A RECURSIVE ALGORITHM FOR ZERNIKE POLYNOMIALS

by

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ABSTRACT

Many applications in optics, such as the diffraction theory of optical aberrations, involves the analysis of a function defined on a rotationally symmetric system, with either a circular or annular pupil. In order to numerically analyze such systems it is typical to expand the given function in terms of a class of orthogonal polynomials. Because of their particular properties, the Zernike polynomials are especially suited for numerical calculations. We develop a recursive algorithm that can be used to generate the Zernike polynomials up to a given order. The algorithm is recursively defined over J where R(J,N) is the Zernike polynomial of degree N obtained by orthogonalizing the sequence r^J, r^{J+2}, ..., r^{J+2N} over (e, l). The terms in the preceding row - the (J-1) row - up to the N+1 term is needed for generating the (J,N)th term. Thus, the algorithm generates an upper left-triangular table. This algorithm has been placed in the computer with the necessary support program also included.
INTRODUCTION

An arbitrary function $W(r, \theta)$, such as a wavefront error function over a circular or annular region, can be expanded in terms of an orthonormal series of orthogonal polynomials. If $W$ is defined over a circular or annular region, it is convenient to expand $W$ in terms of the Zernike polynomials, $Z_n^l$. This

$$W = \sum_{n, l} A_{nl} Z_n^l.$$ 

It can be shown (3) that $Z_n^l = R_n^l(r)e^{il\theta}$ where $R_n^l$ depends only on the radial coordinate and $e^{il\theta}$ depends only on the angular coordinate. Also, $l$ is the minimum exponent of the polynomials $Z_n^l$ and $R_n^l$ and the numbers $n$ and $l$ are either both even or both odd. The radial polynomials $R_n^l$ are of degree $n$ and satisfy the relation

$$R_n^l = R_n^{-l} = R_n^{l}.$$ 

If we write, using only the real part,

$$W(r, \theta) = \sum_{n} \sum_{j=0}^{\infty} A_{nj} R_n^{2j+2} \sin j \theta,$$

the following properties are satisfied.

1. The Zernike polynomials are invariant in form with respect to rotations of axes about the origin (3).

2. The Zernike polynomials are easily related to the classical aberrations (4).
3. The function $W(r, \theta)$ is usually found as a best least-squares fit to a collection of data points. Since the Zernike polynomials are orthogonal over an annular region, the well-known minimum-error property of Fourier expressions shows that each term

$$A_{nj} r^J_{jn+2} \sin J \theta$$

$$\text{ also represents individually a best least-squares fit to the data. Thus, the average amount of each term is given by the magnitude of that term, without the need to do a new least-squares fit.}$$

Because the Zernike polynomials are being applied to an increasing number of physical problems \{1, 2, 5, 6\}, there is an expressed interest in being able to generate the Zernike polynomials up to a given order. In this paper, we develop a numerical method due to Tatian \{8\} for generating the polynomials $r^J_{2n+j}$ over an annular region. A discussion of the derivation of the algorithm is presented in section 2 and a general discussion of the computer program that was written to facilitate this algorithm is discussed in section 3.

Bhatia and Wolf \{3\} have shown that the polynomials $r^J_{2n+j} (r)$ are obtained by orthogonalizing the functions $r^J, r^{J+2}, \ldots, r^{J+2n}$ over the interval $[\epsilon, 1]$. The constant $\epsilon$ represents the inner radius of the annular region and 1
represents the outer radius. Thus $\varepsilon = 0$ represents a circular region. We convert this into the associated problem of orthogonalizing $1, u, \ldots, u^n$ over $[\varepsilon^2, 1]$ with the weight function $u^j$ by substituting $u = r^2$. This follows from

$$
\langle r^{J+21}, r^{J+2k} \rangle = \int_\varepsilon^1 r^{J+21} r^{J+2k} r^3 dr
$$

$$
= \int_\varepsilon^1 r^{2J+21+2k} dr = \int_\varepsilon^1 (r^2)^{J+1+k} dr
$$

$$
= \frac{1}{2} \int_\varepsilon^1 u^{j+1+k} du = \frac{1}{2} = \int_\varepsilon^1 u^1 u^k (u^j du)
$$

First, let us consider the case $J = c$. Then $R_n^O (r) = R_n^O (u)$ is obtained by orthogonalizing the sequence $1, u, \ldots, u^n$ over $[\varepsilon^2, 1]$ with the weight function 1. This however is a Jacobi problem over the shifted interval $[\varepsilon^2, 1]$. Hence,

$$(1) \quad R_{2n}^O (r) = R_n^O (u) = p_n^{(0,0)} (2) \quad \frac{r^2 - \varepsilon^2 - 1}{1 - \varepsilon^2}$$

where $p_n^{(0,0)}$ is the Legendre polynomial of degree $n$. X111-3
Now, let us consider the case when \( J \neq 0 \). We will show that 
\( R_{2n+2}^J \) can be obtained by a recursive method over the variable \( J \).
Suppose we have solved the problem of obtaining \( Q_{n+1}^J \) i.e. orthogonalize \( 1, u, \ldots, u^n \) over \([\varepsilon^1, 1]\) with weight function \( u^{J-1} \) and we want to obtain \( Q_n^J \) over \([\varepsilon^2, 1]\) with weight function \( u^J \). We obtain a relationship between these 2 polynomials by making use of a special case of the following theorems \( \{7\} \).

Theorem 1. Let \( \{p_n(x)\} \) be the orthonormal polynomials associated with the distribution \( d\alpha(x) \) on the interval \( \{a, b\} \). Also, let 
\[
p(x) = c \cdot (x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_1)
\]
be non-negative in this interval. Then the orthogonal polynomials \( \{q_n(x)\} \) associated with the distribution \( p(x) \ d\alpha(x) \) can be represented in terms of the polynomials \( p_n(x) \) as follows:

\[
p(x) q_n(x) =
\]

\[
\begin{vmatrix}
\text{P}_n(x) & \text{P}_{n+1}(x) & \cdots & \text{P}_{n+1}(x) \\
\text{P}_n(x_1) & \text{P}_{n+1}(x_1) & \cdots & \text{P}_{n+1}(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
\text{P}_n(x_e) & \text{P}_{n+1}(x_e) & \cdots & \text{P}_{n+1}(x_e)
\end{vmatrix}
\]
Theorem 2. The following relation holds for any 3 consecutive orthogonal polynomials:

\[ p_n(x) = (A_n X + B_n) p_{n-1}(x) = C_n \cdot p_{n-2}(x) \]

of the highest coefficient of \( p_n(x) \) is denoted by \( k_n \), we have

\[ A_n = \frac{k_n}{k_{n-1}} \quad \text{and} \quad C_n = \frac{A_n}{A_{n-1}} \]

Theorem 3.

\[ p_0(x) p_0(y) + p_1(x) p_1(y) + \ldots + p_n(x) p_n(y) \]

\[ = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y)}{x-y} \]

using Theorem 1 with \( p_n(u) = Q_n^{J-1}(u) \), \( q_n(u) = Q_n^J(u) \) and \( p(u) = u \), we have

\[ 2.1 \ u_Q^J(u) = \frac{1}{Q_n(0)} \begin{vmatrix} Q_n^{J-1}(u) & Q_n^{J-1}(u) \\ Q_n(u) & Q_n^{J-1}(u) \\ Q_n^J(u) & Q_n^{J-1}(u) \end{vmatrix} \]

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We note that this formula becomes indeterminate for \( u = r^2 = 0 \).
But we need the values of \( Q_n^J \) at \( u = 0 \) to calculate the next set of polynomials, so expand equation 2.1 and use Theorem 3 to get

\[
Q_n^J (u) = \frac{h_n^{J-1}}{(1 - \epsilon^2) \sum_{i=0}^{n} \frac{Q_i^{J-1}(o) Q_i^{J-1}(u)}{h_i}}
\]

where

\[
h_n^J = \frac{1}{2} \int_0^t \left( Q_n^J (u) \right)^2 u^J du
\]

is the normalization constant for \( Q_n^J (u) \). By substituting the expression in equation 1 for \( Q_n^J (u) \) into equation 3 and using Theorem 3, we obtain

\[
h_i^J = \frac{-1}{1 - \epsilon^2} \cdot \frac{Q_{i+1}^{J-1}(0)}{Q_i^{J-1}(0)} h_i^{J-1}
\]

converting back to the variable \( r \), we have

\[
R_{2n+J}^J (r) = r^J Q_n^J (r^2).
\]
3. Some comments about the computer program.

The outline for this computer program is:

\[
\begin{align*}
J &= 0 \\
\text{Compute } Q^0_n \\
\text{and } H^0_n \\
\text{For } J = 1 \text{ to } N \\
\end{align*}
\]

Compute 
\[
Q^J_n \text{ and } H^J_n \text{ using } \\
Q^{J-1}_n, \ Q^{J-1}_{n+1} \text{ and } H^{J-1}_n
\]
For a given value of \( N \), the following table shows the values of \( Q_n^J \) and \( H_n^J \) that are generated.

**TABLE 1**

<table>
<thead>
<tr>
<th>( Q_n^J ) or ( H_n^J )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>\ldots</th>
<th>N-2</th>
<th>N-1</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>*</td>
<td>*</td>
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<td>*</td>
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<td>1</td>
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<tr>
<td>2</td>
<td>*</td>
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<td>3</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N-2</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N-1</td>
<td>*</td>
<td>*</td>
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<tr>
<td>N</td>
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</tr>
</tbody>
</table>

**NOTE:** Since \( h_n^J \) is dependent upon knowing \( Q_{n+1}^{J-1} \), each of the above rows are shorter by 1 entry.
The polynomials $Q_n^o$ are directly computed from formula (1) namely,

$$Q_n^o (u) = p_n (o, o) (au + v)$$

where $p_n (o, o)$ is the Legendre polynomial of degree $n$, $a = 2/(1 - \varepsilon^2)$ and $v = -(1 + \varepsilon^2)/(1 - \varepsilon^2)$. Likewise, the constants $H_n^o$ are directly computed from (3); namely,

$$H_n^o = \frac{1}{2} \int_{\varepsilon^2}^1 \left( Q_n^o (u) \right)^2 du.$$

NOTE: The above integral is computed in closed form. This is possible because $Q_n^o (u)$ is a polynomial of degree $n$. This is done via a call to SQPOLY (square the polynomial) and a call to INTGRL (find the integral of a polynomial).

Once the first row is known ($Q_n^o$ and $H_n^o$ for $J = 0$), the recursive algorithm can then be used to compute each succeeding row ($Q_n^J$ and $H_n^J$ for $J = J_0$). The results are then printed via a call to RJN (compute $R_{2n+J}^J = r^2 Q_n^J (r^2)$).
4. Example

A computer run for \( n = 3 \) is shown below.

ENTER N-LARGEST \( R(\emptyset, N) \) DESIRED
?3
\( R(\emptyset, \emptyset) \)
   \( 1.\&ggg \)

\( R(\emptyset, 2) \)
NORM: 3.83767639
   \(-.5487 + 1.\&ggg*R^2 \)

\( R(\emptyset, 4) \)
NORM: 16.4661363\&
   \(.2331 - 1.\&g973*R^2 + 1.\&ggg*R^4 \)

\( R(\emptyset, 6) \)
NORM: 71.94691818
   \(-.g981 = .78g9*R^2 - 1.646g*R^4 + 1.\&ggg*R^6 \)

\( R(1, 1) \)
NORM: 1.9g923153
   \( 1.\&ggg*R^1 + \)

\( R(1, 3) \)
NORM: 8.32586779
   \(-6724*R^1 + 1.\&ggg*R^3 \)

\( R(1, 5) \)
NORM: 35,89625267
   \(.3191*R^1 - 1.2252*R^3 + 1.\&ggg*R^5 \)

\( R(2, 2) \)
NORM: 2.32829g44
   \( 1.\&ggg*R^2 + \)

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R(2, 4)
NORM: 11.84468583
   -.7897**R** 2 + 1.9999**R** 4
* * * * * * * * * * * * * *

R(3,3)
NORM: 2.6873584
1.9999**R** 3 +
* * * * * * * * * * * * * *

* STOP * Ø

A COPY OF THE ABOVE PROGRAM CAN BE OBTAINED FROM THE AUTHOR.
REFERENCES


