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ON THE POWER OUTPUT OF SOME IDEALIZED SOURCE CONFIGURATIONS WITH ONE OR MORE CHARACTERISTIC DIMENSIONS

Harold Levine

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HAROLD LEVINE

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ON THE POWER OUTPUT OF SOME IDEALIZED SOURCE CONFIGURATIONS WITH ONE OR MORE CHARACTERISTIC DIMENSIONS

by

Harold Levine

§1. Introduction

Any roster of theoretical sound wave generators includes the discrete point source linear array, a continuous finite-size linear aggregate, the plane circular piston or annulus and a plane rectangular piston; for the far field directivity pattern of all these configurations, as well as their individual power outputs (a function of the characteristic dimension(s) and wave length, in time-periodic regimes), can be accurately determined if each source distribution possesses a uniform strength and the pistons are flush with a rigid baffle. The abstract nature of source descriptions which postulate a uniformity of motion (e.g., a surface or volume displacement) is evident, although attempts to impose more realistic boundary conditions (while retaining the fiction of an ideally rigid vibrating surface) encounter considerable mathematical difficulties. The radiation problem becomes well nigh intractable, moreover, for real material surfaces whose actual motions cannot be assigned in advance. Such difficulties are left out of consideration in what follows, where the aim is merely to extend and refine the prior analysis of some model sources.

The calculation of power output from a (finite) linear array of equidistant point sources is investigated in §2, with allowance for a relative phase shift and particular focus on the circumstances of small/large individual source separation. A key role is played herein by the estimates found for a

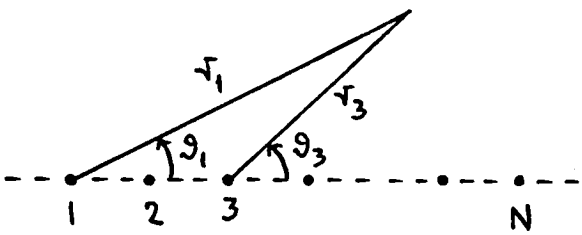
twin-parameter definite integral that involves the Fejer kernel functions, viz.

$$I(a;N) = \int_0^a \left(\frac{\sin Nx}{\sin x}\right)^2 dx$$

where N denotes a (positive) integer; these results also permit a quantitative accounting of energy partition between the principal and secondary lobes of the array pattern. Succeeding sections §3, 4 are concerned with continuously distributed sources along a finite line segment or an open-ended circular cylindrical shell; and estimates for the relatively lower output in the latter configuration are made explicit when the shell radius is small compared to the wave length. The last section, §5, furthers a systematic reduction of divers integrals which characterize the energy output from specific line and strip sources.

§2. A Linear Array of Point Sources

Consider an ensemble of N point sources which are uniformly spaced, at the distance d apart, along a straight line; and assume that the corresponding excitation potentials referred to coordinate origins at the individual source locations,



$$\phi_n = \frac{Q_0}{4\pi r_n} e^{ikr_n} (e^{-i\omega t}), \quad n=1,\ldots,N$$

involve a common strength Q_0 (the maximum rate of volume displacement). The time average radiation intensity for each isolated source is then given by

$$S_n = \frac{1}{2}\rho\omega k \left(\frac{Q_0}{4\pi}\right)^2 \frac{1}{r_n^2}$$

(where ρ specifies the equilibrium density of the ambient medium) and the intensity for the (non-interacting) aggregate has the representation

$$S_N = \frac{1}{2}\rho\omega k \left(\frac{Q_0}{4\pi}\right)^2 \frac{1}{r^2} \left| \sum_{n=1}^{N} A_n e^{i(n-1)kd\cos\vartheta} \right|^2, \quad \vartheta=\vartheta_1, \quad r=r_1 \gg Nd \tag{1}$$

where the factors

$$A_1 = 1, \quad A_2 = e^{i\alpha kd}, \quad A_3 = e^{2i\alpha kd}, \quad \ldots, \quad A_n = e^{i(n-1)\alpha kd} \tag{2}$$

$$\ldots$$

incorporate a fixed phase advance α kd between neighboring and consecutive sources. After the summation in (1) is effected, it follows that

$$S_N = \frac{1}{2}\rho\omega k\left(\frac{Q_0}{4\pi}\right)^2 \frac{1}{r^2}\left(\frac{\sin\frac{Nkd}{2}(\cos\vartheta-\alpha)}{\sin\frac{kd}{2}(\cos\vartheta-\alpha)}\right)^2 \tag{3}$$

and thus the total radiated power assumes the form

$$P = \frac{1}{2}\rho\omega k\left(\frac{Q_0}{4\pi}\right)^2 \cdot 2\pi \int_0^{\pi}\left(\frac{\sin\frac{Nkd}{2}(\cos\vartheta-\alpha)}{\sin\frac{kd}{2}(\cos\vartheta-\alpha)}\right)^2 \sin\vartheta\, d\vartheta$$

$$= C\int_{-\frac{kd}{2}(1+\alpha)}^{\frac{kd}{2}(1-\alpha)}\left(\frac{\sin Nx}{\sin x}\right)^2 dx, \qquad C = \frac{\rho\omega Q_0^2}{8\pi d}. \tag{4}$$

Thiessen and Embleton (1958) cite the latter measure in connection with a proposal to simulate multiple siren noise from suction rolls at a paper mill in terms of a phase-advancing linear source array. An approximate integration scheme suggested by them employs a subdivision of the range and an average value for the rapidly varying function $\sin^2 Nx$, $n \gg 1$, yielding the estimate

$$P/C = 3.07\,N - \frac{1}{2}\left[\cot\frac{kd}{2}(1+\alpha) + \cot\frac{kd}{2}(1-\alpha)\right] \tag{5}$$

on the proviso that α does "not deviate by more than $1 - \frac{\pi}{Nkd}$ from the values $2m\pi/kd$, where m is an integer." In point of fact, the numerical

coefficient 3.07 should be replaced by π and the remaining terms of (5) have a questionable status.

To study the above (power) integral without an initial manner of approximation, it is observed that

$$I(a;N) = \int_0^a \left(\frac{\sin Nx}{\sin x}\right)^2 dx$$

$$= \int_0^a (\sin Nx)^2 \, d\left(-\frac{\cos x}{\sin x}\right)$$

$$= -\frac{\cos a}{\sin a}\sin^2 Na + \frac{N}{2}\int_0^a \frac{\sin(2N+1)x + \sin(2N-1)x}{\sin x}\,dx \tag{6}$$

through an integration by parts. On next utilizing the relation (familiar from the convergence theory of Fourier series)

$$\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} = 1 + 2\sum_{n=1}^{N}\cos nx$$

an exact result follows, namely

$$I(a;N) = -\frac{\cos a}{\sin a}\sin^2 Na + Na + N\sum_{n=1}^{N}\frac{\sin 2na}{n} - \frac{1}{2}\sin 2Na$$

$$= Na + N\sum_{n=1}^{N}\frac{\sin 2na}{n} - \sin Na\,\frac{\sin(N+1)a}{\sin a}. \tag{7}$$

The latter permits a ready determination of I if N and the number of terms in the sum is small; when N is large, on the other hand, the rearrangement

$$\sum_{n=1}^{N} \frac{\sin n\zeta}{n} = \sum_{n=1}^{\infty} \frac{\sin n\zeta}{n} - \sum_{n=N+1}^{\infty} \frac{\sin n\zeta}{n} \tag{8}$$

may be noted, in view of the well-known Fourier series expansion

$$\sum_{n=1}^{\infty} \frac{\sin n\zeta}{n} = F(\zeta) \tag{9}$$

whose sum is an odd (2π-periodic) function, namely

$$F(\zeta) = \frac{\pi - (\zeta - 2m\pi)}{2}, \qquad 2m\pi < \zeta < 2(m+1)\pi \tag{10}$$

and the remainder estimate

$$\sum_{n=N+1}^{\infty} \frac{\sin n\zeta}{n} \simeq \int_{N}^{\infty} \frac{\sin \zeta x}{x}\, dx = \int_{N\zeta}^{\infty} \frac{\sin \tau}{\tau}\, d\tau$$

$$= \frac{\pi}{2} - Si\left\{N(\zeta - 2m\pi)\right\}, \quad N \gg 1, \ |\zeta - 2m\pi| \ll 1 \tag{11}$$

that involves the sine integral function

$$Si(x) = \int_0^x \frac{\sin\tau}{\tau}\,d\tau. \tag{12}$$

In accordance with (7) - (11),

$$\begin{aligned} I(a;N) &= Na + N\sum_{n=1}^{N} \frac{\sin 2na}{n} - \frac{\sin^2 Na}{\sin a}, \quad N \gg 1 \\ &= Na + N\left(\frac{\pi - 2a}{2}\right) - N\left\{\frac{\pi}{2} - Si(2Na)\right\} - \frac{\sin^2 Na}{\sin a}, \quad a \ll 1 \\ &= N\,Si(2Na) - \frac{\sin^2 Na}{a}, \quad N \gg 1,\ a \ll 1. \end{aligned} \tag{13}$$

There is a ready usage for the preceding result in connection with the directivity gain of the considered source array, $g(\alpha)$, defined as the ratio of the maximum far field intensity (at the angle $\theta = \cos^{-1}\alpha$ from its reference line) to the total radiated power/4π ; thus

$$g(\alpha) = \frac{kdN^2}{I\left\{\frac{kd}{2}(1-\alpha);N\right\} + I\left\{\frac{kd}{2}(1+\alpha);N\right\}} \tag{14}$$

and

$$g(1) = \frac{kdN^2}{I(kd;N)}$$

$$= \frac{Nkd}{Si(2Nkd) - \frac{\sin^2 Nkd}{Nkd}}, \quad N \gg 1, \; kd \ll 1 \tag{15}$$

for the specific phase parameter value $\alpha = 1$, on the basis of (13). A passage to the limits $N \to \infty$, $d \to 0$ wherein $Nd \to \ell$ yields the gain function

$$g = \frac{k\ell}{Si(2k\ell) - \frac{\sin^2 k\ell}{k\ell}} \tag{16}$$

which, as verified in §3, befits a uniform line source distribution with a continuous linear phase variation (kx) along its length (ℓ) . If $\alpha = 0$ the corresponding gain function is secured through the mere replacement of k by k/2 and its limiting version is appropriate to a continuous line of in-phase sources with overall span ℓ .

Analogous estimates for values of the phase parameter α between 0 and 1 may be given in the circumstance that $kd \ll 1$ or

$$kd(1-\alpha) = 2m\pi + \frac{\pi}{N}$$

and

$$kd(1+\alpha) = 2M\pi + \delta\pi, \quad \delta \ll 1;$$

in particular,

$$I\left\{\frac{kd}{2}(1-\alpha);N\right\}+I\left\{\frac{kd}{2}(1+\alpha);N\right\}$$

$$=N\pi-N\left\{\pi-Si\left\{Nkd(1-\alpha)\right\}-Si\left\{Nkd(1+\alpha)\right\}\right\}$$

$$-\frac{\sin^2\left\{\frac{Nkd}{2}(1-\alpha)\right\}}{\frac{kd}{2}(1-\alpha)}-\frac{\sin^2\left\{\frac{Nkd}{2}(1+\alpha)\right\}}{\frac{kd}{2}(1+\alpha)},\qquad \begin{matrix} N\gg 1 \\ kd\ll 1\end{matrix} \tag{17}$$

where the first term predominates when the arguments of the sine integrals are large enough to justify the estimate $Si(x)\doteq \pi/2$, $x \gg 1$. The gain function inferred from (14) and (17) in the continuum limit $d \to 0$, $Nd \to \ell$, viz.

$$g(\alpha)=\frac{k\ell}{Si\left\{k\ell(1-\alpha)\right\}+Si\left\{k\ell(1+\alpha)\right\}-\frac{\sin^2\left\{\frac{k\ell}{2}(1-\alpha)\right\}}{\frac{k\ell}{2}(1-\alpha)}-\frac{\sin^2\left\{\frac{k\ell}{2}(1+\alpha)\right\}}{\frac{k\ell}{2}(1+\alpha)}}, \tag{18}$$

extends the prior result (16).

Since a sharp estimate is lacking for the trigonometric sum that appears in the representation (7) of the (power) integral (6) an alternative analysis of the latter becomes warranted. A relation from the theory of Fejer sums of Fourier series has utility in this regard, namely

$$I\left(\frac{\pi}{2};N\right)=\frac{1}{2}I(\pi;N)=\frac{1}{4}I(2\pi;N)=\frac{\pi}{2}N, \tag{19}$$

as do the modified versions

$$I(2m\pi; N) = m\, I(2\pi; N) = 2m\pi N$$

and

$$I\{(2m+1)\pi; N\} = (m+\tfrac{1}{2})\, I(2\pi; N) = (2m+1)\pi N \tag{20}$$

for integer values of m. If, then,

$$0 < a < \pi$$

the rearrangement

$$I(a; N) = \int_0^a \left(\frac{\sin Nx}{\sin x}\right)^2 dx$$

$$= \int_0^a \left(\frac{\sin Nx}{x}\right)^2 dx + \int_0^a \sin^2 Nx \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) dx$$

$$= \int_0^a \left(\frac{\sin Nx}{x}\right)^2 dx + \frac{1}{2}\int_0^a \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) dx - \frac{1}{2}\int_0^a \cos 2Nx \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) dx$$

proves advantageous, inasmuch as the first two integrals are directly reducible while the third can be suitably expanded in a straightforward manner for large values of N; thus

$$\int_0^a \left(\frac{\sin Nx}{x}\right)^2 dx = \int_0^a \sin^2 Nx\, d\left(-\frac{1}{x}\right) = -\frac{\sin^2 Na}{a} + N\int_0^a \frac{\sin 2Nx}{x} dx$$

$$= N\, Si\,(2Na) - \frac{\sin^2 Na}{a}, \tag{21}$$

and

$$\int_0^a \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) dx = \int_0^a d\left(-\frac{\cos x}{\sin x} + \frac{1}{x} \right) = \frac{1}{a} - \frac{\cos a}{\sin a}$$

which yields the representation

$$I(a;N) = N\,\mathrm{Si}(2Na) - \frac{\sin^2 Na}{a} + \frac{1}{a} - \frac{\cos a}{\sin a}$$

$$- \frac{1}{2} \int_0^a \cos 2Nx \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) dx. \tag{22}$$

Evidently the initial pair of terms in (22) duplicates those which make up the prior estimate (13), while the remaining terms are collectively small if $N \gg 1$ and $a \ll 1$. Because the function $\csc^2 x - \frac{1}{x^2}$ is analytic on $0 \leq x \leq a < \pi$ it is permissible to integrate by parts any number of times in the last integral, thereby generating an expansion with reciprocal powers of N, i.e.,

$$\int_0^a \cos 2Nx \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) dx$$

$$= \frac{\sin 2Na}{2N} \left(\csc^2 a - \frac{1}{a^2} \right) - \frac{1}{2N} \int_0^a \sin 2Nx \frac{d}{dx} \left(\csc^2 x - \frac{1}{x^2} \right) dx$$

$$= \frac{\sin 2Na}{2N} \left(\csc^2 a - \frac{1}{a^2} \right) + \frac{1}{2N^2} \cos 2Na \left(\frac{1}{a^3} - \cot a \csc^2 a \right)$$

$$- \frac{1}{4N^2} \int_0^a \cos 2Nx \frac{d^2}{dx^2} \left(\csc^2 x - \frac{1}{x^2} \right) dx, \tag{23}$$

etc.

Suppose, next, that

$$a = 2n\pi + b, \quad |b| < \pi;$$

then, recalling (20),

$$I(a;N) = 2n\pi N + I(b;N) \tag{24}$$

where the second term, $I(b;N)$, is describable in the manner previously detailed. Thus, given the configurational parameter

$$kd = 2n\pi - \delta, \qquad 0 < \delta < \pi$$

an estimate

$$I(kd;N) = 2n\pi N - N\,Si(2N\delta) + \frac{\sin^2 N\delta}{\delta} - \frac{1}{\delta} + \frac{\cos\delta}{\sin\delta}$$

$$+ \frac{\sin 2N\delta}{4N}\left(\csc^2\delta - \frac{1}{\delta^2}\right) + O\left(\frac{1}{N^2}\right)$$

follows; and the related gain in the limit $kd \gg 1$ $(n \gg 1)$, viz.

$$g(1) = \frac{kdN^2}{I(kd;N)} \simeq N\left\{1 + O\left(\frac{1}{n}, \frac{1}{N}\right)\right\}, \quad N, n \gg 1$$

corresponds, as it should, to independent contributions from each of the N sources.

Once the integral (6) has been suitably determined (as a function of the limit a) it becomes possible to specify both the whole power output as well as the lesser amounts radiated into different angular subdomains of

space. For instance, if there is no relative phase advance between sources of the array ($\alpha = 0$ in (4)) the output can be written as

$$P = \rho \frac{\omega^2 Q_0^2}{8\pi c} \left\{ \frac{I(\frac{kd}{2}; N)}{\frac{kd}{2}} \right\}$$

(where $c = \omega/k$ designates the speed of sound) and thus, recalling the particular integrals (20),

$$P = \rho \frac{\omega^2 Q_0^2}{8\pi c} N, \quad kd = n\pi \quad n = 1, 2, \ldots \tag{25}$$

When

$$kd = 2n\pi - \delta, \quad \delta < \pi$$

and use is made of (22) there obtains

$$I(n\pi - \tfrac{\delta}{2}; N) = I(n\pi; N) - I(\tfrac{\delta}{2}; N)$$

$$= n\pi N - N\,Si(N\delta) + \frac{\sin^2(N\delta/2)}{\delta/2} + \cdots, \quad N \gg 1$$

together with the total power estimate

$$P = \rho \frac{\omega^2 Q_0^2}{8\pi c} \frac{1}{n\pi - \frac{\delta}{2}} \left\{ n\pi N - N\,Si(N\delta) - \frac{\sin^2(N\delta/2)}{\delta/2} + \cdots \right\}, \quad N \gg 1.$$

Similarly, when

$$kd = (2n+1)\pi - \delta, \qquad \delta < \pi$$

it follows that

$$I\left\{(n+\tfrac{1}{2})\pi - \tfrac{\delta}{2}; N\right\} = I\left\{(n+\tfrac{1}{2})\pi; N\right\} - I\left(\tfrac{\pi}{2}; N\right) + I\left(\tfrac{\pi-\delta}{2}; N\right)$$

$$= n\pi N + N\,Si\left\{N(\pi-\delta)\right\} - \frac{\sin^2\left(N\frac{\pi-\delta}{2}\right)}{\frac{\pi-\delta}{2}} + \cdots, \quad N \gg 1$$

and

$$P = \rho\,\frac{\omega^2 Q_0^2}{8\pi c}\,\frac{1}{(n+\frac{1}{2})\pi - \frac{\delta}{2}}\left\{n\pi N + N\,Si\left\{N(\pi-\delta)\right\} - \frac{\sin^2\left(N\frac{\pi-\delta}{2}\right)}{\frac{\pi-\delta}{2}} + \cdots\right\},$$

$$N \gg 1.$$

With the current stipulation regarding source phases ($\alpha = 0$) the array is termed a broadside one, for (cf. (3)) its maximal power output appears at right angles to the line of support ($\theta = \pi/2$) ; and since the null directions are specified by the relation

$$\cos\vartheta = \pm\,\frac{2p\pi}{Nkd}, \qquad p = 1, 2, \ldots \tag{26}$$

the angular range of the major lobe (m.l.) is confined within the sector

$$\frac{\pi}{2} - \frac{2\pi}{Nkd} < \vartheta < \frac{\pi}{2} + \frac{2\pi}{Nkd}.$$

Accordingly, the energy radiated into the major lobe has the magnitude

$$P_{m.l.} = \rho \frac{\omega k Q_0^2}{16\pi} \int_{\frac{\pi}{2}-\frac{2\pi}{Nkd}}^{\frac{\pi}{2}+\frac{2\pi}{Nkd}} \left(\frac{\sin\left(\frac{Nkd}{2}\cos\vartheta\right)}{\sin\left(\frac{kd}{2}\cos\vartheta\right)} \right)^2 \sin\vartheta \, d\vartheta$$

$$= \rho \frac{\omega k Q_0^2}{8\pi} \int_{\frac{\pi}{2}-\frac{2\pi}{Nkd}}^{\frac{\pi}{2}} \left(\frac{\sin\left(\frac{Nkd}{2}\cos\vartheta\right)}{\sin\left(\frac{kd}{2}\cos\vartheta\right)} \right)^2 \sin\vartheta \, d\vartheta$$

$$= \rho \frac{\omega^2 Q_0^2}{8\pi c} \frac{I\left(\frac{\pi}{N}; N\right)}{\frac{kd}{2}}$$

$$= \rho \frac{\omega^2 Q_0^2}{8\pi c} \left\{ \frac{2}{kd}\left(N \, Si(2\pi) + \frac{\pi}{3N} + \cdots \right) \right\}, \quad N \gg 1$$

which decreases as kd becomes larger. If $kd = \pi$, i.e., the individual source separation equals a half wave length, the comparative ratio

$$\frac{P_{m.l.}}{P_{total}} = \frac{2}{\pi} Si(2\pi), \quad N \gg 1 \tag{27}$$

has a value slightly less than unity, inasmuch as the asymptotic development

$$Si(x) \simeq \frac{\pi}{2} - \frac{\cos x}{x} - \frac{\sin x}{x^2} + \cdots, \quad x \gg 1 \tag{28}$$

implies that

$$\frac{2}{\pi} Si(2\pi) \simeq 1 - \frac{1}{\pi^2} \doteq 0.89$$

When kd = 2π and the wave length is precisely equal to the separation between neighboring sources of the array a reduced value for the ratio (26), namely

$$\frac{P_{m.l.}}{P_{total}} \doteq \frac{1}{\pi} Si(2\pi), \qquad N \gg 1 \tag{29}$$

makes clear the existence of significant radiation outside the major lobe. Secondary maxima are found by means of the condition

$$\cos\vartheta = \pm \frac{(2p+1)\pi}{Nkd}, \quad p = 0, 1, \ldots \tag{30}$$

(that corresponds with unit absolute magnitude of the factor $\sin(\frac{Nkd}{2} \cos\theta)$ in (3)), and for large N the particular determinations $\theta = 0, \pi$ ($p \doteq N$) follows. The null directions closest to the line of the array satisfy the equation

$$\cos\vartheta = \pm \frac{N-1}{N} = \pm\left(1 - \frac{1}{N}\right)$$

which follows from (26) on choosing $p = N$ and yields the explicit values

$$\vartheta = \sqrt{2/N}$$

and

$$\vartheta = \pi - \sqrt{2/N}$$

respectively. Thus, the total energy radiated (equally) into the pair of secondary lobes whose axial directions are aligned with that of the array has the magnitude

$$P_{s.l.} = \rho \frac{\omega^2 Q_0^2}{8\pi c} \int_0^{\sqrt{2/N}} \left(\frac{\sin(N\pi \cos\vartheta)}{\sin(\pi \cos\vartheta)} \right)^2 \sin\vartheta \, d\vartheta$$

$$= \rho \frac{\omega^2 Q_0^2}{8\pi c} \left\{ \frac{N}{\pi} Si(2\pi) + \cdots \right\}, \quad N \gg 1, \; kd = 2\pi$$

and the ratio

$$\frac{P_{s.l.}}{P_{total}} \doteq \frac{1}{\pi} Si(2\pi) \tag{31}$$

implies (after recalling (29)) a like contribution to the output from the major and secondary lobes.

An array whose radiated intensity peaks along the source line $(\theta = 0)$ is termed end-on or end-fire; and, since the direction of maximum far field amplitude is given by $\cos\theta = \alpha$, it appears that the specific value $\alpha = 1$ (with relative phase advance kd between neighboring sources) characterizes such an array. The half-angle of the major lobe can be deduced from the relation

$$\frac{Nkd}{2}(1 - \cos\vartheta) = \pi$$

with the result that

$$\vartheta \doteq 2\sqrt{\frac{\pi}{Nkd}}, \quad N \gg 1$$

and thus the power which leaves the array in this angular range equals

$$P_{m.l.} = \rho \frac{\omega^2 Q_0^2}{16\pi c} \int_0^{2\sqrt{\pi/Nkd}} \left(\frac{\sin\left(\frac{Nkd}{2}(1-\cos\vartheta)\right)}{\sin\left(\frac{kd}{2}(1-\cos\vartheta)\right)} \right)^2 \sin\vartheta \, d\vartheta$$

$$= \rho \frac{\omega^2 Q_0^2}{8\pi c} \frac{N \, Si(2\pi)}{kd}, \quad N \gg 1.$$

A simple calculation establishes

$$P = \rho \frac{\omega^2 Q_0^2}{8\pi c} \frac{I(kd; N)}{kd}$$

as the total output from this array, whence

$$\frac{P_{m.l.}}{P_{total}} \doteq \frac{N \, Si(2\pi)}{I(kd; N)}, \qquad N \gg 1$$

with the particular values $\frac{2}{\pi} Si(2\pi)$, $\frac{1}{\pi} Si(2\pi)$, and $\frac{1}{2\pi} Si(2\pi)$ if $kd = \pi/2$, π and 2π , respectively. Evidently, the relative importance of the major lobe radiation declines after the distance between adjacent sources exceeds a greater wave length; by contrast, a half wave length provides the corresponding reference scale for the broadside array.

§3. A Continuous Line Source Distribution

If simple sources are continuously distributed along a finite line segment, with uniform magnitude and a linear phase variation, according to the density function

$$\begin{aligned} q(\underline{r},t) &= \mathrm{Re}\left\{ q(\underline{r})\, e^{-i\omega t} \right\} \\ &= \mathrm{Re}\left\{ q_0\, e^{ik\alpha x}\, \delta(y)\delta(z)\, e^{-i\omega t} \right\}, \quad -\frac{\ell}{2} < x < \frac{\ell}{2} \end{aligned} \tag{32}$$

the complex space factor $\phi(\underline{r})$ of the resultant velocity potential satisfies an inhomogeneous equation

$$\begin{aligned} (\nabla^2 + k^2)\,\phi(\underline{r}) &= -\,q(\underline{r}) \\ &= -\,q_0\, e^{ik\alpha x}\, \delta(y)\delta(z), \quad |x| < \ell/2 \\ &= 0, \quad |x| > \ell/2 \end{aligned} \tag{33}$$

and the companion acoustic (over) pressure $p(\underline{r})$ is given by

$$p(\underline{r}) = -i\rho\omega\,\phi(\underline{r}). \tag{34}$$

The outgoing wave solution of (33),

$$\begin{aligned} \phi(x,y,z) &= \frac{1}{4\pi}\int \frac{e^{ik\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, q(x',y',z')\,dx'\,dy'\,dz' \\ &= \frac{q_0}{4\pi}\int_{-\ell/2}^{\ell/2} \frac{e^{ik\sqrt{(x-x')^2+y^2+z^2}}}{\sqrt{(x-x')^2+y^2+z^2}}\, e^{ik\alpha x'}\,dx', \end{aligned} \tag{35}$$

has an asymptotic form

$$\phi \sim \frac{q_0}{4\pi}\frac{e^{ikr}}{r}\int_{-\ell/2}^{\ell/2} e^{ik(\alpha-\cos\vartheta)x'}\,dx', \qquad r=\sqrt{x^2+y^2+z^2}\to\infty, \quad x = r\cos\vartheta$$

$$= \frac{q_0}{4\pi}\frac{e^{ikr}}{r}\cdot 2\,\frac{\sin\left\{\frac{k\ell}{2}(\alpha-\cos\vartheta)\right\}}{k(\alpha-\cos\vartheta)} \tag{36}$$

which underlies expressions for both the far field intensity S and the power output P , namely

$$S = \frac{\omega\rho}{2}\,\mathrm{Im}\left(\phi^*\frac{\partial\phi}{\partial r}\right) \simeq \frac{\rho\omega}{2}\left(\frac{q_0}{4\pi}\right)^2\frac{k\ell^2}{r^2}\left|\frac{\sin\left\{\frac{k\ell}{2}(\cos\vartheta-\alpha)\right\}}{\frac{k\ell}{2}(\cos\vartheta-\alpha)}\right|^2 \tag{37}$$

and

$$P = 2\pi\int_0^{\pi} r^2 S\sin\vartheta\,d\vartheta = \frac{\rho\omega\ell q_0^2}{8\pi}\int_{-\frac{k\ell}{2}(1+\alpha)}^{\frac{k\ell}{2}(1-\alpha)}\frac{\sin^2 x}{x^2}\,dx. \tag{38}$$

Thiessen (1955) cited the latter integral in advance of the discrete point source study referred to earlier and, after invoking the substitution $\sin x = \sqrt{\pi x/2}\,J_{1/2}(x)$ along with an indefinite Bessel function integral formula, gave an exact infinite series expansion for the output which "is not an attractive one to use." However, as soon perceived by Burgess (1956), there is an alternative representation

$$P = \frac{\rho\omega\ell q_0^2}{8\pi}\left\{-\left(\frac{\sin^2\left\{\frac{k\ell}{2}(1-\alpha)\right\}}{\frac{k\ell}{2}(1-\alpha)} + \frac{\sin^2\left\{\frac{k\ell}{2}(1+\alpha)\right\}}{\frac{k\ell}{2}(1+\alpha)}\right) + \int_{-\frac{k\ell}{2}(1+\alpha)}^{\frac{k\ell}{2}(1-\alpha)} \frac{\sin 2x}{x}\,dx\right\}$$

$$= \frac{\rho\omega\ell q_0^2}{8\pi}\left\{Si\left\{k\ell(1-\alpha)\right\} + Si\left\{k\ell(1+\alpha)\right\} - \left(\frac{\sin^2\left\{\frac{k\ell}{2}(1-\alpha)\right\}}{\frac{k\ell}{2}(1-\alpha)} + \frac{\sin^2\left\{\frac{k\ell}{2}(1+\alpha)\right\}}{\frac{k\ell}{2}(1+\alpha)}\right)\right\} \tag{39}$$

[easily found on writing $\frac{dx}{x^2} = d(-\frac{1}{x})$ and effecting a partial integration] that readily lends itself to evaluation or estimation for diverse magnitudes of the parameters $k\ell$ and α .

A typically difficult integration over all spatial directions of the far field intensity need not be the sole basis for calculating the radiated power P ; another procedure (whose counterpart in antenna theory is better known) makes use of an expression for the (local) rate of energy input at the source itself, viz.

$$P = -\frac{\rho\omega}{2}\,\mathrm{Im}\int q(\underline{r})\,\phi^*(\underline{r})\,d\underline{r} \tag{40}$$

which has patent advantages if the domain of $q(\underline{r})$ is compact. Thus, (32), (35), and (40) imply that

$$P = -\frac{\rho\omega}{2}\,\mathrm{Im}\int q_0 e^{ik\alpha x}\,\delta(y)\,\delta(z)\,q_0\int \frac{\exp\left(-ik\sqrt{(x-x')^2+y^2+z^2}\right)}{4\pi\sqrt{(x-x')^2+y^2+z^2}}\,e^{-ik\alpha x'}\,dx'\,dx\,dy\,dz$$

$$= \frac{\rho\omega}{8\pi}q_0^2\int_{-\ell/2}^{\ell/2}\frac{\sin\left\{k|x-x'|-k\alpha(x-x')\right\}}{|x-x'|}\,dx\,dx'$$

and, through the introduction of a new pair of integration variables,

$$\xi = \frac{x - x'}{\sqrt{2}}, \quad \eta = \frac{x + x'}{\sqrt{2}}$$

there obtains

$$P = \frac{\rho\omega q_0^2}{8\pi} \int_{-\ell/\sqrt{2}}^{\ell/\sqrt{2}} d\xi \int_{-\left(\frac{\ell}{\sqrt{2}} - |\xi|\right)}^{\frac{\ell}{\sqrt{2}} - |\xi|} \frac{\sin\left\{\sqrt{2}k|\xi| - \sqrt{2}k\alpha\xi\right\}}{\sqrt{2}\,|\xi|} d\eta$$

$$= \frac{\rho\omega\ell q_0^2}{8\pi} \int_0^1 (1-\tau)\sin(k\ell\tau)\cos(k\ell\alpha\tau)\frac{d\tau}{\tau},$$

with the anticipated outcome (39) after performing the final (τ) integration.

The directivity gain of a line source has previously (§2) been inferred from that of its discrete analogue; and the given result (18) does indeed emerge when the expression (39) is substituted into the formula

$$g(\alpha) = \frac{r^2 S(\vartheta = \cos^{-1}\alpha)}{P/4\pi} = \frac{\frac{\rho\omega}{2}\left(\frac{q_0}{4\pi}\right)^2 k\ell^2}{P/4\pi}.$$

Taking note of the property

$$Si(-x) = -Si(x)$$

and of the asymptotic estimate (28) for Si(x) , the deduction from (39),

$$P \simeq \frac{\rho\omega\ell}{8\pi} q_0^2 \left\{ \frac{1}{k\ell}\left(\frac{1}{\alpha-1} - \frac{1}{\alpha+1}\right) + \frac{1}{(k\ell)^2}\left(\frac{\sin\{k\ell(\alpha-1)\}}{(\alpha-1)^2} - \frac{\sin\{k\ell(\alpha+1)\}}{(\alpha+1)^2}\right)\right\}$$

$$= \frac{\rho c q_0^2}{4\pi\alpha^2}\left\{1 - \frac{\sin k\ell}{k\ell}\cos(k\ell\alpha)\right\}, \qquad \alpha \gg 1 \tag{41}$$

implies a lessened radiation with increasing values of the phase parameter α. It needs to be observed that neither Thiessen (via display curves for P vs. $k\ell$ at different values of α) nor Burgess (in analytical work) have brought out the fact that a trigonometric function depending on $\alpha k\ell$ enters into the power estimate appropriate to a rapidly varying phase regime.

Side lobes in the directivity patterns are manifest for waves whose length is relatively small compared to the overall scale of the continuous line source, whereas longer waves (compared with the spacing between discrete source elements) can be admitted when many of the latter are present.

If the phase variation along the source line has a centered quadratic behavior, as is made explicit by the inhomogeneous term

$$q(\underline{r}) = q_0\, e^{i\beta(kx)^2}\, \delta(y)\,\delta(z), \qquad -\frac{\ell}{2} < x < \frac{\ell}{2} \tag{42}$$

for (33), the corresponding far field velocity potential

$$\phi \sim \frac{q_0}{4\pi}\frac{e^{ikr}}{r}\int_{-\ell/2}^{\ell/2} \exp\left\{-ikx'\cos\vartheta + i\beta k^2 x'^2\right\} dx',$$

$$r \to \infty, \quad x = r\cos\vartheta$$

contains an integral which can be simplified on completing the square in the exponential function; thus

$$\phi \sim \frac{q_0}{4\pi}\frac{e^{ikr}}{r}\frac{e^{-i\cos^2\vartheta/4\beta}}{k\sqrt{\beta}}\int_{-\sqrt{\beta}\frac{kl}{2}-\frac{\cos\vartheta}{2\sqrt{\beta}}}^{\sqrt{\beta}\frac{kl}{2}-\frac{\cos\vartheta}{2\sqrt{\beta}}} e^{i\tau^2}\,d\tau$$

$$= \frac{q_0}{4\pi}\frac{e^{ikr}}{r}\frac{e^{-i\cos^2\vartheta/4\beta}}{k\sqrt{\beta}}\sqrt{\frac{\pi}{2}}\left\{C\left(\sqrt{\beta}\frac{kl}{2}-\frac{\cos\vartheta}{2\sqrt{\beta}}\right)+C\left(\sqrt{\beta}\frac{kl}{2}+\frac{\cos\vartheta}{2\sqrt{\beta}}\right)\right.$$

$$\left.+iS\left(\sqrt{\beta}\frac{kl}{2}-\frac{\cos\vartheta}{2\sqrt{\beta}}\right)+iS\left(\sqrt{\beta}\frac{kl}{2}+\frac{\cos\vartheta}{2\sqrt{\beta}}\right)\right\} \tag{43}$$

with the customary definitions

$$C(x)=\sqrt{\frac{2}{\pi}}\int_0^x \cos\zeta^2\,d\zeta, \qquad S(x)=\sqrt{\frac{2}{\pi}}\int_0^x \sin\zeta^2\,d\zeta$$

$$=-C(-x) \qquad\qquad =-S(-x) \tag{44}$$

of the Fresnel integrals.

The net power output, obtained through an integral over all directions in the far field,

$$P=\frac{\rho c}{2}k^2\lim_{r\to\infty}\left\{2\pi\int_0^\pi r^2|\phi|^2\sin\vartheta\,d\vartheta\right\}$$

is specified by

$$P = \frac{\rho c q_0^2}{16\beta} \int_0^1 \Big[\Big\{ C\Big(\sqrt{\beta}\frac{k\ell}{2} - \frac{\zeta}{2\sqrt{\beta}}\Big) + C\Big(\sqrt{\beta}\frac{k\ell}{2} + \frac{\zeta}{2\sqrt{\beta}}\Big) \Big\}^2 + \Big\{ S\Big(\sqrt{\beta}\frac{k\ell}{2} - \frac{\zeta}{2\sqrt{\beta}}\Big) + S\Big(\sqrt{\beta}\frac{k\ell}{2} + \frac{\zeta}{2\sqrt{\beta}}\Big) \Big\}^2 \Big] d\zeta \tag{45}$$

It is a simple matter, on utilizing the asymptotic estimates,

$$S(x) \simeq \frac{1}{2} - \frac{1}{\sqrt{2\pi}\, x} \cos x^2 + O\Big(\frac{1}{x^2}\Big) \qquad x \to \infty \tag{46}$$

and

$$C(x) \simeq \frac{1}{2} + \frac{1}{\sqrt{2\pi}\, x} \sin x^2 + O\Big(\frac{1}{x^2}\Big)$$

to deduce from (45) that

$$P \simeq \frac{\rho c q_0^2}{8\beta}\left(1 + 4\sqrt{\frac{2}{\pi\beta}}\, \frac{\sin\frac{k\ell}{2}}{(k\ell)^2}\left\{\sin\left(\frac{\beta(k\ell)^2}{4}\right) - \cos\left(\frac{\beta(k\ell)^2}{4}\right)\right\}\right), \qquad \beta \gg 1 \tag{47}$$

so long as $k\ell$ does not assume very small values. The amount of radiation is thus diminished when the magnitude of the phase parameter β increases, and the occurrence of the first reciprocal power of β in the leading term of (47) may be contrasted with the second reciprocal power of α in the prior case of a linear phase variation along the source line. A slightly more intricate handling of (45) is called for in order to confirm the limiting value of the power when $\beta \to 0$; once again use is made of (46) and it appears that

$$\left\{C\left(\frac{\zeta}{2\sqrt{\beta}}+\frac{\sqrt{\beta}kl}{2}\right)+C\left(\frac{\zeta}{2\sqrt{\beta}}-\frac{\sqrt{\beta}kl}{2}\right)\right\}^2$$

$$+\left\{S\left(\frac{\zeta}{2\sqrt{\beta}}+\frac{\sqrt{\beta}kl}{2}\right)-S\left(\frac{\zeta}{2\sqrt{\beta}}-\frac{\sqrt{\beta}kl}{2}\right)\right\}^2$$

$$\simeq 2\frac{\beta}{\pi}\left\{\frac{1}{(\zeta+\beta kl)^2}+\frac{1}{(\zeta-\beta kl)^2}-2\frac{\cos\zeta kl}{\zeta^2-(\beta kl)^2}\right\},\quad \beta\to 0$$

whence

$$P \doteq \frac{\rho c q_0^2}{4\pi}\int_0^1 \frac{1}{\zeta^2}(1-\cos\zeta kl)\,d\zeta$$

$$= \frac{\rho c q_0^2}{4\pi}\left\{kl\,Si(kl)-2\sin^2\left(\frac{kl}{2}\right)\right\},\qquad \beta\to 0$$

in accordance with the result yielded by (39) for $\alpha = 0$.

On adopting (40) as the means of power specification there obtains

$$P = -\frac{\rho c k}{2}q_0^2\,\mathrm{Im}\int_{-l/2}^{l/2} e^{i\beta(x^2-x'^2)}\,\frac{e^{-ik|x-x'|}}{4\pi|x-x'|}\,dx\,dx'$$

$$= -\frac{\rho c k}{2}q_0^2\,\mathrm{Im}\int_{-\frac{l}{\sqrt{2}}}^{\frac{l}{\sqrt{2}}} d\xi\,\frac{e^{-i\sqrt{2}k|\xi|}}{\sqrt{2}|\xi|}\int_{-\left(\frac{l}{\sqrt{2}}-|\xi|\right)}^{\frac{l}{\sqrt{2}}-|\xi|} e^{2i\beta k^2\xi\eta}\,d\eta$$

$$= \frac{\rho c k q_0^2}{4\pi\sqrt{2}}\int_0^{\frac{l}{\sqrt{2}}}\frac{\sin(\sqrt{2}k\xi)}{\xi}\,\frac{\sin\left\{2\beta k^2\xi\left(\frac{l}{\sqrt{2}}-\xi\right)\right\}}{\beta k^2\xi}\,d\xi$$

$$= \frac{\rho c q_0^2}{8\pi\beta}(1+\beta k\ell)\int_0^1 \frac{d\zeta}{\zeta}\sin\left\{k\ell\zeta+\beta(k\ell)^2\zeta-\beta(k\ell)^2\zeta^2\right\}$$

$$-\frac{\rho c q_0^2}{8\pi\beta}(1-\beta k\ell)\int_0^1 \frac{d\zeta}{\zeta}\sin\left\{k\ell\zeta-\beta(k\ell)^2\zeta+\beta(k\ell)^2\zeta^2\right\}$$

$$-\frac{\rho c k\ell q_0^2}{4\pi}\int_0^1\Big\{\sin(k\ell\zeta+\beta(k\ell)^2\zeta-\beta(k\ell)^2\zeta^2) + \sin(k\ell\zeta-\beta(k\ell)^2\zeta+\beta(k\ell)^2\zeta^2)\Big\}d\zeta \tag{48}$$

as an alternative to (45). Estimates are readily secured from (48) when $\beta \to 0$ and $k\ell$ is finite, through series expansions of trigonometric functions, and the consequences

$$P(\beta) = \frac{\rho c q_0^2}{4\pi}\left\{k\ell\, Si(k\ell) - 2\sin^2\left(\frac{k\ell}{2}\right)\right\}$$

$$-\frac{\rho c q_0^2}{4\pi}(k\ell)^2\beta^2\left\{1+\frac{\sin k\ell}{k\ell}-\frac{4}{(k\ell)^2}(1-\cos k\ell)\right\}+O(\beta^4)$$

$\beta \to 0$, $k\ell$ finite

$$P(\beta)-P(0) \doteq -\frac{\rho c q_0^2}{1440\pi}(k\ell)^6\beta^2, \qquad \beta\to 0,\ k\ell\to 0 \tag{49}$$

reveal a lessening of the power output associated with small phase differences between neighboring points of the source distribution. The analogous feature is also present in the case of linear phase variations specified by the parameter α , where the deductions from (39) comprise

$$P(\alpha) - P(0) = -\frac{\rho c q_0^2}{4\pi}\alpha^2\left\{1 - \cos k\ell - \frac{1}{2}k\ell \sin k\ell\right\} + O(\alpha^4)$$

$\alpha \to 0$, $k\ell$ finite (50)

and

$$P(\alpha) - P(0) \doteq -\frac{\rho c q_0^2}{96\pi}(k\ell)^4\alpha^2, \qquad \alpha \to 0,\ k\ell \to 0$$

A preliminary transformation of the power formula (48) becomes appropriate if estimates suitable to large values of β are sought; and the first pertinent relation is given by the integral

$$\int_0^1 \sin(\nu\zeta^2 - \mu\zeta)\,d\zeta = \int_0^1 \sin\left\{\nu\left(\zeta - \frac{\mu}{2\nu}\right)^2 - \frac{\mu^2}{4\nu}\right\}$$

$$= \int_{-\frac{\mu}{2\nu}}^{1-\frac{\mu}{2\nu}} \left\{\sin\nu\tau^2 \cos\frac{\mu^2}{4\nu} - \cos\nu\tau^2 \sin\frac{\mu^2}{4\nu}\right\} d\tau$$

$$= \sqrt{\frac{\pi}{2\nu}}\cos\frac{\mu^2}{4\nu}\left\{S\left(\sqrt{\nu} - \frac{\mu}{2\sqrt{\nu}}\right) + S\left(\frac{\mu}{2\sqrt{\nu}}\right)\right\} \qquad \nu > 0$$

$$-\sqrt{\frac{\pi}{2\nu}}\sin\frac{\mu^2}{4\nu}\left\{C\left(\sqrt{\nu} - \frac{\mu}{2\sqrt{\nu}}\right) + C\left(\frac{\mu}{2\sqrt{\nu}}\right)\right\}. \qquad (51)$$

Further, on designating

$$I(\nu,\mu) = \int_0^1 \frac{\sin(\nu\zeta^2 - \mu\zeta)}{\zeta}\,d\zeta$$

it follows that

$$\frac{\partial I}{\partial \mu} = -\int_0^1 \cos(\nu\zeta^2 - \mu\zeta)\,d\zeta$$

$$= -\sqrt{\frac{\pi}{2\nu}}\cos\frac{\mu^2}{4\nu}\left\{C\left(\sqrt{\nu} - \frac{\mu}{2\sqrt{\nu}}\right) + C\left(\frac{\mu}{2\sqrt{\nu}}\right)\right\}$$

$$-\sqrt{\frac{\pi}{2\nu}}\sin\frac{\mu^2}{4\nu}\left\{S\left(\sqrt{\nu} - \frac{\mu}{2\sqrt{\nu}}\right) + S\left(\frac{\mu}{2\sqrt{\nu}}\right)\right\}$$

and thus

$$\int_0^1 \frac{\sin(\nu\zeta^2 - \mu\zeta)}{\zeta}\,d\zeta = -\frac{\pi}{2} + \sqrt{\frac{\pi}{2\nu}}\int_\mu^\infty \left[\cos\frac{\zeta^2}{4\nu}\left\{C\left(\sqrt{\nu} - \frac{\zeta}{2\sqrt{\nu}}\right) + C\left(\frac{\zeta}{2\sqrt{\nu}}\right)\right\}\right.$$

$$\left. + \sin\frac{\zeta^2}{4\nu}\left\{S\left(\sqrt{\nu} - \frac{\zeta}{2\sqrt{\nu}}\right) + S\left(\frac{\zeta}{2\sqrt{\nu}}\right)\right\}\right]d\zeta \tag{52}$$

inasmuch as

$$I(\nu, \infty) = \lim_{\mu\to\infty}\int_0^1 \frac{\sin(\nu\zeta^2 - \mu\zeta)}{\zeta}\,d\zeta$$

$$= -\lim_{\mu\to\infty}\int_0^\mu \frac{\sin\left(x - \frac{\nu x^2}{\mu}\right)}{x}\,dx = -\frac{\pi}{2}.$$

Integrations by parts, pursuant to writing

$$\cos\frac{\zeta^2}{4\nu}\,d\zeta = \frac{2\nu}{\zeta}\,d\left(\sin\frac{\zeta^2}{4\nu}\right), \quad \sin\frac{\zeta^2}{4\nu}\,d\zeta = -\frac{2\nu}{\zeta}\,d\left(\cos\frac{\zeta^2}{4\nu}\right),$$

permit a sequential refinement of estimates for (52) when $\mu \gg 1$; and this leads, via (48) and (51), to an output characterization which isolates the asymptotically predominant term, $\rho c q_0^2/8\beta$, $\beta \to \infty$, and contains only first powers of the Fresnel integrals [compare (45)]. The explicit construction of a development for P that involves reciprocal powers of $\beta k \ell$ is, nonetheless, a technically laborious matter.

§ 4. A Cylindrical Shell Source

Consider a circular cylindrical shell with negligible wall thickness and open ends, whose two characteristic dimensions are ℓ (the length) and R (the radius), respectively; and suppose that its curved surface is the site of a simple uniform source layer, specified by the density function

$$q(\underline{r}) = \frac{q_0}{2\pi}\,\frac{\delta(r-R)}{r}, \qquad -\frac{\ell}{2} < z < \frac{\ell}{2}, \quad 0 < \vartheta < 2\pi \tag{53}$$

in terms of cylindrical coordinates r, θ, z with origin at the geometrical center of the shell. The velocity potential for this source aggregate

$$\phi(r,z) = \frac{q_0}{8\pi^2}\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} dz' \int_{\vartheta}^{\vartheta+2\pi} \frac{\exp\left\{ik\sqrt{r^2+R^2-2rR\cos(\vartheta-\vartheta')+(z-z')^2}\right\}}{\sqrt{r^2+R^2-2rR\cos(\vartheta-\vartheta')+(z-z')^2}}\,d\vartheta' \tag{54}$$

possesses an evident symmetry around the axis or center line of the shell and leads, when combined with (53) in (40), to the power representation

$$P = \frac{\rho\omega}{8\pi} q_0^2 \int_{-\ell/2}^{\ell/2} G(|z-z'|)\,dz\,dz'$$

where

$$G(|z|) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin\left\{k\sqrt{(z-z')^2+\left(2R\sin\frac{\vartheta}{2}\right)^2}\right\}}{\sqrt{(z-z')^2+\left(2R\sin\frac{\vartheta}{2}\right)^2}}\,d\vartheta. \tag{55}$$

A transformation of integration variables (cf. p. 22) yields

$$\int_{-\ell/2}^{\ell/2} G(|z-z'|)\, dz\, dz' = 2\int_0^{\ell} G(|\zeta|)\left\{\ell-\zeta\right\} d\zeta$$

and thus

$$P = \frac{\rho c k q_0^2}{8\pi^2}\int_0^{2\pi} d\vartheta \int_0^{\ell} (\ell-\zeta)\,\frac{\sin\left\{k\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}\right\}}{\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}}\, d\zeta. \qquad (56)$$

Inasmuch as

$$\int_0^{\infty} \frac{\sin\left\{k\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}\right\}}{\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}}\, d\zeta = \frac{\pi}{2} J_0\left(2kR\sin\frac{\vartheta}{2}\right) \qquad (57)$$

(with the zero-order Bessel function J_0) and

$$\int_0^{\ell} \zeta\,\frac{\sin\left\{k\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}\right\}}{\sqrt{\zeta^2+(2R\sin\vartheta/2)^2}}\, d\zeta = \frac{1}{k}\int_0^{\ell} d\left[-\cos\left\{k\sqrt{\zeta^2+\left(2R\sin\frac{\vartheta}{2}\right)^2}\right\}\right]$$

$$= \frac{1}{k}\left[\cos\left(2kR\sin\frac{\vartheta}{2}\right) - \cos\left\{\sqrt{(k\ell)^2+\left(2kR\sin\frac{\vartheta}{2}\right)^2}\right\}\right]$$

it follows that the output of the source can be recast in the form

$$P = \frac{\rho c k \ell q_0^2}{8\pi} \int_0^{2\pi} \left\{ \frac{\pi}{2} J_0\left(2kR\sin\frac{\vartheta}{2}\right) - \int_{k\ell}^{\infty} \frac{\sin\left\{\sqrt{\mu^2 + \left(2kR\sin\frac{\vartheta}{2}\right)^2}\right\}}{\sqrt{\mu^2 + \left(2kR\sin\frac{\vartheta}{2}\right)^2}} d\mu \right.$$

$$\left. - \frac{1}{k\ell}\left[\cos\left(2kR\sin\frac{\vartheta}{2}\right) - \cos\left\{\sqrt{(k\ell)^2 + \left(2kR\sin\frac{\vartheta}{2}\right)^2}\right\}\right]\right\} d\vartheta \tag{58}$$

The latter expression reduces in the limit $R \to 0$, to

$$P = \frac{\rho c k \ell q_0^2}{4\pi}\left[Si(k\ell) - \frac{\sin^2\left(\frac{k\ell}{2}\right)}{\frac{k\ell}{2}} \right]$$

as befits a line source and, therefore, agrees with an appropriate specialization $(\alpha = 0)$ of (39).

A power estimate is readily forthcoming from (58) if

$$kR \ll 1, \qquad \frac{R}{\ell} \ll 1 \tag{59}$$

i.e., if the radius of the shell is small compared to the acoustic wave length and also the length of the shell. It turns out, by straightforward analysis, that

$$P = \frac{\rho c k \ell}{4\pi} q_0^2 \left[Si(k\ell) - \frac{\sin^2\left(\frac{k\ell}{2}\right)}{\frac{k\ell}{2}} \right.$$

$$\left. -(kR)^2\left\{ \frac{1}{2} Si(k\ell) + \frac{\cos k\ell}{2k\ell} + \frac{\sin k\ell}{2(k\ell)^2} - \frac{1}{k\ell} \right\} + \cdots \right] \tag{60}$$

where the first correction due to the finite radius of the shell involves the product of $(kR)^2$ and a function of $k\ell$, viz.

$$F(k\ell) = \frac{1}{2}\,Si(k\ell) + \frac{\cos k\ell}{2k\ell} + \frac{\sin k\ell}{2(k\ell)^2} - \frac{1}{k\ell}$$

with the (positive) limits

$$F(k\ell) \doteq \frac{\pi}{4}, \qquad k\ell \to \infty$$

and

$$F(k\ell) \doteq \frac{1}{6}k\ell, \qquad k\ell \to 0$$

Since the derivative of F, namely

$$\frac{dF}{dx} = \frac{1}{x^2} - \frac{\sin x}{x^3}$$

does not vanish on the interval $x > 0$ the correction terms in (60) for a small but finite radius have a combined negative value (regardless of the magnitude of $k\ell$), and thus the shell radiates less power than a line of the same length.

After introducing a linear phase factor $e^{ik\alpha z}$ into the source function (53) the power formula can be expressed in the versions

$$P = \frac{\rho c k q_0^2}{16\pi^2}\,\mathrm{Im}\int_0^{2\pi}\int_{-\ell/2}^{\ell/2} \frac{\exp\left\{ik\sqrt{(z-z')^2+(2R\sin\frac{\vartheta}{2})^2}\right\}}{\sqrt{(z-z')^2+(2R\sin\frac{\vartheta}{2})^2}}\, e^{-ik\alpha(z-z')}\, dz\,dz'\,d\vartheta$$

$$= \frac{\rho c k q_0^2}{16\pi^2}\,\mathrm{Im}\int_0^{2\pi} d\vartheta \int_0^{\infty} d\zeta\, \frac{\zeta J_0(2\zeta R\sin\frac{\vartheta}{2})}{\sqrt{\zeta^2-k^2}} \int_{-\ell/2}^{\ell/2} e^{-\sqrt{\zeta^2-k^2}\,|z-z'|-ik\alpha(z-z')}\, dz\,dz' \tag{61}$$

where the second is consequent to a reliance on the integral representation

$$\frac{e^{ik\sqrt{z^2+\rho^2}}}{\sqrt{z^2+\rho^2}} = \int_0^\infty \frac{\zeta J_0(\zeta\rho)}{\sqrt{\zeta^2-k^2}} e^{-\sqrt{\zeta^2-k^2}\,|z|}\, d\zeta \tag{62}$$

Once the integration relative to the variables z , z' are effected and account taken of the complementary specifications

$$\sqrt{\zeta^2-k^2} = \begin{cases} \sqrt{\zeta^2-k^2}, & \zeta > k \\ -i\sqrt{k^2-\zeta^2}, & \zeta < k \end{cases}$$

implicit in (62), it follows that

$$P = \frac{\rho c k q_0^2}{8\pi^2} \int_0^{2\pi} d\vartheta \int_0^k \frac{\zeta J_0(2\zeta R \sin\frac{\vartheta}{2})}{\sqrt{k^2-\zeta^2}} \frac{1}{(\zeta^2-(1-\alpha^2)k^2)^2}$$

$$\cdot \Big\{ ((1+\alpha^2)k^2-\zeta^2)\left[1-\cos k\alpha l \cos(\sqrt{k^2-\zeta^2}\,l)\right] - 2k\alpha\sqrt{k^2-\zeta^2}\sin k\alpha l \sin(\sqrt{k^2-\zeta^2}\,l)\Big\}\, d\zeta \tag{63}$$

and the occurrence of finite limits enables a straightforward reduction of the latter formula, when the inequality $kR \ll 1$ holds, through a power series development of the Bessel function. Thus, on employing the first two terms of the series, i.e.,

$$J_0(z) \doteq 1 - \frac{1}{4}z^2, \qquad z \ll 1$$

the difference between shell and linear source radiated powers, to the lowest order in the radius of the former, is expressed by

$$\delta P = P_{\text{shell}} - P_{\text{line}}$$

$$= -\frac{\rho c k q_0^2 R^2}{8\pi} \int_0^k \frac{\zeta^3}{\sqrt{k^2-\zeta^2}} \frac{1}{\{\zeta^2-(1-\alpha^2)k^2\}^2} \cdot$$

$$\cdot \Big\{ [(1+\alpha^2)k^2-\zeta^2][1-\cos k\alpha\ell \cos(\sqrt{k^2-\zeta^2}\,\ell)] \qquad (64)$$

$$-2k\alpha\sqrt{k^2-\zeta^2}\sin k\alpha\ell \sin(\sqrt{k^2-\zeta^2}\,\ell)\Big\} d\zeta,$$

without restriction on the magnitude of $k\ell$. Given the particular value $\alpha = 0$,

$$(\delta P)_{\alpha=0} = -\frac{\rho c q_0^2}{8\pi}(kR)^2 \int_0^{\pi/2} \frac{\cos^3\varphi}{\sin^2\varphi}\{1-\cos(k\ell\sin\varphi)\}\, d\varphi < 0$$

and, inasmuch as

$$\int_0^{\pi/2} \frac{\cos^3\varphi}{\sin^2\varphi}\{1-\cos(k\ell\sin\varphi)\}\, d\varphi = \int_0^{\pi/2} \frac{\cos\varphi}{\sin^2\varphi}\{1-\cos(k\ell\sin\varphi)\}\, d\varphi$$

$$-\int_0^{\pi/2} \cos\varphi\{1-\cos(k\ell\sin\varphi)\}\, d\varphi$$

$$= \int_0^{\pi/2} d\left(-\frac{\cos\varphi}{\sin\varphi}\right)\left[\cos\varphi\{1-\cos(k\ell\sin\varphi)\}\right] - \int_0^{\pi/2} \cos\varphi\{1-\cos(k\ell\sin\varphi)\}\, d\varphi$$

$$= k\ell \int_0^{\pi/2} \frac{\cos\varphi}{\sin\varphi}\sin(k\ell\sin\varphi)\, d\varphi - k\ell \int_0^{\pi/2} \cos\varphi\sin\varphi\sin(k\ell\sin\varphi)\, d\varphi$$

$$-2\int_0^{\pi/2} \cos\varphi\{1-\cos(k\ell\sin\varphi)\}\, d\varphi$$

$$= k\ell\left\{ Si(k\ell) + \frac{\cos k\ell}{k\ell} + \frac{\sin k\ell}{(k\ell)^2} - \frac{2}{k\ell}\right\}$$

it turns out that

$$(\delta P)_{\alpha=0} = -\frac{\rho c k\ell}{8\pi} q_0^2 (kR)^2 \left\{ Si(k\ell) + \frac{\cos k\ell}{k\ell} + \frac{\sin k\ell}{(k\ell)^2} - \frac{2}{k\ell} \right\}$$

in complete agreement with (60); thus, the shell with a uniform source density radiates a smaller amount of energy than a line with the same density, for all magnitudes of the ratio, length of shell/acoustic wave length.

If the phase parameter α equals unity, then

$$(\delta P)_{\alpha=1} = -\frac{\rho c q_0^2}{8\pi}(kR)^2 \int_0^{\pi/2} \frac{1}{\cos\varphi} \Big\{ (2-\cos^2\varphi)\left[1-\cos k\ell \cos(k\ell \sin\varphi)\right] - 2\sin\varphi \sin k\ell \sin(k\ell\sin\varphi) \Big\} d\varphi$$

$$= -\frac{\rho c q_0^2}{8\pi}(kR)^2 \left\{ \int_0^{2k\ell} \frac{1-\cos\xi}{\xi} d\xi + \frac{\sin 2k\ell}{2k\ell} - 1 \right\} \tag{65}$$

where the dependence on $k\ell$ is contained in a function

$$\bar{F}(2k\ell) = \int_0^{2k\ell} \frac{1-\cos\xi}{\xi} d\xi + \frac{\sin 2k\ell}{2k\ell} - 1$$

$$= \log 2k\ell + \gamma - Ci(2k\ell) + \frac{\sin 2k\ell}{2k\ell} - 1$$

which relates to the cosine integral, Ci x , along with the Euler constant $\gamma = 0.5772 \ldots$, and has the limiting behaviors

$$\bar{F}(2k\ell) \doteq \frac{4}{3}(k\ell)^2, \qquad k\ell \to 0$$

$$\bar{F}(2k\ell) \doteq \log 2k\ell, \qquad k\ell \to \infty$$

Since

$$\frac{d\bar{F}}{dx} = \frac{1}{x}\left(1 - \frac{\sin x}{x}\right) > 0, \quad x > 0$$

the inference $(\delta P)_{\alpha=1} < 0$ may be drawn, and thus a diminished level of radiation is linked, once again, with finite source radius. The respective proportionalities of $(\delta P)_{\alpha=1}$ and $(\delta P)_{\alpha=0}$ to $\log k\ell$ and $k\ell$, when $k\ell \gg 1$, are noteworthy as is also the comparative power ratio

$$\frac{P_{\alpha=1}}{P_{\alpha=0}} = \frac{1}{2}\left\{1 + \left(\frac{1}{2} - \frac{2}{\pi}\frac{\log k\ell}{k\ell}\right)(kR)^2\right\}, \qquad \begin{matrix} k\ell \gg 1 \\ kR \ll 1 \end{matrix}.$$

For large values of α,

$$P \simeq \frac{\rho c k q_0^2}{8\pi^2} \cdot \frac{1}{\alpha^2 k^2} \int_0^{2\pi} d\vartheta \int_0^k \frac{\zeta J_0(2\zeta R \sin\frac{\vartheta}{2})}{\sqrt{k^2-\zeta^2}} \left\{1 - \cos k\alpha\ell \cos\left(\sqrt{k^2-\zeta^2}\,\ell\right)\right\} d\zeta$$

and, in the limit $R \to 0$,

$$P \doteq \frac{\rho c k q_0^2}{8\pi^2} \cdot \frac{2\pi}{\alpha^2 k^2} \int_0^k \frac{\zeta}{\sqrt{k^2-\zeta^2}} \left\{1 - \cos k\alpha\ell \cos\left(\sqrt{k^2-\zeta^2}\,\ell\right)\right\} d\zeta$$

$$= \frac{\rho c q_0^2}{4\pi\alpha^2}\left\{1 - \frac{\sin k\ell}{k\ell}\cos(k\ell\alpha)\right\}, \quad R \to 0, \quad \alpha \gg 1$$

as is fully consistent with (41).

§5. Divers Integrals

Although the power and gain of linear and planar source models have long been a subject for analysis in both the acoustical and electromagnetic contexts, additional details and improvements on specific results are feasible, as shown in the sequel. There is a direct acoustical counterpart of an investigation by Clemmow (1966) on antenna gain and supergain, for example, wherein the resulting integrals can be displayed in a more useful manner; thus, a two-dimensional electric field polarized in the z-direction, say, which is generated by the unidirectional surface current density in the plane $y = 0$, namely

$$\underline{j} = (0, 0, j(x))$$

where (66)

$$j(x) = \begin{cases} j_0, & |x| < \ell/2 \\ 0, & |x| > \ell/2 \end{cases} \qquad j_0 = \text{constant}$$

corresponds with a two-dimensional acoustic field produced by a rigid strip radiator with the normal (y-component) velocity distribution

$$v(x) = \begin{cases} v_0, & |x| < \ell/2 \\ 0, & |x| > \ell/2 \end{cases} \qquad v_0 = \text{constant} \tag{67}$$

in the plane $y = 0$.

If

$$G(\vartheta) = \pi \frac{\text{power radiated per unit angle in direction } \theta}{\text{total power radiated}}$$

defines the gain for radiation into the half-space $0 < \theta < \pi$, $y > 0$, $-\infty < z < \infty$, straightforward analysis reveals that

$$G\left(\frac{\pi}{2}\right) = \pi \Bigg/ \int_0^{\pi} \left\{ \frac{\sin\left(\frac{kl}{2}\cos\vartheta\right)}{\frac{kl}{2}\cos\vartheta} \right\}^2 d\vartheta \tag{68}$$

for either of the source distributions (66) or (67), k denoting the (appropriate) wave number. In accordance with the relation

$$F_0(\beta) = \int_0^{\pi} \left\{ \frac{\sin(\beta\cos\vartheta)}{\cos\vartheta} \right\}^2 d\vartheta = \pi\beta \left\{ \int_0^{2\beta} J_0(\zeta)\,d\zeta - J_1(2\beta) \right\} \tag{69}$$

involving Bessel functions of zero and first orders, an expression for the gain (which Clemmow does not state) follows (compare (16)),

$$G\left(\frac{\pi}{2}\right) = \frac{kl/2}{\int_0^{kl} J_0(\zeta)\,d\zeta - J_1(kl)}, \tag{70}$$

that readily furnishes the respective estimates

$$G\left(\frac{\pi}{2}\right) \begin{array}{ll} \doteq 1, & kl \ll 1 \\ \doteq \frac{kl}{2}, & kl \gg 1 \end{array} \tag{71}$$

Hence, a uniform current or normally vibrating strip produces a gain in the direction transverse to the source plane which rises in proportion to the strip width, provided that the latter exceeds a wave length. An alternative to large scale configurations for surpassing this 'normal' gain, G_0 , has practical merit; and this contemplates variable source distributions acting on

small excitation areas. Consider, in a simplified though illustrative sense, the feasibility of supergain $G > G_0$ with a non-uniform strip velocity

$$v(x) = \left\{ \begin{array}{ll} \sum_{-N}^{N} v_n e^{2in\pi x/\ell} & , \ |x| < \ell/2 \\ 0 & , \ |x| > \ell/2 \end{array} \right\} \tag{72}$$

in place of (67); here the v_n are constants and $v_{-n} = v_n$. The appertaining velocity potential in the radiation field has a cylindrical wave form, i.e.,

$$\phi(r,\vartheta) \simeq f(\vartheta) \frac{e^{ikr}}{\sqrt{kr}}, \qquad r = \sqrt{x^2+y^2} \gg 1/k \qquad 0 < \vartheta < \pi$$

with the angular factor

$$f(\vartheta) = \frac{i}{2}\sqrt{\frac{2}{\pi}}\, e^{-i\pi/4} \int_{-\ell/2}^{\ell/2} e^{-ikx\cos\vartheta}\, v(x)\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, e^{i\pi/4} \frac{\sin\left(\frac{k\ell}{2}\cos\vartheta\right)}{k\cos\vartheta} \left\{ v_0 - 2\left(\frac{k\ell}{2}\right)^2 \cos^2\vartheta \sum_{n=1}^{N} \frac{(-1)^n v_n}{(n\pi)^2 - \left(\frac{k\ell}{2}\cos\vartheta\right)^2} \right\}.$$

Since

$$f\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi}}\, e^{i\pi/4}\, v_0 \frac{\ell}{2}$$

does not depend on the parameters $v_1, v_2, \ldots, v_N$ that selection which minimizes the total radiated power also maximizes the gain in the direction $\pi/2$ (normal to the strip). Clemmow makes a choice

$$v_1 = \eta v_0, \quad v_2 = v_3 = \cdots = v_N = 0$$

and observes that the corresponding gain

$$\frac{1}{G(\frac{\pi}{2})} = \frac{1}{\pi}\frac{1}{|f(\frac{\pi}{2})|^2}\int_0^\pi |f(\vartheta)|^2 d\vartheta$$

$$= \frac{1}{\pi}\int_0^\pi \left| \frac{\sin(\frac{k\ell}{2}\cos\vartheta)}{\frac{k\ell}{2}\cos\vartheta}\left\{1 + \frac{2(\frac{k\ell}{2})^2\cos^2\vartheta}{\pi^2-(\frac{k\ell}{2}\cos\vartheta)^2}\eta\right\}\right|^2 d\vartheta$$

can be rendered a maximum by a suitable (real) determination of η ; and this yields

$$1-\frac{G_0}{G} = \frac{\left(\int_0^\pi \frac{\sin^2(\frac{k\ell}{2}\cos\vartheta)}{\pi^2-(\frac{k\ell}{2})^2\cos^2\vartheta}d\vartheta\right)^2}{\int_0^\pi\left\{\frac{\sin(\frac{k\ell}{2}\cos\vartheta)}{\cos\vartheta}\right\}^2 d\vartheta \int_0^\pi \frac{\cos^2\vartheta\,\sin^2(\frac{k\ell}{2}\cos\vartheta)}{\left\{\pi^2-(\frac{k\ell}{2})^2\cos^2\vartheta\right\}^2}d\vartheta} \tag{73}$$

where G_0 is the normal gain (70).

Estimates for the integrals appearing in (73) are easily secured when $k\ell \ll 1$ and reveal that

$$G \doteq 3G_0, \qquad k\ell \ll 1$$

to the lowest order of approximation. Clemmow also arrived at the complementary result

$$1-\frac{G_0}{G} \simeq \frac{1}{2\pi}\frac{\cos^2(k\ell-\frac{\pi}{4})}{(\frac{k\ell}{2})^3}, \qquad k\ell \gg 1 \tag{74}$$

(which suggests only a small fractional increase of gain over the normal amount) through function theoretic reasoning. It is a more exacting task to obtain fuller details of the (asymptotic) expansions for all the integrals, when $k\ell \gg 1$, by complex variable methods; hence, alternative representations of the integrals, which permit the requisite expansions, may be noted here and attention drawn to the circumstance that similar integrals are encountered in other problems of sound radiation.

One of the three separate integrals in (73), connected with the normal gain (68), has previously been given an alternative form, (69), which enables systematic estimation at both small and large values of the parameter $k\ell$. The remaining pair, say

$$F_1(\beta) = \int_0^\pi \frac{\sin^2(\beta\cos\vartheta)}{\pi^2 - \beta^2\cos^2\vartheta}\, d\vartheta \tag{75}$$

$$F_2(\beta) = \int_0^\pi \frac{\cos^2\vartheta\, \sin^2(\beta\cos\vartheta)}{(\pi^2 - \beta^2\cos^2\vartheta)^2}\, d\vartheta \tag{76}$$

enter jointly into the relation

$$F_2(\beta) = \frac{1}{\beta^2}\int_0^\pi \frac{\pi^2 - (\pi^2 - \beta^2\cos^2\vartheta)}{(\pi^2 - \beta^2\cos^2\vartheta)^2}\, \sin^2(\beta\cos\vartheta)\, d\vartheta$$

$$= \frac{\pi^2}{\beta^2}\int_0^\pi \frac{\sin^2(\beta\cos\vartheta)}{(\pi^2 - \beta^2\cos^2\vartheta)^2}\, d\vartheta - \frac{1}{\beta^2}F_1(\beta)$$

and may be rewritten as

$$F_1(\beta) = -\frac{1}{\pi\beta}\int_0^\pi \frac{\sin^2\{\beta(\cos\vartheta - \frac{\pi}{\beta})\}}{\cos\vartheta - \frac{\pi}{\beta}}\,d\vartheta \tag{77}$$

and

$$F_2(\beta) = \frac{1}{4\pi\beta}\int_0^\pi \cos\vartheta\,\sin^2(\beta\cos\vartheta)\left\{\frac{1}{(\pi-\beta\cos\vartheta)^2} - \frac{1}{(\pi+\beta\cos\vartheta)^2}\right\}d\vartheta$$

$$= \frac{1}{4\pi\beta^3}\int_0^\pi \left[\frac{\sin^2\{\beta(\cos\vartheta-\frac{\pi}{\beta})\}}{(\cos\vartheta-\frac{\pi}{\beta})^2} - \frac{\sin^2\{\beta(\cos\vartheta+\frac{\pi}{\beta})\}}{(\cos\vartheta+\frac{\pi}{\beta})^2}\right]\cos\vartheta\,d\vartheta$$

$$= \frac{1}{2\pi\beta^3}\int_0^\pi \frac{\sin^2\{\beta(\cos\vartheta-\frac{\pi}{\beta})\}}{(\cos\vartheta-\frac{\pi}{\beta})^2}\cos\vartheta\,d\vartheta \tag{78}$$

Define

$$\mathcal{F}_1(\beta,\gamma) = -\frac{1}{\pi\beta}\int_0^\pi \frac{\sin^2\{\gamma(\cos\vartheta-\frac{\pi}{\beta})\}}{\cos\vartheta-\frac{\pi}{\beta}}\,d\vartheta$$

such that

$$\mathcal{F}_1(\beta,\beta) = F_1(\beta), \quad \mathcal{F}_1(\beta,0) = 0;$$

then

$$\frac{\partial\mathcal{F}_1}{\partial\gamma} = -\frac{1}{\pi\beta}\int_0^\pi \sin\left\{2\gamma(\cos\vartheta-\frac{\pi}{\beta})\right\}d\vartheta = \frac{1}{\beta}\sin\left(2\pi\frac{\gamma}{\beta}\right)J_0(2\gamma)$$

and so

$$\mathcal{F}_1(\beta,\gamma) = \frac{1}{2\beta}\int_0^{2\gamma}\sin\frac{\pi\zeta}{\beta}\,J_0(\zeta)\,d\zeta$$

which provides the sought-after result

$$F_1(\beta) = \frac{1}{2\beta}\int_0^{2\beta}\sin\frac{\pi\zeta}{\beta}\,J_0(\zeta)\,d\zeta. \tag{79}$$

Having regard, similarly, for another two-variable function

$$\mathcal{F}_2(\beta,\gamma) = \frac{1}{2\pi\beta^3}\int_0^\pi \cos\vartheta \, \frac{\sin^2\{\gamma(\cos\vartheta - \frac{\pi}{\beta})\}}{(\cos\vartheta - \frac{\pi}{\beta})^2}\, d\vartheta, \qquad \mathcal{F}_2(\beta,\beta) = F_2(\beta)$$

which obeys the differential equation

$$\frac{\partial^2 \mathcal{F}_2}{\partial\gamma^2} = \frac{1}{\beta^3}\sin\left(2\pi\frac{\beta}{\gamma}\right) J_1(2\gamma)$$

it follows, after successive integrations consistent with the conditions

$$\mathcal{F}_2(\beta,0) = \frac{\partial \mathcal{F}_2(\beta,0)}{\partial\gamma} = 0,$$

that

$$\mathcal{F}_2(\beta,\gamma) = \frac{\pi}{2\beta^3}\int_0^{2\gamma}\left\{\left(\frac{\gamma}{\beta} - \frac{1}{2}\frac{\zeta}{\beta}\right)\cos\pi\frac{\zeta}{\beta} - \frac{1}{2\pi}\sin\pi\frac{\zeta}{\beta}\right\} J_0(\zeta)\, d\zeta$$

and

$$F_2(\beta) = \frac{\pi}{2\beta^3}\int_0^{2\beta}\left\{\left(1 - \frac{\zeta}{2\beta}\right)\cos\frac{\pi\zeta}{\beta} - \frac{1}{2\pi}\sin\frac{\pi\zeta}{\beta}\right\} J_0(\zeta)\, d\zeta. \tag{80}$$

Expansions of the functions F_0, F_1, and F_2 which are useful for small values of β can be secured from the respective integrals (69), (79), and (80) on substitution therein of the Bessel function power series. By virtue of the relations

$$\int_0^\infty J_0(a\zeta)\cos b\zeta \, d\zeta = \frac{1}{\sqrt{a^2-b^2}}, \qquad a > b$$

and

$$\int_0^\infty J_0(a\zeta)\sin b\zeta \, d\zeta = 0$$

it is expedient, for large values of β , to employ the representations

$$F_0(\beta) = \pi\beta\left\{1 - \int_{2\beta}^{\infty} J_0(\zeta)\,d\zeta - J_1(2\beta)\right\}$$

$$F_1(\beta) = -\frac{1}{2\beta}\int_{2\beta}^{\infty} \sin\left(\frac{\pi\zeta}{\beta}\right) J_0(\zeta)\,d\zeta$$

and

$$F_2(\beta) = \frac{\pi}{2\beta^3}\left\{\frac{1}{\sqrt{1-\left(\frac{\pi}{\beta}\right)^2}} - \int_{2\beta}^{\infty}\left(\cos\frac{\pi\zeta}{\beta} - \frac{1}{2\pi}\sin\frac{\pi\zeta}{\beta}\right) J_0(\zeta)\,d\zeta\right.$$

$$\left. + \frac{1}{2\beta}\frac{d}{d\mu}\left(\int_{2\beta}^{\infty}\sin\mu\zeta\, J_0(\zeta)\,d\zeta\right)_{\mu=\frac{\pi}{\beta}}\right\}$$

together with asymptotic developments of the Bessel functions, viz.

$$J_n(\zeta) \simeq \sqrt{\frac{2}{\pi\zeta}}\left\{\cos\left(\zeta - \frac{2n+1}{4}\pi\right) - \frac{4n^2-1}{8\zeta}\sin\left(\zeta - \frac{2n+1}{4}\pi\right)\right\}, \quad \begin{matrix}\zeta \gg 1\\ n=0,1\end{matrix}.$$

The estimates

$$F_0(\beta) \simeq \pi\beta\left\{1 - \frac{1}{\sqrt{\pi}}\frac{1}{2\beta^{3/2}}\cos\left(2\beta - \frac{\pi}{4}\right) + \cdots\right\}$$

$$\beta \gg 1$$

$$F_1(\beta) \simeq \frac{1}{2}\frac{\sqrt{\pi}}{\beta^{5/2}}\cos\left(2\beta - \frac{\pi}{4}\right) + \frac{9}{32}\frac{\sqrt{\pi}}{\beta^{7/2}}\sin\left(2\beta - \frac{\pi}{4}\right) + \cdots$$

and

$$F_2(\beta) \simeq \frac{\pi}{2\beta^3}\left\{1+\frac{1}{2\sqrt{\pi\beta}}\sin\left(2\beta-\frac{\pi}{4}\right)+\cdots\right\}$$

follow and their leading terms jointly yield another,

$$1-\frac{G_0}{G} = \frac{\left(F_1\left(\frac{kl}{2}\right)\right)^2}{F_0\left(\frac{kl}{2}\right)F_2\left(\frac{kl}{2}\right)} \simeq \frac{1}{2\pi}\,\frac{\cos^2\left(kl-\frac{\pi}{4}\right)}{(kl/2)^3}, \quad kl \gg 1$$

which confirms (74).

The integrals

$$F_3(\beta,n) = \int_0^{\pi/2} \frac{1}{(1-\beta^2\cos^2\vartheta)^2} \begin{matrix}\sin^2\\ \cos^2\end{matrix}\left\{\frac{n\pi}{2}\beta\cos\vartheta\right\} d\vartheta \tag{81}$$

$\begin{matrix}\sin\\ \cos\end{matrix}$ n: $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$

and

$$F_4(\beta,n) = \int_0^{\pi/2} \frac{\cos^2\vartheta}{(1-\beta^2\cos^2\vartheta)^2} \begin{matrix}\sin^2\\ \cos^2\end{matrix}\left\{\frac{n\pi}{2}\beta\cos\vartheta\right\} d\vartheta \tag{82}$$

occur in a study (Gomperts, 1974) of sound radiation from a rectangular plate whose vibration pattern is one-dimensional and parallel to its free edges, the other pair being hinged; evidently

$$F_4(\beta,2) = \frac{1}{2}\pi^4 F_2(\pi\beta)$$

and

$$F_3(\beta,2) = \frac{1}{2}\pi^2\left\{F_1(\pi\beta)+\pi^2\beta^2 F_2(\pi\beta)\right\}$$

while the reductions of (81), (82) for other integral values of n can be achieved by the means employed in connection with F_1, F_2. Gomperts calls attention to the 'critical frequency' integral

$$Y = \int_0^{N\pi} \frac{\sin^2\zeta}{(N^2\pi^2-\zeta^2)^{5/2}}\, d\zeta \tag{83}$$

and to an estimate

$$Y \simeq \frac{N^{-5/2}}{3\sqrt{2}\,\pi^2}, \qquad N \gg 1 \tag{84}$$

whose derivation is not given; a complete evaluation of this integral commences with the alternative version

$$Y = (N\pi)^{-4} \int_0^{\pi/2} \frac{\sin^2(N\pi\sin\vartheta)}{\cos^4\vartheta}\, d\vartheta$$

and proceeds with successive integration by parts, viz.

$$Y = (N\pi)^{-4} \int_0^{\pi/2} \sin^2(N\pi\sin\vartheta)\; d\left\{\frac{\sin\vartheta}{3}\left(\frac{1}{\cos^3\vartheta} + \frac{2}{\cos\vartheta}\right)\right\}$$

$$= -\frac{1}{3}(N\pi)^{-3} \int_0^{\pi/2} \sin\vartheta \left\{\frac{1}{\cos^2\vartheta} + 2\right\} \sin(2N\pi\sin\vartheta)\, d\vartheta$$

$$= -\frac{2}{3}(N\pi)^{-3} \int_0^{\pi/2} \sin\vartheta \sin(2N\pi\sin\vartheta)\, d\vartheta$$

$$-\frac{1}{3}(N\pi)^{-3} \int_0^{\pi/2} \sin\vartheta \sin(2N\pi\sin\vartheta)\, d\left(\frac{\sin\vartheta}{\cos\vartheta}\right)$$

$$= -\frac{1}{3}(N\pi)^{-3} \int_0^{\pi/2} \sin\vartheta \sin(2N\pi\sin\vartheta)\, d\vartheta$$

$$+\frac{1}{3}(N\pi)^{-2} \int_0^{\pi/2} (1-\cos 2\vartheta) \cos(2N\pi\sin\vartheta)\, d\vartheta$$

$$= -\frac{1}{3}(N\pi)^{-3}\left\{\frac{\pi}{2}J_1(2N\pi)\right\} + \frac{1}{3}(N\pi)^{-2}\left\{\frac{\pi}{2}J_0(2N\pi) - \frac{\pi}{2}J_2(2N\pi)\right\}$$

$$= \frac{\pi}{3}\,\frac{J_0(2N\pi)}{(N\pi)^2} - \frac{\pi}{3}\,\frac{J_1(2N\pi)}{(N\pi)^3}. \tag{85}$$

The estimate (84) is directly inferred from the first (and predominant) term of (85) on utilizing the Bessel function approximation

$$J_0(2N\pi) \sim (N\pi)^{-1/2}\cos\left(2N\pi - \frac{\pi}{4}\right) \doteq (2N\pi)^{-1/2}, \qquad N \gg 1.$$

A pair of integrals relevant to the electromagnetic power output from thin conducting wires are expressed by

$$F_5(\beta) = \int_0^\pi \sin^3\vartheta \; \frac{\cos^2(\beta\cos\vartheta)}{\left\{1-\left(\frac{2\beta}{\pi}\right)^2\cos^2\vartheta\right\}^2}\,d\vartheta \tag{86}$$

and

$$F_6(\beta,\alpha) = \int_0^\pi \sin^3\vartheta \; \frac{\sin^2\{\beta(\alpha-\cos\vartheta)\}}{(\alpha-\cos\vartheta)^2}\,d\vartheta\;; \tag{87}$$

the first (Oppenheimer, 1970) rests on the assumption of a longitudinal current that vanishes (only) at the ends of the wire and varies sinusoidally in between, whereas the second (Papas, 1965) is linked with a hypothetical travelling wave current distribution $(e^{ik\alpha x})$ along the wire. Both integrals involve a parameter $\beta = k\ell/2$ that depends on the free space wave number k and the length ℓ of the wire. Neither of the references contains any analysis bearing on the integrals, although their explicit evaluation is feasible.

The substitution $x = \cos\theta$ converts (86) to

$$F_5(\beta) = \int_{-1}^{1} (1-x^2)\,\frac{\cos^2\beta x}{\left\{1-\left(\frac{2\beta}{\pi}\right)^2 x^2\right\}^2}\,dx$$

$$= \frac{\pi}{2\beta}\int_{1-\frac{2\beta}{\pi}}^{1+\frac{2\beta}{\pi}} \left\{1-\frac{\pi^2}{4\beta^2}(1-\zeta)^2\right\}\frac{\sin^2\left(\frac{\pi\zeta}{2}\right)}{\zeta^2(2-\zeta)^2}\,d\zeta$$

$$= \frac{\pi}{8\beta}\int_{1-\frac{2\beta}{\pi}}^{1+\frac{2\beta}{\pi}} \left\{1-\frac{\pi^2}{4\beta^2}(1-\zeta)^2\right\}\left\{\frac{1}{\zeta^2}+\frac{1}{\zeta}+\frac{1}{2-\zeta}+\frac{1}{(2-\zeta)^2}\right\}\sin^2\left(\frac{\pi\zeta}{2}\right)d\zeta$$

and, after utilizing the differentials $d(-\frac{1}{\zeta}) = \frac{d\zeta}{\zeta^2}$, $d(\frac{1}{2-\zeta}) = \frac{d\zeta}{(2-\zeta)^2}$ to initiate an integration by parts, it is eventually found that

$$F_5(\beta) = \frac{\pi^2}{8\beta}\left(1-\frac{\pi^2}{4\beta^2}\right)\left\{Si(2\beta+\pi)+Si(2\beta-\pi)\right\} - \frac{\pi^2}{8\beta^2}\left\{1+\cos 2\beta\right\}$$

$$+\frac{\pi}{8\beta}\left(1+\frac{\pi^2}{4\beta^2}\right)\left\{\log\left|\frac{2\beta+\pi}{2\beta-\pi}\right| - Ci(2\beta+\pi)+Ci(|2\beta-\pi|)\right\} \tag{88}$$

in terms of the sine and cosine integrals.

The change of variable previously employed for the integral (86) yields, after its application to (87),

$$F_6(\beta,\alpha) = \int_{-1}^{1}\frac{\sin^2\{\beta(x-\alpha)\}}{(x-\alpha)^2}\,dx - \int_{-1}^{1} x^2\,\frac{\sin^2\{\beta(x-\alpha)\}}{(x-\alpha)^2}\,dx = F_6^{(1)} - F_6^{(2)}$$

and

$$d^2F_6^{(1)}/d\beta^2 = 2\cos 2\alpha\beta\,\frac{\sin 2\beta}{\beta}$$

whence

$$F_6^{(1)}(\beta,\alpha) = \beta\left\{ Si(2\beta(1-\alpha)) + Si(2\beta(1+\alpha)) \right\} - \frac{1}{2}\left\{ \frac{1-\cos 2\beta(1-\alpha)}{1-\alpha} + \frac{1-\cos 2\beta(1+\alpha)}{1+\alpha} \right\}$$

in accordance with the particular values $F_6^{(1)}(0,\alpha) = 0$, $\frac{dF_6^{(1)}}{d\beta}(0,\alpha) = 0$; similarly,

$$\frac{d^2F_6^{(2)}}{d\beta^2} = 4\cos 2\alpha\beta \left\{ \frac{\sin 2\beta}{2\beta} + \frac{\cos 2\beta}{2\beta^2} - \frac{\sin 2\beta}{4\beta^3} \right\}$$

and thence, after two integrations,

$$\begin{aligned} F_6^{(2)}(\beta,\alpha) = {} & \left\{ \cos 2\beta - \frac{\sin 2\beta}{2\beta} \right\} \cos 2\alpha\beta + \alpha^2\beta \left\{ Si(2\beta(1+\alpha)) + Si(2\beta(1-\alpha)) \right\} \\ & + \frac{1}{2}(1+\alpha)\left\{ 1 - \cos(2\beta(1-\alpha)) \right\} + \frac{1}{2}(1-\alpha)\left\{ 1 - \cos(2\beta(1+\alpha)) \right\} \\ & - \alpha \int_0^\beta \left\{ \cos(2(1-\alpha)\tau) - \cos(2(1+\alpha)\tau) \right\} \frac{d\tau}{\tau} \end{aligned}$$

describes the appropriate function. Combining the expressions for $F_6^{(1)}$ and $F_6^{(2)}$ there obtains, finally

$$\begin{aligned} F_6(\beta,\alpha) = {} & \alpha \sin 2\beta \sin 2\alpha\beta + \frac{\sin 2\beta}{2\beta} \cos 2\alpha\beta - 2 + \cos 2\beta \cos 2\alpha\beta \\ & + \beta(1-\alpha^2)\left\{ Si(2\beta(1+\alpha)) + Si(2\beta(1-\alpha)) \right\} \qquad (89) \\ & + \alpha\left\{ \log\left| \frac{1-\alpha}{1+\alpha} \right| + Ci(2\beta(1+\alpha)) - Ci(2\beta(1-\alpha)) \right\}. \end{aligned}$$

Newman and Magnus (1959) encounter several definite integrals in their study of maximal antenna gain, with the prototype

$$I_{n,m} = \frac{(-1)^{n+m}}{(kl)^3} \int_{-kl}^{kl} \left\{ (kl)^2 - x^2 \right\} \frac{4l^2 x^2 \sin^2 x}{(x^2-(n\pi)^2)(x^2-(m\pi)^2)} \, dx$$

and they offer a one-term estimate (referring to a long, though not detailed, elementary computation)

$$I_{n,m} \doteq (-1)^{n+m+1} \cdot \frac{4}{k^2} + O(l^{\epsilon-1}), \qquad \begin{array}{l} l \to \infty, \ \epsilon > 0 \\ n, m \text{ fixed.} \end{array}$$

The representation

$$\begin{aligned} I_{n,m} = {} & \frac{8}{lk^3}(-1)^{n+m+1} \left\{ \frac{kl}{2} - \frac{\sin 2kl}{4} \right\} \\ & + \frac{8}{lk^3} \frac{(-1)^{n+m}}{n^2-m^2} \left\{ \frac{n}{4\pi}\left[(kl)^2-(n\pi)^2\right]\left[\log\left|\frac{kl-n\pi}{kl+n\pi}\right| - Ci(|2kl-2n\pi|) \right.\right. \\ & \qquad \left. + Ci(2kl+2n\pi)\right] \\ & \qquad - \frac{m}{4\pi}\left[(kl)^2-(m\pi)^2\right]\left[\log\left|\frac{kl-m\pi}{kl+m\pi}\right| - Ci(|2kl-2m\pi|) \right. \\ & \qquad \left.\left. + Ci(2kl+2m\pi)\right] \right\} \end{aligned} \tag{90}$$

may be noted and its version in the case $m = n$ is likewise available.

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