left[ (\kappa_{III} - \kappa_{IV})\sin 2\sigma \right] \\
   a_{22} &= -\kappa_{II} + 1/2 \left[ (\kappa_{III} + \kappa_{IV}) - (\kappa_{III} - \kappa_{IV})\cos 2\sigma \right] \\
   a_{31} &= \left[ \left( \frac{1}{2} \right) \frac{1}{2} \frac{n_{III}^{(12)}}{n_{III}^{(12)}} \right] - \kappa_{III} \frac{n_{III}^{(1)}}{2} \\
   a_{32} &= \left[ \left( \frac{1}{2} \right) \frac{1}{2} \frac{n_{III}^{(12)}}{n_{III}^{(12)}} \right] - \kappa_{IV} \frac{n_{III}^{(12)}}{2} \\
   b_1 &= \left[ \left( \frac{1}{2} \right) \frac{1}{2} \frac{n_{III}^{(12)}}{n_{III}^{(12)}} \right] - \frac{v_{III}^{(12)}}{2} \left[ (\kappa_{III} + \kappa_{IV}) + (\kappa_{III} - \kappa_{IV})\cos 2\sigma \right] \\
   b_2 &= \left[ \left( \frac{1}{2} \right) \frac{1}{2} \frac{n_{III}^{(12)}}{n_{III}^{(12)}} \right] - \frac{v_{III}^{(12)}}{2} (\kappa_{III} - \kappa_{IV})\sin 2\sigma \\
   b_3 &= \left[ \left( \frac{1}{2} \right) \frac{1}{2} \frac{n_{III}^{(12)}}{n_{III}^{(12)}} \right] - \frac{v_{III}^{(12)}}{2} (\kappa_{III} - \kappa_{IV})\cos 2\sigma
\end{align*}
\]
The number of equations (1.6.34) is not equal to the number of unknowns. Therefore, requirements to this system by which the system will have a solution must be discussed.

Let us consider two cases: (a) the instantaneous contact of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) is a linear-contact; (b) the instantaneous contact of surfaces is a point contact.

In the first case surface \( \Sigma_1 \) is covered with instantaneous contact lines (Fig. 1.6.2,a) and the direction of \( v_{x_{(1)}} \) from point \( M \) to the neighboring one is an indefinite one and the system (1.6.34) must have an infinite number of solutions. In the second case contact points makes on surface \( \Sigma_1 \) a line (Fig. 1.6.2,b), the direction of \( v_{x_{(1)}} \) to the neighboring point is a definite one, and the system (1.6.34) must possess one solution.

It is known from linear algebra that system (1.6.34) possesses an infinite number of solutions if the rank of matrix

\[
\begin{bmatrix}
  a_{11} & a_{12} & b_1 \\
  a_{21} & a_{22} & b_2 \\
  a_{31} & a_{32} & b_3 \\
\end{bmatrix}
\]

(1.6.35)

is equal to one.

That yields

\[
\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{b_1}{b_2}, \quad \frac{a_{21}}{a_{31}} = \frac{a_{22}}{a_{32}} = \frac{b_2}{b_3}
\]

(1.6.36)

Taking into account that \( a_{21} = a_{12} \) equalities (1.6.36) can be represented as:

\[
\frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{a_{31}}{a_{32}} = \frac{b_1}{b_2}, \quad \frac{a_{21}}{a_{31}} = \frac{b_2}{b_3}
\]

(1.6.37, 1.6.38)
Fig: 1.6.2

Directions of Velocity of Contact Point
The system of equalities (1.6.37) provides only two independent equations because
\[ b_1 = a_{31} - v_1^{(12)} a_{11} - v_1^{(12)} a_{12}, \quad b_2 = a_{32} - v_1^{(12)} a_{12} - v_1^{(12)} a_{22} \]

Equality (1.6.38) and
\[ \frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{a_{31}}{a_{32}} \]
provide three equations for definition of \( k_{III}, k_{IV} \) and \( \sigma \):
\[ \tan 2\sigma = \frac{2F}{k_{I} - k_{II} + G} \] (1.6.40)
\[ k_{III} + k_{IV} = k_{I} + k_{II} + S \] (1.6.41)
\[ k_{III} - k_{IV} = \frac{k_{I} - k_{II}}{\cos 2\sigma} + G \] (1.6.42)

Here:
\[ F = \frac{a_{31} a_{32} b_3 + v_1^{(12)} a_{31} + v_1^{(12)} a_{32}}{b_3 + v_1^{(12)} a_{31} + v_1^{(12)} a_{32}} \]
\[ G = \frac{a_{31} - a_{32}^2}{b_3 + v_1^{(12)} a_{31} + v_1^{(12)} a_{32}} \]
\[ S = \frac{a_{31}^2 + a_{32}^2}{b_3 + v_1^{(12)} a_{31} + v_1^{(12)} a_{32}} \]

For the case when surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are in point contact and the system (1.6.34) possesses one solution the rank of matrix (1.6.35) must be equal to two. That yields that the determinant of matrix (1.6.35) must be equal to zero. Consequently,
\[ \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = 0 \] (1.6.43)

Equality (1.6.43) provides an equation.
\[ f(\kappa_I, \kappa_{II}, \kappa_{III}, \kappa_i, \kappa, \sigma) = 0 \] (1.6.44)

which relates the principal curvatures and directions of two surfaces in point contact.

Sample problem 1.6.1. Let us compose equations to define principal curvatures and directions of a spiral bevel gear generated by a cone surface (sample problem 1.5.1). The generating surface \( \Sigma_1 \) is represented by equation (1.5.22).

The relative velocity \( \nu_r^{(1)} \) is represented by the following equations

\[
\begin{bmatrix}
\nu_r^{(1)} \\
\nu_x \\
\nu_y \\
\nu_z
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial x_f}{\partial u} \frac{du}{dt} + \frac{\partial x_f}{\partial \theta} \frac{d\theta}{dt} \\
\frac{\partial y_f}{\partial u} \frac{du}{dt} + \frac{\partial y_f}{\partial \theta} \frac{d\theta}{dt} \\
\frac{\partial z_f}{\partial u} \frac{du}{dt} + \frac{\partial z_f}{\partial \theta} \frac{d\theta}{dt}
\end{bmatrix}
\] (1.6.45)

Equation (1.5.22) and equality (1.6.45) yield

\[
\begin{bmatrix}
\nu_r^{(1)} \\
\nu_x \\
\nu_y \\
\nu_z
\end{bmatrix} =
\begin{bmatrix}
\cos \psi_c \frac{du}{dt} \\
\sin \psi_c \sin(\theta-q+\phi_1) \frac{du}{dt} + u \sin \psi_c \cos(\theta-q+\phi_1) \frac{d\theta}{dt} \\
\sin \psi_c \cos(\theta-q+\phi_1) \frac{du}{dt} - u \sin \psi_c \sin(\theta-q+\phi_1) \frac{d\theta}{dt}
\end{bmatrix}
\] (1.6.46)

The unit normal of generating surface was represented by equations (1.5.24).

It results from (1.5.24) that

\[
\begin{bmatrix}
\hat{n}_r^{(1)} \\
0 \\
\cos \psi_c \cos(\theta-q+\phi_1) \frac{d\theta}{dt} \\
-\cos \psi_c \sin(\theta-q+\phi_1) \frac{d\theta}{dt}
\end{bmatrix}
\] (1.6.47)

Vectors (1.6.46) and (1.6.47) are co-linear for principal directions of surface \( \Sigma_1 \). Consequently,
Equalities (1.6.46) - (1.6.48) yield that

\[
\frac{du}{dt} \frac{d\theta}{dt} = 0
\]  

(1.6.49)

One of the principal directions with unit vector \( \hat{z}_I \) corresponds to \( \frac{du}{dt} = 0 \). The principal curvature

\[
\kappa_I = -\frac{\dot{h}(1)}{v(1)} \cdot \frac{\dot{v}(1)}{v(1)} = -\frac{1}{u \tan \psi_c}
\]  

(1.6.50)

The unit vector \( \hat{z}_I \) can be represented by equation

\[
\hat{z}_I = \left| \frac{v(1)}{\dot{v}(1)} \right| \text{ by } \frac{du}{dt} = 0
\]  

(1.6.51)

Equations (1.6.46) and (1.6.51) yield

\[
\begin{bmatrix}
\hat{z}_I \\
\hat{z}_II
\end{bmatrix} =
\begin{bmatrix}
0 \\
\cos(\theta - \psi_1) \\
-\sin(\theta - \psi_1)
\end{bmatrix}
\]  

(1.6.52)

The second principal direction corresponds to \( \frac{d\theta}{dt} = 0 \). The principal curvature is

\[
\kappa_{II} = 0
\]  

(1.6.53)

and the unit vector of the principal direction is

\[
\begin{bmatrix}
\hat{z}_II
\end{bmatrix} =
\begin{bmatrix}
-\cos \psi_c \\
\sin \psi_c \sin(\theta - \psi_1) \\
\sin \psi_c \cos(\theta - \psi_1)
\end{bmatrix}
\]  

(1.6.54)

A case is suggested when \( \Delta_2 = 0 \), \( \omega(1) = 1 \frac{\text{rad}}{\text{sec}} \). Then:

\[
\omega(2) = \frac{1}{\sin \gamma_2},
\]
\[
[\omega^{(1)}] = \begin{bmatrix}
-1 \\
0 \\
0
\end{bmatrix},
\]

\[
[\omega^{(2)}] = \frac{1}{\sin\gamma_2} \begin{bmatrix}
-sin\gamma_2 \\
0 \\
\cos\gamma_2
\end{bmatrix}
\]

\[
[v^{(12)}] = \cot\gamma_2 \begin{bmatrix}
y \\
x \\
0
\end{bmatrix},
\]

where \((x,y,z)\) are represented by equations (1.5.22), the lower subscript "f" is eliminated. Equations (1.6.40)-(1.6.42) define principal curvatures and directions of tooth surface \(\Sigma_2\) of the generated gear.

Let us define principal curvatures and directions at the mean contact point \(M\) with coordinates \(x=y=0, z=L\). It results from equations (1.5.22) and (1.5.24) that point \(M\) is generated by \(\phi_1=0, \theta_q=90^\circ-\beta\). By \(x=0, y=0\) vector \(v^{(12)}\) is equal to zero. Coefficients \(a_{31}, a_{32}, b_3, F\) and \(S\) are represented by equations:

\[
a_{31} = [n^{(1)}] [\omega^{(12)}] [i_1] = \sin\psi_c \sin\beta \cot\gamma_2 \quad (1.6.55)
\]

Here:

\[
[n^{(1)}] = \begin{bmatrix}
\sin\psi_c \\
\cos\psi_c \cos\beta \\
\cos\psi_c \sin\beta
\end{bmatrix}
\]

\[
[\omega^{(12)}] = \begin{bmatrix}
0 \\
0 \\
-cot\gamma_2
\end{bmatrix}
\]

\[
i_1 = \begin{bmatrix}
0 \\
\sin\beta \\
-cos\beta
\end{bmatrix}
\]
\[
a_{32} = \begin{bmatrix} n^{(1)} \omega^{(12)}_1 \end{bmatrix} = \cos \beta \cot \gamma_2
\] (1.6.56)

Here:

\[
\begin{bmatrix} \psi_c \\
\sin \psi_c \cos \beta \\
\sin \psi_c \sin \beta
\end{bmatrix}
\]

\[
b_3 = \begin{bmatrix} n^{(1)} \omega^{(2)} \end{bmatrix} - \begin{bmatrix} n^{(1)} \omega^{(1)} \end{bmatrix} = -L \sin \psi_c \cot \gamma_2
\] (1.6.56.a)

Here:

\[
\begin{bmatrix} \omega^{(1)} \\
\nu^{(2)} \end{bmatrix} = \begin{bmatrix} \omega^{(1)} \end{bmatrix} = \begin{bmatrix} \omega^{(1)} \end{bmatrix}
\]

\[
\begin{bmatrix} i_f \\
j_f \\
k_f \\
-1 \\
0 \\
0 \\
0 \\
L
\end{bmatrix} ; \begin{bmatrix} \omega^{(2)} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
L
\end{bmatrix}
\]

\[
2F = \frac{2a_{31}a_{32}}{b_3} = \frac{\sin 2\beta \cot \gamma_2}{L}
\] (1.6.57)

\[
G = \frac{a_{31}^2 - a_{32}^2}{b_3} = \frac{(\sin^2 \psi_c \sin^2 \beta - \cos^2 \beta) \cot \gamma_2}{L \sin \psi_c}
\] (1.6.58)

\[
S = \frac{a_{31}^2 + a_{32}^2}{b_3} = \frac{(\sin^2 \psi_c \sin^2 \beta + \cos^2 \beta) \cot \gamma_2}{L \sin \psi_c}
\] (1.6.59)

Equation (1.6.50) yields that at point M

\[
\kappa_I = -\frac{1}{u \tan \psi_c} = -\frac{\cos \psi_c}{r_c}
\] (1.6.60)

It results from equations (1.6.40)-(1.6.42) and (1.6.57)-(1.6.59) that

\[
\tan 2\varphi = \frac{2F}{\kappa_I - \kappa_{II} + G}
\]
Equations (1.6.61)-(1.6.63) define the principal curvatures and directions of the generated surface of spiral bevel gear at the main contact point M.

These equations may be applied for bevel gears with straight teeth, too. For this case \( \beta = 0, \frac{1}{r_c} = 0, \kappa_I = \kappa_{II} = 0 \) because the generating surface is a plane. Equations (1.6.61)-(1.6.63) yield

\[
\tan 2\sigma = 0, \kappa_{III} = 0, \kappa_{iv} = -\frac{\cot \gamma_2}{L \sin \psi_c} \tag{1.6.64}
\]

1.7. Contact Ellipse

The bearing contact of spiral bevel and hypoid gears is checked on a test-machine under a small load. The bearing contact depends on the contact ellipse of tooth surfaces which are considered as elastic ones.

There is a typical problem in the theory of elasticity: (a) the magnitudes of contact forces and mechanical properties of surface materials are given; (b) the principal curvatures and directions of surfaces at their contact point are known. Methods known from the theory of elasticity permit to define the approach of surfaces, the size and location of contact ellipse.
To appraise conditions of tooth contact it is more reasonable to consider as given the approach of surfaces under the action of load. Then, the size and location of instantaneous contact ellipse can be defined as a result of a simple geometric solution. The magnitude of surface approach is known from experiments.

Fig. 1.7.1 shows surfaces \( \Sigma_1 \) and \( \Sigma_2 \) in tangency at point M. The unit normal and the tangent plane are designated by \( \vec{n} \) and \( t-t \). The deformed surfaces are shown by dotted lines. The areas of deformation are \( K_1ML_1 \) for surface \( \Sigma_1 \) and \( K_2ML_2 \) for surface \( \Sigma_2 \).

Let us choose points \( N(\rho, \xi^{(1)}) \) and \( N'(\rho, \xi^{(2)}) \) where \( \rho \) is the distance from M and \( \xi^{(i)} \) \((i=1,2)\) is the distance from the tangent plane. As a result of deformation, body 1 will be displaced in a direction opposite the unit normal \( \vec{n} \) by \( \delta_1 \) (Fig. 1.7.1, Fig. 1.7.2); body 2 will be displaced in the opposite direction by \( \delta_2 \). The approach of both bodies is \( \delta = \delta_1 + \delta_2 \).

The approach of bodies is accompanied with their elastic deformation. It is necessary to distinguish the displacement of a body point with the body given by \( \delta_i \) \((i=1,2)\), and a displacement relative to the body resulting from elastic deformation.

Let us define the new location \( N_2 \) of point N. With the body point 1 will displace by \( \delta_1 \) and get the position \( N_1 \). Due to elastic deformation which is equal to \( f_1 \) point N will be displaced from \( N_1 \) to \( N_2 \). The distance \( \xi \) between point \( N_2 \) and the tangent plane \( t-t \) is represented by the following equation

\[
\xi = \xi^{(1)} - \delta_1 + f_1 \tag{1.7.1}
\]

The resulting position of point \( N' \) of body 2 is \( N'_2 \). The distance \( \xi \) between point \( N'_2 \) and the tangent plane \( t-t \) is represented by equation

\[
\xi = \xi^{(2)} + \delta_2 - f_2 \tag{1.7.2}
\]
Fig. 1.7.1

Surfaces Σ₁ and Σ₂ in Tangency - Before and After Deformation
Fig. 1.7.2
Displacements of Surfaces $\Sigma_1$ and $\Sigma_2$
Due to the approach of bodies and their deformation, points $N$ and $N'$ must coincide and

$$\xi^{(1)} - \delta_1 + f_1 = \xi^{(2)} + \delta_2 - f_2$$

(1.7.3)

Equality (1.7.3) yields

$$\left| \xi^{(1)} - \xi^{(2)} \right| = \delta_1 + \delta_2 - (f_1 + f_2)$$

(1.7.4)

Equation (1.7.4) is observed at all points of the area of deformation.

Without this area

$$\left| \xi^{(1)} - \xi^{(2)} \right| > \delta = \delta_1 + \delta_2$$

(1.7.5)

The right part of equation (1.7.4) is larger than zero because $\delta_1>f_1$, $\delta_2>f_2$. Therefore the left part of equation (1.7.4) represents the absolute magnitude of the difference between $\xi^{(1)}$ and $\xi^{(2)}$.

Within the area of deformation

$$\left| \xi^{(1)} - \xi^{(2)} \right| \leq \delta$$

(1.7.6)

Equation

$$\left| \xi^{(1)} - \xi^{(2)} \right| = \delta$$

(1.7.7)

corresponds to the edge of deformation area. Equation (1.7.7) defines the line which limits the area of deformation.

Let us correlate $\xi^{(i)}$ with surface $\Sigma_i$ curvatures. Surface $\Sigma_i$ is represented by equation

$$\mathbf{r} = \mathbf{r}(u, \theta)$$

(1.7.8)

Curve $MM'$ (Fig. 1.7.3) on a surface $\Sigma$ is represented by equation

$$\mathbf{r} = \mathbf{r}[u(s), \psi(s)]$$

(1.7.9)

where $s$ is the length of an arc.

Let us designate by $\Delta s = MM'$ the arc length and by $\Delta \mathbf{r} = MM'$ the increment of vector-radius $\mathbf{r}$. The increment $\Delta \mathbf{r}$ can be expressed by Taylor-Series Expansion.

54
Fig. 1.7.3

Tooth Surface and Tangent Plane
\[ M'M' = \Delta \mathbf{r} = \frac{dr}{ds} \Delta s + \frac{d^2r}{ds^2} \left( \frac{\Delta s}{2} \right)^2 + \frac{d^3r}{ds^3} \left( \frac{\Delta s}{3!} \right)^3 + \ldots, \quad (1.7.10) \]

where

\[
\frac{dr}{ds} = \frac{\partial r}{\partial u} du \frac{\partial r}{\partial \theta} \frac{ds}{ds} + \frac{\partial r}{\partial \theta} \frac{ds}{ds} \frac{\partial r}{\partial \theta} \left( \frac{du}{ds} \right)^2 + \\
\frac{d^2r}{ds^2} \frac{du}{ds} \frac{d\theta}{ds} + \frac{d^2r}{d\theta^2} \left( \frac{d\theta}{ds} \right)^2 \quad \text{and so on.} 
\]

Let us draw a plane \( \Pi \) tangent to the surface \( \Sigma \) at point \( M \) and then draw from point \( M' \) a perpendicular \( MP' \) to \( \Pi \). Vector \( PM' \) which is parallel to surface unit normal \( n \) represents the deflexion of point \( M' \) from the tangent plane \( \Pi \). This deflexion is

\[ PM' = \kappa n \quad (1.7.11) \]

Here: \( \kappa > 0 \) if directions of \( PM' \) and \( n \) coincide.

Equalities

\[ MM' = \Delta \mathbf{r}, \quad MM' = MP + PM' = MP + \kappa n \]

yield

\[ MP + \kappa n = \frac{dr}{ds} \Delta s + \frac{d^2r}{ds^2} \left( \frac{\Delta s}{2} \right)^2 + \frac{d^3r}{ds^3} \left( \frac{\Delta s}{3!} \right)^3 + \ldots \quad (1.7.12) \]

Because vectors \( MP \) and \( n \), \( \frac{dr}{ds} \) and \( n \) make right angles the scalar product

\[ (MP + \kappa n) n \quad (1.7.13) \]

yields

\[ \kappa = \frac{d^2r}{ds^2} \frac{1}{2!} \frac{\left( \frac{\Delta s}{2} \right)^2}{2!} + \frac{d^3r}{ds^3} \frac{\left( \frac{\Delta s}{3!} \right)^3}{3!} + \ldots \quad (1.7.14) \]

Up to members of third order \( \kappa \) is represented by the equation

\[ \kappa = \frac{d^2r}{ds^2} n \frac{\left( \frac{\Delta s}{2} \right)^2}{1.2} \quad (1.7.15) \]
It is known from differential geometry that
\[ \frac{d^2 r}{ds^2} \cdot n = \kappa, \]  
(1.7.16)
where \( \kappa \) is the surface curvature in normal section.

Equations (1.7.15) and (1.7.16) yield
\[ \kappa = \frac{\Delta s^2}{2} \]  
(1.7.17)

Let us express \( \Delta s \) in terms of components of the coordinate system
\( \eta, \zeta \) and \( \ell \) (Fig. 1.7.3); axes \( \eta \) and \( \zeta \) are located on the tangent plane \( \Pi \).
\[ \Delta s^2 = \eta^2 + \zeta^2 = \rho^2, \]  
(1.7.18)
where \( \rho = MP \).

It results from (1.7.17) and (1.7.18) that
\[ \kappa = 1/2 \kappa \rho^2 \]  
(1.7.19)

The surface normal curvature can be expressed by principal curvatures and angle \( q \) (Fig. 1.7.3) made by \( MD \) and \( MP \), where \( MD \) is the principal direction with principal curvature \( \kappa_1 \).
\[ \kappa = \kappa_1 \cos^2 q + \kappa_1 \sin^2 q = \kappa_1 \cos^2 (\mu - \sigma) + \kappa_1 \sin^2 (\mu - \sigma) \]  
(1.7.20)
Equations (1.7.19) and (1.7.20) yield
\[ 2\kappa = \rho^2 \left[ \kappa_1 \cos^2 (\mu - \sigma) + \kappa_1 \sin^2 (\mu - \sigma) \right] \]  
(1.7.21)

Figure 1.7.4 shows a plane tangent to surfaces \( \Sigma_1 \) and \( \Sigma_2 \) at point \( M \) of their contact; \( MD_1 \) and \( MD_2 \) with unit vectors \( i_1^{(1)} \) and \( i_1^{(2)} \) are principal directions of \( \Sigma_1 \) and \( \Sigma_2 \) with principal curvatures \( \kappa_1^{(1)} \) and \( \kappa_1^{(2)} \), \( MP \) defines a common normal section of surfaces \( \Sigma_1 \) and \( \Sigma_2 \). Deflections of points of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) from the tangent plane \( T \) (Fig. 1.7.3) are represented by equations
\[ 2\ell^{(1)} = \rho^2 \left[ \kappa_1^{(1)} \cos^2 (\mu - \alpha^{(1)}) + \kappa_1^{(1)} \sin^2 (\mu - \alpha^{(1)}) \right] \]  
(1.7.22)
Fig. 1.7.4

Location of Contact Ellipse

Fig. 1.7.5
\[ 2\lambda(2) = \rho^2 \left[ \kappa_I^1 \cos^2(\mu - \alpha(1)) + \kappa_{II}^1 \sin^2(\mu - \alpha(2)) \right] \]  

(1.7.23)

At the edge of the area of deformation equation (1.7.7) must be held. Equations (1.7.22), (1.7.23) and (1.7.7) yield

\[ \rho^2 \left[ \kappa_I^1 \cos^2(\mu - \alpha(1)) + \kappa_{II}^1 \sin^2(\mu - \alpha(1)) - \kappa_{II}^2 \cos^2(\mu - \alpha(2)) \right. \]
\[ \left. - \kappa_{II}^2 \sin^2(\mu - \alpha(2)) \right] = \pm 2\delta \]

(1.7.24)

Let us transform equation (1.7.24) taking into account that

\[ \rho^2 = \eta^2 + \zeta^2, \cos \mu = \frac{\eta}{\rho}, \sin \mu = \frac{\zeta}{\rho} \]

(1.7.25)

It results from (1.7.24) and (1.7.25) that

\[ \eta^2 (\kappa_I^1 \cos^2(\alpha(1)) + \kappa_{II}^1 \sin^2(\alpha(1)) - \kappa_{II}^2 \cos^2(\alpha(2)) - \kappa_{II}^2 \sin^2(\alpha(2))) + \]
\[ \zeta^2 (\kappa_I^1 \sin^2(\alpha(1)) + \kappa_{II}^1 \cos^2(\alpha(1)) - \kappa_I^2 \sin^2(\alpha(2)) - \kappa_{II}^2 \cos^2(\alpha(2))) + \]
\[ \eta \zeta (g_1 \sin 2\alpha(1) - g_2 \sin 2\alpha(2)) = \pm 2\delta, \]

(1.7.26)

where

\[ g_1 = \kappa_I^1 - \kappa_{II}^1, \quad g_2 = \kappa_I^2 - \kappa_{II}^2 \]

Let us designate \( \alpha(2) - \alpha(1) = \sigma \) (Fig. 1.7.4). The angle \( \alpha(1) \) defining the location of \( MD_1 \) - the principal direction with principal curvature \( \kappa_I \) - can be chosen in an arbitrary way, particularly the way that

\[ g_1 \sin 2\alpha(1) - g_2 \sin 2\alpha(2) = 0 \]

(1.7.27)

Equation (1.7.27) and equation

\[ \alpha(2) = \alpha(1) + \sigma \]

(1.7.28)

yield

\[ \tan 2\alpha(1) = \frac{g_2 \sin 2\sigma}{g_1 - g_2 \cos 2\sigma} \]

(1.7.29)

It results from equations (1.7.26) and (1.7.27) that

\[ B\eta^2 + A\zeta^2 = \pm \delta \]

(1.7.30)
Here:

\[ A = \frac{1}{4} \left[ \kappa_e^{(1)} - \kappa_e^{(2)} - (g_1^2 - 2g_1g_2\cos 2\sigma + g_2^2) \right] \]  \hspace{1cm} (1.7.31)

\[ B = \frac{1}{4} \left[ \kappa_e^{(1)} - \kappa_e^{(2)} + (g_1^2 - 2g_1g_2\cos 2\sigma + g_2^2) \right], \]  \hspace{1cm} (1.7.32)

where

\[ \kappa_e^{(1)} = \kappa_1^{(1)} + \kappa_1^{(2)} \quad \text{and} \quad \kappa_e^{(2)} = \kappa_1^{(2)} + \kappa_1^{(2)} \]

Equation (1.7.30) confirms that the projection of the area of deformation on the tangent plane is an ellipse with lengths of major and minor axes of 2a and 2b (Fig. 1.7.5), where

\[ a = \left( \begin{pmatrix} \delta \\ A \end{pmatrix} \right)^{1/2}, \quad b = \left( \begin{pmatrix} \delta \\ B \end{pmatrix} \right)^{1/2} \]  \hspace{1cm} (1.7.33)

Equations (1.7.29), (1.7.30)-(1.7.33) define the size and direction of contact ellipse with known values of \( \delta \) and principal curvatures of surfaces.

Sample problem 1.7.1. Surfaces of spiral bevel gears being in point contact are considered. There are given:

\[ \kappa_1^{(1)} = 0.004122047, \quad \kappa_1^{(2)} = -0.000292913, \]

\[ \kappa_1^{(2)} = -0.001513779, \quad \kappa_1^{(2)} = -0.000279921, \]

the angle \( \sigma \) made by principal directions with \( \kappa_1^{(1)} \) and \( \kappa_1^{(2)} \) is equal to 12.47°. The approach of surfaces \( \delta = 0.00787401 \). It is necessary to define the size and direction of contact ellipse.

Equations (1.7.29) and (1.7.31-1.7.33) yield

\[ \alpha^{(1)} = -7.95°, \quad a = 0.539370078, \quad b = 0.035826771 \]

angle \( \alpha^{(1)} \) is made by axis \( 0\eta \) and principal direction with curvature \( \kappa_1^{(1)} \).

By positive value of \( \alpha^{(1)} \) this angle is counted from axis \( 0\eta \) counter-clockwise. (Fig. 1.7.4).
2. GEOMETRY OF SPIRAL BEVEL GEARS

2.1 Introduction

Spiral bevel gears which are used in practice are normally generated with approximately conjugated tooth surfaces by using special machines and tool settings. The geometry of spiral bevel gears is not defined until these special settings are calculated; and the geometry of spiral bevel gears with all machine and tool settings is a very complicated one.

There are some important reasons why simplified mathematical models of the geometry of spiral bevel gears must be developed. These models can be applied as a basis for designers and researchers to solve the Herzian contact stress problem and define dynamic capacity and contact fatigue life, to develop the theory of lubrication of tooth surfaces. Dynamic load capacity and surface fatigue life was considered by J. Coy, D. P. Townsend, and E. Zaretzky for spur and helical gears [11]. The proposed geometric models of spiral bevel gears will enable researchers to extend this work to these gears, too.

The offered models of the geometry of spiral bevel gears are based on an assumption that tooth surfaces are conjugated ones. The aim to use special machine settings is dictated by the attempt to generate conjugated surfaces. Therefore the mentioned assumption is not in contradiction with the practice.

The basic idea of generation of conjugated surfaces of spiral bevel gears is grounded on the following principles:

(1) Two generating surfaces \( \Sigma_F \) and \( \Sigma_k \) are considered being in tangency along a line.

(2) Surfaces \( \Sigma_F \) and \( \Sigma_k \) are rigidly connected with each other in the process of an imaginary generation of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) of the pinion and the member gear. It is supposed that surface \( \Sigma_F \) generates surface \( \Sigma_1 \)
of pinion teeth and surface $\Sigma_k$ generates surface $\Sigma_2$ of member-gear teeth.

(3) There are three axes of instantaneous rotation which correspond:
(a) to the meshing of $\Sigma_F$ and $\Sigma_1$ in the process of generation of $\Sigma_1$;
(b) to the meshing of $\Sigma_k$ and $\Sigma_2$ in the process of generation of $\Sigma_2$;
(c) to the meshing of surfaces $\Sigma_1$ and $\Sigma_2$. All three mentioned axes of rotation must coincide with each other.

(4) The contact of tooth surfaces $\Sigma_1$ and $\Sigma_2$ is localized because generating surfaces $\Sigma_F$ and $\Sigma_k$ does not coincide with each other (they have a common line only).

There are two kinds of bearing contact of spiral bevel gears applied in practice. The first one corresponds to the motion of the contact ellipse across the tooth (Fig. 2.1.1,a), the second one to the motion along the tooth (Fig. 2.1.1,b). Accordingly, two mathematical models of the geometry of spiral bevel gears corresponding to the mentioned cases will be proposed.

2.2. Geometry I: The Line of Action

Generating surfaces $\Sigma_F$ and $\Sigma_k$ are two cone surfaces (Fig. 2.2.1) which are in tangency along the generatrix $AB$.

Let us imagine that generating surfaces being rigidly connected with each other rotate about axis $x_F$ (Fig. 2.2.2) with angular velocity $\omega^{(d)}$ ($d = F, k$) while gears 1 and 2 rotate about axes $Oa$ and $Ob$ with angular velocities $\omega^{(1)}$ and $\omega^{(2)}$. Axis $z_F$ is the instantaneous axis of rotation because angular velocities $\dot{\omega}^{(1)}, \dot{\omega}^{(2)}$ and $\dot{\omega}^{(d)}$ are related by the following equations

$$\dot{\omega}^{(1d)} = \lambda \dot{k}_F,$$  \(2.2.1\)

where

$$\dot{\omega}^{(1d)} = \dot{\omega}^{(1)} - \dot{\omega}^{(d)}; \quad \dot{\omega}^{(12)} = \dot{\omega}^{(1d)},$$ \(2.2.2\)

where

$$\dot{\omega}^{(12)} = \dot{\omega}^{(1)} - \dot{\omega}^{(2)}$$
Fig. 2.1.1

Two Types of Bearing Contacts

63
Fig. 2.2.1

Generating Cone Surfaces
Fig. 2.2.2

\[ \omega^{(2)} \]

\[ \omega^{(1)} \]

\[ \omega^{(12)} \]

\[ \gamma_1 \]

\[ \gamma_2 \]

PITCH CONE
Vectors \( \omega^{(1)}, \omega^{(d)}, \) and \( \omega^{(2)} \) are represented by the following equations

\[
\begin{align*}
\omega^{(1)} &= \omega^{(1)}(-\sin \gamma_{1}, \cos \gamma_{1}) \\
\omega^{(d)} &= \omega^{(d)}(-\sin \gamma_{1}, \cos \gamma_{1}) \\
\omega^{(2)} &= \omega^{(2)}(-\sin \gamma_{2}, \cos \gamma_{2}),
\end{align*}
\]

where \( \gamma_{1} \) and \( \gamma_{2} \) are pitch cone angles.

Equations (2.2.1)-(2.2.5) yield

\[
\begin{align*}
\omega^{(1)} &= \frac{\omega^{(d)}}{\sin \gamma_{1}} \\
\omega^{(2)} &= \omega^{(1)} \frac{\sin \gamma_{1}}{\sin \gamma_{2}} = \omega^{(1)} \frac{\sin \gamma_{1}}{\sin \gamma_{2}}
\end{align*}
\]

The generating surface \( \Sigma_{d}(d = F,k) \) can be represented by equations which are analogous to (1.5.22)

\[
\begin{align*}
x^{(d)} &= r_{d} \cot \psi_{c} - u_{d} \cos \psi_{c} \\
y^{(d)} &= u_{d} \sin \psi_{c} \sin \tau_{d} - b_{d} \sin(q_{d} - \phi_{d}) \\
z^{(d)} &= u_{d} \sin \psi_{c} \cos \tau_{d} + b_{d} \cos(q_{d} - \phi_{d}),
\end{align*}
\]

where \( \tau_{d} = \theta_{d} - q_{d} + \phi_{d} \)

Here: \( (u_{d}, \theta_{d}) \) are generating surface coordinates, \( \phi_{d} \) is the angle of rotation about axis \( x_{f} \); \( \psi_{c} \) is the shape angle of head-cutter blades; \( r_{d}, b_{d} \) and \( q_{d} \) are parameters of tool settings (Fig. 1.5.4).

The surface normal is represented by equations

\[
N_{f} = \frac{\partial f}{\partial \phi} \times \frac{\partial f}{\partial u} =
\]
\[
\begin{pmatrix}
    i_f & j_f & k_f \\
    3x_f & 3y_f & 3z_f \\
    \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \theta} \\
    \frac{\partial}{\partial u} & \frac{\partial}{\partial u} & \frac{\partial}{\partial u}
\end{pmatrix} =
\]

\[u_d \sin \psi_c (\sin \psi_c i_f + \cos \psi_c \sin \tau_d j_f + \cos \psi_c \cos \tau_d k_f), \quad (2.2.9)\]

where \(\tau_d = \theta_d - q_d + \phi_d\).

The surface unit normal is represented by equation

\[\eta_f = \frac{N_f}{|N_f|} =
\sin \psi_c i_f + \cos \psi_c \sin \tau_d j_f + \cos \psi_c \cos \tau_d k_f\]

(by \(u_d \sin \psi_c \neq 0\)) \quad (2.2.10)

To define the line of action of gears 1 and 2 let us imagine that all four surfaces - \(\Sigma_p, \Sigma_k, \Sigma_1\), and \(\Sigma_2\) - are in meshing. Surfaces \(\Sigma_p\) and \(\Sigma_k\) are rigidly connected with each other and are in tangency along the generatrix AB (Fig. 2.2.1). Surfaces \(\Sigma_p\) and \(\Sigma_1\) are in linear contact and lines of instantaneous contact cover these surfaces. The same statement is true for surfaces \(\Sigma_k\) and \(\Sigma_2\). Fig. 2.2.3 shows surface \(\Sigma_d (d = p, k)\) covered with instantaneous contact lines; the location of contact lines on the surface depends on the angle \(\phi_d\) of rotation.

Surfaces \(\Sigma_1\) and \(\Sigma_2\) can be in point-contact only. Contact points of these surfaces move along the common generatrix AB (Fig. 2.2.3, Fig. 2.2.1) while all four surfaces - \(\Sigma_p, \Sigma_k, \Sigma_1\), and \(\Sigma_2\) - are in meshing. The line of action of surfaces \(\Sigma_1\) and \(\Sigma_2\) is the locus of contact points represented in coordinate system \(S_f\) by equations

\[r_f^{(d)} = r_f^{(d)} (u_d, \theta_d, \phi_d) \quad (2.2.11)\]
Fig. 2.2.3

Instantaneous Contact Lines on Generating Surface
Equation (2.2.11) was represented in terms of components \( x_f, y_f, z_f \) by equations (2.2.8). The surface normal \( N_f \) and unit normal \( n_f \) were represented by equations (2.2.9) and (2.2.10).

Vector \( \mathbf{v}_f^{(F1)} \) is represented by equation

\[
\mathbf{v}_f^{(F1)} = \mathbf{w}^{(F1)} \times \mathbf{r}_f^{(F)} =
\begin{vmatrix}
\mathbf{i}_f & \mathbf{j}_f & \mathbf{k}_f \\
\omega_{fx}^{(F1)} & \omega_{fy}^{(F1)} & \omega_{fz}^{(F1)} \\
x_f & y_f & z_f
\end{vmatrix}
\]  

Equations (2.2.1), (2.2.3) and (2.2.4) yield that by \( d = F \)

\[
\omega^{(F1)} = \omega^{(F)} - \omega^{(1)} = - \omega^{(1)} \cos \gamma_1 k_f = - \omega^{(F)} \cot \gamma_1 k_f
\]  

It results from equations (2.2.8), (2.2.9), (2.2.14) and (2.2.15) that

\[
N_f^{(F1)} \cdot \mathbf{v}_f^{(F1)} = \omega^{(F)} \cot \gamma_1 \left( y_f n_{fx} - x_f n_{fy} \right) = \omega^{(F)} \cot \gamma_1 \left( u_f - r_c \cot \psi_c \cos \psi_c \right) \sin \tau_F - b_F \sin \psi_c \sin (q_F - \phi_F) = 0,
\]

Where \( \tau_F = \theta_F - q_F + \phi_F \).

Equation (2.2.16) yields that

\[
(u_f - r_c \cot \psi_c \cos \psi_c) \sin (\theta_F - \phi_F) - b_F \sin \psi_c \sin (q_F - \phi_F) = 0
\]  

Similarly, equation (2.2.13) can be expressed as

\[
(u_k - r_k \cot \psi_c \cos \psi_c) \sin (\theta_k - \phi_k) - b_k \sin \psi_c \sin (q_k - \phi_k) = 0
\]
At contact points of surfaces $\Sigma_1$ and $\Sigma_2$ the following equations must be observed

\[
\begin{align*}
X_f &= r_k \cot \psi_c - u_k \cos \psi_c = r_F \cot \psi_c - u_F \cos \psi_c \quad (2.2.19) \\
y_f &= u_k \sin \psi_c \sin \tau_k - b_k \sin(q_k - \phi_k) = \left\{ \begin{array}{l}
u_p \sin \psi_c \sin \tau_F - b_p \sin(q_F - \phi_F) \\
z_f &= u_k \sin \psi_c \cos \tau_k + b_k \cos(q_k - \phi_k) = \left\{ \begin{array}{l}
u_p \sin \psi_c \cos \tau_F + b_p \cos(q_F - \phi_F) \end{array} \right. \quad (2.2.21)
\end{align*}
\]

Here: $\tau_d = \theta_d - q_d + \phi_d(d=F,k)$

Parameters $u_d, \tau_d(d=F,k)$ are related by equations (2.2.17) and (2.2.18); $\phi_k = \phi_F$ because generating surfaces $\Sigma_k$ and $\Sigma_F$ are rigidly connected and rotate with the same angular velocity.

After elimination of $u_k$ and $u_F$ the system of equations (2.2.17)-(2.2.21) yields a system of two equations

\[
\begin{align*}
r_k \sin(q_k - \phi_k) & = r_F \sin(q_F - \phi_F) \\
b_k \sin \theta_k & = r_k \sin \tau_k - b_k \sin(q_k - \phi_k) \cos \psi_c \cos \tau_k \\
b_F \sin \tau_F & = r_F \sin \tau_F - b_F \sin(q_F - \phi_F) \cos \psi_c \cos \tau_F \quad (2.2.23)
\end{align*}
\]

These equations will be observed for all values of $\phi_d$ if machine settings will satisfy the following equations

\[
\begin{align*}
\phi_k &= \phi_F, \quad \theta_k - q_k = \theta_F - q_F, \quad b_k \sin \theta_k = b_F \sin \theta_F, \\
r_k \frac{b_k \sin q_k}{\cos \beta} &= r_F \frac{b_F \sin q_F}{\cos \beta}, \quad (2.2.24)
\end{align*}
\]

where $\beta = 90^\circ - (\theta_k - q_k) = 90^\circ - (\theta_F - q_F)$
The geometrical interpretation of equations (2.2.24) is represented by Fig. 2.2.4.

The line of action of surfaces $\Sigma_1$ and $\Sigma_2$ is represented by equations

$$x_f = r_d - b_d \frac{\sin(q_d - \phi_d)}{\sin r_d} \sin \psi_c \cos \psi_c,$$

$$y_f = \frac{\sin r_d}{\tan \psi_c} x_f,$$

$$z_f = \frac{b_d \sin \theta_d}{\sin r_d} + \frac{\cos r_d}{\tan \psi_c} x_f,$$

where

$$r_d = \theta_d - q_d + \phi_d, \quad d = F, k, \phi_k = \phi_F.$$

Equations (2.2.25) represent coordinates of the line of action as functions $x_f(\phi_d), y_f(\phi_d), z_f(\phi_d)$.

2.3. Geometry I: Contact Point Path on Surface $\Sigma_i$ (i=1,2)

Contact point path on surface $\Sigma_i$ (i=1,2) is a locus of points of contact represented in coordinate system $\Sigma_i$ rigidly connected with gear i.

Fig. 2.3.1 shows coordinate systems $S_f$ and $S_h$ rigidly connected with the frame and system $S_1$ rigidly connected with gear 1. The coordinate transformation by transition from $S_f$ to $S_1$ is represented by matrix equation (Fig. 2.3.1)

$$[r_1] = [L_{1h}] [L_{hf}] [r_f] =$$

$$\begin{bmatrix}
\cos \phi_1 & \sin \phi_1 & 0 \\
-sin \phi_1 & \cos \phi_1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \gamma_1 & 0 & \sin \gamma_1 \\
0 & 1 & 0 \\
-sin \gamma_1 & 0 & \cos \gamma_1
\end{bmatrix}
\begin{bmatrix}
x_f(\phi_d) \\
y_f(\phi_d) \\
z_f(\phi_d)
\end{bmatrix}$$
Fig. 2.2.4
Parameters of Machine Settings

\[ O_c^{(K)} M = r_K, \quad O_c^{(F)} M = r_F \]

\[ O_f^{(K)} O_c = b_K, \quad O_f^{(F)} O_c = b_F \]

\[ O_f B = b_K \sin \theta_K = b_F \sin \theta_F \]
Fig. 2.3.1

Coordinate Systems Associated with Gear 1
Here: $x_f(\phi_d)$, $y_f(\phi_d)$ and $z_f(\phi_d)$ are functions represented by equations (2.2.25). The angle of rotation $\phi_1$ of gear 1 and the angle of rotation of generating gear are related by the equation which is analogous to equation (2.2.6)

$$\psi_1 = \frac{\phi_d}{\sin \gamma_1} \quad (2.3.2)$$

Fig. 2.3.2 shows coordinate systems $S_f$ and $S_p$ rigidly connected with the frame and coordinate system $S_2$ rigidly connected with gear 2. The coordinate transformation is represented by matrix equality

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} L_{2p} \\ L_{pf} \end{bmatrix} \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix}$$

$$= \begin{bmatrix} \cos \phi_2 & -\sin \phi_2 & 0 \\ \sin \phi_2 & \cos \phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \gamma_2 & 0 & -\sin \gamma_2 \\ 0 & 1 & 0 \\ \sin \gamma_2 & 0 & \cos \gamma_2 \end{bmatrix} \begin{bmatrix} x_f(\phi_d) \\ y_f(\phi_d) \\ z_f(\phi_d) \end{bmatrix}$$

$$= \begin{bmatrix} x_f(\phi_d) \cos \phi_2 \cos \gamma_2 - y_f(\phi_d) \sin \phi_2 + z_f(\phi_d) \cos \phi_2 \sin \gamma_2 \\ x_f(\phi_d) \sin \phi_2 \cos \gamma_2 + y_f(\phi_d) \cos \phi_2 - z_f(\phi_d) \sin \phi_2 \sin \gamma_2 \\ x_f(\phi_d) \sin \gamma_2 - z_f(\phi_d) \cos \gamma_2 \end{bmatrix}, \quad (2.3.3)$$

where

$$\phi_2 = \frac{\phi_d}{\sin \gamma_2} \quad (2.3.4)$$

Matrix equality (2.3.3) and equations (2.2.25) and (2.3.4) represent the contact point path on the surface $\Sigma_2$ of gear 2.
Fig. 2.3.2

Coordinate Systems Associated with Gear 2
2.4 Geometry I: The Instantaneous Contact Ellipse

The size and direction of the instantaneous contact ellipse may be obtained by the equations given in Items 1.6 and 1.7.

The solution of this problem can be divided into three stages: (1) the determination of principal curvatures of surfaces $\Sigma_1$ and $\Sigma_2$, (2) the determination of the principal directions of surfaces $\Sigma_1$ and of $\Sigma_2$, and (3) the determination of contact ellipse.

** Principle Curvatures and Directions of Surface $\Sigma_1$ **

Surface $\Sigma_1$ is generated by cone surface $\Sigma_F$. Principal directions and curvatures of $\Sigma_F$ are represented by the following equations (see sample problem 1.6.1):

\[
\begin{align*}
\mathbf{i}^{(F)}_I &= \frac{\partial \mathbf{r}_F}{\partial \theta} = \begin{bmatrix} 0 \\ \cos \tau_F \\ -\sin \tau_F \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(\beta - \phi_F) \\ -\cos(\beta - \phi_F) \end{bmatrix}, \\
\kappa_I^{(F)} &= -\frac{1}{u_F \tan \psi_c}, \\
\end{align*}
\]

\[
\begin{align*}
\frac{-\cos(\beta - \phi_F)}{b_F \sin \psi_c \tan \psi_c \sin(\phi_F - \phi) + r_F \cos \psi_c \cos(\beta - \phi_F)} \\
\mathbf{i}^{(F)}_{II} &= \frac{\partial \mathbf{r}}{\partial \psi_c} = \begin{bmatrix} -\cos \psi_c \\ \sin \psi_c \sin \tau_F \\ \sin \psi_c \cos \tau_F \end{bmatrix} = \begin{bmatrix} -\cos \psi_c \\ \sin \psi_c \cos(\beta - \phi_F) \\ \sin \psi_c \sin(\beta - \phi_F) \end{bmatrix}, \\
\kappa_{II}^{(F)} &= 0 \\
\end{align*}
\]

The principal curvatures and directions of $\Sigma_1$ are represented by equations analogical to equations (1.6.40)-(1.6.42)

\[
\tan \gamma^{(1)} = \frac{2F^{(1)}}{\kappa_I^{(F)} + G^{(1)}}
\]
\[ \kappa_1^{(1)} + \kappa_{II}^{(1)} = \kappa^{(F)}_I + S^{(1)} \quad (2.4.6) \]

\[ \kappa_1^{(1)} - \kappa_{II}^{(1)} = \frac{\kappa^{(F)}_I + G^{(1)}}{\cos 2\sigma^{(1)}} \quad (2.4.7) \]

Here: \( \kappa_1^{(1)} \) and \( \kappa_{II}^{(1)} \) are principal curvatures of surface \( \Sigma_1; \sigma^{(1)} \) is the angle made by the directions of principal curvatures \( \kappa^{(F)}_I \) and \( \kappa^{(1)}_I \). Coefficients \( p^{(1)}, s^{(1)} \) and \( G^{(1)} \) are functions of \( \phi_F \) and represented by equations

\[ F^{(1)} = \frac{a_{31}a_{32}}{b^3 + \nu^{(F1)}_I a_{31} + \nu^{(F1)}_{II} a_{32}} \quad (2.4.8) \]

\[ G^{(1)} = \frac{a_{31}^2 - a_{32}^2}{b^3 + \nu^{(F1)}_I a_{31} + \nu^{(F1)}_{II} a_{32}} \quad (2.4.9) \]

\[ S^{(1)} = \frac{a_{31}^2 + a_{32}^2}{b^3 + \nu^{(F1)}_I a_{31} + \nu^{(F1)}_{II} a_{32}} \quad (2.4.10) \]

\[ a_{31}^{(1)} = \left[ n^{(F)} \right]_1 \left[ (F) \right]_1 - \kappa^{(F)}_I \nu^{(F1)}_I \quad (2.4.11) \]

\[ a_{32}^{(1)} = \left[ n^{(F)} \right]_2 \left[ (F) \right]_2 - \kappa^{(F)}_I \nu^{(F1)}_{II} \quad (2.4.12) \]

\[ b_{3}^{(1)} = \left[ \right. \left[ n^{(F)} \right]_2 \left[ (F) \right]_2 - \left[ n^{(F)} \right]_2 \left[ (F) \right]_1 \left. \right] \quad (2.4.13) \]

\[ n^{(F)} = \begin{bmatrix} \sin \psi_c \\ \cos \psi_c \cos (\beta - \phi_F) \\ \cos \psi_c \sin (\beta - \phi_F) \end{bmatrix} \quad (2.4.14) \]

\[ \omega^{(F1)} = \begin{bmatrix} 0 \\ 0 \\ -\omega^{(F)} \cot \gamma_1 \end{bmatrix} \quad (2.4.15) \]
To simplify equations for $v^{(F1)}$ and $a_{31}$ let us note that

$$b_F = r_F \frac{\cos \beta}{\sin q_F}; \quad r_F \cos(\beta - \phi_F) - b_F \sin(q_F - \phi_F) =$$

$$r_F \frac{\sin \phi_F \cos(\beta - \phi_F)}{\sin q_F}; \quad b_F \sin \psi_c \tan \psi_c \sin(q_F - \phi_F)$$

$$+ r_F \cos \psi_c \cos(\beta - \phi_F) = r_F \frac{\cos \beta \sin(q_F - \phi_F) + \cos^2 \psi_c \sin \psi_c \cos(\beta - q_F)}{\sin q_F \cos \psi_c}$$

After that $v^{(F1)}$ can be represented by the following equation

$$[v^{(F1)}] = r_F \frac{\omega^{(F)} \cot \gamma_1 \cos \psi_c \sin \phi_F}{\sin q_F} \begin{bmatrix} \cos(\beta - \phi_F) \cos \psi_c \\ -\sin \psi_c \\ 0 \end{bmatrix} \quad (2.4.16)$$

Vectors $I^{(F)}_1$ and $I^{(F)}_2$ were represented by equations (2.4.1) and (2.4.3).

Equations (2.4.11)-(2.4.16), (2.4.1) and (2.4.3) yield

$$a_{31}^{(1)} = \omega^{(F)} \cot \gamma_1 \sin \psi_c \sin(\beta - \phi_F).$$

$$a_{32}^{(1)} = \omega^{(F)} \cot \gamma_1 \cos(\beta - \phi_F) \quad (2.4.17)$$

$$a_{32}^{(1)} = \omega^{(F)} \cot \gamma_1 \cos(\beta - \phi_F) \quad (2.4.18)$$

$$b_3^{(1)} = -L \left(\omega^{(F)}\right)^2 \cot \gamma_1 \cos \beta \sin \psi_c \quad (2.4.19)$$

$$v_1^{(F1)} a_{31}^{(1)} = -r_F (\omega^{(F)} \cot \gamma_1 \sin \psi_c \cos \psi_c) \cos \beta.$$

$$\frac{\sin \phi_F \sin(\beta - \phi_F) \sin(q_F - \phi_F)}{\sin q_F \left[\cos \beta \sin(q_F - \phi_F) + \cos^2 \psi_c \sin \phi_F \cos(\beta - \phi_F)\right]} \quad (2.4.20)$$
Equations (2.4.2), (2.4.5)-(2.4.10) and (2.4.17)-(2.4.21) represent the principal directions and principal curvatures of surface \( \Sigma_{1} \). At the mean contact point the principal directions and curvatures are represented by the following equations:

\[
\tan 2\sigma^{(1)} = \frac{\sin \psi_{c} \sin 2\beta}{\frac{L}{2F} \tan \gamma_{1} \sin 2\psi_{c} + \sin^{2} \beta \sin^{2} \psi_{c} - \cos^{2} \beta} \tag{2.4.22}
\]

\[
\kappa_{I}^{(1)} + \kappa_{II}^{(1)} = -\frac{\cos \psi_{c}}{r_{F}} - \frac{\cot \gamma_{1} \left( \sin^{2} \beta \sin^{2} \psi_{c} + \cos^{2} \beta \right)}{L \sin \psi_{c}} \tag{2.4.23}
\]

\[
\kappa_{I}^{(1)} - \kappa_{II}^{(1)} = -\frac{\cos \psi_{c}}{r_{F}} + \frac{\cos \gamma_{1} \left( \sin^{2} \beta \sin^{2} \psi_{c} - \cos^{2} \beta \right)}{L \sin \psi_{c} \cos 2\sigma^{(1)}} \tag{2.4.24}
\]

Now, let us define principal curvatures and directions of surface \( \Sigma_{2} \) generated by surface \( \Sigma_{\kappa} \). They are represented by equations analogical to equations (2.4.5)-(2.4.7)

\[
\tan 2\sigma^{(2)} = \frac{2F^{(2)}}{\kappa_{I}^{(2)} + G^{(2)}} \tag{2.4.25}
\]

\[
\kappa_{I}^{(2)} + \kappa_{II}^{(2)} = \kappa_{I}^{(2)} + S^{(2)} \tag{2.4.26}
\]

\[
\kappa_{I}^{(2)} - \kappa_{II}^{(2)} = \frac{\kappa_{I}^{(2)} + G^{(2)}}{\cos 2\sigma^{(2)}} \tag{2.4.27}
\]

79
To define functions $F_1^{(2)}(\phi_k), G_1^{(2)}(\phi_k)$ and $S_1^{(2)}(\phi_k)$ ($\phi_k - \phi_F$) it is sufficient to change subscripts "F" for "k" and "1" for "2" in expressions (2.4.17)-(2.4.21).

The principal curvature $\kappa_1^{(k)}$ of surface $\Sigma_k$ is represented by equation analogous to (2.4.2)

$$\kappa_1^{(k)} = \frac{\cos(\beta - \phi_k)}{-b_k \sin \psi \tan \psi_c \sin(\phi_k - \phi_k) + r_k \cos \psi_c \cos(\beta - \phi_k)}$$  \hspace{1cm} (2.4.28)

Equations (2.4.28), (2.4.2) and (2.2.22) yield that

$$\frac{1}{\kappa_1^{(k)}} = \frac{1}{\kappa_1^{(F)}} - \frac{r_k - r_F}{\cos \psi_c}$$  \hspace{1cm} (2.4.29)

Equations (2.4.25)-(2.4.27) and (2.4.28) represent principal curvatures and directions of surface $\Sigma_2$.

On the third stage of solution the size and direction of instantaneous contact ellipse is to be obtained. Equations (1.7.30)-(1.7.34) are to be applied for this aim.

2.5. GEOMETRY II: GENERATING SURFACES

Fig. 2.5.1 shows two generating surfaces $\Sigma_k$ and $\Sigma_p$ rigidly connected with each other. These surfaces are in tangency along their common circle $C$ of radius $r_d$ (Fig. 2.5.1). Surface $\Sigma_k$ is a cone surface represented in the coordinate system by equations (2.2.19)-(2.2.21). Surface $\Sigma_p$ is a surface of revolution. It is generated by the revolution of an arc $m-m$ of a circle of radius $\rho$ about axis $x_a$ (Fig. 2.5.2,a). The arc $m-m$ is represented in the auxiliary coordinate system $S_a(x_a, y_a, z_a)$ by equations.
Fig. 2.5.1

Generating Surfaces: Conical Surface and Surface of Revolution

81
Fig. 2.5.2
$$x_a = \rho (\sin \psi_F - \sin \psi_c)$$

$$y_a = 0$$

$$z_a = \rho (\cos \psi_F - \cos \psi_c) + r_F$$

Surface $$\Sigma_F$$ is represented in coordinate system $$S_c(x_c, y_c, z_c)$$ by matrix equality

$$\begin{bmatrix} \mathbf{r}_c \\ \mathbf{r}_a \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{ca} \end{bmatrix} \begin{bmatrix} \mathbf{r}_a \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{r}_c \\ \mathbf{r}_a \end{bmatrix} = \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix}$$

Here: $$\psi_F$$ and $$\theta_F$$ are surface $$\Sigma_F$$ coordinates. The coordinate transformation by transition from $$S_c(x_c, y_c, z_c)$$ to $$S_f(x_f, y_f, z_f)$$ (Fig. 1.5.4) is represented by matrix equality

$$\begin{bmatrix} \mathbf{r}_f \\ \mathbf{r}_c \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{fc} \end{bmatrix} \begin{bmatrix} \mathbf{r}_c \end{bmatrix}$$

Expressions analogous to (1.5.9) and (1.5.15) yield

$$\begin{bmatrix} \mathbf{M}_{fc} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q_F-\phi_F) & -\sin(q_F-\phi_F) & -b \sin(q_F-\phi_F) \\ 0 & \sin(q_F-\phi_F) & \cos(q_F-\phi_F) & b \cos(q_F-\phi_F) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It results from expressions (2.5.3)-(2.5.5) that the generating surface $$\Sigma_F$$ is represented in coordinate system $$S_f$$ by equations

$$x_f^{(F)} = \rho (\sin \psi_F - \sin \psi_c)$$

$$y_f^{(F)} = \left[ \rho (\cos \psi_F - \cos \psi_c) + r_F \right] \sin \tau_F + b \sin(q_F-\phi_F)$$

$$z_f^{(F)} = \left[ \rho (\cos \psi_F - \cos \psi_c) + r_F \right] \cos \tau_F + b \cos(q_F-\phi_F)$$
where
\[ \tau_F = \theta_F - (q_F - \phi_F) \]

The surface normal is represented by equation
\[ N^{(F)} = \frac{\partial x_F}{\partial \psi_F} \times \frac{\partial x_F}{\partial \psi_F} = \]
\[ \begin{vmatrix}
  i^F & j^F & k^F \\
  \rho \cos \psi_F & -\rho \sin \psi_F \sin \tau_F & -\rho \sin \psi_F \cos \tau_F \\
  0 & A \cos \tau_F & -A \sin \tau_F \\
\end{vmatrix} = \]
\[ A \rho \sin \psi_F i^F + A \rho \cos \psi_F \sin \tau_F j^F + A \rho \cos \psi_F \cos \tau_F k^F \] (2.5.7)

Here:
\[ A = \rho (\cos \psi_F - \cos \psi_C) + r_F \]

The surface unit normal \( n^{(F)} \) is represented by equation
\[ n^{(F)} = \frac{N^{(F)}}{|N^{(F)}|} = \sin \psi_F i^F + \cos \psi \left( \sin \tau_F j^F + \cos \tau_F k^F \right) \] (2.5.8)

The generating surface \( \Sigma_k \) and its unit normal are represented by equations (2.2.8) and (2.2.10) with subscript \( d=k \).

By \( \psi_F = \psi_C - r_F - r_k \), \( u_k = \frac{r_k}{\sin \psi_C} \) surfaces \( \Sigma_F \) and \( \Sigma_k \) are in tangency along the circle of radius \( r_k = r_F \).

2.6 Geometry II: The Line of Action

The law of meshing of surfaces \( \Sigma_k \) and \( \Sigma_2 \) was represented by the equation [see(2.2.18)]
\[ (u_k - r_k \cot \psi_C \cos \psi_C \sin(\theta_k - q_k + \phi_k)) \]
\[ - b_k \sin \psi_C \sin(q_k - \phi_k) = 0 \] (2.6.1)

At contact points of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) parameter
\[ u_k \sin \psi_C = r_k \] (2.6.2)
Equations (2.6.1) and (2.6.2) yield
\[ r_k \sin [\theta_k - (q_k - \phi_k)] - b_k \sin (q_k - \phi_k) = f(\theta_k, \phi_k) = 0 \] (2.6.3)

This equation relates the surface parameter \( \theta_k \) with the angle of rotation \( \phi_k \). By \( \frac{\partial f}{\partial \theta_k} \neq 0 \) this equation represents in implicit form a function \( \theta_k(\phi_k) \).

Equations (2.2.8), (2.6.2) and (2.6.3) yield that the line of action can be represented that way
\[ x_f = 0, \ y_f = 0, \ z_f = r_k \cos \left[ \theta_k - (q_k - \phi_k) \right] + b_k \cos (q_k - \phi_k) = z_f(\phi_k) \] (2.6.4)

where angles \( \left[ \theta - (q_k - \phi_k) \right] \) and \( (q_k - \phi_k) \) are related by (2.6.3).

Contact point paths on surface \( \Sigma_1 \) and \( \Sigma_2 \) can be defined the same way mentioned in item 2.3.

2.7. Geometry II: The Instantaneous Contact Ellipse.

The principal curvatures and directions of surface \( \Sigma_2 \) generated by surface \( \Sigma_k \) were defined in item 2.4 by equations (2.4.20)-(2.4.21). For surface \( \Sigma_2 \) with geometry II coefficients \( a_{31}^{(2)}, a_{32}^{(2)}, b_3^{(2)}, f^{(2)}, g^{(2)} \) and \( S^{(2)} \) are represented by following equations

\[ a_{31}^{(1)} = -\omega(k) \cot \gamma_2 \sin \psi_k \cos \tau_k \] (2.7.1)

\[ a_{32}^{(2)} = -\omega(k) \cot \gamma_2 \sin \tau_k \] (2.7.2)

\[ b_3^{(2)} = r_k(\omega(k))^2 \cot \gamma_2 \sin \psi_k \left[ \frac{\cos \tau_k \sin q_k + \cos \beta \cos (q_k - \phi_k)}{\sin q_k} \right] \] (2.7.3)

\[ f^{(2)} = \frac{a_{31} a_{32}}{b_3} = \frac{\sin q_k \cot \gamma_2 \cos \tau_k \sin \tau_k}{r_k \left[ \cos \tau_k \sin q_k + \cos \beta \cos (q_k - \phi_k) \right]} \] (2.7.4)

\[ g^{(2)} = \frac{a_{31} - a_{32}}{b_3} = \frac{(\sin^2 \psi \cos^2 \tau_k - \sin^2 \tau_k) \sin q_k \cot \gamma_2}{r_k \left[ \cos \tau_k \sin q_k + \cos \beta \cos (q_k - \phi_k) \right]} \] (2.7.5)
\[ S(2) = \frac{a_{31}^2 + a_{32}^2}{b_3} = \frac{(\sin^2 \psi_c \cos^2 \tau_k + \sin^2 \tau_k) \sin q_k \cot \gamma_2}{r_k \left[ \cos \tau_k \sin q_k + \cos \beta \cos (q_k - \phi_k) \right]} \]  

(2.7.6)

Parameters \( \theta_k \) and \( \phi_k \) are related by equation (2.6.3).

Now, let us define principal curvatures and directions of surface \( \Sigma_1 \) generated by \( \Sigma_F \). To solve this problem we must first define principal directions and curvatures of surface \( \Sigma_F \).

It is easy to verify that principal directions of surface \( \Sigma_F \) correspond to \( \frac{d\psi_F}{dt} = 0 \) and to \( \frac{d\theta_F}{dt} = 0 \) and that principal curvatures are represented by equations

\[ \kappa_1^{(F)} = -\frac{\cos \psi_F}{\rho (\cos \psi_F - \cos \psi_c) + r_F} \]  

(2.7.7)

\[ \kappa_2^{(F)} = -\frac{1}{\rho} \]  

(2.7.8)

At the point of contact of surfaces \( \Sigma_F \) and \( \Sigma_1 \) the principal curvature is

\[ \kappa_1^{(F)} = -\frac{\cos \psi_c}{r_F} \]  

(2.7.9)

because at this point \( \psi_F = \psi_c \).

Principal curvatures and directions of surface \( \Sigma_1 \) are represented by equations

\[ \tan 2\sigma^{(1)} = \frac{2\sigma^{(1)}}{\kappa_1^{(1)} - \kappa_2^{(1)}} \]  

(2.7.10)

\[ \kappa_1^{(1)} + \kappa_2^{(1)} = \kappa_1^{(F)} + \kappa_2^{(F)} + g^{(1)} \]  

(2.7.11)

\[ \kappa_1^{(1)} - \kappa_2^{(1)} = \frac{\kappa_1^{(F)} - \kappa_2^{(F)} + g^{(1)}}{\cos 2\sigma^{(1)}} \]  

(2.7.12)
Here:
\[ f^{(1)} = f^{(2)}, \quad s^{(1)} = s^{(2)}, \quad g^{(1)} = g^{(2)} \]

The size and direction of instantaneous contact ellipse are defined the same way which was mentioned in item 1.7.
3. METHODS TO CALCULATE GEAR DRIVE KINEMATICAL ERRORS.

3.1. Introduction

It is well known that errors of manufacturing and assemblage of gears induce kinematical errors in gear-drives. These errors can be represented by a function

\[ \Delta \phi_2 (\phi_1, \Delta \theta) \],

(3.1.1)

where \( \phi_1 \) is the angle of rotation of the driving gear 1,

\[ \Delta \theta = (\Delta q_1, \Delta q_2, \ldots) \]

(3.1.2)

is the vector of errors;

\[ \Delta \phi_2 = \phi_2^a - \phi_2 \]

(3.1.3)

is the kinematical error of the gear drive represented as the difference of theoretical and actual angles of rotation of the driven gear.

In this part of the report two methods to calculate the function (3.1.1) are presented: the first one is a numerical computer method and the second one is worked out as an approximate method but with a possibility to obtain relatively simple results which are in most cases in an analytical form.

3.2. The Computer Method.

In the process of motion tooth surfaces \( \Sigma_1 \) and \( \Sigma_2 \) must be in continuous tangency. It was demonstrated (see item 1.1) that following equations are to be observed (see equations (1.1.12) and (1.1.13)).

\[ \mathbf{r}_f^{(1)}(u_1, \theta_1, \phi_1) = \mathbf{r}_f^{(2)}(u_2, \theta_2, \phi_2) \]

(3.2.1)

\[ \mathbf{n}_f^{(1)}(u_1, \theta_1, \phi_1) = \mathbf{n}_f^{(2)}(u_2, \theta_2, \phi_2) \]

(3.2.2)

Here: \( \mathbf{r}_f^{(i)} \) and \( \mathbf{n}_f^{(i)} \) are the position vectors and normals of surfaces \( \Sigma_i \) as defined in coordinate system \( S_f \) rigidly connected with the frame; \( u_i, \theta_i \) are the surface coordinates, \( \phi_i \) are the angles of gear rotation.
Here it is assumed that errors of manufacturing and assemblage did not appear.

For gears with errors represented by vectors $\Delta Q_1$ and $\Delta Q_2$ following equations of tangency must be observed instead of equations (3.2.1) and (3.2.2)

$$r_f^{(1)}(u_1, \theta_1, \phi_1, \Delta Q_1) = r_f^{(2)}(u_2, \theta_2, \phi_2, \Delta Q_2)$$ (3.2.3)

$$n_f^{(1)}(u_1, \theta_1, \phi_1, \Delta Q_1) = n_f^{(2)}(u_2, \theta_2, \phi_2, \Delta Q_2)$$ (3.2.4)

Equations (3.2.3) and (3.2.4) yield the function

$$\phi_2(\phi_1, \Delta Q_1, \Delta Q_2) = \phi_2^0(\phi_1) + \Delta \phi_2(\phi_1, \Delta Q_1, \Delta Q_2)$$ (3.2.5)

Here: $\phi_2^0(\phi_1)$ is the theoretical function yielded by equations (3.2.1) and (3.2.2).

Equations (3.2.3) and (3.2.4) also yield the functions

$$u_i(\phi_1, \Delta Q_1, \Delta Q_2), \theta_i(\phi_1, \Delta Q_1, \Delta Q_2) (i=1,2)$$ (3.2.6)

Functions

$$r_i(u_1, \theta_1), u_i(\phi_1, \Delta Q_1, \Delta Q_2), \theta_i(\phi_1, \Delta Q_1, \Delta Q_2) (i=1,2)$$ (3.2.7)

represent the path of contact points on surface $\Sigma_i$ corresponding to gear meshing with errors of manufacturing and assemblage.

Functions

$$r_i(u_1, \theta_1), u_i^0(\phi_1), \theta_i^0(\phi_1)$$ (3.2.8)

represent the path of contact point on surface $\Sigma_i$ correspondent to the meshing without errors. Comparison of functions (3.2.8) and (3.2.7) yields the change of contact point path induced by errors.

Let us consider the detailed solution of equations (3.2.1)-(3.2.2) and (3.2.3)-(3.2.4).
Vector-equations (3.2.1) and (3.2.2) yield only five independent scalar
equations because $|n_f^{(1)}| = |n_f^{(2)}|:

$$f_j(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \quad (j=1,2,\ldots,5)$$ (3.2.9)

It is assumed that

$$\{f_1, f_2, f_3, f_4, f_5\} \in C^1$$

and that the system of equations (3.2.9) is satisfied by a set of parameters

$$P = (u_1^*, \theta_1^*, \phi_1^*, u_2^*, \theta_2^*, \phi_2^*)$$ (3.2.10)

and surfaces $\Sigma_1$ and $\Sigma_2$ are in tangency at a point $M_0$. Surfaces $\Sigma_1$ and $\Sigma_2$ will be in point contact in the neighborhood of $M_0$ if by the set of parameters $P$ the following inequality is held

$$D(f_1, f_2, f_3, f_4, f_5) = \begin{vmatrix}
\frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \phi_2} \\
\frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \phi_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \phi_2} \\
\frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \phi_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \phi_2} \\
\frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \phi_1} & \frac{\partial f_4}{\partial u_2} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \phi_2} \\
\frac{\partial f_5}{\partial u_1} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \phi_1} & \frac{\partial f_5}{\partial u_2} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \phi_2}
\end{vmatrix} \neq 0$$ (3.2.11)

Then in the neighborhood of $P$ equations (3.2.9) provide functions

$$\{u_1(\phi_1), \theta_1(\phi_1), u_2(\phi_1), \theta_2(\phi_1), \phi_2(\phi_1)\} \in C^1$$ (3.2.12)

Function $\phi_2(\phi_1)$ represents the ideal law of motion transformation.

Mostly, $\phi_2(\phi_1)$ is a linear function.

Equations (3.2.3)-(3.2.4) also yield a system of five independent
equations

$$\psi_j(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2, \Delta Q) = 0$$ (3.2.13)

By the same value of $\phi_1$ this system is satisfied by a set of parameters

$$P' = (u_1^{**}, \theta_1^{**}, \phi_1^{**}, u_2^{**}, \theta_2^{**}, \phi_2^{**})$$ (3.2.14)

which is different from the set $P$ represented by (3.2.10).
System of equations (3.2.13) can yield functions
\[ u_1(\phi_1, \Delta Q), \theta_1(\phi_1, \Delta Q), u_2(\phi_1, \Delta Q), \theta_2(\phi_1, \Delta Q), \phi_2(\phi_1, \Delta Q) \in C^1 \] (3.2.15)
in the neighborhood of \( P' \) if at \( P' \) the following inequality is held
\[ \frac{\partial D(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)}{\partial (u_1, \theta_1, u_2, \theta_2, \psi_2)} \neq 0 \] (3.2.16)

Function \( \phi_2(\phi_1, \Delta Q) \) represents the real law of motion transformation.

Kinematical errors of the gear-drive are represented by function
\[ \Delta \phi_2 = \phi_2(\phi_1) - \phi_2(\phi_1, \Delta Q) \] (3.2.17)

The demonstrated method can provide not only the kinematical errors induced by errors \( \Delta Q \) but new contact point path on the surface \( \Sigma_1 \), too. (see functions (3.2.7)).

The solution of a system of five non-linear equations is a hard problem and needs iterations. To save computer time a more effective way of solution was recently proposed by F. Litvin and YE. Gutman [12]. The principle of this method follows:

The system of equations (3.2.13) can be represented as follows
\[ f_1(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2, A, H_1, H_2) = 0 \] (3.2.18)
\[ f_2(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2, A, H_1, H_2) = 0 \] (3.2.19)
\[ f_3(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2, A, H_1, H_2) = 0 \] (3.2.20)
\[ f_4(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \] (3.2.21)
\[ f_5(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \] (3.2.22)

Equations (3.2.18)-(3.2.20) are yielded by vector equation (3.2.3) and equations (3.2.21)-(3.2.22) by vector equation (3.2.4). Parameters \( A, H_1 \) and \( H_2 \) are linear measurements which represent the shortest distance between gear axes of rotation and axial settings of gears (Fig. 3.2.1).
FIG. 3.21
Axial Settings of Gears: $H_1$, $H_2$ and $A$
Let us suppose that points \( M_1(u_1, \theta_1) \) and \( M_2(u_2, \theta_2) \) of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) are chosen. By a set of given parameters \((u_1, \theta_1, u_2, \theta_2)\) system of equations (3.2.21) and (3.2.22) becomes a system of two equations in two unknowns which may be expressed as

\[
F_1(\phi_1, \phi_2) = 0 \tag{3.2.23}
\]

\[
F_2(\phi_1, \phi_2) = 0 \tag{3.2.24}
\]

After that, a system of three equations must be solved

\[
A-K_1(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \tag{3.2.25}
\]

\[
H_1-K_2(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \tag{3.2.26}
\]

\[
H_2-K_3(u_1, \theta_1, \phi_1, u_2, \theta_2, \phi_2) = 0 \tag{3.2.27}
\]

The method of solution of the two systems of equations (3.2.23)-(3.2.24) and (3.2.25)-(3.2.26) is an iterative procedure. By computation one of four varied parameters \((u_1, \theta_1, u_2, \theta_2)\) is fixed and the three others must be changed that way that two mentioned above systems of equations are to be satisfied.

The advantage of the proposed method is the opportunity to divide the system (3.2.18)-(3.2.22) of five equations into two subsystems -- of two and one of three equations -- and solve them separately.

3.3. Approximate Method

Accuracy of gear drives investigated by the above computer method can be defined as a rule only numerically and this is a certain disadvantage of this method. Therefore, in addition to the computer method an approximate method with the opportunity to obtain results analytically is proposed.
Figure 3.3.1 shows two surfaces $\Sigma_1$ and $\Sigma_2$ which are in tangency at point $M$. Points $M_1$ and $M_2$ of these surfaces coincide with each other at $M$, position vectors $\mathbf{x}^{(1)}_f$ and $\mathbf{x}^{(2)}_f$ drawn from $O_f$ and surface unit normals $\mathbf{n}^{(1)}_f$ and $\mathbf{n}^{(2)}_f$ coincide at $M$, too. Surfaces $\Sigma_1$ and $\Sigma_2$ rotate about axes I-I and II-II and angles of rotation $\phi_1$ and $\phi_2^*$ correspond to the positions of surfaces shown in Fig. 3.3.1. It is supposed initially that $\Sigma_1$ and $\Sigma_2$ are manufactured and assembled without errors. Due to errors surfaces $\Sigma_1$ and $\Sigma_2$ cannot be in tangency by the same values of $\phi_1$ and $\phi_2^*$ - either a clearance will appear between these surfaces or the surfaces will interfere with each other. Figure 3.3.2 shows that surfaces $\Sigma_1$ and $\Sigma_2$ are not in tangency: points $M_1$ and $M_2$ do not coincide with each other, $\mathbf{x}^{(1)}_f \neq \mathbf{x}^{(2)}_f$ and $\mathbf{n}^{(1)}_f \neq \mathbf{n}^{(2)}_f$. To get surfaces $\Sigma_1$ and $\Sigma_2$ in tangency it is sufficient to rotate one of the surfaces by an additional small angle. It is more preferable to hold the position of surface $\Sigma_1$ and to rotate surface $\Sigma_2$ until it contacts $\Sigma_1$. Then the additional angle of rotation $\Delta\phi_2$ will represent the change of theoretical value $\phi_2^*$ induced by errors of manufacturing and assemblage. It can be predicted that $\Delta\phi_2$ is a function of the vector $\Delta\mathbf{q}$ and changes in the process of motion. So

$$\Delta\phi_2 = f(\phi_1, \Delta\mathbf{q}).$$

(3.3.1)

The definition of function (3.3.1) can be based on the equations of kinematical relations discussed in Item 1.1.

Because tooth surfaces $\Sigma_1$ and $\Sigma_2$ are to be in continuous contact the following vector equations must be observed

$$dr^{(1)}_f = dr^{(2)}_f$$

(3.3.2)

$$dn^{(1)}_f = dn^{(2)}_f$$

(3.3.3)
FIG. 3.3.1
Contacting Tooth Surfaces
FIG. 3.3.2

Tooth Surfaces with Clearance Induced by Errors
It results from equations (3.3.2) and (3.3.3) that

\[
\begin{align*}
\frac{ds}{dt} + \frac{ds}{\tau} &= ds + ds \\
\frac{dn}{t} + \frac{dn}{\tau} &= dn + dn
\end{align*}
\]  

(3.3.4) \hspace{1cm} (3.3.5)

Here: \(ds\) is the displacement of the contact point of surface \(\Sigma_i\) \((i=1,2)\) in transfer motion (with the surface); \(ds\) is the contact point displacement in relative motion (relative to the surface); notations of \(dn\) and \(dn\) have the same meanings for the tip of the unit normal vectors; subscript "f" is dropped for simplification.

Equations (3.3.4) and (3.3.5) are similar to equations (1.1.35) and (1.1.36).

Errors of manufacturing and assemblage induce that the theoretical contact point changes its position. To hold surfaces in tangency following equations must be observed

\[
\begin{align*}
\frac{ds}{t} + \frac{ds}{\tau} + \frac{ds}{q} &= ds + ds + ds \\
\frac{dn}{t} + \frac{dn}{\tau} + \frac{dn}{q} &= dn + dn + dn
\end{align*}
\]  

(3.3.6) \hspace{1cm} (3.3.7)

Here: the subscript "q" corresponds to the displacement induced by errors.

It is necessary to emphasize that not only angular errors but linear errors also induce \(dn\).

It was mentioned above that interference of surfaces or their clearance can be compensated by rotation of surface \(\Sigma_2\) only. Therefore, \(\frac{ds}{t}=0\) and \(\frac{dn}{t}=0\) and

\[
\begin{align*}
\frac{ds}{r} + \frac{ds}{q} &= ds + ds + ds \\
\frac{dn}{r} + \frac{dn}{q} &= dn + dn + dn
\end{align*}
\]  

(3.3.8) \hspace{1cm} (3.3.9)

It was demonstrated in item 1.1 that

97
\[
\frac{d_s^{(2)}}{t} = \frac{d\phi^{(2)}}{t} \times \rho^{(2)} \quad (3.3.10)
\]

where
\[
\rho^{(2)} = \frac{N^{(2)}}{M^{(2)}}
\]
is a vector drawn from an arbitrary point \(N^{(2)}\) of axis rotation to the contact point \(M^{(2)}\). (Fig. 1.2.1).

Then, (see item 1.1),
\[
\frac{dn^{(i)}}{t} = \frac{d\phi^{(i)}}{t} \times \hat{n}^{(i)} \quad (3.3.11)
\]

Here: vector \(\hat{d}_\phi^{(i)}\) is similar to vector \(\omega^{(i)}\) and is directed along the axis of rotation according to the direction of rotation
\[
d\phi^{(i)} = \omega^{(i)} dt, \quad (3.3.12)
\]

where \(t\) is time.

Let us compose following scalar products
\[
n \cdot (ds^{(1)}_{tr} + ds^{(1)}_q) = n \cdot (ds^{(2)}_{tr} + ds^{(2)}_r + ds^{(2)}_q) \quad (3.3.13)
\]
\[
n \cdot (dn^{(1)}_{tr} + dn^{(1)}_q) = n \cdot (dn^{(2)}_{tr} + dn^{(2)}_r + dn^{(2)}_q), \quad (3.3.14)
\]

where \(n\) is the common unit normal of surfaces.

Vectors \(ds^{(1)}_{tr}\) and \(ds^{(2)}_{tr}\) belong to the common tangent plane \(T\) (Fig. 3.3.1). Therefore,
\[
n \cdot ds^{(i)}_{tr} = 0 \quad (i=1,2) \quad (3.3.15)
\]

Equations (3.3.13), (3.3.10) and (3.3.15) yield
\[
\left[ \frac{d\phi^{(2)}}{t} \rho^{(2)} \right]_n = \left( \frac{ds^{(1)}_{q}}{t} - \frac{ds^{(2)}_{q}}{t} \right)_n \quad (3.3.16)
\]

It is easy to be verified that both parts of equation (3.3.14) are equal to zero identically. Indeed, vectors \(dn^{(i)}_{r}\) belong to the tangent plane and therefore
It results from equation (3.3.11) that
\[ \mathbf{n} \cdot \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} = 0 \]  

(3.3.17)

Vector \( \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} \) \((i=1,2)\) can be represented the expression
\[ \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} = \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} \times \mathbf{n} \]

(3.3.19)

where \( \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} \) is a vector represented by the angular error.

Therefore
\[ \mathbf{n} \cdot \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} = \left[ \mathbf{n} \quad \mathbf{d}_{\mathbf{n} \_\_ q}^{(i)} \times \mathbf{n} \right] = 0 \]

(3.3.20)

Equation (3.3.16) is the basic equation for the determination of kinematical errors of gear drives. Its application will be demonstrated in the following items.

3.4. Kinematical Errors of Spiral Bevel Gears Induced by Their Eccentricity

Gear eccentricity occurs when a gear's geometrical axis does not coincide with its axis of rotation (Fig. 3.4.1). By rotation the geometrical axis of a gear generates a cylindrical surface of radius \( \Delta e \). The vector of eccentricity \( \Delta e \) is represented by a vector of constant magnitude which rotates about gear axis.

The initial position of vector \( \Delta e \) (the position at the beginning of motion) is given by the angle \( \alpha \) and its current position by angle \((\phi + \alpha)\) (Fig. 3.4.2).

Fig. 3.4.2 shows coordinate systems \( S_1(x_1, y_1, z_1) \) and \( S_f \) rigidly connected with gear 1 and the frame; the coordinate system \( S_h \) is an auxiliary one which is also rigidly connected with the frame. The driving gear 1 rotates about axis \( z_h \). The position of \( \Delta e_{\_l} \) in coordinate system \( S_1 \) is given by the angle \( \alpha_{\_l} \) made by \( \Delta e_{\_l} \) and axis \( x_1 \). The current position of \( \Delta e_{\_l} \) in coordinate
FIG. 3.4.1

Cylinder Generated by Geometrical Axis of Eccentric Gear
FIG. 3.4.2

Coordinate Systems Associated with Gear 1
system $S_f$ (or $S_p$) is defined by the angle $(\phi_1 + \alpha_1)$. Vector $\Delta e_f^{(1)}$ is represented by the matrix equation

$$
\begin{bmatrix}
\Delta e_f^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
L_{fh}
\end{bmatrix}
\begin{bmatrix}
\Delta e_h^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
\cos \gamma_1 & 0 & \sin \gamma_1 & \Delta e_1 \cos(\phi_1 + \alpha_1) \\
0 & 1 & 0 & -\Delta e_1 \sin(\phi_1 + \alpha_1) \\
-sin \gamma_1 & 0 & \cos \gamma_1 & 0
\end{bmatrix}
$$

(3.4.1)

Matrix equality (3.4.1) yields

$$
\begin{bmatrix}
\Delta e_f^{(1)}
\end{bmatrix}
= 
\begin{bmatrix}
\Delta e_1 \cos(\phi_1 + \alpha_1) \cos \gamma_1 \\
-\Delta e_1 \sin(\phi_1 + \alpha_1) \\
-\Delta e_1 \cos(\phi_1 + \alpha_1) \sin \gamma_1
\end{bmatrix}
$$

(3.4.2)

The vector of eccentricity of the driven gear 2 $\Delta e^{(2)}$ can be defined the same way. Fig. 3.4.3 shows coordinate systems $S_2$ and $S_f$ rigidly connected with gear 2 and the frame. Coordinate system $S_p$ is also rigidly connected with the frame.

Vector $\Delta e_f^{(2)}$ is represented by the matrix equation

$$
\begin{bmatrix}
\Delta e_f^{(2)}
\end{bmatrix}
= 
\begin{bmatrix}
L_{fp}
\end{bmatrix}
\begin{bmatrix}
\Delta e_p^{(2)}
\end{bmatrix}
= 
\begin{bmatrix}
\cos \gamma_2 & 0 & -\sin \gamma_2 & \Delta e_2 \cos(\phi_2 + \alpha_2) \\
0 & 1 & 0 & \Delta e_2 \sin(\phi_2 + \alpha_2) \\
\sin \gamma_2 & 0 & \cos \gamma_2 & 0
\end{bmatrix}
$$

(3.4.3)

It results from matrix equality (3.4.3) that

$$
\begin{bmatrix}
\Delta e_f^{(2)}
\end{bmatrix}
= 
\begin{bmatrix}
\Delta e_2 \cos(\phi_2 + \alpha_2) \cos \gamma_2 \\
\Delta e_2 \sin(\phi_2 + \alpha_2) \\
\Delta e_2 \cos(\phi_2 + \alpha_2) \sin \gamma_2
\end{bmatrix}
$$

(3.4.4)

Kinematical errors induced by gear’s eccentricities are defined by an equation similar to (3.3.16):
FIG 3.4.3

Coordinate Systems Associated with Gear 2
where \( \Delta e_\ell^{(1)} \) and \( \Delta e_\ell^{(2)} \) are represented by matrices (3.4.2) and (3.4.4); \\
\( \Delta \Phi^{(2)}_f \) (Fig. 3.4.3) is represented by matrix \\

\[
\begin{bmatrix}
\Delta \Phi^{(2)}_f \\
\end{bmatrix} = 
\begin{bmatrix}
\begin{bmatrix}
\cos \gamma_2 & 0 & -\sin \gamma_2 \\
0 & 1 & 0 \\
\sin \gamma_2 & 0 & \cos \gamma_2
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\Delta \Phi^P \\
\end{bmatrix}
\]

Vector \( \rho_f^{(2)} \) represents the position vector of a point which belongs to the line of action and \( n_f \) represents the unit normal of the contacting surfaces at their point of tangency.

Equations (3.4.5) and (3.4.6) yield

\[
\Delta \Phi^P_2 = \frac{n_x \Sigma \Delta e_x + n_y \Sigma \Delta e_y + n_z \Sigma \Delta e_z}{-y \cos \gamma_2 n_x + (x \cos \gamma_2 + z \sin \gamma_2)n_y -y \sin \gamma_2 n_z}
\]

Here: \( \Sigma \Delta e_x = \Delta e_x^{(1)} - \Delta e_x^{(2)} \), \( \Sigma \Delta e_y = \Delta e_y^{(1)} - \Delta e_y^{(2)} \), \( \Sigma \Delta e_z = \Delta e_z^{(1)} - \Delta e_z^{(2)} \). The subscript "f" was dropped in equation (3.4.7). The unit normal was represented by equations (2.2.10)

\[
n_f = \sin \psi c + \cos \psi c \sin \tau d i_f + \cos \psi c \cos \tau d k_f =
\]

\[
\sin \psi c i_f + \cos \psi c \left[ \cos (\beta - \phi_d) j_f + \sin (\beta - \phi_d) k_f \right] =
\]

\[
\sin \psi c i_f + \cos \psi c \left[ \cos (\beta - \phi_1 \sin \gamma_1) j_f + \sin (\beta - \phi_1 \sin \gamma_1) k_f \right]
\]

Equations (3.4.7) by \( \phi_1 = 0 \) represent the surface unit normal at the point of intersection of the tooth surface with the generatrix of the pitch cone.

Coordinates \( x, y, z \) of a current point of line action were represented:

(a) by equations (2.2.25) for spiral bevel gears with geometry I; (b) by equations (2.6.4) for spiral bevel gears with geometry II.
In the process of meshing of one pair of teeth the angle of rotation \( \phi_1 \) changes in the interval \([\mp \pi/N_1, \pi/N_1]\), where \( N_1 \) is the number of teeth of gear 1. Considering \( \phi_1 \sin \gamma_1 \) as negligible the unit surface normal can be represented by the equation

\[
\mathbf{n}_f = \sin \psi_1 \mathbf{z}_f + \cos \psi_1 (\cos \beta \mathbf{j}_f + \sin \beta \mathbf{k}_f)
\]

(3.4.9)

With the same assumption for \( \phi_1 \sin \gamma_1 \) it can be taken that \( x_f = 0, y_f = 0, z_f = L \)

(3.4.10)

Equations (3.4.7), (3.4.9) and (3.4.10) yield

\[
\Delta \phi_2 (\phi_1) = \frac{n_x \Sigma \Delta e_x + n_y \Sigma \Delta e_y + n_z \Sigma \Delta e_z}{L \sin \gamma_2 \cos \psi_1 \cos \beta}
\]

(3.4.11)

Here:

\[
n_x \Sigma \Delta e_x + n_y \Sigma \Delta e_y + n_z \Sigma \Delta e_z = a_1 \sin(\phi_1 + \alpha_1) + b_1 \cos(\phi_1 + \alpha_1) + a_2 \sin(\phi_2 + \alpha_2) + b_2 \cos(\phi_2 + \alpha_2)
\]

(3.4.12)

Here:

\[
a_1 = -\Delta e_1 \cos \psi_1 \cos \beta; \quad b_1 = \Delta e_1 (\cos \gamma_1 \sin \psi_1 - \sin \gamma_1 \cos \psi_1 \sin \beta)
\]

\[
a_2 = -\Delta e_2 \cos \psi_1 \cos \beta; \quad b_2 = \Delta e_2 (\cos \gamma_2 \sin \psi_1 + \sin \gamma_2 \cos \psi_1 \sin \beta)
\]

\[
\phi_2 = \frac{N_1}{N_2}
\]

(3.4.13)

It results from equations (3.4.12) that kinematical errors of spiral bevel gears can be represented as the sum of four harmonics. The period of two harmonics coincides with the period of revolution of gear 1; the period of the other two harmonics coincides with the period of revolution of driven gear (of gear 2).

The function \( \Delta \phi_2 (\phi_1) \) as defined by equation (3.4.11) is a smoothed function. In reality this function breaks by changing teeth in meshing. This break can be discovered if the function \( \Delta \phi_2 (\phi_1) \) is defined by equation (3.4.7).
Equation (3.4.11) can be applied for spur gears, too. By $L \sin \gamma_2 = r_2$, $\beta = 0$, $\sin \gamma_1 = \sin \gamma_2 = 0$ equations (3.4.11) and (3.4.12) yield:

$$\Delta \phi_2(\phi_1) = \frac{\Delta e_1 \sin(\psi_c - \phi_1 - \alpha_1) + \Delta e_2 \sin(\psi_c + \alpha_2 + \phi_1)}{r_2 \cos \psi_c}, \quad (3.4.14)$$

where $r_2$ is the pitch radius of gear 2.

Parameters $\alpha_1$ and $\alpha_2$ influence the distribution of function $\Delta \phi_2(\phi_1)$ in the positive and negative areas. For a drive with $N_1 = N_2$, $\alpha_2 = \pi + \alpha_1$ and $\Delta e_2 = \Delta e_1$ the function $\Delta \phi_2(\phi_1) = 0$. In other words, kinematical errors induced by eccentricities $\Delta e_1$ and $\Delta e_2$ are compensated completely.

3.5 Kinematical Errors Induced by Misalignment

There are following kinds of misalignment (Fig. 2.2.2): (a) displacement of a gear in direction of positive or negative axis $x_f$; (b) axial displacement of gear 1 in direction of its axis $Oa$; (c) axial displacement of gear 2 in direction of axis $Ob$; (d) an error of the angle made by axes $Oa$ and $Ob$.

Let us suppose that gear 1 is displaced in the direction of negative axis $x_f$ by

$$\Delta s_{(1)}^{(1)} = -A_{\alpha_f}$$

Equations (3.3.16) and (3.5.1) yield

$$\left[ \Delta \phi_{(2)}^{(2)} \right] = -\Delta s_{(1)}^{(1)}$$

(3.5.2)

It results from (3.5.2) that

$$\Delta \phi_2(\phi_d) = -\frac{\Delta A \sin \psi_c}{y \cos \gamma_2 x + (x \cos \gamma_2 + z \sin \gamma_2) y - y \sin \gamma_2 z}$$

(3.5.3)
Here: \( \phi_d = \phi_1 \sin \gamma_1 - \phi_2 \sin \gamma_2 \) is the angle of rotation of the generating gear; \( x, y, z \) are coordinates of the line of action represented by equations (2.2.25) and (2.6.4) for spiral bevel gears with geometry I and II, respectively.

Now, let us consider a case when gear 1 is displaced in the direction of negative axis \( y_f \) at

\[
\Delta s^{(1)}(q) = -\Delta E_{j_f}
\]

By analogy with equation (3.5.3) it will be

\[
\Delta \phi_2(\phi_d) = \frac{\Delta E_n \gamma}{-y \cos \gamma_2 n_x + (x \cos \gamma_2 + z \sin \gamma_2) n_y - y \sin \gamma_2 n_z}
\]

The variation of the angle made by gear axes \( O_a \) and \( O_b \) can be represented as a result of rotation of one of the gears about axis \( y_f' \), for instance, gear 1. The vector of rotation is

\[
\Delta \delta = \Delta \delta j_f
\]

and the displacement of contact point is represented by equation

\[
\Delta s^{(1)}(q) = \Delta \delta \times \rho,
\]

where \( \rho \) is the radius-vector drawn from \( O_f \) to the point of action.

Kinematical errors induced by \( \Delta s^{(1)} \) are represented by equation

\[
\begin{bmatrix} \Delta \phi(2) q \end{bmatrix} = \begin{bmatrix} \Delta \delta \rho n \end{bmatrix}
\]

Equation (3.5.8) yields

\[
\Delta \phi_2(\phi_d) = \frac{(z n_x - x n_z) \Delta \delta}{-y \cos \gamma_2 n_x + (x \cos \gamma_2 + z \sin \gamma_2) n_y - y \sin \gamma_2 n_z}
\]

Equations (3.5.3), (3.5.5) and (3.5.9) can be simplified for spiral bevel gears with geometry II taking into account that in this case \( x = 0 \),
Equations proposed in this item can be applied for approximate determination of kinematical errors induced by incorrect methods of generation of spiral bevel gears and for determination of machine settings to compensate such errors.

It was mentioned in item 2.1 that a correct meshing of spiral bevel gears can be gotten by coinciding three axes of instantaneous rotation. In reality these axes do not coincide and therefore kinematical errors represented by equation (3.5.9) appear by $\Delta \delta$ equal to the sum of dedendum angles of the two gears.

To compensate these errors corrections of machine settings for cutting the pinion are used. These corrections are pinion displacements represented by equation

$$\Delta s_{(1)} = \Delta E \gamma_f + \Delta L \kappa_f.$$  \hspace{1cm} (3.5.10)

where $\Delta E$ and $\Delta L$ are algebraic values.

Equations (3.3.15) and (3.5.10) yield

$$\Delta \phi_\zeta(\phi_d) =$$

$$\frac{\Delta E \gamma_y + \Delta L \gamma_z}{-y \cos \gamma_y n_x + (x \cos \gamma_y + z \sin \gamma_y) n_y - y \sin \gamma_y n_z}.$$  \hspace{1cm} (3.5.11)

To compensate kinematical errors (3.5.9) the following function

$$f(\phi_d) =$$

$$\frac{\Delta E \gamma_y + \Delta L \gamma_z - (x n_x - x n_z) \Delta \delta}{-y \cos \gamma_y n_x + (x \cos \gamma_y + z \sin \gamma_y) n_y - y \sin \gamma_y n_z}.$$  \hspace{1cm} (3.5.12)

must be minimized.

Let us represent function $f(\phi_d)$ as a difference of two functions as follows:
\[ f(\phi_d) = f_1(\phi_d) - f_2(d) \] (3.5.13)

Here

\[ f_2(\phi_d) = \frac{(z n_x - x n_z) \Delta \delta}{-y \cos \gamma_2 n_x + (x \cos \gamma_2 + z \sin \gamma_2) n_y - y \sin \gamma_2 n_z} \] (3.5.14)

is the function of errors, and

\[ f_1(\phi_d) = \frac{\Delta E n_x + \Delta L n_z}{-y \cos \gamma_2 n_x + (x \cos \gamma_2 + z \sin \gamma_2) n_y - y \sin \gamma_2 n_z} \] (3.5.15)

is the compensating function which is applied in order to compensate the

kinematical errors induced by \( \Delta \delta \) as a result of an incorrect method of gear generation.

Let us define derivatives \( \frac{df_1}{d\phi_d} \) and \( \frac{df_2}{d\phi_d} \) at the main contact point at

which \( y=0, x=0, z=L \) for gears with geometry I and geometry II.

**Geometry I.** Projections of the surface unit normal were represented by

equations (2.2.10)

\[ n_x = \sin \psi_c \]
\[ n_y = \cos \psi_c \sin(\theta_d - q_d + \phi_d) \]
\[ n_z = \cos \psi_c \cos(\theta_d - q_d + \phi_d) \] (3.5.16)

where \( \psi_c, \theta_d \) and \( q_d \) are constant parameters and \( \theta_d - q_d = 90^\circ - \beta \).

Coordinates of contact point were represented by equations (2.2.25)

\[
x = \left[ r_d - b_d \frac{\sin(q_d - \phi_d)}{\cos(\beta - \phi_d)} \right] \sin \psi_c \cos \psi_c \\
y = \frac{\cos(\beta - \phi_d)}{\tan \psi_c} \\
z = \frac{b_d \sin \theta_d}{\cos(\beta - \phi_d)} \frac{\sin(\beta - \phi_d)}{\tan \psi_c} x
\] (3.5.17)
At the main contact point \( \phi_d = 0, x = y = 0, z = L \). Equations (3.5.16) and (3.5.17) yield that at the main contact point

\[
\frac{dn_x}{d\phi_d} = 0, \quad \frac{dn_y}{d\phi_d} = \cos \psi_c \sin \beta, \quad \frac{dn_z}{d\phi_d} = -\cos \psi_c \cos \beta 
\] (3.5.18)

\[
\frac{dx}{d\phi_d} = \frac{b_d \sin \theta_d}{\cos^2 \beta} \sin \psi_c \cos \psi_c = \frac{L}{\cos \beta} \sin \psi_c \cos \psi_c 
\] (3.5.19)

\[
\frac{dy}{d\phi_d} = L \cos^2 \psi_c 
\] (3.5.20)

\[
\frac{dz}{d\phi_d} = -L \sin^2 \psi_c \tan \beta 
\] (3.5.21)

At the main contact point the derivative \( \frac{df_2}{d\phi_d} \) is represented by equation

\[
\frac{df_2}{d\phi_d} = \left[ \frac{dz}{d\phi_d} \frac{n_x - dx}{d\phi_d} \frac{n_z}{d\phi_d} - \frac{L \sin \gamma_2 n_y}{L \sin^2 \gamma_2 n^2 y} \right] \Delta \delta 
\] (3.5.22)

Equations (3.5.22), (3.5.16) and (3.5.19)-(3.5.21) yield

\[
\frac{df_2}{d\phi_d} = -\tan \beta \tan \psi_c \frac{\sin \gamma_2 \cos \beta \Delta \delta}{L \sin \gamma_2 \cos \beta} 
\] (3.5.23)

Equation (3.5.15), (3.5.16) and (3.5.19)-(3.5.21) yield that

\[
\frac{df_1}{d\phi_d} = \frac{\Delta E}{d\phi_d} \frac{dn_y}{d\phi_d} + \frac{\Delta L}{d\phi_d} \frac{dn_z}{d\phi_d} = \frac{\Delta E \sin \beta - \Delta L \cos \beta}{L \sin \gamma_2 \cos \beta} 
\] (3.5.24)

Kinematical errors will be compensated in the neighborhood of the main contact point if
This requirement is satisfied by

$$\frac{\Delta E \sin \beta \cdot \Delta L \cos \beta}{L} + \Delta \delta \tan \beta \tan \psi_c = 0$$  \hspace{1cm} (3.5.26)$$

A requirement that functions $f_1(\phi_d)$ and $f_2(\phi_d)$ must be equal at the main contact point yields

$$\frac{\Delta E \cos \beta + \Delta L \sin \beta}{L} - \Delta \delta \tan \psi_c = 0$$  \hspace{1cm} (3.5.27)$$

It results from equations (3.5.26) and (3.5.27) that

$$\frac{\Delta E}{L} = \frac{\tan \psi_c \cos 2 \beta}{\cos \beta} \Delta \delta$$  \hspace{1cm} (3.5.28)$$

$$\frac{\Delta L}{L} = 2 \tan \psi_c \sin \beta \Delta \delta$$  \hspace{1cm} (3.5.29)$$

Equations (3.5.28) and (3.5.29) provide approximate magnitudes of machine settings for spiral bevel gears.

For spiral bevel gears with geometry II functions (3.5.14) and (3.5.15) will be the following ones.

$$f_2(\phi_d) = \frac{\Delta \delta n_x}{\sin \gamma_2 n_y} = \frac{\tan \psi_c}{\sin \gamma_2 \cos \beta} \Delta \delta$$  \hspace{1cm} (3.5.30)$$

$$\frac{df_2}{d\phi_d} = -\frac{\cot \psi_c \tan \psi_c}{\sin \beta} \Delta \delta$$  \hspace{1cm} (3.5.31)$$

$$f_1(\phi_d) = \frac{\Delta E n_x + \Delta L n_z}{z \sin \gamma_2 n_y} = \frac{\Delta E \cos \beta + \Delta L \sin \beta}{L \sin \gamma_2 \cos \beta}$$  \hspace{1cm} (3.5.32)$$
\[ \frac{df_1}{d\phi_d} = \frac{1}{L \sin \gamma_2} \left[ \Delta L \left( \frac{\cot \varphi_d}{\cos \beta \sin \beta} - 1 \right) - \Delta E \cot \beta \right] \] (3.5.33)

Requirements that at the main contact point

\[ f_1(\phi_d) = f_2(\phi_d), \quad \frac{df_1}{d\phi_d} = \frac{df_2}{d\phi_d} \]

yield

\[ \frac{\Delta L}{L} = (\cos \beta - \sin \beta \tan \varphi_d) \tan \psi_c \Delta \delta \] (3.5.34)

\[ \frac{\Delta L}{L} = (\sin \beta + \cos \beta \tan \varphi_d) \tan \psi_c \Delta \delta \] (3.5.35)

4. CONCLUSION

a. General kinematic relations for conjugate gear tooth surfaces are proposed. The proposed equations relate the motions of: (a) points of contact and (b) surface unit normals. The equations above are applied to define: (a) relations between principal curvatures and directions for two gear tooth surfaces which are in mesh, (b) kinematical errors induced by errors of manufacturing and assemblage.

b. Two mathematical models of geometry of spiral bevel gears are proposed. Models above correspond to the motion of contact point across and along the tooth surface.

c. The bearing contact of spiral bevel gears for both models is determined. A computer program for this has been worked out.

d. Method to investigate kinematical errors of spiral bevel gears is worked out.
LIST OF SYMBOLS

Section 1

- **a**
  - half the length of major axis

- **A**
  - auxiliary function used in Eq. (1.7.30) represented by Eq. (1.7.31)

- **b**
  - half the length of the minor axis

- **B**
  - auxiliary function defined by Eq. (1.7.32)

- **C**
  - shortest distance between axis of rotation

- **f_i**
  - elastic deformation of surface $\Sigma_i$

- **g_1 = \kappa^{(1)}_I - \kappa^{(1)}_{II}**
  - auxiliary function to determine size of contact ellipse

- **g_2 = \kappa^{(2)}_I - \kappa^{(2)}_{II}**
  - auxiliary function to determine size of contact ellipse

- **N**
  - distance of point N from tangent plane t t

- **[L_{ij}]**
  - projection transformation matrix

- **M_o**
  - point of contact of tooth surfaces

- **[M_{ij}]**
  - coordinate transformation matrix; transformation from $S_j$ to $S_1$

- **n^{(i)}_{abs}**
  - absolute velocity of the end of unit normal

- **n^{(i)}_1(u_1, \theta_1)**
  - unit normal vector to surface $\Sigma_1$

- **(n_x^{(i)}, n_y^{(i)}, n_z^{(i)})**
  - projections of $n^{(i)}$ in coordinate system $S_f$

- **n^{(i)}_r**
  - relative velocity of the end of unit normal vector $n_1$

- **n^{(i)}_{tr}**
  - transfer velocity of the end of unit normal vector $n_1$

- **N, N_1, N_2**
  - point on surface $\Sigma_1$

- **N'**
  - new position of point N after displacement

- **N''**
  - final position of point N after displacement and elastic deformation

- **N'_2**
  - final position of $N'$ after displacement and elastic deformation
\( N_i \)  
normal vector to surface \( \Sigma_i \)

\( \vec{r}_i(u_i, \theta_i) \)  
position vector describing surface \( \Sigma_i \) with surface coordinate \((u_i, \theta_i)\)

\( S_i(x_i, y_i, z_i) \)  
coordinate system \( i \)

\( t-t \)  
tangent plane to surface \( \Sigma_1 \) and \( \Sigma_2 \)

\( V_{abs}^{(i)} \)  
absolute velocity of contact point on surface \( \Sigma_i \)

\( V_{tr}^{(i)} \)  
relative velocity of contact point on surface \( \Sigma_i \)

\( V_{tr}^{(i)} \)  
transfer velocity of contact point on surface \( \Sigma_i \)

\( V_{tr}^{(i)} \)  
transfer velocities of points on surface \( \Sigma_i \) in coordinate system \( i \)

\( V_1^{(2)} = V_2^{(2)} - V_1^{(1)} \)  
relative velocity of point 2 with respect to point 1

\( (x_f^{(i)}, y_f^{(i)}, z_f^{(i)}) \)  
Cartesian coordinates of contact point on surface \( \Sigma_i \) as expressed in coordinate system \( S_f \)

\( \alpha^{(1)} \)  
angle made by axis \( \eta \) and \( i_1^{(1)} \)

\( \alpha^{(2)} \)  
angle made by axis \( \eta \) and \( i_1^{(2)} \)

\( \gamma \)  
angle of crossing of axis of rotation

\( \delta \)  
approach of surface \( \Sigma_1 \) and \( \Sigma_2 \)

\( \delta_1 \)  
displacement of surface \( \Sigma_1 \) when \( \Sigma_1 \) and \( \Sigma_2 \) are in meshing

\( \delta_2 \)  
displacement of surface \( \Sigma_2 \)

\( (\hat{\imath}_I^{(1)}, \hat{\jmath}_II) \)  
unit vectors along principal direction of surface \( \Sigma_1 \)

\( (\hat{\imath}_I^{(2)}, \hat{\jmath}_II) \)  
unit vector along principal direction of surface \( \Sigma_2 \)

\( \kappa_I^{(1)} \)  
principal curvatures of surface \( \Sigma_1 \)

\( \kappa_I^{(2)} \)  
principal curvatures of surface \( \Sigma_2 \)

\( \kappa^{(1)} = \kappa_I^{(1)} + \kappa_{II}^{(1)} \)  
auxiliary function

\( \kappa^{(2)} = \kappa_{II}^{(2)} + \kappa_{II}^{(2)} \)  
auxiliary function

\( \rho \)  
distance of points \( N \) and \( N' \) from point \( M_0 \)

\( \sigma \)  
angle between \( \hat{\imath}_I^{(1)} \) and \( \hat{\jmath}_II^{(1)} \)
$\Sigma_i$  
surface $i$

$\phi_i$  
angle of rotation of gear $i$

$\omega_f^{(i)}$  
angular velocity of surface $\Sigma_i$

**Section 2**  \((i=1,2)\) \((d=f,k)\)

$a_{31}^{(l)}$  
auxiliary function defined by Eq. (2.4.11)

$a_{32}^{(l)}$  
auxiliary function defined by Eq. (2.4.12)

$b_3^{(l)}$  
auxiliary function defined by Eq. (2.4.13)

$b_d$  
a parameter of tool setting

$f^{(l)}$  
auxiliary function used to compute the principal direction of surface $\Sigma_i$

$G^{(l)}$  
auxiliary function used to compute the principal curvatures of surface $\Sigma_i$

$[L_{ij}]$  
projection transformation matrix

$[M_{ij}]$  
coordinate transformation matrix; transformation from $S_j$ to $S_i$

$n_f$  
surface unit normal

$N_f^{(d)}$  
surface normal to surface $d$

$q_d$  
a parameter of tool setting

$r_f^{(d)}$  
locus of contact point on surface $d$

$r_d$  
a parameter of tool setting

$s^{(l)}$  
auxiliary function used to compute principal curvature of surface $\Sigma_i$

$S_a (x_a, y_a, z_a)$  
auxiliary coordinate system

$S_c (x_c, y_c, z_c)$  
coordinate system used to represent surface $\Sigma_f$ in geometry II

$S_h$  
coordinate system rigidly connected with frame

$S_i$  
coordinate system rigidly connected with gear $i$

$u_d$  
generating surface coordinate
$v_f^{(F1)}$  relative velocity of a contact point on surface $\Sigma_f$ with respect to contact point on surface $\Sigma_1$ 

$v_f^{(K2)}$  relative velocity of a contact point on surface $\Sigma_K$ with respect to contact point on surface $\Sigma_2$ 

$(x_f, y_f, z_f)$  coordinates of the line of action of surface $\Sigma_i$ 

$(x_f^{(d)}, y_f^{(d)}, z_f^{(d)})$  components of the equations of the generating surface $\Sigma^{(d)}$ 

$\beta$  $90^\circ - (\phi_d - \phi_q)$ see Eq. (2.2.24) 

$\gamma_i$  half of pitch cone angles of gear $i$ 

$\phi_d$  generating surface coordinate 

$i_1^{(d)}$  unit vector representing the first principal direction of surface $d$ 

$i_1^{(d)}$  unit vector representing the second principal direction of surface $d$ 

$\kappa_i^{(d)}$  principal curvature $I$ of surface $d$ 

$\kappa_{II}^{(d)}$  principal curvature of II of surface $d$ 

$\theta_i^{(d)}$  angle between $i_1^{(d)}$ and $D_1$ positive clockwise 

$\Sigma_d$  tool surface $d$ 

$\Sigma_1$  generated surface of pinion and gear 

$\tau_d$  $\theta_d - (q_d - \phi_d)$ auxiliary function 

$\phi_d$  angle of rotation of generating surface about axis $x_f$ 

$\phi_i$  angle of rotation of gear $i$ 

$\psi_c$  shape angle of head-cutter blades 

$\omega^{(d1)}$  relative angular velocity of contact point on surface $d$ with respect to contact point on surface $1$ 

$\omega^{(d)}$  angular velocity of surface $d$ 

$\omega^{(l)}$  angular velocity of gear $i$
Section 3

$\Delta A$  gear displacement

$\Delta E$  machine setting

$\Delta e^{(i)}$  eccentricity vector of gear \( i \)

$\Delta L$  machine setting

$M_i$  contact point on surface $\Sigma_i$

$n_f^{(i)}$  unit normal vector of surface $\Sigma_i$

$\Delta Q$  vector of errors

$Q_i$  components of vector of errors

$\Xi_f^{(1)}$  position vector of point on surface $\Sigma_i$

$d\delta_{q}^{(i)}$  displacement vector of contact point due to kinematical errors

$\alpha_i$  angular position of eccentricity vector

$\Delta \delta$  sum of dedendum angles of gears 1 and 2

$\Sigma_i$  surface \( i \)

$\Delta \phi_2$  kinematical error function

$\phi_2^o$  theoretical value of gear 2 angle of rotation

$\phi_2$  actual value of gear 2 angle of rotation
REFERENCES


Spiral bevel gears have widespread applications in the transmission systems of aircraft. Major requirements in the field of helicopter transmissions are: (a) improved life and reliability, (b) reduction in overall weight (i.e., a large power to weight ratio) without compromising the strength and efficiency during the service life, (c) reduction in the transmission noise. The first two parts of this report deal with tooth contact geometry. In this report, a novel approach to the study of the geometry of spiral bevel gears and to their rational design is proposed. The nonconjugate tooth surfaces of spiral bevel gears are, in theory, replaced (or approximated) by conjugated tooth surfaces. These surfaces can be generated: (a) by two conical surfaces and (b) by a conical surface and a revolution. Although these conjugated tooth surfaces are simpler than the actual ones, the determination of their principal curvatures and directions is still a complicated problem. Therefore, a new approach, to the solution of these is proposed in this report. In this approach, direct relationships between the principal curvatures and directions of the tool surface and those of the generated gear surface are obtained. With the aid of these analytical tools, the Hertzian contact problem for conjugate tooth surfaces can be solved. These results are eventually useful in determining compressive load capacity and surface fatigue life of spiral bevel gears. In the third part of this report, a general theory of kinematical errors exerted by manufacturing and assembly errors is developed. This theory is used to determine the analytical relationship between gear misalignments and kinematical errors. This is important to the study of noise and vibration in geared systems.