(NASA-TM-83949) AN APPLICATICN OF THE BRAUNBECK METHOD TO THE MAGGI-RUBINOWICZ FIELD REPRESENTATION (NASA) 24 p HC A02/MF A01 CSCL 20F G3/74

N82-31108

Unclas 30365

NASA

Technical Memorandum 83949

AN APPLICATION OF THE BRAUNBECK METHOD TO THE MAGGI-RUBINOWICZ FIELD REPRESENTATION

Robert Meneghini

JUNE 1982

National Aeronautics and
Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

TM 83949

AN APPLICATION OF THE BRAUNBEK METHOD TO THE MAGGI-RUBINOWICZ FIELD REPRESENTATION

Robert Meneghini

June 1982

GODDARD SPACE FLIGHT CENTER
Greenbelt, Maryland

AN APPLICATION OF THE BRAUNBEK METHOD TO THE MAGGI-RUBINOWICZ FIELD REPRESENTATION

ABSTRACT

The Braunbek method is applied to the generalized vector potential associated with the Maggi-Rubinowicz representation. Under certain approximations, an asymptotic evaluation of the vector potential is obtained. For observation points away from caustics or shadow boundaries, the field derived from this quantity is the same as that determined from the geometrical theory of diffraction on a singly diffracted edge ray. The paper concludes with an evaluation of the field for the simple case of a plane wave normally incident on a circular aperture, showing that the field predicted by the Maggi-Rubinowicz theory is continuous across the shadow boundary.

AN APPLICATION OF THE BRAUNBEK METHOD TO THE MAGGI-RUBINOWICZ FIELD REPRESENTATION

INTRODUCTION

The Maggi-Rubinowicz (M-R) representation(1-3) for the field is derived by means of Stokes' theorem which transforms an open surface integral, the integrand of which has zero divergence, into a line integral around the boundary of the surface and certain singular contributions interior to this boundary. This type of field representation is usually applied under a physical optics approximation, which, in the case of diffraction by an aperture, is taken to denote the approximation by which the actual field within the aperture is replaced by the unperturbed incident field. There is nothing in the theory, however, that requires this assumption.

Another common high frequency approximation is the Braunbek method(4-5), which takes the field near an edge point as that which would exist if the edge were replaced by a half plane oriented in the plane of the aperture with its edge lying along the tangent vector at the point of interest. An application of this method to the various surface integral representations for the field has been shown to produce results that closely resemble those predicted by the geometrical theory of diffraction (GTD)(6-7). Since the M-R theory is, in a sense, only a restatement of certain surface integral representations, there is reason to expect that the Braunbek method should also serve to illuminate the relationships between the M-R theory and GTD.

In this paper, the Braunbek method is applied to the generalized vector potential of M-R(1). Under certain approximations, a closed form expression for this quantity is obtained. With this vector potential, an asymptotic evaluation of the field is carried out for observation points away from caustics and shadow boundaries. It is found that the expression for the field is identical to a result derived by Keller et al.(6). In the last section of the paper, an explicit calculation of the M-R field is carried out for a simple case, showing that the field is continuous across the shadow boundary.

THE RAYLEIGH INTEGRAL REPRESENTATION

Consider a thin screen in the $z = 0$ plane with an aperture A. The tangent to the rim of the aperture is taken to be continuous. If the sources of the incident field are located in the $z < 0$ half space, and if the field is zero on the screen, then the Rayleigh representation for the field in the $z > 0$ half space is[8]

$$u(\underline{x}) = 2 \int_A u(\underline{x}') \nabla' G \cdot \hat{z}\, ds' \tag{1}$$

The region of integration is the aperture; $\underline{x}$, $\underline{x}'$ are, respectively, the position vectors of the observer and of a "source" point in the aperture. The primed derivatives are to be taken with respect to the "source coordinates". The quantity G is the Green's function, where, for an $e^{-i\omega t}$ time convention

$$G = e^{ikr}/4\pi r \tag{2}$$

and

$$\nabla' G = -\hat{r}\,(ik - 1/r)\, G \tag{3}$$

$$\hat{r} = (\underline{x} - \underline{x}')/|\underline{x} - \underline{x}'| \tag{4}$$

$$r = |\underline{x} - \underline{x}'| \tag{5}$$

The quantity k in (2) and (3) is the wavenumber which equals $2\pi/\lambda$ where λ is the wavelength of the incident field.

The M-R analogue of (1) is[8]

$$u(\underline{x}) = \oint (\underline{W} - \underline{W}_I) \cdot \hat{\ell}\, d\ell' + \sum_i \oint \underline{W} \cdot \hat{\ell}\, d\ell' \tag{6}$$

The path of integration of the first term, the boundary diffraction integral, is taken in a counter-clockwise sense (when viewed from $z > 0$) around the rim of the aperture. The second term represents the integrations around the singularities of $\underline{W}$, taken in a clockwise sense. Following Miyamoto and Wolf[1] the singularities of $\underline{W}$ are assumed to be isolated poles, finite in number. A corresponding term that accounts for the singularities in $\underline{W}_I$ is absent in (6) because $\underline{W}_I$ is never singular over the aperture[8].

The generalized vector potentials for the observation point and its image, W, W_I, are given by[1]

$$\underline{W} = -G\,\hat{s} \times \int_0^\infty e^{ikt}\,\nabla' u(\underline{x}' + t\hat{s})\,dt + \underline{W}_\infty \tag{7}$$

$$\underline{W}_I = -G\,\hat{s}_I \times \int_0^\infty e^{ikt}\,\nabla' u(\underline{x}' + t\,\hat{s}_I)dt + (\underline{W}_\infty)_I \tag{8}$$

where

$$\hat{s} = -\hat{r} \tag{9}$$

$$\hat{s}_I = -\hat{r}_I = \hat{s} - 2(\hat{s} \cdot \hat{z})\,\hat{z} \tag{10}$$

The end point contributions to the vector potentials, $\underline{W}_\infty$ and $(\underline{W}_\infty)_I$ are defined by Miyamoto and Wolf[1]. In this paper they will be neglected. The effect these additional terms might have on the solution is discussed later on.

Since the Braunbek method is valid only in the vicinity of the edge, it is applied only to the first term of (6). For the second term of (6), the vector potential corresponding to the physical optics approximation will be used.

APPLICATION OF THE BRAUNBEK METHOD TO THE VECTOR POTENTIAL

For aperture diffraction, the physical optics approximation takes the field in the aperture to be equal to the unperturbed incident field. For the same problem, the Braunbek method assumes the field near a point ℓ on the aperture rim as that which would exist by replacing the edge by a half plane oriented in the plane of the screen with its edge aligned along the tangent vector at ℓ.

In this paper, the incident field, u_o, is approximated in the vicinity of an edge point ℓ by $u_o = A(\ell)\, e^{ik\psi(\ell)}$. The phase $\psi(\ell)$ is assumed to satisfy the eikonal equation

$$\nabla'\psi \cdot \nabla'\psi = 1 \tag{11}$$

so that a unit vector $\hat{p}$ can be defined by $\nabla'\psi = \hat{p}$. Again, following Keller et al.[6], the incident field at ℓ is associated with a plane wave propagating the direction $\hat{p}$ with an amplitude $A(\ell)$.

The solution to this half plane problem, with u = 0, on the plane, can be written in the form

(Appendix)

$$u = u^i - u^r \tag{12}$$

$$u^{\overset{i}{r}} = \pi^{-1/2} A(\ell) e^{i(k\psi(\ell) - \pi/4)} e^{-ik[\rho \cos(\phi \mp \phi_o) \sin\theta_o - z \cos\theta_o]} F_c(\alpha_{\underset{2}{1}}) \tag{13}$$

$$\alpha_1 = -\epsilon^i \gamma_1 \tag{14}$$

$$\alpha_2 = -\epsilon^r \gamma_2 \tag{15}$$

$$\gamma_{\underset{2}{1}} = (2k\, \rho \sin\theta_o)^{1/2} \left| \cos \tfrac{1}{2}(\theta \mp \theta_o) \right| \tag{16}$$

$$\epsilon^{\overset{i}{r}} = \operatorname{sgn}\left(\cos \tfrac{1}{2}(\phi \mp \phi_o)\right) \tag{17}$$

$$\operatorname{sgn}(x) = \begin{matrix} +1 & x > 0 \\ -1 & x < 0 \end{matrix} \tag{18}$$

$$F_c(x) = \int_x^\infty e^{it^2}\, d\tau$$

The notation follows that of James(9) and is defined in the Appendix.

To use (12) through (17) in (7) and (8) it is convenient to first express the above formulas in terms of quantities independent of a particular coordinate system. At each point ℓ on the rim, we choose the orthogonal vectors $(\hat{\ell}, \hat{n}, \hat{z})$ where $\hat{\ell}$ is the tangential unit vector of the rim at ℓ (in a counter-clockwise sense when viewed from $z > 0$) and $\hat{n}$ is the unit vector from ℓ directed into the aperture. Defining $\hat{\xi}$ to be a unit vector from ℓ to an arbitrary point and t the associated distance along the vector to this point, then

$$\hat{\xi} \cdot \hat{z} = -\sin\theta \sin\phi \tag{19}$$

$$\hat{\xi} \cdot \hat{n} = -\sin\theta \cos\phi \tag{20}$$

$$\hat{\xi} \cdot \hat{\ell} = \cos\theta \tag{21}$$

$$\hat{p} \cdot \hat{z} = \sin\theta_o \sin\phi_o \tag{22}$$

$$\hat{p} \cdot \hat{n} = \sin\theta_o \cos\phi_o \tag{23}$$

$$\hat{p} \cdot \hat{\ell} = \cos\theta_o \tag{24}$$

and

$$\rho = t \sin\theta \tag{25}$$

$$z = t \cos\theta \tag{26}$$

where

$$\hat{p} = \nabla' \psi \tag{27}$$

Expressing the variables in (12) through (17) in terms of the quantities $\hat{\xi}$, $\hat{p}$, $\hat{\ell}$, $\hat{n}$, $\hat{z}$, t, then

$$u^i(\hat{\xi}, t) = \pi^{-1/2} A\, e^{i(k\psi - \pi/4)}\, e^{ik\, t\, \hat{p} \cdot \underline{\underline{I}} \cdot \hat{\xi}}\, F_c(\alpha_1) \tag{28}$$

$$u^r(\hat{\xi}, t) = \pi^{-1/2} A\, e^{i(k\psi - \pi/4)}\, e^{ik\, t\, \hat{p} \cdot \hat{\underline{\underline{I}}} \cdot \hat{\xi}}\, F_c(\alpha_2) \tag{29}$$

where

$$\underline{\underline{I}} = \hat{\ell}\hat{\ell} + \hat{n}\hat{n} + \hat{z}\hat{z} \tag{30}$$

$$\hat{\underline{\underline{I}}} = \hat{\ell}\hat{\ell} + \hat{n}\hat{n} - \hat{z}\hat{z} \tag{31}$$

$$\alpha_1(\hat{\xi}, t) = \epsilon^i(\hat{\xi})\, \gamma_1(\hat{\xi}, t) \tag{32}$$

$$\alpha_2(\hat{\xi}, t) = \epsilon^r(\hat{\xi})\, \gamma_2(\hat{\xi}, t) \tag{33}$$

$$\gamma_1(\hat{\xi}, t) = (kt)^{1/2}\, (T(\hat{\xi}) - \hat{p} \cdot \underline{\underline{I}}_t \cdot \hat{\xi})^{1/2} \tag{34}$$

$$\gamma_2(\hat{\xi}, t) = (kt)^{1/2}\, (T(\hat{\xi}) - \hat{p} \cdot \hat{\underline{\underline{I}}}_t \cdot \hat{\xi})^{1/2} \tag{35}$$

$$\begin{aligned} \epsilon^{\substack{i \\ r}}(\hat{\xi}) = \operatorname{sgn} \Big\{ & \operatorname{sgn}(\hat{\xi} \cdot \hat{z})\, ((\hat{\xi} \cdot \underline{\underline{I}}_t \cdot \hat{\xi})^{1/2} - \hat{\xi} \cdot \hat{n})^{1/2}\, ((\hat{p} \cdot \underline{\underline{I}}_t \cdot \hat{p})^{1/2} + \hat{p} \cdot \hat{n})^{1/2} \\ & \pm ((\hat{\xi} \cdot \underline{\underline{I}}_t \cdot \hat{\xi})^{1/2} + \hat{\xi} \cdot \hat{n})^{1/2}\, ((\hat{p} \cdot \underline{\underline{I}}_t \cdot \hat{p})^{1/2} - \hat{p} \cdot \hat{n})^{1/2} \Big\} \end{aligned} \tag{36}$$

with

$$T(\hat{\xi}) = (\hat{p} \cdot \underline{\underline{I}}_t \cdot \hat{p})^{1/2}\, (\hat{\xi} \cdot \underline{\underline{I}}_t \cdot \hat{\xi})^{1/2} \tag{37}$$

$$\underline{\underline{I}}_t = \underline{\underline{I}} - \hat{\ell}\hat{\ell} \tag{38}$$

$$\hat{\underline{\underline{I}}}_t = \hat{\underline{\underline{I}}} - \hat{\ell}\hat{\ell} \tag{39}$$

To compute the $\nabla' u$ appearing in the integral of (7) and (8), a spherical coordinate system is chosen with the origin at a point ℓ on the rim with the polar axis in the $\hat{\ell}$ direction. In this sytem, the arguments of $\nabla' u$, $\underline{x}' + t\,\hat{s}$ and $\underline{x}' + t\,\hat{s}_I$ can be written as $t\hat{\xi}$ with $\hat{\xi}$ evaluated at $\hat{s}$ and $\hat{s}_I$, respectively. However, we will continue to write the arguments of u as $\hat{\xi}$, t since these quantities do not always appear together. In spherical coordinates

$$\nabla' u(\hat{\xi}, t) = \frac{\partial u}{\partial t}\hat{\xi} + \frac{1}{t}\frac{\partial u}{\partial \theta}\hat{\theta} + \frac{1}{t \sin\phi}\frac{\partial u}{\partial \phi}\hat{\phi} \tag{40}$$

where θ is the polar angle measured from the $\hat{\ell}$ vector to $\hat{\xi}$ and $\hat{\phi}$ is the azimuthal angle measured between the $-\hat{n}$ direction and the projection of $\hat{\xi}$ onto the $\hat{n}$–$\hat{z}$ plane. For an observer on the $\hat{\ell}$ axis, a positive ϕ angle is measured in a counterclockwise sense from the $-\hat{n}$ axis. Note that these are the same angles as defined by (19)-(21). Equation (6) shows that only the $\hat{\ell}$ component of $(\underline{W} - \underline{W}_I)$ is needed to compute the field. Since $\hat{\xi} \times \hat{\xi} = 0$, $\hat{\xi} \times \hat{\theta} \cdot \hat{\ell} = 0$ and $\hat{\xi} \times \hat{\phi} \cdot \hat{\ell} = \sin\theta$, then for the vector potential $\underline{W}$,

$$[\hat{\xi} \times \nabla' u(\xi, t)] \cdot \hat{\ell} = \frac{1}{t} \frac{\partial u(\hat{\xi}, t)}{\partial \phi} \tag{41}$$

so that

$$\underline{W} \cdot \hat{\ell} = -G \int_0^{\infty} \frac{e^{ikt}}{t} \frac{\partial u(\hat{\xi}, t)}{\partial \phi}\Big|_{\hat{\xi}=\hat{s}} dt \tag{42}$$

A similar simplification can be carried out for $\underline{W}_I$. Therefore,

$$(\underline{W} - \underline{W}_I) \cdot \hat{\ell} = -G \int_0^{\infty} \frac{e^{ikt}}{t} \left(\frac{\partial u}{\partial \phi}(\hat{\xi},t)\Big|_{\hat{\xi}=\hat{s}} - \frac{\partial u}{\partial \phi}(\hat{\xi},t)\Big|_{\hat{\xi}=\hat{s}_I} \right) dt \tag{43}$$

In computing $\frac{\partial}{\partial \phi} u$, the following relationships are used

$$\frac{\partial}{\partial \phi} (\hat{p} \cdot \underline{\underline{I}} \cdot \hat{\xi}) = \hat{p} \cdot (\underline{\underline{I}}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{44}$$

$$\frac{\partial}{\partial \phi} (\hat{p} \cdot \hat{\underline{\underline{I}}} \cdot \hat{\xi}) = \hat{p} \cdot (\hat{\underline{\underline{I}}}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{45}$$

$$\frac{\partial}{\partial \phi} \alpha_1(t\hat{\xi}) = \frac{\epsilon^i(\hat{\xi})\, kt}{2\alpha_1} \hat{p} \cdot (\underline{\underline{I}}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{46}$$

$$\frac{\partial}{\partial \phi} \alpha_2(t\hat{\xi}) = \frac{\epsilon^r(\hat{\xi})\, kt}{2\alpha_2} \hat{p} \cdot (\hat{\underline{\underline{I}}}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{47}$$

which give

$$\frac{\partial}{\partial \phi} u(\hat{\xi},t) = \pi^{-1/2} A\, e^{i(k\psi - \pi/4)} \cdot \Bigg\{ ikt \left[B(\hat{\xi})\, e^{ikt\, P(\hat{\xi})}\, F_c(\alpha_1) - C(\hat{\xi})\, e^{ikt\, Q(\hat{\xi})}\, F_c(\alpha_2) \right] \tag{48}$$
$$- \frac{kt}{2} \left[\frac{B(\hat{\xi})}{\gamma_1} e^{ikt\, P(\hat{\xi}) + i\alpha_1^2} \epsilon^i(\hat{\xi}) - \frac{C(\hat{\xi})}{\gamma_2} e^{ikt\, Q(\hat{\xi}) + i\alpha_2^2} \epsilon^r(\hat{\xi}) \right] \Bigg\}$$

with

$$P(\hat{\xi}) = \hat{p} \cdot \underline{\underline{I}} \cdot \hat{\xi} \tag{49}$$

$$Q(\hat{\xi}) = \hat{p} \cdot \hat{\underline{I}} \cdot \hat{\xi} \tag{50}$$

$$B(\hat{\xi}) = \hat{p} \cdot (\underline{I}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{51}$$

$$C(\hat{\xi}) = \hat{p} \cdot (\hat{\underline{I}}_t \times \hat{\ell}) \cdot \hat{\xi} \tag{52}$$

In finding the integrand of (43) from (48), a considerable amount of simplification is possible by using the following relationships between $\hat{s}$ and $\hat{s}_I$

$$\hat{p} \cdot \underline{I} \cdot \hat{s} = \hat{p} \cdot \hat{\underline{I}} \cdot \hat{s}_I \,;\; \hat{p} \cdot \underline{I} \cdot \hat{s}_I = \hat{p} \cdot \hat{\underline{I}} \cdot \hat{s} \tag{53}$$

$$\hat{p} \cdot (\underline{I}_t \times \hat{\ell}) \cdot \hat{s} = -\hat{p} \cdot (\hat{\underline{I}}_t \times \hat{\ell}) \cdot \hat{s}_I \,;\; \hat{p} \cdot (\underline{I}_t \times \hat{\ell}) \cdot \hat{s}_I = -\hat{p} \cdot (\hat{\underline{I}}_t \times \hat{\ell}) \cdot \hat{s} \tag{54}$$

$$\epsilon^i (\hat{s}) = -\epsilon^r (\hat{s}_I) \tag{55}$$

$$\epsilon^i (\hat{s}_I) = -\epsilon^r (\hat{s}) \tag{56}$$

$$\alpha_1 (\hat{s},t) = -\alpha_2 (\hat{s}_I,t) \tag{57}$$

$$\alpha_1 (\hat{s}_I,t) = -\alpha_2 (\hat{s},t) \tag{58}$$

Then (43) becomes

$$\begin{aligned}(\underline{W} - \underline{W}_I) \cdot \hat{\ell} = \frac{\pi^{-1/2} A e^{ik(\psi+r)} e^{-i\pi/4}}{4\pi} \cdot \Bigg\{ & 2ik \int_0^\infty e^{ikt} \left[B(\hat{s})\, e^{ikt\,P(\hat{s})}\, F(\alpha_1) \right. \\ & \left. - C(\hat{s})\, e^{ikt\,Q(\hat{s})}\, F(\alpha_2) \right] dt + k \int_0^\infty e^{ikt} \left[\frac{B(\hat{s})}{\gamma_1} e^{ikt\,P(\hat{s}) + i\alpha_1^2} \epsilon^i(\hat{s}) \right. \\ & \left. - \frac{C(\hat{s})}{\gamma_2} e^{ikt\,Q(\hat{s}) + i\alpha_2^2}\, \epsilon^r(\hat{s}) \right] dt \Bigg\}\end{aligned} \tag{59}$$

where

$$F(x) = \int_0^x e^{it^2}\, dt \tag{60}$$

$\hat{P}(s)$, $\hat{Q}(s)$ $\hat{B}(s)$, $\hat{C}(s)$, are given by (49)-(53) with $\hat{s}$ replacing $\hat{\xi}$. In (59) the arguments of α_1 and α_2 are $\hat{s}$ and t. Therefore α_1 and α_2 can be evaluated from (32)-(36) by replacing $\hat{\xi}$ with $\hat{s}$.

It is worth mentioning that the application of the Braunbek method here is somewhat different than in Keller et al.[6] where the method was used to obtain an approximation of the field near the edge in the plane of the aperture. Here, the method is also used to approximate the field near the edge, but the vector potential requires that the field be evaluated along the path of the vectors $\hat{s}$ and $\hat{s}_I$. These vectors will be in the z = 0 plane only for observation points on the aperture or screen. One consequence of this difference is the retention of the ϵ^i and ϵ^r quantities in (59). In Keller et al.[6], $\epsilon^i = 1$ and $\epsilon^r = -1$, irrespective of the location of the observer.

To find a closed form expression for the vector potential $(\underline{W} - \underline{W}_I) \cdot \hat{\ell}$, (59) is evaluated by using the first term in an asymptotic expansion for large k. Since the Braunbek approximation is valid only in the vicinity of an edge (i.e. near t = 0), the upper limit in the integrals of (59) will be ignored. In fact, ignoring the upper limit is more than just a matter of convenience since the first two terms of (59) do not converge. This problem is not necessarily a deficiency of the M-R theory but perhaps a consequence of having ignored the $\underline{W}_\infty$ and $(\underline{W}_\infty)_I$ contributions to vector potential. The results obtained by Miyamoto and Wolf[1] for a convergent spherical wave (which is similar to the Braunbek method in the sense that neither satisfies the Sommerfeld radiation condition in three dimensions) suggest the possibility that the full expression for the vector potential would converge and even admit an evaluation of higher order terms in the asymptotic series. Since we have not been able to compute $\underline{W}_\infty$ and $(\underline{W}_\infty)_I$, we obtain an expression for the first two terms by an integration by parts, evaluating the result only at the lower limit, and using the formula[6, 10]

$$\int_0 \frac{e^{ik\,h(t)}}{t^{1/2}}\, g(t)\, dt \sim \frac{\pi^{1/2}\, g(o)\, e^{ikh(o) + i\pi/4}}{(k\, h'(o))^{1/2}} \tag{61}$$

The last two integrals of (59) are of the same form as (61) and can be evaluated immediately.

From the results of the calculations and simplifications, the vector potential can be written

$$(\underline{W} - \underline{W}_I) \cdot \hat{\ell} = \frac{A(\ell)\, e^{ik(\psi(\ell) + r)}}{4\pi r} \left(1 + T(\hat{s}) + (\hat{p} \cdot \hat{\ell})(\hat{s} \cdot \hat{\ell})\right)^{1/2} \left[\frac{\epsilon^i(\hat{s})\, \hat{p} \cdot (\underline{I}_t \times \hat{\ell}) \cdot \hat{s}}{(T(\hat{s}) - \hat{p} \cdot \underline{I}_t \cdot \hat{s})^{1/2}\, (1 + \hat{p} \cdot \underline{I} \cdot \hat{s})} - \frac{\epsilon^r(\hat{s})\, \hat{p} \cdot (\underline{I}_t \times \hat{\ell}) \cdot \hat{s}}{(T(\hat{s}) - \hat{p} \cdot \underline{I}_t \cdot \hat{s})^{1/2}\, (1 + \hat{p} \cdot \underline{I} \cdot \hat{s})} \right] \tag{62}$$

This formula can now be compared with the first term in the asymptotic expansion of $(\underline{W} - \underline{W}_I) \cdot \hat{\ell}$ under the physical optics approximation,[2,8]

$$(\underline{W} - \underline{W}_I) \cdot \hat{\ell}\rfloor_{P.O.} = \frac{A(\ell)\, e^{ik(\psi(\ell) + r)}}{4\pi r} \left[\frac{\hat{s} \times \hat{p} \cdot \hat{\ell}}{1 + \hat{s} \cdot \hat{p}} - \frac{\hat{s}_I \times \hat{p} \cdot \hat{\ell}}{1 + \hat{s}_I \cdot \hat{p}} \right] \tag{63}$$

ORIGINAL PAGE
OF POOR QUALITY

The comparison is made easier by noting that

$$\hat{p} \cdot (\underline{\underline{I}}_t \times \hat{\ell}) \cdot \hat{s} = (\hat{s} \times \hat{p}) \cdot \hat{\ell} \tag{64}$$

$$\hat{p} \cdot (\hat{\underline{\underline{I}}}_t \times \hat{\ell}) \cdot \hat{s} = (\hat{s}_I \times \hat{p}) \cdot \hat{\ell} \tag{65}$$

$$(1 + \hat{p} \cdot \underline{\underline{I}} \cdot \hat{s}) = 1 + \hat{p} \cdot \hat{s} \tag{66}$$

$$(1 + \hat{p} \cdot \hat{\underline{\underline{I}}} \cdot \hat{s}) = 1 + \hat{p} \cdot \hat{s}_I \tag{67}$$

where

$$\hat{s}_I = \hat{s} - 2\,(\hat{s} \cdot \hat{z})\,\hat{z} \tag{10}$$

Therefore, the first terms in (62) and (63) differ by the factor $\epsilon^i(\hat{s}) \left\{[1 + T(\hat{s}) + (\hat{p} \cdot \hat{\ell})(\hat{s} \cdot \hat{\ell})] / [T(\hat{s}) - \hat{p} \cdot \underline{\underline{I}}_t \cdot \hat{s}]\right\}^{1/2}$; the second terms by the factor $\epsilon^r(\hat{s}) \left\{[1 + T(\hat{s}) + (\hat{p} \cdot \hat{\ell})(\hat{s} \cdot \hat{\ell})] / [T(\hat{s}) - \hat{p} \cdot \hat{\underline{\underline{I}}}_t \cdot \hat{s}]\right\}^{1/2}$.

COMPARISON OF M-R AND GTD

As a check on the vector potential given by (62), the boundary diffraction term of (6), u_B, is evaluated for large k when the observer is not close to a shadow boundary or caustic, i.e.

$$u_B(\underline{x}) = \oint (\underline{W} - \underline{W}_I) \cdot \hat{\ell}\, d\ell' \tag{68}$$

where $(\underline{W} - \underline{W}_I) \cdot \hat{\ell}$ is given by (62).

Recognizing that the saddle points of (68) occur when

$$\frac{\partial}{\partial \ell'} (\psi + r) = 0 \tag{69}$$

then since $\partial\psi/\partial\ell' = \hat{\ell} \cdot \hat{p}$ and $\partial r/\partial\ell' = \hat{s} \cdot \hat{\ell}$ the saddle points are given by points on the rim $\ell = \ell_{si}$ for which $\hat{\ell} \cdot \hat{p} = -\hat{\ell} \cdot \hat{s}$. Using the formula for an isolated first order saddle point gives[11]

$$u_B(\underline{x}) \sim \sum_i \left[\frac{(2\pi)^{1/2}\, e^{i\pi/4\, \mathrm{sgn}\,(\psi'' + r'')}\, (\underline{W} - \underline{W}_I) \cdot \hat{\ell}}{(k\,|\,\psi'' + r''\,|)^{1/2}}\right]_{\ell = \ell_{si}} \tag{70}$$

where

$$\psi'' + r'' = \frac{\partial^2 \psi}{\partial \ell'^2} + \frac{\partial^2 r}{\partial \ell'^2} \tag{71}$$

To compare (70) with the result of Keller et al.[6], $(\underline{W} - \underline{W}) \cdot \hat{\ell}$ is evaluated at $\hat{\ell} \cdot \hat{p} = -\hat{\ell} \cdot \hat{s}$ and the result is expressed in terms of the angles $\phi_0, \phi, \theta_0, \theta$ defined by (19) - (27). In making this transformation, it is important to note that the vector $\hat{\xi}$ in (19) - (21) is taken to be $\hat{r}$, the unit vector from the rim to the observer, where $\hat{r} = -\hat{s}$.

This procedure gives

$$(\underline{W} - \underline{W}_I) \cdot \hat{\ell} = \frac{A\, e^{ik(\psi + r)}}{4\pi r} [\epsilon^i(\hat{s})\, \mathrm{sgn}(\sin\tfrac{1}{2}(\phi_o - \phi))\, \sec\tfrac{1}{2}(\phi - \phi_o) \tag{72}$$

$$+ \epsilon^r(\hat{s})\, \mathrm{sgn}(\sin\tfrac{1}{2}(\phi_o + \phi))\, \sec\tfrac{1}{2}(\phi + \phi_o)]$$

where both sides are evaluated at $\hat{\ell} \cdot \hat{p} = -\hat{\ell} \cdot \hat{s}$.

Since only observation points in the $z > 0$ half space are considered, it is not difficult to show that

$$\epsilon^i(\hat{s})\ \mathrm{sgn}\,(\sin\tfrac{1}{2}(\phi_o - \phi)) = -1 \tag{73}$$

$$\epsilon^r(\hat{s})\ \mathrm{sgn}\,(\sin\tfrac{1}{2}(\phi_o + \phi)) = 1 \tag{74}$$

To account for a different convention used by Keller et al.[6], let $\phi_o = \alpha + \pi/2$ and $\phi = \theta + \pi/2$. Substituting these equations into (72) and using (73), (74) gives

$$u_B(\underline{x}) = -\left\{ \frac{A\, e^{ik(\psi + r)}\, e^{i\pi/4\, \mathrm{sgn}(\psi'' + r'')}}{2\,(2\pi k\, |\, \psi'' + r''\, |)^{1/2}}\, [\sec\tfrac{1}{2}(\theta - \alpha) + \csc\tfrac{1}{2}(\theta + \alpha)] \right\}, \tag{75}$$

This is the same expression obtained by applying the Braunbek approximation directly to the Rayleigh representation and is also the field predicted by GTD on rays singly diffracted from an edge[6].

ASYMPTOTIC EVALUATION OF THE M-R REPRESENTATION FOR A SIMPLE CASE

Although we have been unable to show in general that the vector potential given by (62) leads to fields that are continuous across the shadow boundary, there are special cases where this continuity can be shown explicitly. It should be mentioned that under a physical optics approximation, Otis, et al.[12] and Takenaka et al.[13] have obtained asymptotic evaluations of the M-R representation for an on-axis gaussian beam incident on a circular aperture. Their work has shown the field to be everywhere well behaved. Here we will treat the simpler case of a plane wave normally incident on a circular aperture of radius a, but use the vector potential given by (62).

For a normally incident plane wave of unit amplitude, (62) reduces to

ORIGINAL PAGE IS
OF POOR QUALITY

$$(\underline{W} - \underline{W}_I) \cdot \hat{\ell} = -\frac{e^{ikr}}{4\pi r} (1 + (\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2})^{1/2} \cdot \left[\frac{(\hat{r} \times \hat{z} \cdot \hat{\ell})\, \epsilon^i(-\hat{r})}{((\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2} + \hat{r} \cdot \hat{z})^{1/2} (1 - \hat{r} \cdot \hat{z})} - \frac{(\hat{r}_I \times \hat{z} \cdot \hat{\ell})\, \epsilon^r(-\hat{r})}{((\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2} + \hat{r}_I \cdot \hat{z})^{1/2} (1 - \hat{r}_I \cdot \hat{z})} \right] \quad (76)$$

where $\hat{s} = -\hat{r}$ and $\hat{r}_I = \hat{r} - 2(\hat{r} \cdot \hat{z})\, \hat{z}$.

The total field in the x-z plane is

$$u(\underline{x}) = \oint (\underline{W} - \underline{W}_I) \cdot \hat{\ell}\, d\ell' + e^{ikz} [U(x + a) - U(x - a)] \quad (77)$$

where the second term represents the geometrical optics field. The boundary diffraction integral can be written as the sum of I_1 and I_2,

$$I_1 + I_2 = \oint (\underline{W} - \underline{W}_I) \cdot \hat{\ell}\, d\ell' \quad (78)$$

where

$$I_1 = -\frac{a}{4\pi} \int_{-\pi/2}^{3\pi/2} e^{ikr} f_1(\phi') \frac{\hat{r} \times \hat{z} \cdot \hat{\phi}}{1 - \hat{r} \cdot \hat{z}} d\phi' \quad (79)$$

$$I_2 = \frac{a}{4\pi} \int_{-\pi/2}^{3\pi/2} e^{ikr} f_2(\phi') \frac{\hat{r}_I \times \hat{z} \cdot \hat{\phi}}{1 - \hat{r}_I \cdot \hat{z}}\, d\phi' \quad (80)$$

and

$$\hat{\phi} = \hat{\ell} \quad (81)$$

$$f_1(\phi') = \frac{\epsilon^i(-\hat{r}) (1 + (\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2})^{1/2}}{r ((\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2} + \hat{r} \cdot \hat{z})^{1/2}} \quad (82)$$

$$f_2(\phi') = \frac{\epsilon^r(-\hat{r}) (1 + (\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2})^{1/2}}{r ((\hat{r} \cdot \underline{I}_t \cdot \hat{r})^{1/2} + \hat{r}_I \cdot \hat{z})^{1/2}} \quad (83)$$

$$\epsilon^i(-\hat{r}) = 1 \quad (84)$$

The expression for I_2 can be evaluated by means of the standard formula for an isolated saddle point. The saddle points are determined from the equation $\hat{\ell} \cdot \nabla' r = -\hat{\ell} \cdot \hat{r} = 0$. Since $\hat{\ell} = -\hat{x} \sin\phi' + \hat{y} \cos\phi'$ and $\hat{r} = (\hat{x}(x - a \cos\phi') - \hat{y}\, a \sin\phi' + \hat{z} z)/r$ then saddle points occur at $\phi_{s_1} = 0$ and $\phi_{s_2} = \pi$. Noting that $\epsilon^r[-\hat{r}(\phi_{s_1})] = \mathrm{sgn}(a - x)$ and $\epsilon^r[-\hat{r}(\phi_{s_2})] = \mathrm{sgn}(a + x)$, a straightforward calculation gives

$$I_2 \sim \frac{1}{2}\left(\frac{a}{\pi k}\right)^{1/2}\left[\frac{e^{ikr_1 + i\pi/4}\,|a+x|}{(x\,(r_1-z))^{1/2}\,(r_1+z)} + \frac{e^{ikr_2 - i\pi/4}\,|a-x|}{(x\,(r_2-z))^{1/2}\,(r_2+z)}\right] \tag{85}$$

with

$$r_1 = ((a-x)^2 + z^2)^{1/2} \tag{86}$$

$$r_2 = ((a+x)^2 + z^2)^{1/2} \tag{87}$$

By expanding (85) about the small parameter $x \mp a$, it can be shown that I_2 is continuous across the shadow boundaries at $x = \pm a$.

Turning to the expression for I_1, it can be shown that the steepest descent path may be close to the pole at $\hat{r} \cdot \hat{z} = 1$. Therefore, the standard formula no longer applies[11]. In the following development we make use of the fact that the field is symmetric about the z-axis. Therefore, it is sufficient to consider only observation points in the x-z plane for which $x > 0$. Since the saddle points of I_1 are the same as for I_2, then $\phi_{s_1} = 0$ and $\phi_{s_2} = \pi$. In the complex ϕ-plane, the steepest descent path (SDP) which passes through ϕ_{si} is given by $\mathrm{Im}\,(i\,r\,(\phi)) = \mathrm{Im}\,(i\,r\,(\phi_{si}))$ where Im denotes the imaginary part. Graphs of the steepest descent paths are given by Takenaka et al.[13]

Solving the equation $\hat{r} \cdot \hat{z} = 1$ for ϕ yields poles at

$$\phi_{p_1} = i\,\mathrm{Cosh}^{-1}\,q \tag{88}$$

$$\phi_{p_2} = -\,i\,\mathrm{Cosh}^{-1}\,q \tag{89}$$

where

$$q = (x^2 + a^2)/2ax \tag{90}$$

Because of the presence of r in the denominator of the integral, I_1 also has algebraic singularities at

$$\phi_{bi} = \pm\, i\,\mathrm{Cosh}\,q' \tag{91}$$

$$q' = (x^2 + a^2 + z^2)/2ax \tag{92}$$

For $z >> a$, these contributions can be ignored.

In the deformation from the original integration path to the steepest descent paths, no poles are crossed so the contributions from the residues that occur for the case of a gaussian beam[13] are absent in this case. Furthermore, the poles can be located only in the vicinity of the SDP_1 (since we have chosen $x > 0$). The integration along SDP_2 therefore can be evaluated in the same manner as I_2.

Using the formula derived by Felsen and Marcuvitz[11] (with a modification to account for the presence of two poles) then

$$4\pi I_1 \sim -a\, e^{ikr(\phi_{s_1})}\left\{ \sum_{i=1,2} \left[\operatorname{sgn}(\operatorname{Im} b_i)\, 2iR_i\pi^{1/2}\, e^{-k\, b_i^2}\, Q(-i\, b_i\, k^{1/2} \operatorname{sgn}(\operatorname{Im} b_i)) + \frac{R_i}{b_i}\left(\frac{\pi}{k}\right)^{1/2} \right] + \left(\frac{\pi}{k}\right)^{1/2} h_1\, f_1(\phi_{s_1}) \left(\frac{\hat{r}\times\hat{z}\cdot\hat{\phi}}{1-\hat{r}\cdot\hat{z}}\right)_{\phi_{s_1}} \right\}$$

$$- a\, e^{ikr(\phi_{s_2})} \left(\frac{\pi}{k}\right)^{1/2} h_2\, f_1(\phi_{s_2}) \left(\frac{\hat{r}\times\hat{z}\cdot\hat{\phi}}{1-\hat{r}\cdot\hat{z}}\right)_{\phi_{s_2}} \tag{93}$$

where

$$Q(x) = \int_x^\infty e^{-t^2}\, dt \tag{94}$$

The quantity R_i is the residue of $f_1(\phi)(\hat{r}\times\hat{z}\cdot\hat{\phi})/(1-\hat{r}\cdot\hat{z})$ at the pole ϕ_{pi} $(i = 1, 2)$ and

$$h_i = (-2/i|r''(\phi_{si})|)^{1/2} \qquad i = 1, 2 \tag{95}$$

$$b_i = [i\,(r(\phi_{s_1}) - r(\phi_{pi}))]^{1/2} \qquad i = 1, 2 \tag{96}$$

where the argument of b_i is to be chosen the same as the argument of the quantity $(\phi_{pi} - \phi_{s_1})/h_1$ as ϕ_{pi} approaches ϕ_{s_1} [11].

To compute R_i, the following result is used[14] for the residue of $t(\phi)/p(\phi)$

$$\operatorname{Res}\left(\frac{t(\phi)}{p(\phi)}\right)\Bigg|_{\phi_p} = t(\phi_p)/p'(\phi_p) \tag{97}$$

where t and p are analytic at ϕ_p and $t(\phi_p) \neq 0$, $p(\phi_p) = 0$ and $p'(\phi_p) \neq 0$. This gives

$$R_1 = i \,\mathrm{sgn}\,(x - a)/a \tag{98}$$

$$R_2 = -i \,\mathrm{sgn}\,(x - a)/a \tag{99}$$

The results for b_1, b_2 are

$$b_1 = e^{i\pi/4}\,\eta \tag{100}$$

$$b_2 = -e^{i\pi/4}\,\eta \tag{101}$$

where

$$\eta = [r(\phi_{s_1}) - r(\phi_p)]^{1/2} \tag{102}$$

We have written $r(\phi_p)$ since $r(\phi_{p_1}) = r(\phi_{p_2})$.

The substitution of these results into (93), shows that two terms of the summation are identical and equal to

$$-\frac{\pi^{1/2}\,\mathrm{sgn}\,(x-a)}{a}\left[\quad 2\,e^{-ik\eta^2}\,Q\left(\sqrt{k}\,\eta(1-i)/\sqrt{2}\right) - e^{i\pi/4}\big/k^{1/2}\eta\right] \tag{103}$$

Using the fact that[11]

$$Q\left[\sqrt{k}\,\eta(1-i)/\sqrt{2}\right] = e^{-i\pi/4}\int_{\sqrt{k}\eta}^{\infty} e^{it^2}\,dt \tag{104}$$

and evaluating the remaining terms of (93), then

$$\begin{aligned} I_1 \sim \frac{\mathrm{sgn}\,(x-a)}{2\pi^{1/2}}&\left[2e^{ikz - i\pi/4}\int_{\sqrt{k}\eta}^{\infty} e^{it^2}\,dt - \frac{1}{k^{1/2}\eta}\,e^{ikr_1 + i\pi/4}\right] \\ &- \frac{1}{2}\left(\frac{a}{\pi k}\right)^{1/2}\left[\frac{e^{ikr_1 + i\pi/4}}{(x\,(r_1 + z))^{1/2}}\left(\frac{a-x}{r_1 - z}\right) + \frac{e^{ikr_2 - i\pi/4}}{(x(r_2 + z))^{1/2}}\left(\frac{a+x}{r_2 - z}\right)\right] \end{aligned} \tag{105}$$

where r_1, r_2 are given by (86) and (87) and η by (102).

From (78), (85), and (105) the total field, for $x > 0$, is

$$u(\underline{x}) \sim I_1 + I_2 + e^{ikz}\,[U(x + a) - U(x - a)] \tag{106}$$

To refer to the four terms of (105), let $I_1 = I_{1a} + I_{1b} + I_{1c} + I_{1d}$. As a check on (106), notice that as x approaches a, i.e., as the observer approaches the shadow boundary, $I_{1b} + I_{1c} = 0$ and as the observer crosses the shadow boundary, the discontinuity in I_{1a} exactly cancels the discontinuity in the geometrical optics field. Since I_2 and the remaining term of I_1 are continuous, this implies that the total field is continuous across the shadow boundary. For points away from the shadow boundary ($k^{1/2}\eta >> 1$), $I_{1a} + I_{1b}$ is $o(k^{-1/2})$ and the I_2 and the remaining terms of I_1 yield the standard asymptotic result.

Points on the z-axis, i.e. the axis passing through the center of the aperture, correspond to caustics of the diffracted field and equation (106) is not valid on or near such points. For such points, the procedure described by Takenaka et al.[13] could be followed. However, we will not pursue the problem here.

CONCLUSIONS

In an attempt to establish a correspondence between the M-R and GTD theories, an approximate form of the vector potential has been derived through the use of the Braunbek method. For the aperture diffraction problem, the two theories predict similar results for observation points away from caustics and shadow boundaries. To examine the behavior of the M-R results near the shadow boundary an explicit calculation was performed for the simple case of a plane wave incident upon a circular aperture. That the M-R field was continuous across the shadow boundary in this case, suggests that the vector potential given by (62) might yield reliable results near a shadow boundary for a more general incident wave.

ORIGINAL PAGE IS
OF POOR QUALITY

Appendix: Diffraction from a half-plane

If an incident plane wave of the form

$$u_o^i = A\, e^{-jk\psi}\, e^{jk\rho \cos(\phi-\phi_o)\sin\theta_o}\, e^{-jkz\cos\theta_o} \tag{A1}$$

is incident on a half plane the edge of which is aligned with the z-axis and oriented in the x-z plane ($x > 0$), and if the total field, u, on the screen is zero, then the field at a point $\underline{x}$ is

$$u = u^i - u^r \tag{A2}$$

where

$$u^i = U(\epsilon^i)\, u_o^i + u_d^i \tag{A3}$$

$$u^r = U(\epsilon^r)\, u_o^r + u_d^r \tag{A4}$$

and

$$u_d^{\substack{i\\r}} = -A\, e^{-jk\psi}\, \epsilon^{\substack{i\\r}}\, K_-[(2k\rho\sin\theta_o)^{1/2}\, |\cos\tfrac{1}{2}(\phi \mp \phi_o)|]\, e^{-jk[\rho\sin\theta_o + z\cos\theta_o]} \tag{A5}$$

$$\epsilon^{\substack{i\\r}} = \mathrm{sgn}\,[\cos\tfrac{1}{2}(\phi \mp \phi_o)] \tag{A6}$$

$$K_-(\gamma) = \left(\frac{j}{k}\right)^{1/2} e^{j\gamma^2} \int_\gamma^\infty e^{-jt^2}\, dt \tag{A7}$$

$$U(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{A8}$$

An $e^{j\omega t}$ time convention is used to obtain these results. Apart from some slight modifications – in particular, the addition of the factor $A\, e^{-jk\psi}$ which provides an amplitude and phase reference for the Braunbek method – the above formulas were obtained from James[9].

From equation (A1), the direction of the incident wave, $\hat{p}$, is

$$\hat{p} = -\cos\phi_o \sin\theta_o\, \hat{x} - \sin\phi_o \sin\theta_o\, \hat{y} + \cos\theta_o\, \hat{z} \tag{A9}$$

The azimuthal angle ϕ_o is measured in a counterclockwise sense from the x-axis to the projection of the vector $-\hat{p}$ onto the x-y plane. The polar angle θ_o is measured between the $\hat{z}$ and $\hat{p}$ vectors. On the other hand, the coordinates of an observation point $\underline{x}$ with respect to an origin on the edge of the half-plane is

$$\underline{x} = \rho\cos\phi\, \hat{x} + \rho\sin\phi\, \hat{y} + z\hat{z} \tag{A10}$$

where

$$\rho = |\underline{x}| \sin\theta \tag{A11}$$

$$z = |\underline{x}| \cos\theta \tag{A12}$$

The angle ϕ is measured in a counterclockwise sense from the x-axis to the projection of the vector $\underline{x}$ onto the x-y plane; θ is the angle between $\hat{z}$ and $\underline{x}$.

From (A3) and (A7)

$$u^i = A\, e^{-jk\psi}\, e^{jk[\rho \cos(\phi-\phi_o) \sin\theta_o - z \cos\theta_o]} \left[U(\epsilon^i) - \pi^{-1/2}\, \epsilon^i\, e^{j\pi/4} \int_{\gamma_1}^{\infty} e^{-jt^2}\, dt \right] \tag{A13}$$

where γ_1 is given by the argument of K_- in (A5) using the upper sign.

Since

$$1 = \pi^{-1/2}\, e^{j\pi/4} \int_{-\infty}^{\infty} e^{-jt^2}\, dt \tag{A14}$$

the expression in the brackets of (A13) can be rewritten as

$$\pi^{-1/2}\, e^{j\pi/4} \int_{\alpha_1}^{\infty} e^{-jt^2}\, dt \tag{A15}$$

where

$$\alpha_1 = -\,\epsilon^i\, \gamma_1 \tag{A16}$$

so that

$$u^i = A\, \pi^{-1/2}\, e^{-jk\psi + j\pi/4}\, e^{jk[\rho\cos(\phi-\phi_o) \sin\theta_o - z \cos\theta_o]} \int_{\alpha_1}^{\infty} e^{-jt^2}\, dt \tag{A17}$$

Similarly

$$u^r = A\, \pi^{-1/2}\, e^{-jk\psi + j\pi/4}\, e^{jk[\rho\cos(\phi+\phi_o) \sin\theta_o - z \cos\theta_o]} \int_{\alpha_1}^{\infty} e^{-jt^2}\, dt \tag{A18}$$

where

$$\alpha_2 = -\,\epsilon^r\, \gamma_2 \tag{A19}$$

and where γ_2 is the argument of K_- in (A5) using the lower sign.

To account for the $e^{-i\omega t}$ convention used in the text, we substitute $j = -i$ into (A17) and (A18). This gives equations (13) of the text.

REFERENCES

1. K. Miyamoto and E. Wolf, "Generalization of the Maggi-Rubinowicz Theory of the Boundary Diffraction Wave-Part I", J. Opt. Soc. Am., 52, 615-625 (1962).

2. K. Miyamoto and E. Wolf, "Generalization of the Maggi-Rubinowicz Theory of the Boundary Diffraction Wave-Part II", J. Opt. Soc. Am., 52, 626-637 (1962).

3. A. Rubinowicz, "The Miyamoto-Wolf Diffraction Wave", Progress in Optics, IV, 195-240 (1964).

4. W. Braunbek, "Neue Näherungsmethode für die Beugung am ebenen Schrim", Z. Physik, 127, 381-390 (1950).

5. C. J. Boukamp, "Diffraction Theory", Repts. Progr. in Phys., 17, 35-100 (1954).

6. J. B. Keller, R. M. Lewis, and B. D. Seckler, "Diffraction by an Aperture II", J. Appl. phys. 28, 570-579 (1957).

7. P. Wolfe, "A New Approach to Edge Diffraction", SIAM J. Appl. Math. 15, 1434-1469 (1967).

8. E. W. Marchand and E. Wolf, "Boundary Diffraction Wave in the Domain of the Rayleigh-Kirchhoff Diffraction Theory", J. Opt. Soc. Am., 52, 761-767, (1962).

9. G. L. James, Geometrical Theory of Diffraction (Peregrinus, Stevenage, Herts., England, 1976) Ch. 5.

10. A. Erdelyi, Asymptotic Expansions (Dover, NY, 1956). Chap. 2.

11. F. B. Felsen and N. Marcuvitz, Radiation and Scattering of Waves (Prentice-Hall, NJ, 1973) Ch. 4.

12. G. Otis, J. L. Lachambre, J. W. Y. Lit, and P. Lavigne, "Diffracted Waves in the Shadow Boundary Region", J. Opt. Soc. Am. 67, 551-562 (1977).

13. T. Takenaka, M. Kakeya, and O. Fukumitsu, "Asymptotic Representation of the Boundary Diffraction Wave for a Gaussian Beam Incident on a Circular Aperture", 70, 1323-1328 (1980).

14. R. V. Churchill, Complex Variables and Applications, (McGraw-Hill, NY, 1960) Chap. 7.