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Lubrication of Rigid Ellipsoidal Solids

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LUBRICATION OF RIGID ELLIPSOIDAL SOLIDS

There have been relatively few theoretical studies of the lubrication of rolling contacts compared with the attention given to plain thrust and journal bearings. Certainly a great deal of development effort has been put into gear, ball, and roller bearing lubrication since all are machine elements of great technological and commercial importance. But understanding of their lubricating mechanism is still far from complete. The huge pressures involved in the elliptical contact generated between a rolling element and a track influence in a rather complicated way the properties of the lubricant and the shapes of the contacting surfaces.

The first requirement therefore is to develop a basic solution to the problem of the lubrication of rigid ellipsoidal solids with an isoviscous, incompressible fluid. A solution to this problem is presented in this chapter, and the results provide a foundation for the full appreciation of the elastohydrodynamic theory of elliptical contacts to be presented later. The influence of the variation of viscosity with pressure and

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the effect of the deformation of surfaces under pressure are considered in Chapter 7.

Most studies of the hydrodynamic lubrication of nominal point and line contacts have concentrated on minimum-film-thickness predictions for either a ball near a plane or a cylinder near a plane in which side leakage is neglected. But the full range of geometries between the two extremes has only recently been studied by Brewe, et al. (1979). Kapitza (1955) presented an early and elegant analysis of this problem in which he generated a minimum-film-thickness formula for a nonrotating sphere floating in a sea of lubricant above a plane surface sliding at a given velocity. This analytical solution represents a rare and outstanding example of a successful mathematical approach to the solution of the second-order differential equation presented by Osborne Reynolds. Solutions taking account of side leakage were developed analytically for the special case of a ball on a plane by using a clever substitution. One disadvantage of the Kapitza approach is that it adopted the half-Sommerfeld boundary condition, which violates the requirement of flow continuity at the cavitation boundary. However, the effect of this assumption on film thickness prediction is not always serious. The Brewe, et al. (1979) work is used extensively in this chapter because it uses the Reynolds cavitation boundary conditions and is applicable to the complete range of geometries of the contacting solids.
The material presented in this chapter not only provides an introduction to classical hydrodynamic lubrication as it relates to nonconformal contacts, but it also introduces a minimum-film-thickness formula that has a number of direct applications: circular-arc thrust bearing pads; industrial coating processes in which paint, emulsion, or protective coatings are applied to sheet or film materials passing between rollers; and very lightly loaded cylindrical roller bearings.

6.1 Equation for Pressure Distribution

From the previous chapter the appropriate form of the Reynolds equation when the viscosity is assumed to be constant is given as

$$\frac{\partial}{\partial x} \left( h^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^3 \frac{\partial p}{\partial y} \right) = 12u_0 \frac{\partial h}{\partial x}$$  \hspace{1cm} (5.45)

where

$$u = \frac{u_a + u_b}{2}$$

It is convenient to nondimensionalize with respect to the effective radius \( R_x \); that is,

$$x = \frac{x}{R_x}, \quad y = \frac{y}{R_x}, \quad H = \frac{h}{R_x}, \quad \text{and} \quad \frac{p_{R_x}}{\eta_0 u}$$  \hspace{1cm} (6.1)

In terms of these dimensionless variables equation (5.45) becomes
\[
\frac{\partial}{\partial x} \left( H^3 \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( H^3 \frac{\partial p}{\partial y} \right) = 12 \frac{\partial H}{\partial x}
\]  \hspace{1cm} (6.2)

The film thickness between two rigid ellipsoidal solids as shown in Figure 2.18 can be written as

\[ h = h_0 + S \] \hspace{1cm} (6.3)

where

- \( S \) = geometrical separation of the solids
- \( h_0 \) = central (minimum) film thickness

Using equation (2.35) and rewriting equation (6.3) in dimensionless form give

\[ H = H_0 + \frac{x^2}{2} + \frac{y^2}{2\alpha_a} \] \hspace{1cm} (6.4)

where

\[ \alpha_a = \frac{R_y}{R_x} \] \hspace{1cm} (6.5)

\[ H_0 = \frac{h_0}{R_x} \] \hspace{1cm} (6.6)

In equation (6.4) the parabolic approximation is used in defining the geometrical separation of the undeformed solids.

The solution of the Reynolds equation (6.2) consists of a homogeneous and a particular solution; that is,

\[ P = P_h + P_p \] \hspace{1cm} (6.7)
for which \( P_h \) is a solution to the homogeneous equation and the condition that \( P_h = -P_p \) at the boundaries leads to

\[
\frac{\partial}{\partial X} \left( H^3 \frac{\partial P_h}{\partial X} \right) + \frac{\partial}{\partial Y} \left( H^3 \frac{\partial P_h}{\partial Y} \right) = 0
\]

(Kapitza 1955) was the first to recognize that the particular solution for the pressure is simply proportional to \( X/H^2 \), or

\[
P_p = -\frac{4\lambda aX}{H^2}
\]

(6.9)

where

\[
\lambda_b = \frac{1}{1 + 2/3a}
\]

(6.10)

In the preceding equation \( \lambda_b \) is the side-leakage factor established by Archard and Cowking (1965-66) and can be verified by inserting \( P_p \) back into equation (6.2). If we define \( P_h(X,Y) = 4\lambda_b Q(X,Y) \) by using equation (6.7), we can express the full solution as

\[
P = 4\lambda_b \left( -\frac{X}{H^2} + Q \right)
\]

(6.11)

In general the homogeneous solution \( P_h \) is an unknown function of \( X \) and \( Y \). Consequently, the pressure distribution must be determined numerically for a given speed, viscosity, geometry, and film thickness. The numerical solution is
normally achieved by the Gauss-Seidel iterative method with overrelaxation.

A variable-mesh nodal structure like the one shown in Figure 6.1 is used to provide close spacing in and around the pressure peak. In this figure the inlet is to the left and the outlet is to the right. The variable mesh helps to minimize the errors that can occur because of large gradients in the high-pressure region. The grid spacing of the coordinates X and Y varies depending on the anticipated pressure distribution. That is, for a very highly peaked and localized pressure distribution, the dimensionless fine mesh spacing is normally about 0.002, and the coarse mesh about 0.1. For a relatively flat pressure distribution, the fine mesh would be about 0.005, and the coarse mesh about 0.13.

6.2 Boundary Conditions

At the inlet the pressure is taken as zero at a distance sufficiently far from the center of the contact to give fully flooded conditions. Similarly, at a sufficient distance from the center of the contact to the sides of the nodal structure, the pressure is also taken to be zero. The boundary condition in the outlet is not as straightforward since it is necessary to take account of cavitation.
In the outlet region there is a tendency to form subambient pressures, which lead to disruption of the lubricating film by cavitation. The usual form of cavitation in lubricating films is the liberation of dissolved gases. Mineral oils contain between 8 and 10 percent of dissolved air. When the pressure in the oil film falls below ambient, some of the air is liberated in the form of bubbles. This tends to maintain the oil film pressure near the level of the saturation pressure. For most lubrication conditions the saturation and ambient pressures will be almost equal. These observations suggest that the pressure in the cavitated region of lubricating films will be approximately constant and near to the atmospheric or ambient pressure.

The approach adopted by Kapitza (1955) in defining the outlet boundary condition was to ignore the negative pressures, that is, to employ the half-Sommerfeld boundary condition. Kapitza's solution has the appeal of simplicity, but it does not satisfy the flow continuity conditions at the cavitation boundary, namely, that the pressure gradient normal to the cavitation boundary must be zero. To insist that $P = \partial P/\partial N = 0$ at $X = 0$ would be overspecifying the problem mathematically. It is possible, however, to insist that $P = \partial P/\partial N = 0$ at the cavitation boundary and to locate the interface in such a position that the well-known Reynolds boundary condition is satisfied.
6.3 Load Capacity

Once the pressure distribution for the appropriate cavitation boundary conditions has been determined numerically, we can express the load capacity as

\[ F = \iint p \, dx \, dy \quad (6.12) \]

By making use of equation (6.11) we can write this equation in dimensionless form as

\[ F = 4 \eta_0 u R \lambda_b \iint \left( -\frac{X}{H^2} + Q \right) dx \, dy \quad (6.13) \]

The central film thickness can be isolated from the integrand by introducing the following transformations:

\[
\begin{cases}
X = X_t (2H_0)^{1/2} \\
Y = Y_t (2a_H H_0)^{1/2}
\end{cases} \quad (6.14)
\]

If we assume the homogeneous solution to transform in the same manner as the particular solution, we obtain

\[ F = 8 \lambda_b \eta_0 u R \left( \frac{2 a_a}{H_0} \right)^{1/2} \iint \left[ \frac{-X_t}{\left(1 + X_t^2 + Y_t^2\right)^2} + Q(X_t, Y_t) \right] dX_t \, dY_t \quad (6.15) \]
Kapitza (1955) refers to this integral as the reduced hydrodynamic lift \( L_t \). Thus

\[
L_t = \int \int \left[ \frac{-X_t}{(1 + X_t^2 + Y_t^2)^2} + Q(X_t, Y_t) \right] dX_t \, dY_t \tag{6.16}
\]

### 6.4 Film Thickness Formula

The reduced hydrodynamic lift was found by Kapitza (1955) to equal \( \pi/2 \) by assuming \( Q = 0 \) and integrating over the half-space of positive pressures. For the Reynolds boundary conditions the limits depend on the shape of the cavitated region and hence on the geometry. Consequently we seek an additional geometrical effect to modify and generalize Kapitza's solution. The central (minimum) film thickness can be expressed as a function of the load, speed, geometry, and fluid viscosity by rearranging equation (6.15) and writing

\[
H_{\text{min}} = H_0 = 128 \alpha_a \left( \frac{\lambda_b \rho u R_x}{F} L_t \right)^2 \tag{6.17}
\]

The ratio of dimensionless speed to dimensionless load can be defined as
\( \frac{U}{W} = -\frac{\rho u R}{F} \)  \hspace{1cm} (6.18)

and equation (6.17) becomes

\[ H_{\text{min}} = H_0 = 128 \alpha_a \left( \lambda_t L_t \frac{U}{W} \right)^2 \]  \hspace{1cm} (6.19)

The integrand of equation (6.16) thus becomes a function of the geometry represented by \( \alpha_a \) and the central film thickness \( H_0 \). Consequently \( L_t = L_t(\alpha_a, H_0) \), and this results in a transcendental equation for \( H_0 \).

It is necessary to determine \( L_t \) as a function of the geometry alone, or

\[ L_t = L_t(\alpha_a) \quad \text{if} \quad x_t^2 \ll \frac{1}{2H_0}; \quad y_t^2 \ll \frac{1}{2H_0} \]  \hspace{1cm} (6.20)

Once the hydrodynamic load-carrying capacity \( F \) has been obtained from the numerically determined pressure distribution, \( L_t \) can be evaluated for various geometries. A curve fit can then be used to show the effect of geometry on \( L_t \). The following relationship provides a good representation of the numerical results:

\[ L_t = 0.131 \tan^{-1} \left( \frac{\alpha_a}{2} \right) + 1.683 \]  \hspace{1cm} (6.21)

Inserting equation (6.21) into equation (6.19) gives the following general expression for minimum film thickness between any rigid ellipsoidal solids, ranging from two spheres in nominal...
point contact to infinitely long cylinders in nominal line contact:

\[
H_{\text{min}} = H_0 = 128 \alpha_a \left\{ \frac{\lambda_b U}{W} \left[ 0.131 \tan^{-1} \left( \frac{\alpha_b}{2} \right) + 1.683 \right] \right\}^2
\]  
(6.22)

6.5 Comparison Between Different Theories for the Lubrication of Rigid Ellipsoidal Solids

The minimum-film-thickness equation derived by Kapitza (1955) using the half-Sommerfeld cavitation boundary condition is

\[
(H_{\text{min}})_K = 128 \alpha_a \left( \frac{\pi \lambda_b U}{W} \right)^2
\]  
(6.23)

From equations (6.19) and (6.23) we can write

\[
\frac{(H_{\text{min}})_K}{H_{\text{min}}} = \left( \frac{\pi}{2L_L} \right)^2
\]  
(6.24)

By using equations (6.21) and (6.24) we find that the film thickness as obtained from equation (6.22) is 11 to 21 percent greater than that obtained from Kapitza's (1955) solution (equation (6.23)), with the smallest difference occurring for a ball-on-plane contact. The alteration of the pressure distribution due to the Reynolds cavitation boundary conditions is responsible for this influence of contact geometry on minimum film
thickness. These differences between the results of Kapitza (1955) and Brewe, et al. (1979) are illustrated graphically in Figure 6.2. For the entire range of the radius ratio $R_x/R_y$ the dimensionless film thickness is higher for the Brewe, et al. (1979) results than for the Kapitza (1955) results.

Both the Kapitza (1955) analysis and the numerical solutions of Brewe, et al. (1979) resulted in an exponent of 2 for $U/W$ in the dimensionless film thickness equation. Dalmaz and Godet (1973) also used the Reynolds boundary conditions for a ball-on-plane configuration, and they reported a comparable exponent of 1.77. This lower exponent has been discussed by Brewe, et al. (1979), and it appears to be due to starvation effects resulting from the inlet boundary condition for both the analytical and experimental (Dalmaz and Godet, 1973) situations. The sophisticated apparatus used by Dalmaz and Godet simultaneously measured the film thickness, traction, and speed between a steel ball and a glass plate in pure sliding. A 30-mm-diameter ball, which turned around one of its axes, was immersed in an oil bath and thus carried the lubricant into the contact area through viscous lifting. An optical interferometric technique was used to measure the oil film thickness.

Experimental results obtained by Dalmaz and Godet (1973) under lightly loaded, isoviscous conditions for pure sliding of a ball on a plane are compared with the theoretical results obtained from equation (6.22) in Figure 6.3. The grouping of
dimensionless parameters in this figure is the same as that used by Dalmaiz and Godet (1973) and was first introduced by Thorp and Gohar (1972). The theoretical results obtained from equation (6.22) are in excellent agreement with the experimental data for the lower range of $H_0/WG$, but at higher speeds the measured film thicknesses fall below the theoretical predictions, probably as a result of lubricant starvation in the inlet region. For comparison, predictions based on the theory of Kapitza (1955) and the calculations of Dalmaiz and Godet (1973) have also been included in Figure 6.3.

6.6 Pressure Distribution Between Ellipsoidal Solids

Three-dimensional contour plots of the pressure distribution for values of $a_a$ of 1.00 and 36.54 are presented in Figure 6.4. The shape of the cavitation boundary is clearly evident in this figure. As $a_a$ becomes large, the cavitation boundary tends to straighten out and is accompanied by decreasing changes in $L_t$. The scale along the $Y$ axis in Figure 6.4 has been magnified about three times to improve the resolution. Consequently the differences in the shapes of the cavitation boundary are actually subdued in Figure 6.4.

Isobar plots for three radius ratios $a_a$ of 25.29, 8.30, and 1.00 are shown in Figure 6.5. The center of contact is represented by an asterisk. The pressure peak builds up in the
entrance region, which is located to the left of the center of contact and is indicated by a cross. Since the isobars in each case are evenly spaced, the pressure gradients can be easily envisaged. Note that, as the radius ratio $a_d$ increases, the steeper pressure gradients are found predominantly in the rolling direction. This implies that the amount of side leakage decreases as $a_d$ increases. A decrease in side leakage is reflected in an increase in the value of $\lambda_b$. For nominal line contact $\lambda_b = 1$, and for the largest value of $a_d$ (25.29) recorded in Figure 6.5 the corresponding value of $\lambda_b$ is 0.974.

6.7 Closure

The influence of geometry on the hydrodynamic lubrication of rigid, ellipsoidal solids has been investigated in this chapter. The study has been restricted to conjunctions fully immersed in lubricant (i.e., fully flooded). The effect of geometry on film thickness was determined numerically by varying the radius ratio $a_d$ from 1 (a ball-on-plane or ball-on-ball configuration) to 36 (a ball in a conforming groove). Pressure-viscosity effects were not considered. It was found that the minimum film thickness had the same speed, viscosity, and load dependence as found by Kapitza in his classical analytical solution to the problem. When the Reynolds cavitation boundary con-
dition was incorporated in the analysis, an additional geometrical effect was introduced into the film thickness equation. The derived film thickness equations can be compared as follows:

Reynolds boundary condition:

\[ H_0 = 128 \alpha_a \left( \frac{\lambda_b U}{W} \left[ 0.131 \tan^{-1} \left( \frac{\alpha_a}{2} \right) + 1.683 \right] \right)^2 \]

Half-Sommerfeld boundary condition:

\[ H_0 = 128 \alpha_a \left( \frac{\lambda_b U \pi}{W} \right)^2 \]

With the Reynolds boundary condition the minimum film thickness has been found to be 11 to 21 percent greater than that obtained when using the half-Sommerfeld boundary condition adopted by Kapitza.
SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>constant used in equation (3.113)</td>
</tr>
<tr>
<td>A*, B*, C*</td>
<td>relaxation coefficients</td>
</tr>
<tr>
<td>D*, L*, M*</td>
<td>drug area of ball, m²</td>
</tr>
<tr>
<td>A_v</td>
<td>semimajor axis of contact ellipse, m</td>
</tr>
<tr>
<td>a</td>
<td>semimajor axis of contact ellipse, m</td>
</tr>
<tr>
<td>a̅</td>
<td>a/2m</td>
</tr>
<tr>
<td>B</td>
<td>total conformity of bearing</td>
</tr>
<tr>
<td>b</td>
<td>semiminor axis of contact ellipse, m</td>
</tr>
<tr>
<td>b̅</td>
<td>b/2m</td>
</tr>
<tr>
<td>C</td>
<td>dynamic load capacity, N</td>
</tr>
<tr>
<td>C_v</td>
<td>drag coefficient</td>
</tr>
<tr>
<td>C_1,...,C_B</td>
<td>constants</td>
</tr>
<tr>
<td>c</td>
<td>19,609 N/cm² (28,440 lbf/in²)</td>
</tr>
<tr>
<td>c̅</td>
<td>number of equal divisions of semimajor axis</td>
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<tr>
<td>D</td>
<td>distance between race curvature centers, m</td>
</tr>
<tr>
<td>D̅</td>
<td>material factor</td>
</tr>
<tr>
<td>D̅</td>
<td>defined by equation (5.63)</td>
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<tr>
<td>D_e</td>
<td>Deborah number</td>
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<tr>
<td>d</td>
<td>ball diameter, m</td>
</tr>
<tr>
<td>d̅</td>
<td>number of divisions in semiminor axis</td>
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<tr>
<td>d_o</td>
<td>overall diameter of bearing (Figure 2.13), m</td>
</tr>
<tr>
<td>d_b</td>
<td>bore diameter, m</td>
</tr>
<tr>
<td>d_e</td>
<td>pitch diameter, m</td>
</tr>
<tr>
<td>d_e̅</td>
<td>pitch diameter after dynamic effects have acted on ball, m</td>
</tr>
<tr>
<td>d_i</td>
<td>inner-race diameter, m</td>
</tr>
<tr>
<td>d_o</td>
<td>outer-race diameter, m</td>
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$E$  modulus of elasticity, N/m$^2$

$E'$ effective elastic modulus, $2\left(\frac{1 - \nu_a^2}{E_a} + \frac{1 - \nu_b^2}{E_b}\right)$, N/m$^2$

$E_a$ internal energy, m$^2$/s$^2$

$E_b$ processing factor

$E_1$ internal energy, m$^2$/s$^2$

$e_1$ elliptic integral of second kind with modulus $(1 - 1/k^2)^{1/2}$

$e$ approximate elliptic integral of second kind

$E_1$ dispersion exponent

$F$ normal applied load, N

$F^*$ normal applied load per unit length, N/m

$\tilde{F}$ lubrication factor

$\bar{F}$ integrated normal applied load, N

$F_c$ centrifugal force, N

$F_{\text{max}}$ maximum normal applied load (at $\psi = 0$), N

$F_r$ applied radial load, N

$F_t$ applied thrust load, N

$F_\psi$ normal applied load at angle $\psi$, N

$\mathcal{F}$ elliptic integral of first kind with modulus $(1 - 1/k^2)^{1/2}$

$\mathcal{F}$ approximate elliptic integral of first kind

$f$ race conformity ratio

$f_b$ rms surface finish of ball, m

$f_r$ rms surface finish of race, m

$G$ dimensionless materials parameter, $\nu E$

$G^*$ fluid shear modulus, N/m$^2$

$G^*$ hardness factor

$g$ gravitational constant, m/s$^2$
$g_E$ dimensionless elasticity parameter, $W^{2/3}/U^2$

$g_V$ dimensionless viscosity parameter, $Gh^2/U^2$

$H$ dimensionless film thickness, $h/R_x$

$A$ dimensionless film thickness, $H(W/U)^2 = F^2 h/u^2 n_0 R_x^3$

$H_c$ dimensionless central film thickness, $h_c/R_x$

$H_{c,s}$ dimensionless central film thickness for starved lubrication condition

$H_f$ frictional heat, N m/s

$H_{min}$ dimensionless minimum film thickness obtained from EHL elliptical-contact theory

$H_{min,r}$ dimensionless minimum film thickness for a rectangular contact

$H_{min,s}$ dimensionless minimum film thickness for starved lubrication condition

$H_c$ dimensionless central film thickness obtained from least-squares fit of data

$H_{min}$ dimensionless minimum film thickness obtained from least-squares fit of data

$H_{c}$ dimensionless central-film-thickness - speed parameter, $H_c U^{-0.5}$

$H_{min}$ dimensionless minimum-film-thickness - speed parameter, $H_{min} U^{-0.5}$

$H_0$ new estimate of constant in film thickness equation

$h$ film thickness, m

$h_c$ central film thickness, m

$h_i$ inlet film thickness, m
$h_m$  film thickness at point of maximum pressure, where $\frac{dp}{dx} = 0$, m

$h_{\text{min}}$  minimum film thickness, m

$h_0$  constant, m

$I_d$  diametral interference, m

$I_p$  ball mass moment of inertia, m N s$^2$

$I_t$  integral defined by equation (3.75)

$I_r$  integral defined by equation (3.76)

$J$  function of $k$ defined by equation (3.8)

$J^*$  mechanical equivalent of heat

$\bar{J}$  polar moment of inertia, m N s$^2$

$k$  load-deflection constant

$k$  ellipticity parameter, a/b

$\tilde{k}$  approximate ellipticity parameter

$k_f$  thermal conductivity, N/s °C

$K_f$  lubricant thermal conductivity, N/s °C

$L$  fatigue life

$L_a$  adjusted fatigue life

$L_t$  reduced hydrodynamic lift, from equation (6.21)

$L_1, ..., L_4$  lengths defined in Figure 3.11, m

$L_{10}$  fatigue life where 90 percent of bearing population will endure

$L_{50}$  fatigue life where 50 percent of bearing population will endure

$\lambda$  bearing length, m

$\bar{\lambda}$  constant used to determine width of side-leakage region

$M$  moment, Nm
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$M_g$</td>
<td>gyrosopic moment, Nm</td>
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<td>$M_p$</td>
<td>dimensionless load-speed parameter, $W^0.75$</td>
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<tr>
<td>$M_s$</td>
<td>torque required to produce spin, N m</td>
</tr>
<tr>
<td>$m$</td>
<td>mass of ball, $N s^2/m$</td>
</tr>
<tr>
<td>$m^*$</td>
<td>dimensionless inlet distance at boundary between fully flooded and starved conditions</td>
</tr>
<tr>
<td>$\tilde{m}$</td>
<td>dimensionless inlet distance (Figures 7.1 and 9.1)</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>number of divisions of semimajor or semiminor axis</td>
</tr>
<tr>
<td>$m_w$</td>
<td>dimensionless inlet distance boundary as obtained from Wedeven, et al. (1971)</td>
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<tr>
<td>$N$</td>
<td>rotational speed, rpm</td>
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<tr>
<td>$n$</td>
<td>number of balls</td>
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<tr>
<td>$n^*$</td>
<td>refractive index</td>
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<td>$\bar{n}$</td>
<td>constant used to determine length of outlet region</td>
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<td>$P$</td>
<td>dimensionless pressure</td>
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<tr>
<td>$P_{D}$</td>
<td>dimensionless pressure difference</td>
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<td>diametral clearance, m</td>
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<td>maximum pressure within contact, $3F/2wab$, $N/m^2$</td>
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<td>$Q$</td>
<td>solution to homogeneous Reynolds equation</td>
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<tr>
<td>$Q_{m}$</td>
<td>thermal loading parameter</td>
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<td>$Q_{\bar{Q}}$</td>
<td>dimensionless mass flow rate per unit width, $q_{n0}/\rho_0E'R^2$</td>
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<td>$q_f$</td>
<td>reduced pressure parameter</td>
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<td>$q_x$</td>
<td>volume flow rate per unit width in $x$ direction, $m^2/s$</td>
</tr>
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\( q_y \) volume flow rate per unit width in \( y \) direction, \( m^2/s \)

\( R \) curvature sum, \( m \)

\( R_a \) arithmetical mean deviation defined in equation (4.1), \( m \)

\( R_c \) operational hardness of bearing material

\( R_x \) effective radius in \( x \) direction, \( m \)

\( R_y \) effective radius in \( y \) direction, \( m \)

\( r \) race curvature radius, \( m \)

\( r_{ax}, r_{bx} \) radii of curvature, \( m \)

\( r_{ay}, r_{by} \) radii of curvature, \( m \)

\( r_c, \phi_c, z \) cylindrical polar coordinates

\( r_s, \theta_s, \phi_s \) spherical polar coordinates

\( \bar{r} \) defined in Figure 5.4

\( S \) geometric separation, \( m \)

\( S^* \) geometric separation for line contact, \( m \)

\( S_0 \) empirical constant

\( s \) shoulder height, \( m \)

\( T \) \( \tau_0/p_{max} \)

\( \tilde{T} \) tangential (traction) force, \( N \)

\( T_m \) temperature, \( °C \)

\( T_b^* \) ball surface temperature, \( °C \)

\( T_f^* \) average lubricant temperature, \( °C \)

\( \Delta T^* \) ball surface temperature rise, \( °C \)

\( T_1 \) \( (\tau_0/p_{max})_{k=1} \)

\( T_v \) viscous drag force, \( N \)

\( t \) time, \( s \)

\( t_a \) auxiliary parameter

\( u_B \) velocity of ball-race contact, \( m/s \)
velocity of ball center, m/s

dimensionless speed parameter, $\eta_0 U / E^* R_x$

surface velocity in direction of motion, $(u_a + u_b)/2$, m/s

number of stress cycles per revolution

sliding velocity, $u_a - u_b$, m/s

surface velocity in transverse direction, m/s

dimensionless load parameter, $F / E^* R^2$

surface velocity in direction of film, m/s

dimensionless coordinate, $x/R_x$

dimensionless coordinate, $y/R_x$

dimensionless grouping from equation (6.14)

external forces, N

constant defined by equation (3.48)

viscosity pressure index, a dimensionless constant

coordinate system

pressure-viscosity coefficient of lubrication, m$^2$/N

radius ratio, $R_y / R_x$

contact angle, rad

free or initial contact angle, rad

iterated value of contact angle, rad

curvature difference

viscous dissipation, N/m$^2$ s

total strain rate, s$^{-1}$

elastic strain rate, s$^{-1}$

viscous strain rate, s$^{-1}$
\( \gamma_a \) flow angle, deg
\( \delta \) total elastic deformation, m
\( \delta^* \) lubricant viscosity temperature coefficient, \(^\circ\text{C}^{-1}\)
\( \delta_D \) elastic deformation due to pressure difference, m
\( \delta_r \) radial displacement, m
\( \delta_t \) axial displacement, m
\( \delta_x \) displacement at some location \( x \), m
\( \delta_e \) approximate elastic deformation, m
\( \delta_k \) elastic deformation of rectangular area, m
\( \varepsilon \) coefficient of determination
\( \varepsilon_1 \) strain in axial direction
\( \varepsilon_2 \) strain in transverse direction
\( \zeta \) angle between ball rotational axis and bearing centerline (Figure 3.10)
\( \zeta_a \) probability of survival
\( \eta \) absolute viscosity at gauge pressure, N s/m\(^2\)
\( \overline{\eta} \) dimensionless viscosity, \( \eta/\eta_0 \)
\( \eta_0 \) viscosity at atmospheric pressure, N s/m\(^2\)
\( \eta_\infty \) 6.31\times10^{-5} \text{ N s/m}^2 (0.0631 \text{ cP})
\( \phi \) angle used to define shoulder height
\( \Lambda \) film parameter (ratio of film thickness to composite surface roughness)
\( \lambda \) equals 1 for outer-race control and 0 for inner-race control
\( \lambda_a \) second coefficient of viscosity
\( \lambda_b \) Archard-Cowking side-leakage factor, \((1 + 2/3 \alpha_a)^{-1}\)
\( \lambda_c \) relaxation factor
\( \mu \)  
coefficient of sliding friction

\( \mu^* \)

Poisson's ratio

\( v \)
divergence of velocity vector, \((au/ax) + (av/ay) + (aw/az)\), \(s^{-1}\)

\( \rho \)
lubricant density, \(N \, s^2/m^4\)

\( \bar{\rho} \)
dimensionless density, \(\rho/\rho_0\)

\( \rho_0 \)
density at atmospheric pressure, \(N \, s^2/m^4\)

\( \sigma \)
normal stress, \(N/m^2\)

\( \sigma_1 \)
stress in axial direction, \(N/m^2\)

\( \tau \)
shear stress, \(N/m^2\)

\( \tau_0 \)
maximum subsurface shear stress, \(N/m^2\)

\( \tau_e \)
shear stress, \(N/m^2\)

\( \tau_L \)
equivalent stress, \(N/m^2\)

\( \phi \)
limiting shear stress, \(N/m^2\)

\( \phi^* \)

ratio of depth of maximum shear stress to semiminor axis of contact ellipse

\( \phi_{1} \)

(\(\phi\))_{k=1}

\( \phi \)
auxiliary angle

\( \phi_T \)
thermal reduction factor

\( \psi \)
angular location

\( \psi_L \)
limiting value of \(\psi\)

\( \Omega_i \)
absolute angular velocity of inner race, rad/s

\( \Omega_o \)
absolute angular velocity of outer race, rad/s

\( \omega \)
angular velocity, rad/s

\( \omega_B \)
angular velocity of ball-race contact, rad/s

\( \omega_B \)
angular velocity of ball about its own center, rad/s
\( \omega_c \)  
angular velocity of ball around shaft center, rad/s  

\( \omega_s \)  
ball spin rotational velocity, rad/s  

Subscripts:
- a: solid a  
- b: solid b  
- c: central  
- bc: ball center  
- IE: isoviscous-elastic regime  
- IR: isoviscous-rigid regime  
- i: inner race  
- K: Kapitza  
- min: minimum  
- n: iteration  
- o: outer race  
- PVE: piezoviscous-elastic regime  
- PVR: piezoviscous-rigid regime  
- r: for rectangular area  
- s: for starved conditions  
- x,y,z: coordinate system  

Superscript:
- (\( \approx \)): approximate
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Figure 6.1 - Variable nodal structure used for numerical calculations.

Figure 6.2 - Film thickness results of Kapitza (1955) and Brewe, et al. (1979) for range of $R_d/R_y$ and $U_W = 1 \times 10^{-3}$. 
Figure 6.3. - Theoretical and experimental results.
Figure 6.4 - Three-dimensional representations of pressure distributions as viewed from exit region, illustrating cavitation boundary.
Figure 6.5. - Pressure contours for three radius ratios.