A RECURSIVE ALGORITHM FOR ZERNIKE POLYNOMIALS

by

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ABSTRACT

Many applications in optics, such as the diffraction theory of optical aberrations, involves the analysis of a function defined on a rotationally symmetric system, with either a circular or annular pupil. In order to numerically analyze such systems it is typical to expand the given function in terms of a class of orthogonal polynomials. Because of their particular properties, the Zernike polynomials are especially suited for numerical calculations. We develop a recursive algorithm that can be used to generate the Zernike polynomials up to a given order. The algorithm is recursively defined over J where R(J,N) is the Zernike polynomial of degree N obtained by orthogonalizing the sequence r^J, r^{J+2}, ..., r^{J+2N} over (ε, l). The terms in the preceding row - the (J-1) row - up to the N+1 term is needed for generating the (J,N)th term. Thus, the algorithm generates an upper left-triangular table. This algorithm has been placed in the computer with the necessary support program also included.
INTRODUCTION

An arbitrary function $W(r, \theta)$, such as a wavefront error function over a circular or annular region, can be expanded in terms of an orthonormal series of orthogonal polynomials. If $W$ is defined over a circular or annular region, it is convenient to expand $W$ in terms of the Zernike polynomials, $Z_n^l$. This

$$W = \sum_{n, l}^{\infty} a_n^l Z_n^l.$$  

It can be shown [3] that $Z_n^l = R_n^l(r)e^{il\theta}$ where $R_n^l$ depends only on the radial coordinate and $e^{il\theta}$ depends only on the angular coordinate. Also, $l$ is the minimum exponent of the polynomials $Z_n^l$ and $R_n^l$ and the numbers $n$ and $l$ are either both even or both odd. The radial polynomials $R_n^l$ are of degree $n$ and satisfy the relation

$$R_n^l = R_{n-l} = R_n^{|l|}.$$  

If we write, using only the real part,

$$W(r, \theta) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} a_{nj} R_n^j R_{n+2}^j \sin j \theta \cos j \theta$$  

the following properties are satisfied.

1. The Zernike polynomials are invariant in form with respect to rotations of axes about the origin [3].

2. The Zernike polynomials are easily related to the classical aberrations [4].
3. The function \( W(r, \theta) \) is usually found as a best least-squares fit to a collection of data points. Since the Zernike polynomials are orthogonal over an annular region, the well-known minimum-error property of Fourier expressions shows that each term

\[
A_{nj} r^J jn+2 \sin J \theta \cos J \theta
\]

also represents individually a best least-squares fit to the data. Thus, the average amount of each term is given by the magnitude of that term, without the need to do a new least-squares fit.

Because the Zernike polynomials are being applied to an increasing number of physical problems \( \{1, 2, 5, 6\} \), there is an expressed interest in being able to generate the Zernike polynomials up to a given order. In this paper, we develop a numerical method due to Tatian \( \{8\} \) for generating the polynomials \( R^J_{2n+j} \) over an annular region. A discussion of the derivation of the algorithm is presented in section 2 and a general discussion of the computer program that was written to facilitate this algorithm is discussed in section 3.

Bhatia and Wolf \( \{3\} \) have shown that the polynomials \( R^J_{2n+j} (r) \) are obtained by orthogonalizing the functions \( r^J, r^{J+2}, \ldots, r^{J+2n} \) over the interval \( \{\epsilon, 1\} \). The constant \( \epsilon \) represents the inner radius of the annular region and 1
represents the outer radius. Thus $\epsilon = 0$ represents a circular region. We convert this into the associated problem of orthogonalizing $1, u, \ldots, u^n$ over $[\epsilon^2, 1]$ with the weight function $u^j$ by substituting $u = r^2$. This follows from

$$\left\langle r^{J+2l}, r^{J+2k} \right\rangle = \int_{\epsilon}^{1} r^{J+2l} r^{J+2k} r \, dr$$

$$= \int_{\epsilon}^{1} r^{2(J+2l+2k)} r \, dr = \int_{\epsilon}^{1} (r^2)^{J+1+k} r \, dr$$

$$= \frac{1}{2} \int_{\epsilon^2}^{1} u^{j+1+k} \, du = \frac{1}{2} \int_{\epsilon}^{1} u^{l} u^{k} (u^j \, du)$$

First, let us consider the case $J = 0$. Then $R^0_{2n}(r) = R^0_n(u)$ is obtained by orthogonalizing the sequence $1, u, \ldots, u^n$ over $[\epsilon, 1]$ with the weight function 1. This however is a Jacobi problem over the shifted interval $[\epsilon^2, 1]$. Hence,

(1) $R^0_{2n}(r) = R^0_n(u) = p^{(0,0)}_n$ (2) $\frac{r^2 - \epsilon^2 - 1}{1 - \epsilon}$

where $p^{(0,0)}_n$ is the Legendre polynomial of degree $n$. 

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Now, let us consider the case when \( J \neq 0 \). We will show that \( R^J_{2n+2} \) can be obtained by a recursive method over the variable \( J \).

Suppose we have solved the problem of obtaining \( Q^J_n \) (\( u \)) i.e.

orthogonalize \( 1, u, \ldots, u^n \) over \( [e^2, 1] \) with weight function \( u^{J-1} \)

and we want to obtain \( Q^J_n (J) \); orthogonalize \( 1, u, \ldots, u^n \) over \( [e^2, 1] \) with weight function \( u^J \). We obtain a relationship between these 2 polynomials by making use of a special case of the following theorems [7].

Theorem 1. Let \( \{p_n(x)\} \) be the orthonormal polynomials associated with the distribution \( d\alpha(x) \) on the interval \( (a, b) \). Also, let

\[
p(x) = c \cdot (x - x_1) \cdot (x - x_2) \ldots (x - x_l)
\]

be non-negative in this interval. Then the orthogonal polynomials \( \{q_n(x)\} \) associated with the distribution \( p(x) \ d\alpha(x) \) can be represented in terms of the polynomials \( p_n(x) \) as follows:

\[
p(x) \ p_n(x) = \begin{vmatrix}
P_n(x) & P_{n+1}(x) & \cdots P_{n+1}(x)
P_n(x_1) & P_{n+1}(x_1) & \cdots P_{n+1}(x_1)
\vdots & \vdots & \ddots & \vdots 
P_n(x_l) & P_{n+1}(x_l) & \cdots P_{n+1}(x_l)
\end{vmatrix}
\]
Theorem 2. The following relation holds for any 3 consecutive orthogonal polynomials:

\[ p_n(x) = (A_n x + B_n) p_{n-1}(x) = C_n \cdot p_{n-2}(x) \]

of the highest coefficient of \( p_n(x) \) is denoted by \( k_n \), we have

\[ A_n = \frac{k_n}{k_{n-1}} \quad \text{and} \quad C_n = \frac{A_n}{A_{n-1}} \]

Theorem 3.

\[ p_0(x) p_0(y) + p_1(x) p_1(y) + \cdots + p_n(x) p_n(y) \]

\[ = \frac{k_n}{k_{n+1}} \left( p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y) \right) \]

using Theorem 1 with \( p_n(u) = Q_n^{J-1}(u) \), \( q_n(u) = Q_n^J(u) \) and \( p(u) = u \), we have

\[
2.1 \ u_Q^J(u) = \frac{1}{Q_n(o)} \begin{vmatrix}
Q_n^{J-1}(u) & Q_n^{J-1}(o) \\
Q_n^J(u) & Q_n^{J-1}(o)
\end{vmatrix}
\]

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We note that this formula becomes undeterminate for $u = r^2 = 0$. But we need the values of $Q^J_n$ at $u = 0$ to calculate the next set of polynomials, so expand equation 2.1 and use Theorem 3 to get

$$Q^J_n (u) = \frac{\nu^{J-1}}{(1-\epsilon^2)^{J-1} Q^J_n(0)} \sum_{i=0}^{n} \frac{Q^{J-1}_i(0)}{h_i} \frac{Q^{J-1}_i(u)}{h_i}$$

where

$$h^J_n = \frac{1}{2} \int \left\{ Q^J_n(u) \right\}^2 u^J \, du$$

is the normalization constant for $Q^J_n(u)$. By substituting the expression in equation 1 for $Q^J_n(u)$ into equation 3 and using Theorem 3, we obtain

$$h^J_i = \frac{-1}{1 - \epsilon^2} \frac{Q^{J-1}_{i+1}(0)}{Q^{J-1}_i(0)} h^{J-1}_i$$

converting back to the variable $r$, we have

$$R^J_{2n+J}(r) = r^J Q^J_n(r^2).$$
3. Some comments about the computer program.

The outline for this computer program is:

\[\begin{array}{c}
J=0 \\
\text{Compute } Q_n^0 \text{ and } H_n^0 \\
\text{For } J = 1 \text{ to } N \\
\text{Compute } Q_n^J \text{ and } H_n^J \text{ using } Q_n^{J-1}, Q_{n+1}^{J-1} \text{ and } H_n^{J-1}
\end{array}\]
For a given value of $N$, the following table shows the values of $Q_n^J$ and $H_n^J$ that are generated.

**TABLE 1**

<table>
<thead>
<tr>
<th>$Q_n^J$ or $H_n^J$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>N-2</th>
<th>N-1</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>*</td>
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<td>*</td>
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<tr>
<td>1</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
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<td>*</td>
<td></td>
<td></td>
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<tr>
<td>2</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>...</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N-2</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N-1</td>
<td>*</td>
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<tr>
<td>N</td>
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</tr>
</tbody>
</table>

NOTE: Since $h_n^J$ is dependent upon knowing $Q_{n+1}^{J-1}$, each of the above rows are shorter by 1 entry.
The polynomials $Q_n^o$ are directly computed from formula (1) namely,

$$Q_n^o (u) = p_n (o, o) (au + v)$$

where $p_n (o, o)$ is the Legendre polynomial of degree $n$, $a = 2 / (1 - \varepsilon^2)$ and $v = -(1 + \varepsilon^2) / (1 - \varepsilon^2)$. Likewise, the constants $H_n^o$ are directly computed from (3); namely,

$$H_n^o = \frac{1}{2} \int_{\varepsilon^2}^{1} \left( Q_n^o (u) \right)^2 du.$$ 

NOTE: The above integral is computed in closed form. This is possible because $Q_n^o (u)$ is a polynomial of degree $n$. This is done via a call to SQPOLY (square the polynomial) and a call to INTGRL (find the integral of a polynomial).

Once the first row is known ($Q_n^j$ and $H_n^j$ for $J = 0$), the recursive algorithm can then be used to compute each succeeding row ($Q_n^j$ and $H_n^j$ for $J = J_o$). The results are then printed via a call to RJN (compute $R_{2n+J}^J = r^2 Q_n^J (r^2)$).

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4. Example

A computer run for \( n = 3 \) is shown below.

\[
\text{ENTER N-LARGEST } R(\emptyset, N) \text{ DESIRED}
\]

\( ?3 \)

\( R(\emptyset, \emptyset) \)

\[
1.\text{gggg}
\]

\[
\text{NORM: } 3.83767639
\]

\[
-0.5487 + 1.\text{gggg}R^{**} 2
\]

\( R(\emptyset, 2) \)

\[
R(\emptyset, 4) \]

\[
\text{NORM: } 16.4661363\emptyset
\]

\[
-0.2331 - 1.\emptyset973R^{**} 2 + 1.\text{gggg}R^{**} 4
\]

\( R(\emptyset, 6) \)

\[
\text{NORM: } 71.94691818
\]

\[
-0.981 = 0.7899R^{**} 2 - 1.6469R^{**} 4 + 1.\text{gggg}R^{**} 6
\]

\( R(1, 1) \)

\[
\text{NORM: } 1.9\emptyset923153
\]

\[
1.\text{gggg}R^{**} 1 +
\]

\( R(1, 3) \)

\[
\text{NORM: } 8.32586779
\]

\[
-0.6724R^{**} 1 + 1.\text{gggg}R^{**} 3
\]

\( R(1, 5) \)

\[
\text{NORM: } 35.89625267
\]

\[
0.3191R^{**} 1 - 1.2252R^{**} 3 + 1.\text{gggg}R^{**} 5
\]

\( R(2, 2) \)

\[
\text{NORM: } 2.32829\emptyset44
\]

\[
1.\text{gggg}R^{**} 2 +
\]

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R(2, 4)
NORM: 11.84463583
- .7897R** 2 + 1.68385R** 4
* * * * * * * * * * * * *

R(3, 3)
NORM: 2.6873584
1.68385R** 3 +
* * * * * * * * * * * * *

* STOP *

A COPY OF THE ABOVE PROGRAM CAN BE OBTAINED FROM THE AUTHOR.
References


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