General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

Produced by the NASA Center for Aerospace Information (CASI)
(a) along and (b) across the tooth length are considered. The paper is supplied with numerical examples.

**NOMENCLATURE**

\( \Delta A(i) \) \quad \text{axial displacement of gear } i

\( b_d \) \quad \text{machine setting parameter}

\( \Delta e(i) \) \quad \text{magnitude of gear eccentricity vector } \Delta e(i) \text{ of gear } i

\( \Delta E \) \quad \text{machine setting}

\( (i,j,k) \) \quad \text{unit vectors of coordinate system } S_f

\( L \) \quad \text{cone distance measured from apex to mean contact point}

\( \Delta L \) \quad \text{machine setting}

\( n \) \quad \text{surface unit normal}

\( (n_x, n_y, n_z) \) \quad \text{components of } n

\( q_d \) \quad \text{machine setting parameter}

\( r \) \quad \text{position vector locating contact point}

\( r_d \) \quad \text{mean head cutter radius}

\( d_s(i) \) \quad \text{displacement of contact point due to errors of gear } i

\( S_f \) \quad \text{fixed coordinate system}

\( S_d \) \quad \text{coordinate system fixed to generating gear}

\( (x,y,z) \) \quad \text{components of } r

\( a_1 \) \quad \text{initial position of eccentricity vector } \Delta e(i)

\( \beta \) \quad \text{gear spiral angle}

\( \gamma_i \) \quad \text{pitch angle of gear } i

\( \Delta s \) \quad \text{sum of gear dedendum angles}

\( \Delta \alpha_i \) \quad \text{dedendum angle of gear } i

\( \omega_d \) \quad \text{surface coordinate of generating surfaces}

\( \varepsilon_i \) \quad \text{surface of gear tooth } i
An analytical method was derived for determining the kinematic errors in spiral-bevel gear trains caused by the generation of nonconjugate surfaces, by axial displacements of the gear assembly, and by eccentricity of the assembled gears. Such errors are induced during manufacturing and assembly. Two mathematical models of spiral-bevel gears were included in the investigation. One model corresponded to the motion of the contact ellipse across the tooth surface (geometry I) and the other along the tooth surface (geometry II). The following results were obtained:

1. Kinematic errors induced by errors of manufacture may be minimized by applying special machine settings. The original error may be reduced by an order of magnitude. The procedure is most effective for geometry II gears.

2. When trying to adjust the bearing contact pattern between the gear teeth for geometry I gears, it is more desirable to shim the gear axially; for geometry II gears, shim the pinion axially.

* Professor of Mechanical Engineering, University of Illinois at Chicago Circle, Chicago, Illinois 60680; Member ASME.
+ Research Assistant, University of Illinois at Chicago Circle, Chicago, Illinois 60680; Associate Member ASME.
++Propulsion Laboratory, AVRAUCOM Research and Technology Laboratories, Lewis Research Center, Cleveland, Ohio 44135; Member ASME.
++Lewis Research Center, Cleveland, Ohio 44135; Fellow ASME.
3. The kinematic accuracy of spiral-bevel drives are most sensitive to eccentricities of the gear and less sensitive to eccentricities of the pinion. The precision of mounting accuracy and manufacture are most crucial for the gear, and less so for the pinion.

**INTRODUCTION**

Kinematic errors of spiral bevel gear trains are induced (a) by the applied methods of their generation, and (b) by errors in manufacture and assembly. In practice the generated tooth surfaces are not conjugate and thus result in kinematic errors. To reduce these errors special machine and tool settings must be applied during spiral bevel gear manufacture.

Problems of gear precision were solved by Litvin [1]* and Baxter [2]. Gear-train noise as a result of kinematic errors was investigated by Townsend, Coy, and Hatvani [3].

The new solution to the problem of spiral-bevel gear precision presented in this paper is based on the following principles: (a) the real (nonconjugate) tooth surfaces are replaced by conjugate surfaces; (b) these surfaces are put into mesh by modeling the errors of manufacture and assembly; (c) the influence of these errors on gear-train kinematic errors is studied using the new method of investigation applied in this paper. The investigation of kinematic errors includes (a) the determination of kinematic errors caused by the applied methods of tooth generation, (b) the determination of approximate machine settings used to compensate the kinematic errors resulting from such methods, and (c) the determination of kinematic errors exerted by gear eccentricity and by axial displacements of gears during their assembly. Two models of spiral-bevel geometry [4], corresponding to the contact point path directed

*Numbers in brackets designate References at end of paper.
Consider that the contact of gear-tooth surfaces $\Sigma_1$ and $\Sigma_2$ is localized and that they are in contact at a point at every instant. The location of theoretical contact point $M$ and the direction of the common surface unit normal are given in coordinate system $S_f$ rigidly connected to the frame. The position vector locating the contact point and the unit normal vector are denoted as $r = \overline{OM}$ and $n$, respectively. Suppose now that some errors of manufacture and assembly occur. Due to these errors, tooth surfaces $\Sigma_1$ and $\Sigma_2$ are no longer in tangency — either they interfere with each other or a clearance exists between them (Fig. 1). To bring the two surfaces into tangency, once again, it is sufficient to rotate one of them (the output gear 2) by a small additional angle $\Delta \varphi_2$. The kinematic error function $\Delta \varphi_2$ as a function of gear 1 rotation angle $\varphi_1$ may be found by applying the following equation [5]:

$$[\Delta \varphi_2 n] = (\zeta ds_{q}^{(1)} - \zeta ds_{q}^{(2)}) \cdot n$$ (1)

Here, $\zeta ds_{q}^{(i)}$ represent small changes in the position of the contact point due to errors of manufacture and assembly of gear "i" (where $i = 1,2$).

Equation (1) is applied to two spiral-bevel geometries. Geometry I corresponds to the contact point path running across the length of the gear teeth (Fig. 2(a)). Gears with this geometry are generated using two tool cones (generating surfaces) which are rigidly connected and in tangency along...
a common cone genatrix, line L (Fig. 2(b)). In the process of meshing, the contact point moves through space along line L. Geometry II corresponds to the contact point path running along the length of the gear teeth (Fig. 3(a)). These gears are generated by tool surfaces which are a cone and a surface of revolution which are in tangency along a circle L (Fig. 3(b)). In the process of meshing the contact point moves through space along circle L. Figures 4 and 5 show the coordinate systems applied to express the equations for the contact point path and surface unit normal vectors for both geometry I and geometry II gears. Coordinate system \( S_f \) is rigidly connected to the frame and system \( S_d \) is rigidly connected to the generating gear. Auxiliary coordinate system \( S_c \) (Fig. 5) is also rigidly connected to the generating gear (and to system \( S_d \)).

Expression of the equations is based on the following principles [1]:

(a) Two generating surfaces are rigidly connected and in tangency along a line L. These surfaces form the two generating gears - surface \( \Sigma_A \) generates gear 1 and surface \( \Sigma_B \) generates gear 2. (b) The four gears that are in mesh - the two generating gears and the two generated gears - all have the same instantaneous axis of rotation, axis \( z_f \). (c) Generated surfaces \( \Sigma_1 \) and \( \Sigma_2 \) always contact each other at a point that belongs to line L, while their common normal intersects axis \( z_f \), the instantaneous axis of rotation. (d) The line of action is the locus of contact points (contact point path) of surfaces \( \Sigma_1 \) and \( \Sigma_2 \) represented in coordinate system \( S_f \).

On the basis of the above principles, there results the following equations for the position vector \( r(\varphi_d) \) of a point on the line of action (contact point path) and the surface unit normal \( n(\varphi_d) \):

\[
r(\varphi_d) = x(\varphi_d)\hat{i} + y(\varphi_d)\hat{j} + z(\varphi_d)\hat{k}
\]
\[ n(\vartheta_d) = n_x(\vartheta_d) j + n_y(\vartheta_d) j + n_z(\vartheta_d) k \]

Here \( \vartheta_d \) is the angle of rotation of the generating gear.

For Geometry I:

\[
\begin{align*}
    x &= r_d - b_d \frac{\sin(q_d - \vartheta_d)}{\sin \tau_d} \sin \psi \cos \psi \\
    y &= \frac{\sin \tau_d}{\tan \psi_c} x \\
    z &= \frac{b_d \sin \vartheta_d}{\sin \tau_d} + \frac{\cos \tau_d}{\tan \psi_c} x \\
    n_x &= \sin \psi_c \\
    n_y &= \cos \psi_c \sin \tau_d \\
    n_z &= \cos \psi_c \cos \tau_d \\
    \tau_d &= \vartheta_d - q_d + \varphi_d; \quad \vartheta_d - q_d = 90^\circ - \beta
\end{align*}
\]

Here, \( \psi_c \) is the gear pressure angle; \( q_d \) and \( b_d = 0 \) are parameters of machine settings; \( r_d = \rho_c M \) is the mean head cutter radius; \( \beta \) is the gear spiral angle.

For Geometry II:

\[
\begin{align*}
    x &= 0 \\
    y &= 0 \\
    z &= r_d \cos \tau_d + b_d \cos(q_d - \vartheta_d) \\
    n_x &= \sin \psi_c \\
    n_y &= \cos \psi_c \sin \tau_d \\
    n_z &= \cos \psi_c \cos \tau_d
\end{align*}
\]
Here \( \phi_d \) is constant, and \( \phi_d \) and \( \phi_d \) are related by

\[
\begin{align*}
\phi_d \sin \tau_d - b_d \sin(q_d - \phi_d) &= 0
\end{align*}
\]

(7)

The line of action of Geometry II gears coincides with axis \( z_f \).

**EVALUATION OF KINEMATIC ERROR FUNCTIONS**

For most gears, kinematic error functions \( \Delta \phi_2(\phi_d) \) (defined by eq. (1)) are piecewise (noncontinuous) functions with discontinuities at the points of changing gear teeth. To evaluate these functions one must examine (a) the range of the kinematic error function over the mesh of one gear tooth, and (b) the size of the jump at the points of function discontinuities. Note that the magnitudes of the error functions are of secondary importance—more important are the changes in these functions which are given by (a) and (b) above. Large changes and jumps in kinematic error functions are a source of excessive tooth surface wear, vibrations, noise, and the impact loading of gear teeth. Because of these maladies it is important to understand and evaluate the nature of the kinematic errors caused by gear manufacture and different types of mounting errors.

To apply the general equation (1), a sample gearset was chosen as follows:

- \( N_1 \) no teeth gear 1 = 20
- \( N_2 \) no teeth gear 2 = 40
- \( M_{12} \) \( \omega_1/\omega_2 = 2.0 \)
- \( \varepsilon \) shaft angle = 90°
- \( \psi_c \) pressure angle = 20°
- \( \beta \) mean spiral angle = 35°
- \( L \) cone distance – apex-to-main contact point = 4 in.
- \( r_A \) mean radius of head cutter for \( \varepsilon_A = 4 \) in.
- \( q_A \) (see Fig. 5) = 62.5°
b_A = (see Fig. 5) = 3.6939 in.
γ_1  pitch angle gear 1 = 26.57°
γ_2  pitch angle gear 2 = 63.43°

Angle of rotation for one tooth of gear 1: -9° ≤ γ_1 ≤ 9°

Kinematic Errors Due to Methods of Tooth Generation

Consider Figs. 4 and 5. Conjugate gear tooth surfaces may be generated if
axis-x_f is the common axis of rotation of the generating surfaces, and if
axis-z_f is the instantaneous axis of rotation belonging to both the generating
surfaces and the generated tooth surfaces. In practice, when generating gears 1
and 2, the axes of rotation of the generating surfaces make angles of Δ1 and
Δ2, respectively, with axis x_f (Fig. 6). Here Δ_i (i = 1, 2) are the
dedendum angles of the respective gears. To simulate these manufacturing errors,
rotate the generating surface of gear 1 (with respect to that of gear 2) by an
amount

$$Δδ = Δδj = (Δ1 + Δ2)j$$

Therefore the displacement of the theoretical contact point due to errors may
be represented as

$$ds(1) = Δδ x r = Δδ x (zi - xk)$$

Then, from equations (7) and (1)

$$Δϕ_2(θ_d) = \frac{(zn_x - zn_z)Δδ}{-y cos γ_2n_x + (x cos γ_2 + z sin γ_2)n_y - y sin γ_2n_z}$$

Here θ_d = θ_1 sin γ_1; γ_2 is the pitch angle of gear 2. This equation
was applied to the example gearset with approximate dedendum angles calculated by
\[ \tan \gamma_1 = \frac{2.5 \sin \gamma_1}{N} \]

resulting in

\[ \Delta \theta = \Delta_1 + \Delta_2 = 0.0559 + 0.0558 = 0.1117 \text{ rad} \]

Fig. 7 shows the plot of \( \Phi_2 \) vs. \( \Phi_1 \) due to errors of manufacture for both geometry I and geometry II. Both curves are nearly linear but with opposite slopes. During the mesh of one pinion tooth the kinematic error function changes by approximately 14 to 19 arc-minutes. Now the objective is to minimize these kinematic errors by applying special machine settings. These machine settings are represented by translating the pinion (gear 1) by the amount

\[ \Delta s_1(1) = \Delta E_{j} + \Delta L_{k} \]  \hspace{1cm} (11)

This yields that the total kinematic error function due to \( \Delta \theta, \Delta E, \) and \( \Delta L \) is given by

\[ \Phi_2(\varphi_d) = \frac{\Delta E_{y} + \Delta L_{z} - (\Delta n_{x} - \Delta n_{z}) \Delta \theta}{-y \cos \gamma_2 n_{x} + (x \cos \gamma_2 + z \sin \gamma_2) n_{y} - y \sin \gamma_2 n_{z}} \]  \hspace{1cm} (12)

To find the appropriate values of \( \Delta E \) and \( \Delta L \), two conditions are imposed on equation (12):

\[ \Phi_2 = 0 \]  \hspace{1cm} (13)

\[ \frac{d(\Phi_2)}{\varphi_1} = 0 \]  \hspace{1cm} (14)

at the "midpoint" of gear tooth 1 rotation, \( \varphi_1 = 0 \). Applying equations (12) to (14) and (2) to (7) results in
Geometry I:

\[
\frac{\Delta E}{L} = \frac{\tan \psi_c \cos 2\beta}{\cos \beta} \Delta \delta \tag{15}
\]

\[
\frac{\Delta L}{L} = 2 \tan \psi_c \sin \beta \Delta \delta \tag{16}
\]

Geometry II:

\[
\frac{\Delta E}{L} = (\cos \beta - \sin \beta \tan \varphi_0) \tan \psi_c \Delta \delta \tag{17}
\]

\[
\frac{\Delta L}{L} = (\sin \beta + \cos \beta \tan \varphi_0) \tan \psi_c \Delta \delta \tag{18}
\]

For the example gearset

Geometry I:

\[\Delta E = 0.0679 \text{ in.}\]
\[\Delta L = 0.1866 \text{ in.}\]

Geometry II:

\[\Delta E = -0.0460 \text{ in.}\]
\[\Delta L = 0.3492 \text{ in.}\]

The plot of kinematic errors after the application of machine settings is shown in Fig. 8. As may be observed the \( \Delta \psi_2 \) function is now of near-parabolic shape and the range of the original error function is reduced between 10 and 15 times. Since the range of error is smaller for gears of geometry II, it is concluded that compensating for errors by adjusting machine settings is most effective for gears of geometry II. Thus the application of special machine settings is very effective in the reduction of the kinematic errors caused by the method of generation applied in practice. Fig. 9 shows the kinematic error function \( \Delta \psi_2(\varphi_1) \) plotted over the mesh of several gear teeth. Notice that, at the points of changing gear teeth, there is no jump in the value of this function. However, there are discontinuities in its slope.
Kinematic Errors Due to Axial Displacements of Gears

In practice, during the mounting of spiral-bevel gears and during their testing on Gleason Works machines, it is common to change their axial positions in order to correct the location and size of the bearing contact pattern between the gear teeth [6]. Such axial displacements can induce kinematic errors in the gear train which remain unnoticed unless the relations between the angles of gear rotation are examined. It is for this reason that it is important to investigate the nature of kinematic errors which accompany such mounting changes.

Changes in the axial position of gear "i" are represented by vectors $\Delta A(i)$ ($i = 1, 2$). Vectors $\Delta A(i)$ point out from the apex of their respective pitch cones. The kinematic error function for this case is given by

$$
\Delta \theta_2(\psi_1) = \frac{(\Delta A(1) \sin \gamma_1 + \Delta A(2) \sin \gamma_2)n_x + (\Delta A(1) \cos \gamma_1 - \Delta A(2) \cos \gamma_2)n_z}{-(y \cos \gamma_2)n_x + (x \cos \gamma_2 + z \sin \gamma_2)n_y - (y \sin \gamma_2)n_z}
$$

(19)

The kinematic error functions which are induced by axial displacements of the pinion and gear (gear 1 and 2, respectively) are shown in Fig. 10. Taking into account that the main criterion for evaluating a kinematic error function is not so much its magnitude, but rather, the amount it changes during the mesh cycle, the following conclusion is true for the example gearset: When trying to improve the bearing contact for geometry I gears it is more desirable to displace the gear axially; for geometry II gears it is more desirable to displace the pinion axially.

Kinematic Errors Due to Gear Eccentricities

A gear is said to be eccentric when its geometric axis (the axis about which it rotates during cutting) does not coincide with its axis of rotation
during operation. Gear eccentricity may be induced both by errors of manufacture and assembly. Denoting the magnitudes of gear eccentricities by $\Delta e_i(1)$ ($i = 1, 2$), the displacement of the contact point is given by the vectors $\Delta e_i(1)$ which rotate about gear axis "i". Further, the original position of the eccentricity vector is given by angles $\alpha_i(i = 1, 2)$. Applying equation (1) results in

$$\Delta e_i(\varphi_i) = \frac{n_x \Delta e_x + n_y \Delta e_y + n_z \Delta e_z}{-(y \cos \gamma_2)n_x + (x \cos \gamma_2 + z \sin \gamma_2)n_y - (y \sin \gamma_2)n_z}$$ (20)

where

$$\Delta e = \Delta e_{(1)}(1) - \Delta e_{(2)}(2); (k = x, y, z)$$

and

$$\begin{align*}
\Delta e_x &= \Delta e_1 \cos(\varphi_1 + \alpha_1) \cos \gamma_1 + \Delta e_2 \cos(\varphi_2 + \alpha_2) \cos \gamma_2 \\
\Delta e_y &= -\Delta e_1 \sin(\varphi_1 + \alpha_1) - \Delta e_2 \sin(\varphi_2 + \alpha_2) \\
\Delta e_z &= -\Delta e_1 \cos(\varphi_1 + \alpha_1) \sin \gamma_1 - \Delta e_2 \cos(\varphi_1 + \alpha_1) \sin \gamma_2
\end{align*}$$ (21)

To simplify expressions (20) and (21), one may approximate by considering the angle of rotation of the generating gear for one tooth as small: $\varphi_1 \sin \gamma_1 = 0$ so that the kinematic error function is now given by

$$\Delta \psi_2(\varphi_1) = \frac{a_1 \sin(\varphi_1 + \alpha_1) + b_1 \cos(\varphi_1 + \alpha_1) + a_2 \sin(\varphi_2 + \alpha_2) + b_2 \cos(\varphi_2 + \alpha_2)}{L \sin \gamma_2 \cos \gamma_2 \cos \theta}$$ (22)

where

$$\varphi_2 = \frac{N_1}{N_2} \varphi_1$$
Unlike the case of spur gears [5] the actual kinematic error function (eq. (20)) has discontinuities at the points where gear teeth change during meshing. However, the overall shape of this function is given by equation (22). This approximate function is a sum of four harmonics: The period of two of them coincides with the period of revolution of gear 1, and two of them with the period of revolution of gear 2. Figure 11 shows the results of plotting functions (20) and (22) for both geometry I and geometry II gears. Here

\[ \Delta e_1 = 0.002 \text{ in.} \quad \alpha_1 = \alpha_2 = 0^\circ \]

\[ \Delta e_2 = 0.0 \text{ in.} \]

Figure 12 shows the smooth kinematic error function for

\[ \Delta e_1 = 0.002 \text{ in.} \quad \alpha_1 = 0^\circ \]

\[ \Delta e_2 = 0.002 \text{ in.} \quad \alpha_2 = 180^\circ \]

It is clear that the angle of the gear "contribution" to the kinematic error function is somewhat larger than the pinion contribution. Thus, in general, the bearings and overall mounting precision of the gear are more crucial to spiral-bevel gear-drive accuracy than those of the pinion.

The calculations for the examples presented in the paper are summarized in the Appendix.

**SUMMARY OF RESULTS**

An analytical method was derived for determining the kinematic errors in spiral-bevel gear trains caused by the generation of non-conjugate surfaces, by...
axial displacements of the gear assembly, and by eccentricity of the assembled gears. Such errors are induced during manufacturing and assembly. Two mathematical models of spiral-bevel gears were included in the investigation. One model corresponded to the motion of the contact ellipse across the tooth surface (geometry I) and the other along the tooth surface (geometry II). The following results were obtained:

1. Kinematic errors induced by errors of manufacture may be minimized by applying special machine settings. The original error may be reduced by an order of magnitude. The procedure is most effective for geometry II gears.

2. When trying to adjust the bearing contact pattern between the gear teeth for geometry I gears, it is more desirable to shim the gear axially; for geometry II gears, shim the pinion axially.

3. The kinematic accuracy of spiral-bevel drives are most sensitive to eccentricities of the gear and less sensitive to eccentricities of the pinion. The precision of mounting accuracy and manufacture are most crucial for the gear, and less so for the pinion.
### APPENDIX - SUMMARY OF EXAMPLE CALCULATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_1$</td>
<td>Number of pinion teeth</td>
<td>20</td>
</tr>
<tr>
<td>$N_2$</td>
<td>Number of gear teeth</td>
<td>40</td>
</tr>
<tr>
<td>$M_{12}$</td>
<td>Angular velocity ratio $= N_2/N_1$</td>
<td>2</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Shaft angle</td>
<td>90°</td>
</tr>
<tr>
<td>$\gamma_c$</td>
<td>Pressure angle</td>
<td>20°</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Mean spiral angle</td>
<td>35°</td>
</tr>
<tr>
<td>$L$</td>
<td>Cone distance - apex to main contact point</td>
<td>4 in.</td>
</tr>
<tr>
<td>$r_A$</td>
<td>Mean head cutter radius for pinion</td>
<td>4 in.</td>
</tr>
<tr>
<td>$q_A$</td>
<td>Machine setting (see Fig. 5(c))</td>
<td>62.5°</td>
</tr>
<tr>
<td>$b_A$</td>
<td>Machine setting (see Fig. 5(c))</td>
<td>3.6939 in.</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>Pitch angle of pinion</td>
<td>26.57°</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>Pitch angle of gear</td>
<td>63.43°</td>
</tr>
<tr>
<td>$\Delta_1$</td>
<td>Dedendum angle of pinion</td>
<td>0.0559 rad.</td>
</tr>
<tr>
<td>$\Delta_2$</td>
<td>Dedendum angle of gear</td>
<td>0.0558 rad.</td>
</tr>
</tbody>
</table>

To obtain the kinematic error functions, the formulas defining them must be evaluated for different values of $\varphi_1$ corresponding to the rotation of the pinion. For kinematic errors caused by methods of tooth generation and by axial displacements the period of the kinematic error function is equal to the pitch angle of the pinion. For errors caused by gear eccentricity the error function has a period which depends on the angular velocity ratio $M_{12}$ (see ref. [5]).
Range of pinion rotation

\[-360^\circ \leq \varphi_1 \leq \frac{360^\circ}{2N_1}\]

\[-9^\circ \leq \varphi_1 \leq 9^\circ\]

All kinematic error functions are evaluated here for

\[\varphi_1 = 3^\circ\]

\[\varphi_A = \varphi_1 \sin \gamma_1 = 1.3419^\circ\]

\[\varphi_A = \begin{cases} 
\text{Geom I: } 90 + q_A - \beta \\
\text{Geom II: } \sin^{-1} \left( \frac{b_A \sin(q_A - \varphi_A)}{r_A} \right) + q - \varphi 
\end{cases}\]

\[\varphi_A = \theta_A - q_A + \varphi_A\]

Contact Point and Unit Normal

\[
\begin{aligned}
\{x\} & \quad \text{Geom I: eqs. (2)} \quad \{0.03620\} \\
\{y\} & \quad \text{Geom I: eqs. (2)} \quad \{0.08285\} \text{ in.} \\
\{z\} & \quad \text{Geom I: eqs. (2)} \quad \{3.99\} \\
\{n_x\} & \quad \text{Geom I: eqs. (3)} \quad \{0.3420\} \text{ in.} \\
\{n_y\} & \quad \text{Geom I: eqs. (3)} \quad \{0.7822\} \text{ in.} \\
\{n_x\} & \quad \text{Geom I: eqs. (3)} \quad \{0.5208\} \\
\end{aligned}
\]

Kinematic Errors Due to Methods of Tooth Generation

\[\Delta \delta = \Delta_1 + \Delta_2 = 0.1117 \text{ rad}\]

Before machine settings (see Fig. 7)

\[\Delta \phi_2 \text{ eq. (10)}\]

\[\begin{aligned}
\text{Geom I: } & -3^\circ 7' 43'' \\
\text{Geom II: } & -3^\circ 13' 11''
\end{aligned}\]
Machine settings

Geom I: $\Delta E$ eq. (15) $0.0679 \text{ in.}$
$\Delta L$ eq. (16) $0.1866 \text{ in.}$

Geom II: $\Delta E$ eq. (17) $-0.0460 \text{ in.}$
$\Delta L$ eq. (18) $0.3492 \text{ in.}$

After machine settings (see Fig. 8)

$\Delta \varphi_2$ eq. (12)

Geom I: $-6^\circ$
Geom II: $4^\circ$

Kinematic Errors Due to Axial Displacements

Pinion only (see Fig. 10(a)):
Given $\Delta A^{(1)} = 0.20 \text{ in.}$ $\Delta A^{(2)} = 0$

$\Delta \varphi_2$ eq. (19)

Geom I: $2^\circ 34^\prime 30^\prime$
Geom II: $2^\circ 38^\prime 20^\prime$

Gear only: (see Fig. 10(b))
Given $\Delta A^{(1)} = 0$ $\Delta A^{(2)} = 0.20 \text{ in.}$

$\Delta \varphi_2$ eq. (19)

Geom I: $18^\circ 13^\prime$
Geom II: $14^\circ 23^\prime$

Kinematic Errors Due to Gear Eccentricity

Since for eccentric gears the kinematic error function $\Delta \varphi_2(\varphi_1)$ changes from tooth to tooth, one must take into account which tooth of the gear is in mesh. Specifically, $\varphi_1^*$ denotes the total angle of rotation of the pinion and is the value to be used in equation (22) where $\varphi_1$ appears. Here

$$\varphi_1^* = (n - 1) \frac{360^\circ}{N_1} + \frac{\varphi_A}{\sin \gamma_1}$$

$$\varphi_2 = \varphi_1^*/M_{12}$$
where \( n = 1, 2, \ldots, N_1 \) is the pinion tooth number being considered and \( \phi_2 \) is the total angle of rotation of the gear.

Given

\[
\begin{align*}
\gamma_1 &= \gamma_2 = 0 \\
\Delta e_1 &= 0.002 \text{ in.} \\
\Delta e_2 &= 0
\end{align*}
\]

For \( \phi_A = 1.3419^\circ \), \( N = 4 \), \( \phi_1 = 57^\circ \), \( \phi_2 = 28.5^\circ \), and applying equations (20) and (21) (see figs. 11(a) and (b))

\[
\Delta \phi_2 = \text{Geom I: } -1' 32'' \\
\Delta \phi_2 = \text{Geom II: } -1' 29''
\]

For the smooth kinematic error function, apply equations (22) and (23), such that

\[
\begin{align*}
a_1 &= -1.5395 \times 10^{-3} \text{ in.} \\
b_1 &= 1.2963 \times 10^{-4} \text{ in.} \\
a_2 &= 0.0 \\
b_2 &= 0
\end{align*}
\]

Using equation (22) with \( \phi_1 = 57^\circ \) results in (fig. 11(c))

\[
\Delta \phi_2 = -1' 31''
\]

REFERENCES

Figure 1. - Tooth surfaces with clearance induced by errors.

Figure 2. - Geometry I: bearing contact and generating surfaces.
Figure 3. - Geometry II: bearing contact and generating surfaces.

Figure 4. - Applied coordinate systems.
Figure 5. - Applied coordinate systems.

Figure 6. - Axis of rotation of generating gear and member gear.
Figure 7. - Kinematic error function due to manufacture without special machine settings.

Figure 8. - Kinematic error function due to manufacture with special machine settings.
Figure 9. - Kinematic error function for the mesh of several gear teeth.

Figure 10. - Kinematic error functions due to axial pinion/gear displacements.