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16. Abstract
The long period variations of the first eight planets in the solar system are studied. First, the Lagrangian solution is calculated and then the long period terms with fourth order eccentricities and inclinations are introduced into the perturbation function. A second approximation was made taking into account the short period terms' contribution, namely the perturbations of first order with respect to the masses. Special attention was paid to the determination of the integration constants.

The relative importance of the different contributions is shown. It is useless for example to introduce the long period terms of fifth order if no account has been taken of the short period terms. Meanwhile, the terms that have been neglected would not introduce large changes in the integration constants. Even so, the calculation should be repeated with higher order short period terms and fifth order long periods.

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LONG PERIODIC TERMS IN THE SOLAR SYSTEM

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SUMMARY [English language summary from the original text]

We have studied the long period variations of the eight planets of the solar system (Pluto is excluded). We first calculated the Lagrange solution. We then introduced the long period terms of fourth order in excentricities and inclinations in the disturbing function. In a second approximation we took into account the contribution of the short period terms which provide the perturbations of the first order with respect to the masses. We have paid special attention to the problem of the determination of the integration constants.

We began with the expansion of the disturbing function R [formula (1)]. We used the variables $h = e \sin \varpi$, $K = \cos \varpi$, $p = \sin \frac{i}{2} \cos \Omega$, $q = \sin \frac{i}{2} \cos \Omega$ and obtained expression (3) for the disturbing function and the equations of Lagrange (4).

In the Lagrange method, one retains only the second order terms of the quantities h, k, p, q of the so called long period part of the disturbing function. The resolution of the system of differential equations thus obtained gives the solution of Lagrange (5). The corresponding integration constants are given in Tables 2, 3, 4 and 5.

We later introduced the long period terms of the disturbing function, of fourth order in the quantities h, k, p, q. These terms give rise to third order terms in the Eq. (6) for the variables h_u, for example. We then substitute numerically the Lagrange solution

*Numbers in the margin indicate pagination in the foreign text.

in these third order terms and hence obtain the form (8) of the equation for dh_u/dt.

In a second approximation, we also introduced the short period terms of the disturbing function. The masses are substituted numerically and the terms thus found are indentical in form to those arising from long period terms of fourth order of the disturbing function and are directly added to the Eq. (8).

To solve the systems of Eq. (8) and (9), we used the Krylov-Bogolioubov method, which consists in seeking a solution of the form (11) with a modification of the frequencies given by (12). Through (12) and derivation of (11) we obtain (13). In addition, the substitution of (11) into (8) and (9) gives (14), so that we get the two expressions (13) and (14) for dh_u/dt and dk_u/dt; their third order parts are given in (15) by identification. It is then possible to determine the quantities $M_{u,\psi,\theta}$, $N_{u,\psi,\theta}$, B_j and C_j introduced in (11) and (12).

The solutions are given by (16) and (17) and in Tables 8 to 13.

The comparison between Tables 3 and 8 shows that the integration constants have been greatly modified, particularly for the planets Mercury, Venus, Earth and Mars. This is due to the importance of third order terms for these planets. Table 9 gives the modifications B_i and C_i of the frequencies as well as the new values of these frequencies: $\tilde{g}_i = g_i + B_i$; $\tilde{s}_i = s_i + C_i$. Tables 10 and 11 show the amplitude of the Lagrange solution calculated with the new constants; Tables 12 and 13 show the amplitudes $M_{u,\psi,\theta}$ and $N_{u,\psi,\theta}$ of the arguments of higher order.

This work displays the relative importance of the different contributions: it is, for example, useless to introduce the long period terms of fifth order if one has not taken into account the short period terms. We have included the major contributions; the neglected terms would not introduce large modifications of the

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constants of integration. However, the calculation should be repeated including long period terms of fifth order and short period terms of higher order.

Key words: planetary theory, secular perturbations

There have been several studies of long period terms in the solar system. Stockwell, Harzer (1895), Hill (1897), and more recently Brouwer and van Woerkom (1950) and Anolik et al. (1969).

Brouwer and van Woerkom calculated the Lagrange solution for the eight planets and in particular investigated the Jupiter-Saturn case. This was a continuation of the work of Hill, who had determined a mean perturbation function on the basis of Le Verrier's findings. /142
Brouwer and van Woerkom used this perturbation function, which had been extended to sixth order excentricities and inclinations, for Jupiter-Saturn. It is difficult, however, to determine the accuracy of their result because Hill empirically established some of the coefficients.

Anolik et al. dealt with the eight planet case by introducing all the perturbation function's long period terms up to fourth order excentricities and inclinations.

Our goal was to evaluate the significance of the various long period terms according to their origin. We too dealt with only the eight planet problem. Pluto was neglected for several reasons. First of all, the generally accepted mass of Pluto, which previously had been 1/360,000 the solar mass, is now 1/1,8000,000 with a large uncertainty:

$$\frac{m_{\odot}}{m_p} = 1800000 \pm 600000 .$$

Moreover, Pluto's radius vector can be less than Neptune's, with the result that expansions in α, the ratio of semimajor axes, of the perturbation function, are no longer convergent. Finally, the introduction of Pluto's influence causes the appearance of very large resonances between Neptune and Pluto whose physical character is unclear.

Lastly, we calculated the eight planet Lagrangian solution and then introduced the perturbation function's fourth order terms as well as the contribution of the short period terms of first order with respect to the masses. In addition, we particulary concentrated on the problem of determining the integration constants because of the significance of the terms modifying the Lagrangian solution.

The expansions of perturbation function R that we used are those constructed by Chapront at the Bureau des Longitudes. They take the fomr of analytical expansions in e and sin i/2, where e represents the excentricity and i the inclination of the orbital plane relative to the plane of origin.

$$R = \sum_{r,s,t,u,j} Q(\alpha) e_I^r e_E^s \times \left(\sin\frac{i_I}{2}\right)^t \times \left(\sin\frac{i_E}{2}\right)^u \cos\Phi_j \tag{1}$$

with

$$\Phi_j = j_1\lambda_I + j_2\lambda_E + j_3\varpi_I + j_4\varpi_E + j_5\Omega_I + j_6\Omega_E .$$

λ being the planet's longitude, ϖ the argument of the perihelion, and Ω the argument of the node. The subscript I refers to the inside planet and E to the outside one. The perturbation function's long period portion is that part for which λ_I and λ_E are absent, i.e. in which $j_1 = j_2 = 0$. The summation with respect to the small quantities e_I, e_E, sin $i_{I/2}$, sin $i_{E/2}$ is done starting with zero order terms and then 2, 3,...

The orbit's descriptive elements (the semimajor axis a, the excentricity e, the inclination i, the node argument Ω, and the perihelion argument ϖ are those of Newcomb. These elements are expressed relative to the 1850.0 ecliptic averaged over short periods, which will serve as our point of departure (t = 0 for 1850.0) in determining the integration constants of the sought after solutions. Also, we used more recent values for the masses of Venus, Earth, Mars, and Saturn than the ones Newcomb used.

The mean motions n_1 are the average observed values. The semi-

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major axes a_1 are related to the values of n_1 by the expression $n_1^2 a_1^3$ = constant. Now, we need for the mean motions values from which the secular perturbations have been removed. We therefore calculated the secular perturbations δn on the basis of Chapront's and Simon's work concerning the construction of planetary theory with secular terms.

In the end, we used for each planet the value n of the mean motion defined by:

$$n = n_1 - \delta n$$

and the value a of the semimajor axis obtained by

$$n^2 a^3 = \text{constant.}$$

We have assembled the elements adopted for the eight planets in Table 1.

Table 1
Planetary Elements for 1850.0

Planet	a_1 ("/yr)	a ("/yr)	n_1 (AU)	n (AU)	e	ϖ	i	Ω	$m_\odot/m$
Mercury	5381016.3893	5381023.1732	0.3870986713	0.3870983460	0.20560396	75 07'19".37	7 00'07".00	46 33'12".24	6000000
Venus	2106641.4171	2106651.7631	0.7233322169	0.7233298487	0.00684458	129 27 34.5	3 23 35.26	75 19 47.41	408500
Earth	1295977.4496	1295975.6094	1.000000021	1.000000968	0.01677126	100 21 36.30	0		328900
Mars	689050.9354	689059.2817	1.5236914428	1.5236791387	0.09326685	333 17 52.37	1 51 02.42	48 24 03.40	3099000
Jupiter	109256.63954	109263.05033	5.202803945	5.202600424	0.04825382	11 54 26.72	1 18 41.81	98 55 58.16	1047.355
Saturn	43996.20414	43885.62112	9.538843653	9.554827367	0.05606075	90 06 39.53	2 29 39.26	112 20 51.38	3498
Uranus	15426.0928	15384.0851	19.18228185	19.21710613	0.0469055	168 15 46.9	0 46 20.54	73 14 08.0	22869
Neptune	7864.698	7843.328	30.057342	30.111791	0.0085082	43 19 43.7	1 47 01.81	130 08 00.2	19314

[Commas in tabulated material are equivalent to decimal points.]

We chose the following variables to analyze our problem:

$$h = e \sin\varpi, \quad p = \sin\tfrac{i}{2}\sin\Omega, \qquad k = e\cos\varpi, \quad q = \sin\tfrac{i}{2}\cos\Omega. \tag{2}$$

This choice was made in order to avoid the appearance of quantities expressed in e and i in the denominators of the Lagrangian equations. Such quantities could cancel each other out. In addition, this is necessary for the resolving process because in this way the solutions /143 are expressed formally through the use of these variables and, in the algorithm of solution's construction, the second members always retain the same polynomial form.

The change in variables defined by (2) yields in the perturbation function in form (1) an expression of the form:

$$R = \sum S(\alpha) h_I^{r_1} h_E^{r_2} k_I^{s_1} k_E^{s_2} p_I^{t_1} p_E^{t_2} q_I^{u_1} q_E^{u_2} \cos(i_1 \lambda_I + i_2 \lambda_E) \tag{3}$$

where the summation is extended to such exponential values that $r_1 + r_2 + s_1 + s_2 + t_1 + t_2 + u_1 + u_2 \leq \omega$, where ω is the order at which it is desired to limit the calculations.

For the variables defined in (2) the Lagrange equations are written:

$$\begin{aligned}
\frac{dh}{dt} &= \frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial k} - \frac{h(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} + \frac{kp}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} + \frac{kq}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q} \\
\frac{dk}{dt} &= -\frac{(1-e^2)^{1/2}}{na^2}\frac{\partial R}{\partial h} - \frac{k(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\frac{\partial R}{\partial \lambda} - \frac{hp}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{hq}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q} \\
\frac{dp}{dt} &= -\frac{1}{4na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial q} - \frac{p}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{pk}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{ph}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial k} \\
\frac{dq}{dt} &= -\frac{1}{4na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial p} - \frac{q}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial \lambda} - \frac{qk}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial h} + \frac{qh}{2na^2(1-e^2)^{1/2}}\frac{\partial R}{\partial k} \\
\frac{1}{a}\frac{da}{dt} &= \frac{2}{na^2}\frac{\partial R}{\partial \lambda} \\
\frac{d\lambda}{dt} &= n - \frac{2}{na}\frac{\partial R}{\partial a} + \frac{(1-e^2)^{1/2}}{na^2[1+(1-e^2)^{1/2}]}\left(h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right) + \frac{1}{2na^2(1-e^2)^{1/2}}\left(p\frac{\partial R}{\partial p} + q\frac{\partial R}{\partial q}\right)
\end{aligned} \tag{4}$$

where $e^2 = h^2 + k^2$.

LAGRANGIAN METHOD

For a planet of subscript u perturbed by the seven other planets of subscript v, the perturbation function is written:

$$R_u = \sum_{v<u} \mu \frac{m_v}{a_u} \bar{R}_{uv} + \sum_{v>u} \mu \frac{m_v}{a_v} \bar{R}_{uv}$$

The first summation is extended to the planets inside the one under consideration, the second to the planets outside. We use the following notation:

$\bar{R}_{ij} = a_j/\Delta$ (Δ = distance of the two planets)

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and

$$\mu = \frac{n_u^2 a_u^3}{1+m_u} = \frac{n_v^2 a_v^3}{1+m_v}.$$

Limited to the second order, $\bar{R}_{uv}$ has the following expression: /144

$$\bar{R}_{uv} = C_{uv} + A_{uv}(h_u^2 + k_u^2 + h_v^2 + k_v^2) - 4A_{uv}(p_u^2 + q_u^2 + p_v^2 + q_v^2)$$
$$+ B_{uv}(k_u k_v + h_u h_v) + 8A_{uv}(q_u q_v + p_u p_v)$$

where C_{uv}, A_{uv}, B_{uv} are functions of $\alpha_{uv} = a_u/a_v$, which is constant here. We thus obtain, with the notation:

$$[u,v] = \frac{n_u \alpha_{uv} m_v}{1+m_u} \text{ if } v>u,$$

$$[u,v] = \frac{n_u m_v}{1+m_u} \text{ if } v<u,$$

$$\frac{dh_u}{dt} = + \sum_{v \neq u} [u,v](2A_{uv}k_u + B_{uv}k_v),$$

$$\frac{dk_u}{dt} = - \sum_{v \neq u} [u,v](2A_{uv}h_u + B_{uv}h_v),$$

$$\frac{dp_u}{dt} = - \sum_{v \neq u} [u,v](2A_{uv}q_u - 2A_{uv}q_v),$$

$$\frac{dq_u}{dt} = + \sum_{v \neq u} [u,v](2A_{uv}p_u - 2A_{uv}p_v).$$

This system is written in matrix form as:

$$\frac{dH}{dt} = E \times K, \quad \frac{dK}{dt} = -E \times H,$$

$$\frac{dP}{dt} = I \times Q, \quad \frac{dQ}{dt} = -I \times P,$$

where H is the column vector with components (h_{Me}, h_V, ..., h_N), K the column vector (k_{Me}, k_V, ..., k_N), P the column vector (p_{Me}, p_V, ..., p_N), and Q the column vector (q_{Me}, q_V, ..., q_N). The subscripts Me, V, ..., N represent Mercury, Venus, ..., Neptune, respectively. E and I are the matrices of the linear systems in excentricities and inclinations respectively.

The conventional resolution of the two Lagrangian systems gives the eigenvalues:

g_i, i = 1, 2, ..., 8 for the excentricities;
s_i, i = 1, 2, ..., 8 for the inclinations.

One of the eigenvectors in the system of inclinations in zero. We will assume $s_5 = 0$.

We determine the eigenvalues λ_{ij} associated with g_j, and μ_{ij} associated with s_j, which gives the Lagrangian solution:

$$\left.\begin{aligned} h_i &= \sum_{j=1}^{8} \lambda_{ij} M_j \sin(g_j t + \beta_j) \\ k_i &= \sum_{j=1}^{8} \lambda_{ij} M_j \cos(g_j t + \beta_j) \\ p_i &= \sum_{j=1}^{8} \mu_{ij} N_j \sin(s_j t + \delta_j) \\ q_i &= \sum_{j=1}^{8} \mu_{ij} N_j \cos(s_j t + \delta_j). \end{aligned}\right\} \qquad (5)$$

Lastly, we calculate the 32 constants of integration M_j, β_j, N_j, δ_j from the values of h, k, p, q at t = 0.

We have assembled the eigenvalues g and s in Table 2, and in Table 3 we show the 32 constants of integration M, β, N, δ. Lastly, Table 4 gives the amplitudes of the Lagrangian solution multiplied by 10^8: $\lambda_{ij} M_j \times 10^8$ for the excentricities, and similarly in Table 5, $\mu_{ij} N_j \times 10^8$ for the inclinations. Frequencies g and s are expressed in seconds per year. λ_{ih}, μ_{ij}, M_j, N_j are dimensionless numbers.

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Table 2
Frequencies in "/yr
(Lagrangian Solution)

i	g	s
1	+ 5.461369	− 5.199958
2	+ 7.346581	− 6.571387
3	+17.331295	−18.746205
4	+18.004584	−17.636111
5	+ 3.711401	0
6	+22.286552	−25.741176
7	+ 2.701787	− 2.904326
8	+ 0.633116	− 0.677520

Table 3
Constants of Integration
(Lagrangian Solution)

i	M	β	N	δ
1	0.18141040	87 11'11".37	0.06274851	18 15'28".76
2	0.01909712	192 40 29.79	0.00506380	316 20 04.81
3	0.01056860	332 56 51.67	0.01222166	254 17 05.60
4	0.07300403	316 06 34.81	0.02519918	295 40 38.75
5	0.04319426	27 48 34.76	0.01383974	106 08 44.50
6	0.04837743	127 51 04.46	0.00786789	126 19 31.43
7	0.03140786	106 19 25.39	0.00880286	313 42 55.41
8	0.00923780	66 09 18.45	0.00588386	201 00 52.94

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Table 4

$\lambda_{ij}M_j \times 10^8$. Amplitudes of Lagrangian Solution

i \ j	1	2	3	4	5	6	7	8
Mercury	18141040	−2318196	155897	−170495	2414071	11364	62612	727
Vénus	631873	1909712	−1275337	1497153	1624499	−55447	61654	1106
Earth	405895	1490407	1056860	−1492703	1624452	247593	65295	1284
Mars	66283	264664	3017351	7300403	1870379	1617034	86626	2053
Jupiter	−703	−1055	−95	−55	4319426	−1563040	218379	5968
Saturn	−627	−1088	−750	−840	3404356	4837743	199229	6736
Uranus	271	265	44	46	−4384597	−181906	3140786	141054
Neptune	4	10	3	3	160418	−13561	−338902	923780

Table 5

$\mu_{ij}N_j \times 10^8$. Amplitudes of Lagrangian Solution

i \ j	1	2	3	4	5	6	7	8
Mercury	6274851	−1781583	204668	58171	1383974	13920	−166549	−72367
Vénus	591896	506380	−1341594	−343391	1383974	6022	−95883	−66215
Earth	426404	408232	1222166	226117	1383974	140699	−86614	−64906
Mars	90534	90894	−1794150	2519918	1383974	482917	−62850	−61488
Jupiter	−1038	−655	−9	−88	1383974	−315878	−47877	−58499
Saturn	−1328	−925	−241	−916	1383974	786799	−39034	−56388
Uranus	1112	477	20	86	1383974	−34790	880286	54715
Neptune	28	27	2	9	1383974	−3858	−103566	588386

INTRODUCTION OF HIGHER ORDER TERMS

We are now going to introduce the perturbation function's long period terms of fourth order h, k, p, q, as well as the perturbation function's short order terms.

Fourth Order Long Period Terms

By differentiation, these terms yield third order terms, and the Lagrangian equation for variable h_u, for example, then has the following form:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u,v]\{2A_{uv}k_u + B_{uv}k_v - P_{uv}(h_u, h_v, k_u, k_v, p_u, p_v, q_u, q_v)\} \qquad (6)$$

where P_{uv} is a homogeneous third degree polynomial.

Into polynomial P_{uv} we substitute the Lagrangian solution (5), whose numerical values are given in Tables 2, 3, 4, and 5:

$$h_u = \sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j, \quad k_u = \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j,$$
$$p_v = \sum_{j=1}^{8} \mu_{vj} N_j \sin\theta_j, \quad q_v = \sum_{j=1}^{8} \mu_{vj} N_j \cos\theta_j,$$

where we have made $\psi_j = g_j t + \beta_j$ and $\theta_j = s_j t + \delta_j$. Therefore only the numerical values of amplitudes $\lambda_{uj} M_j$ and $\mu_{uj} N_j$ appear in this calculation.

Among the values of the i and j subscripts of arguments ψ_i ($i = 1, 2, \ldots, 8$) and θ_j ($j = 1, 2, \ldots, 8$), such a substitution makes combinations appear in which at most only three values of subscripts i and j are involved. For example, there will be combinations of the type $(\psi_1 + \theta_2 - \theta_4)$, $(2\psi_5 - \psi_6)$.

The expression $\sum_{v \neq u} [u, v] P_{uv}$ therefore has the form:

$$\sum_{v \neq u} [u, v] P_{uv} = \sum_{\substack{i_1, i_2, \ldots, i_8 \\ j_1, j_2, \ldots, j_8}} \xi_{u, i_1, \ldots, i_8, j_1, \ldots, j_8} \cos(i_1 \psi_1 + \cdots + i_8 \psi_8 + j_1 \theta_1 + \cdots + j_8 \theta_8) \tag{7}$$

where $\xi_{u, i_1 \ldots i_8, j_1 \ldots j_8}$ is a numerical coefficient.

The summation over integers i and j is such that:

$$\sum_{m=1}^{8} |i_m| + \sum_{m=1}^{8} |j_m| = 1 \text{ or } 3$$

We will designate that:

$$(\psi, \theta) = i_1 \psi_1 + i_2 \psi_2 + \cdots + i_8 \psi_8 - j_1 \theta_1 + j_2 \theta_2 + \cdots + j_8 \theta_8$$

and hence equation (7) takes on the form:

$$\sum_{v \neq u} [u, v] P_{uv} = \sum_{\psi, \theta} \xi_{u, \psi, \theta} \cos(\psi, \theta).$$

We also make:

$$\varepsilon = +1 \text{ if } \sum_{m=1}^{8} i_m - \sum_{m=1}^{8} j_m = +1,$$ /146

$$\varepsilon = -1 \text{ if } \sum_{m=1}^{8} i_m + \sum_{m=} i_m = -1.$$

Equation (6) is then written:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uv}k_u + B_{uv}k_v\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi,\theta). \tag{8}$$

Substituting the Lagrangian solution into the equation in dk_u/dt similarly yields:

$$\frac{dk_u}{dt} = -\sum_{v \neq u} [u, v]\{2A_{uv}h_u + B_{uv}h_v\} - \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi,\theta). \tag{9}$$

We also calculate:

$$\frac{dp_u}{dt} = -\sum_{v \neq u} [u, v]\{2A_{uv}q_u - 2A_{uv}q_v\} + \sum_{\psi,\theta} \eta_{u,\psi,\theta} \cos(\psi,\theta).$$

$$\frac{dq_u}{dt} = \sum_{v \neq u} [u, v]\{2A_{uv}p_u - 2A_{uv}p_v\} - \sum_{\psi,\theta} \varepsilon \times \eta_{u,\psi,\theta} \sin(\psi,\theta).$$

Short Period Terms

Substituting the Lagrangian solution into the short period part of the perturbation function yields only short period terms that are first order with respect to the masses. It is only with the second mass order that we come across long period terms again.

This time we have to consider for each planet the complete /147
system of Lagrange equations (4), which we will write for a planet of subscript u in the form:

$$\left.\begin{aligned} \frac{dh_u}{dt} &= F_{h_u}, & \frac{dk_u}{dt} &= F_{k_u}. \\ \frac{dp_u}{dt} &= F_{p_u}, & \frac{dq_u}{dt} &= F_{q_u}. \\ \frac{1}{a_u}\frac{da_u}{dt} &= F_{a_u}, & \frac{d\lambda_u}{dt} &= n_u + F_{\lambda_u}. \end{aligned}\right\} \qquad (10)$$

We determine the short period effects argument by argument. For a short period argument $i\lambda_u + j\lambda_v$, i and j being given integers, the functions F have the form:

$$F_c \cos(i\lambda_u + j\lambda_v) + F_s \sin(i\lambda_u + j\lambda_v)$$

where F_c and F_s are polynomials in h_u, k_u, q_u, h_v, k_v, p_v, q_v, whose coefficients are functions of $\alpha_{uv} = a_u/a_v$, n_u and n_v. (In the special case in which one of the two integers i, j is zero, i.e., in the case in which the short period argument takes on the form $i\lambda_u$, the functions F_c and F_s depend on h_v, k_v, p_v, q_v, n_v for v = 1, 2, ..., 8 and on the seven quantities $\alpha_{uv} = a_u/a_v$ for $u \neq v$.)

We therefore substitute the Lagrangian solution into equations (10), which after integration yield a short period increase in the Lagrangian solution. Then by doing a Taylor expansion of the second members of equations (10), we obtain second order terms with respect to mass after substituting the first order that we have just found. We will retain only the second terms' long period parts.

Since the masses are always substituted for numerically, the terms thus found in the second members of the Lagrangian equations have the same form as those coming directly from the perturbation function's fourth order long period terms.

In contrast to the case of the perturbation function's long periods, for which we kept the fourth period terms, the criterion for choosing short period terms is numerical. What we did was to retain the beginning of the h, k, p, q expansion of all the arguments causing changes in the Lagrangian solution frequencies of more than 10^{-3} "/yr.

RESOLUTION OF THE SYSTEMS OF DIFFERENTIAL EQUATIONS

We saw that the contribution of the short period terms took on the same form as the terms coming directly from the perturbation function. We therefore have to resolve a system of differential equations having the form:

$$\frac{dh_u}{dt} = \sum_{v \neq u} [u,v]\{2A_{uv}k_u + B_{uv}k_v\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta}\cos(\psi,\theta). \tag{8}$$

$$\frac{dk_u}{dt} = -\sum_{v \neq u} [u,v]\{2A_{uv}h_u + B_{uv}h_v\} - \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta}\sin(\psi,\theta) \tag{9}$$

as well as a similar system for variables p_u and q_u.

For that, we are going to use the Krylov-Bogolyubov method. This method consists of finding a solution of the form:

$$\left.\begin{aligned} h_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{u,\psi,\theta}\sin(\psi,\theta) \\ k_u &= \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{u,\psi,\theta}\cos(\psi,\theta) \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} p_u &= \sum_{j=1}^{8} \mu_{uj} N_j \sin\theta_j + \sum_{\psi,\theta} N_{u,\psi,\theta}\sin(\psi,\theta) \\ q_u &= \sum_{j=1}^{8} \mu_{uj} N_j \cos\theta_j + \sum_{\psi,\theta} N'_{u,\psi,\theta}\cos(\psi,\theta) \end{aligned}\right\}$$

with

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$$\left.\begin{aligned} \frac{d\psi_j}{dt} &= g_j + B_j \\ \frac{d\theta_j}{dt} &= s_j + C_j. \end{aligned}\right\} \tag{12}$$

By differentiating system (11) and taking account of (12), we obtain:

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$$\left.\begin{aligned}\frac{dh_u}{dt} &= \sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j \times (g_j + B_j) + \sum_{\psi,\theta} (g.s) M_{u,\psi,\theta} \cos(\psi,\theta)\\ \frac{dk_u}{dt} &= -\sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j \times (g_j + B_j) - \sum_{\psi,\theta} (g.s) M'_{u,\psi,\theta} \sin(\psi,\theta)\end{aligned}\right\} \qquad (13)$$

where

$$(g.s) = i_1 g_1 + i_2 g_2 + \cdots + i_8 g_8 + j_1 s_1 + j_2 s_2 + \cdots + j_8 s_8 .$$

Furthermore, equations (8) and (9) yield, by plugging in (11):

$$\left.\begin{aligned}\frac{dh_u}{dt} &= \sum_{v\neq u} [u,v]\left\{2A_{uv}\left[\sum_{j=1}^{8} \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{u,\psi,\theta} \cos(\psi,\theta)\right]\right.\\ &\quad \left. + B_{uv}\left[\sum_{j=1}^{8} \lambda_{vj} M_j \cos\psi_j + \sum_{\psi,\theta} M'_{v,\psi,\theta} \cos(\psi,\theta)\right]\right\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi,\theta)\\ \frac{dk_u}{dt} &= -\sum_{v\neq u} [u,v]\left\{2A_{uv}\left[\sum_{j=1}^{8} \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta)\right]\right.\\ &\quad \left. + B_{uv}\left[\sum_{j=1}^{8} \lambda_{vj} M_j \sin\psi_j + \sum_{\psi,\theta} M_{v,\psi,\theta} \sin(\psi,\theta)\right]\right\} - \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi,\theta) .\end{aligned}\right\} \qquad (14)$$

Hence, we have two expressions, (13) and (14), for dh_u/dk and dk_u/dt. We make equal their parts that are of third order with respect to variables h, k, p, and q:

$$\left.\begin{aligned}&\sum_{j=1}^{8} B_j \lambda_{uj} M_j \cos\psi_j + \sum_{\psi,\theta} (g.s) M_{u,\psi,\theta} \cos(\psi,\theta)\\ &\quad = \sum_{v\neq u} [u,v]\left\{2A_{uv} \sum_{\psi,\theta} M'_{u,\psi,\theta} \cos(\psi,\theta) + B_{uv} \sum_{\psi,\theta} M'_{v,\psi,\theta} \cos(\psi,\theta)\right\} + \sum_{\psi,\theta} \xi_{u,\psi,\theta} \cos(\psi,\theta)\\ &\sum_{j=1}^{8} B_j \lambda_{uj} M_j \sin\psi_j + \sum_{\psi,\theta} (g.s) M'_{u,\psi,\theta} \sin(\psi,\theta)\\ &\quad = \sum_{v\neq u} [u,v]\left\{2A_{uv} \sum_{\psi,\theta} M_{u,\psi,\theta} \sin(\psi,\theta) + B_{uv} \sum_{\psi,\theta} M_{v,\psi,\theta} \sin(\psi,\theta)\right\} + \sum_{\psi,\theta} \varepsilon \times \xi_{u,\psi,\theta} \sin(\psi,\theta) .\end{aligned}\right\} \qquad (15)$$

The method now consists of establishing argument by argument identities within each order. Two cases arise:

1) The case in which the argument (ψ,θ) is equal to ψ_j, i.e. we have:

$$\sum_{m=1}^{8} |i_m| + \sum_{m=1}^{8} |j_m| = 1 .$$

Here, we once again come across the Lagrangian solution arguments, and establishing identities between coefficients makes it possible to determine the new frequency values. The solutions then are expressed in Fourier series of these new arguments. Establishing the identities yields:

$$\lambda_{uj} M_j B_j + g_j M'_{u,\psi_j} = \sum_{v \neq u} [u,v](2A_{uv} M'_{u,\psi_j} + B_{uv} M'_{v,\psi_j}) + \xi_{u,\psi_j}$$

and

$$\lambda_{uj} M_j B_j + g_j M_{u,\psi_j} = \sum_{v \neq u} [u,v](2A_{uv} M_{u,\psi_j} + B_{uv} M_{v,\psi_j}) + \xi_{u,\psi_j}.$$

For a given ψ_j, subtraction of these two equations furnishes: /149

$$\left(\sum_{v \neq u} [u,v] \times 2A_{uv} + g_j\right)(M'_{u,\psi_j} - M_{u,\psi_j}) + \sum_{v \neq u} [u,v] B_{uv}(M'_{v,\psi_j} - M_{v,\psi_j}) = 0.$$

The fact that and $[u,v]B_{uv}$ are not zero means that $M_{u,\psi_j} = M'_{u,\psi_j}$ whatever u and j are.

We can then write:

$$\left(\sum_{v \neq u} [u,v] \times 2A_{uv} - g_j\right) M_{u,\psi_j} + \sum_{v \neq u} [u,v] B_{uv} M_{v,\psi_j} = \lambda_{uj} M_j B_j - \xi_{u,\psi_j}.$$

In the first member of this expression, we once again come across matrix E of the Lagrangian system. Subtracting the eigenvalue g_j from the principal diagonal means that the M_{u,ψ_j} values (u = 1, 2, ..., 8) will not be independent. We then let:

$$M_{j,\psi_j} = 0$$

This is an arbitrary step in the Krylov-Bogolyubov method. It reduces to changing variables over the integration constants M_j, N_j. The choice of $M_{j,\psi_j} = 0$ does not specify the solution but imposes the choice of a certain type of expansion for the coefficients of arguments ψ_j.

Having made this choice, we then have for each ψ_j, j fixed, a system of eight equations in eight unkowns: B_j and M_{u,ψ_j}, $(u \neq j)$.

2) Case in which the argument (ψ,θ) is random, i.e. such that:

$$\sum_{m=1}^{8}|i_m|+\sum_{m=1}^{8}|j_m|=3.$$

Establishing argument by argument identities in equations (15) yields:

$$\sum_{v\neq u}[u,v](2A_{uv}M'_{u,\psi,\theta}+B_{uv}M'_{v,\psi,\theta})-(g,s)M_{u,\psi,\theta}=-\xi_{u,\psi,\theta}$$

and

$$\sum_{v\neq u}[u,v](2A_{uv}M_{u,\psi,\theta}+B_{uv}M_{v,\psi,\theta})-(g,s)M'_{u,\psi,\theta}=-\varepsilon\times\xi_{u,\psi,\theta}.$$

By subtraction, we obtain $M_{u,\psi,\theta}=\varepsilon M'_{u,\psi,\theta}$ whatever u and argument (ψ,θ) is.

We can then write:

$$\left[\sum_{v\neq u}[u,v]\times 2A_{uv}-\varepsilon(g,s)\right]M_{u,\psi,\theta}+\sum_{v\neq u}[u,v]B_{uv}M_{v,\psi,\theta}=-\varepsilon\xi_{u,\psi,\theta}.$$

This time there is no arbitrary step and the resolution of the system of eight equations in eight unknowns gives for each argument (ψ,θ) the eight values $M_{u,\psi,\theta}$ $(u = 1, 2, \ldots, 8)$.

We therefore have expansions of h_u and k_u:

$$\left.\begin{aligned} h_u &= \sum_{j=1}^{8}\lambda_{uj}M_j\sin\psi_j+\sum_{\psi,\theta}M_{u,\psi,\theta}\sin(\psi,\theta)\\ k_u &= \sum_{j=1}^{8}\lambda_{uj}M_j\cos\psi_j+\sum_{\psi,\theta}\varepsilon M_{u,\psi,\theta}\cos(\psi,\theta)\end{aligned}\right\}\qquad(16)$$

And similarly we find:

$$\left.\begin{aligned} p_u &= \sum_{j=1}^{8}\mu_{uj}N_j\sin\theta_j+\sum_{\psi,\theta}N_{u,\psi,\theta}\sin(\psi,\theta)\\ q_u &= \sum_{j=1}^{8}\mu_{uj}N_j\cos\theta_j+\sum_{\psi,\theta}\varepsilon N_{u,\psi,\theta}\cos(\psi,\theta).\end{aligned}\right\}\qquad(17)$$

RESULTS AND DETERMINATION OF THE CONSTANTS OF INTEGRATION

In Tables 6 and 7 we give the integration constants and frequencies of the solutions obtained solely from the second order h, k, p, q long period terms. Comparison of tables 3 and 6 show how great the contribution of the perturbation function's fourth order long period terms is, especially for Mercury, Venus, Earth, and Mars. Table 7 contains the frequency modifications B_i anc C_i, as well as the frequencies' new values: $\tilde{g}_i = g_i + B_i$, $\tilde{s}_i = s_i + C_i$.

Table 6
Constants of Integration (According to the Solution Based on the Perturbation Function's Second and Fourth Order Long Periods)

i	M_i	β_i	N_i	δ_i
1	0,18454867	83 32'24".16	0,05887664	11 51'44".62
2	0,01864278	191 26 15.44	0,00323843	302 59 26.61
3	0,01204101	318 18 13.08	0,00967327	248 33 14.44
4	0,06311073	307 01 46.20	0,03227762	275 46 53.61
5	0,04297488	27 17 43.13	0,01384057	106 07 34.95
6	0,04842782	127 29 49.50	0,00786457	125 49 11.16
7	0,03210686	100 44 43.97	0,00880119	316 00 16.25
8	0,00932559	64 54 28.20	0,00592089	201 11 07.71

[Commas in tabulated material are equivalent to decimal points.]

Table 7
Modifications and New Frequencies in "/yr
(According to the Solution Based on the Perturbation Function's Second and Fourth Order Long Periods)

i	B_i	$\tilde{g}_i$	C_i	$\tilde{s}_i$
1	−0,258373	+ 5,202996	−0,443985	− 5,643943
2	−0,000721	+ 7,345860	−0,220510	− 6,791897
3	−0,130032	+17,201263	−0,152704	−18,898909
4	−0,169168	+17,835416	−0,231362	−17,867473
5	+0,018087	+ 3,729488	0	0
6	+0,322115	+22,608667	−0,606907	−26,348083
7	+0,078361	+ 2,780148	−0,083205	− 2,987531
8	+0,009180	+ 0,642296	−0,009136	− 0,686656

[Commas in tabulated material are equivalent to decimal points.]

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Table 8
Constants of Integration (Complete Solution)

i	M_i	β_i	N_i	δ_i
1	0.17791613	87 03′02″09	0.05962772	12 08 17″15
2	0.02104749	193 35 04.97	0.00315338	305 13 17.19
3	0.00988829	319 43 16.64	0.01002295	249 01 59.07
4	0.06115173	307 48 39.56	0.03130279	277 56 06.16
5	0.04341616	28 30 11.60	0.01383939	106 09 11.57
6	0,04814727	127 42 54.52	0.00785328	125 38 34.23
7	0.03126134	114 46 31.58	0.00880088	316 17 35.94
8	0.00899181	72 05 25.03	0.00588806	201 17 15.59

[Commas in tabulated material are equivalent to decimal points.]

Table 9
Modifications and New Frequencies in "/yr (Complete Solution)

i	B_i	$\bar{g}_i$	C_i	$\bar{s}_i$
1	−0,262290	+ 5,199079	−0.410979	− 5.610937
2	−0,000490	+ 7,346091	−0,199640	− 6.771027
3	−0,110749	+17,220546	−0.083094	−18.529299
4	−0,147321	+17.857263	−0.182658	−17.818769
5	+0.495804	+ 4.207205	0	0
6	+3,930206	+26,216758	−0,525894	−26.267070
7	+0,363394	+ 3.065181	−0.095511	− 2.999837
8	+0,034747	+ 0,667863	−0.013911	− 0.691431

[Commas in tabulated material are equivalent to decimal points.]

Tables 8 and 9 give the constants of integration and the frequencies for the complete solutions, i.e. the solutions that take the short periods into consideration. By comparing tables 6 and 8, we can see that the integration constants are once more greatly altered. /151
Comparison of Tables 7 and 9 show that while the short period terms hardly change the frequencies related to the inside planets, the g_5 and g_6 frequencies on the contrary are changed to a much greater extent by the short period terms than by the perturbation function's fourth order long period terms.

The modification of the constants of integration originates in the magnitude of the nonlinear terms found in particular in the expressions of the elements related to the inside planets. Of course, we began by calculating these terms by numerical substitution of the Lagrangian solution in Tables 4 and 5. We then determined an analytic form of the expressions found so as to calculate the new integration /153 constants. With the help of this analytic form and by making a first order Taylor expansion about the first values of the integration constants, we obtained integration constants of sufficient accuracy after several iterations.

Table 10

$\lambda_{ij}M_j \times 10^8$. Lagrangian Solution Amplitudes

i \ j	1	2	3	4	5	6	7	8
Mercury	17791613	-2554951	145862	-142815	2426473	11310	62320	708
Venus	619702	2104749	-1193243	1254088	1632845	-55183	61366	1076
Earth	398077	1642622	988829	-1250360	-1632798	246415	64990	1250
Mars	65007	291694	2823123	6115173	1879988	1609341	86222	1998
Jupiter	-689	-1163	-89	-46	4341616	-1555604	217360	5809
Saturn	-615	-1200	-702	-703	3421845	4814727	198300	6557
Uranus	266	293	41	38	-4407122	-181041	3126134	137298
Neptune	3	11	3	2	161243	-13497	-337321	899181

Table 11

$\mu_{ij}N_j \times 10^8$. Lagrangian Solution Amplitudes

i \ j	1	2	3	4	5	6	7	8
Mercury	5962772	-1109444	167847	72261	1383939	13894	-166511	-72418
Venus	562458	315338	-1100237	-426565	1383939	6010	-95861	-66262
Earth	405197	254218	1002295	280886	1383939	140436	-86594	-64951
Mars	86031	56602	-1471377	3130279	1383939	482015	-62835	-61532
Jupiter	-986	-408	-7	-109	1383939	-315287	-47866	-58541
Saturn	-1262	-576	-197	-1138	1383939	785328	-39025	-56429
Uranus	1057	297	16	107	1383939	-34725	880088	54753
Neptune	27	16	1	12	1383939	-3851	-103543	588806

Lastly, we give the totality of our solution in Tables 8 to 13. /154 Hence, Table 8 contains the 32 integration constants. Table 9 gives the B_i and C_i frequency modifications as well as the frequencies' new values: $\tilde{g}_i = g_i + B_i$; $\tilde{s}_i = s_i + C_i$. The amplitudes of the Lagrangian solution corresponding to the new constants are given in Table 10 for

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$\lambda_{ij}M_j \times 10^8$ and in Table 11 for $\mu_{ij}N_j \times 10^8$. Finally, Tables 12 and 13 contain the amplitudes $M_{u,\psi,\theta}$ and $N_{u,\psi,\theta}$ of the higher order arguments (ψ,θ). When computing these terms, we retained only the arguments whose amplitudes are higher than 10^{-4} for the planets Mercury, Venus, Earth, and Mars, and 10^{-6} for Jupiter, Saturn, Uranus, and Neptune. The zeros found in Tables 12 and 13 are amplitudes less than the retained percisions.

Table 12
$M_{u,\psi,\theta} \times 10^6$

Argument	Mercury	Vénus	Earth	Mars
ψ_7	151	0	0	0
$\psi_6 - \theta_3 + \theta_6$	0	0	0	− 199
$\psi_6 - \theta_3 + \theta_4$	0	0	0	453
$\psi_6 - \theta_4 + \theta_6$	0	0	0	287
ψ_6	0	981	− 1064	− 13379
$\psi_6 + \theta_3 - \theta_4$	0	0	0	762
$\psi_5 - \psi_6 - \psi_7$	0	0	0	134
$\psi_5 - 2\theta_1$	254	0	0	0
$\psi_5 - \theta_1 - \theta_2$	− 114	0	0	0
$\psi_5 - \theta_1 + \theta_7$	102	0	0	0
$\psi_5 - \theta_1 + \theta_2$	− 275	0	0	0
$\psi_5 - \theta_3 + \theta_4$	150	0	0	0
ψ_5	14547	2918	2280	1092
$\psi_5 + \theta_1 - \theta_2$	− 2172	0	0	0
$\psi_5 + \theta_1 - \theta_3$	0	− 216	182	503
$\psi_5 + \psi_6 - \psi_7$	0	0	0	105
$2\psi_5 - \psi_7$	− 118	0	0	0
$\psi_4 - \psi_5 + \psi_7$	0	0	0	− 180
$\psi_4 - \theta_1 + \theta_4$	1644	0	0	0
$\psi_4 - \theta_1 + \theta_3$	− 598	0	0	0
$\psi_4 - \theta_3 - \theta_4$	0	0	0	− 177
$\psi_4 - \theta_3 + \theta_6$	0	0	0	− 240
$\psi_4 - \theta_3 + \theta_4$	0	− 536	276	6177
$\psi_4 - 2\theta_4$	0	0	0	176
$\psi_4 - \theta_4 + \theta_6$	0	0	0	370
ψ_4	− 120	456	− 503	0
$\psi_4 + \theta_4 - \theta_6$	0	0	0	− 354
$\psi_4 + \theta_3 - \theta_4$	− 519	4126	− 3367	− 14852
$\psi_4 + \theta_3 - \theta_6$	0	0	0	211
$\psi_4 + \psi_5 - \psi_7$	0	0	0	118
$2\psi_4 - \psi_6$	0	0	0	− 286
$\psi_3 - 2\psi_4$	0	465	− 378	− 726
$\psi_3 - \psi_4 - \psi_6$	0	0	0	− 133
$\psi_3 - \psi_4 + \psi_6$	0	0	0	184
$\psi_3 - \theta_1 + \theta_4$	396	0	0	0
$\psi_3 - \theta_1 + \theta_3$	312	0	0	0
$\psi_3 - \theta_2 + \theta_3$	− 237	0	0	0
$\psi_3 - \theta_3 + \theta_4$	− 275	2457	− 2354	8864
$\psi_3 - \theta_4 + \theta_6$	0	0	0	148
ψ_3	0	0	0	− 323

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Table 12 (cont.)

Argument	Mercury	Venus	Earth	Mars
$\psi_3+\theta_4-\theta_6$	0	0	0	− 162
$\psi_3+\theta_3-\theta_4$	115	− 901	857	− 1422
$\psi_3+\theta_3-\theta_6$	0	0	0	102
$\psi_3+\psi_5-\psi_7$	0	0	0	137
$\psi_3+\psi_4-\psi_6$	0	0	0	− 144
$2\psi_3-\psi_4$	0	− 691	599	388
$\psi_2-2\theta_1$	− 374	0	0	0
$\psi_2-\theta_1-\theta_2$	175	0	0	0
$\psi_2-\theta_1+\theta_3$	− 137	0	0	0
$\psi_2-\theta_1+\theta_2$	− 4711	− 261	− 174	0
ψ_2	2947	0	− 151	0
$\psi_2+\theta_3-\theta_4$	0	111	0	0
$\psi_2+\theta_1-\theta_2$	− 875	0	0	0
$\psi_2+\theta_1-\theta_3$	105	0	0	0
$\psi_2+\theta_1-\theta_7$	355	0	0	0
$\psi_2+\psi_3-\psi_4$	205	0	0	0
$\psi_1-2\psi_2$	− 361	0	0	0
$\psi_1-\psi_2-\psi_5$	− 3340	− 131	0	0
$\psi_1-\psi_2+\psi_5$	− 194	0	0	0
$\psi_1-\psi_3+\psi_4$	− 591	0	0	0
$\psi_1-2\theta_1$	2231	0	0	0
$\psi_1-\theta_1-\theta_2$	− 1022	0	0	0
$\psi_1-\theta_1-\theta_3$	350	0	0	0
$\psi_1-\theta_1-\theta_4$	147	0	0	0
$\psi_1-\theta_1+\theta_7$	399	0	0	0
$\psi_1-\theta_1+\theta_5$	− 146	0	0	0
$\psi_1-\theta_1+\theta_4$	212	0	0	0
$\psi_1-\theta_1+\theta_3$	480	0	0	0
$\psi_1-\theta_1+\theta_2$	− 8302	0	0	0
$\psi_1-2\theta_2$	119	0	0	0
$\psi_1-\theta_2+\theta_3$	− 139	0	0	0
$\psi_1-\theta_3+\theta_4$	− 605	0	0	0
ψ_1	0	− 812	− 650	0
$\psi_1+\theta_3-\theta_4$	603	0	0	0
$\psi_1+\theta_2-\theta_3$	147	0	0	190
$\psi_1+\theta_1-\theta_2$	6786	1372	1007	173
$\psi_1+\theta_1-\theta_3$	− 505	164	− 124	135
$\psi_1+\theta_1-\theta_4$	− 209	0	0	− 1087
$\psi_1+\theta_1-\theta_5$	146	0	0	0
$\psi_1+\theta_1-\theta_7$	− 316	0	0	0
$\psi_1+\psi_3-\psi_4$	645	0	0	0
$\psi_1+\psi_2-\psi_5$	258	0	0	0
$2\psi_1-\psi_2$	1544	0	0	0
$2\psi_1-\psi_3$	− 109	0	0	0
$2\psi_1-\psi_4$	107	0	0	0
$2\psi_1-\psi_5$	− 824	− 438	− 337	0

Argument	Jupiter	Saturn	Uranus	Neptune
$\psi_8-\theta_7+\theta_8$	− 2	− 2	35	5
ψ_8	0	− 3	271	0
$\psi_8+\theta_7-\theta_8$	0	0	1	− 1
$\psi_7-2\psi_8$	0	0	0	− 1
$\psi_7-\theta_5+\theta_7$	0	0	3	0
$\psi_7-\theta_6+\theta_7$	0	− 2	2	0
$\psi_7-2\theta_7$	0	0	4	− 1
$\psi_7-\theta_7-\theta_8$	0	0	− 4	2
$\psi_7-\theta_7+\theta_8$	0	0	8	− 4
$\psi_7-2\theta_8$	0	0	1	0

Argument	Jupiter	Saturn	Uranus	Neptune
ψ_7	− 174	− 198	0	− 619
$\psi_7+\theta_7-\theta_8$	0	0	5	92
$\psi_7+\theta_6-\theta_7$	0	0	− 2	0
$\psi_7+\theta_5-\theta_7$	0	0	− 3	0
$2\psi_7-\psi_8$	0	0	− 4	3
$\psi_6-2\psi_7$	0	0	− 7	0
$\psi_6-\psi_7+\psi_8$	0	− 1	0	0
$\psi_6-\theta_5-\theta_6$	0	5	0	6
$\psi_6-2\theta_6$	− 9	15	0	0
$\psi_6-\theta_6-\theta_7$	0	0	1	0
$\psi_6-\theta_6+\theta_7$	0	0	− 1	0
$\psi_6-2\theta_7$	0	0	− 1	0
$\psi_6-\theta_7+\theta_8$	0	− 1	0	0
ψ_6	− 594	0	192	33
$\psi_6+\theta_6-\theta_7$	− 1	0	22	3
$\psi_6+\theta_6-\theta_8$	0	0	1	0
$\psi_6+\theta_5-\theta_6$	0	− 5	0	0
$\psi_6+\psi_7-\psi_8$	0	1	0	0
$2\psi_6-\psi_7$	− 23	78	2	0
$2\psi_6-\psi_5$	0	3	0	0
$\psi_5-2\psi_6$	298	− 962	27	0
$\psi_5-\psi_6-\psi_7$	− 113	348	− 49	0
$\psi_5-\psi_6-\psi_8$	− 1	4	− 1	0
$\psi_5-\psi_6+\psi_8$	0	0	− 1	0
$\psi_5-\psi_6+\psi_7$	0	− 6	− 36	0
$\psi_5-2\psi_7$	− 7	− 7	− 440	34
$\psi_5-\psi_7-\psi_8$	0	0	− 3	− 35
$\psi_5-\psi_7-\psi_8$	− 1	− 1	− 13	− 33
$\psi_5-2\psi_5$	0	0	− 1	1
$\psi_5-\theta_5-\theta_7$	0	0	− 5	0
$\psi_5-\theta_5+\theta_6$	− 1	3	0	0
$\psi_5-2\theta_6$	7	− 5	0	0
$\psi_5-\theta_6-\theta_7$	0	0	2	0
$\psi_5-\theta_6-\theta_7$	− 1	6	− 5	0
$\psi_5-2\theta_7$	0	1	− 8	1
$\psi_5-\theta_7-\theta_8$	0	0	4	− 2
$\psi_5-\theta_7+\theta_8$	0	0	− 3	4
$\psi_5-2\theta_5$	0	0	− 2	0
ψ_5	0	257	5257	463
$\psi_5+\theta_7-\theta_8$	1	1	15	2[illegible]
$\psi_5+\theta_6-\theta_7$	0	0	5	0
$\psi_5+\theta_5-\theta_6$	1	− 3	0	0
$\psi_5+\theta_5-\theta_7$	0	0	5	0
$\psi_5+\theta_3-\theta_4$	0	0	− 1	0
$\psi_5+\psi_7-\psi_8$	0	0	2	− 6
$\psi_5+\psi_6-\psi_7$	− 111	349	15	− 1[illegible]
$\psi_5+\psi_6-\psi_8$	− 1	4	0	0
$2\psi_5-\psi_6$	− 10	− 50	41	0
$2\psi_5-\psi_7$	− 66	− 46	511	− 43
$2\psi_5-\psi_6$	0	0	2	[illegible]
$\psi_4-2\psi_6$	− 1	3	0	[illegible]
$\psi_4-\theta_3-\theta_4$	0	− 1	0	[illegible]
ψ_4	− 2	8	0	[illegible]
$\psi_4+\theta_3-\theta_4$	0	3	0	[illegible]
$\psi_3-2\psi_6$	0	3	0	[illegible]
ψ_3	− 2	7	0	[illegible]
ψ_2	− 1	0	1	[illegible]
ψ_1	− 1	− 1	2	[illegible]

Table 13

N_{u} [illegible] $\times 10^6$

Argument	Mercury	Vénus	Earth	Mars
θ_4	0	0	0	−180
[illegible]θ_4	306	−1138	997	0
[illegible]$\theta_3 - 2\theta_4$	0	0	0	1101
θ_3	275	0	0	−1838
$2\theta_3 - \theta_4$	0	457	−378	−207
$3\theta_2 - \theta_3 + \theta_4$	175	0	0	0
$5\theta_2$	−4695	0	0	0
$\theta_1 - 2\theta_2$	172	0	0	0
$\theta_1 - \theta_3 + \theta_4$	166	0	0	0
θ_1	0	2637	1988	429
$2\theta_1 - \theta_2$	−879	0	0	0
$\nu_5 - \nu_6 - \theta_6$	−105	0	0	0
$\nu_5 - \nu_7 + \theta_3$	0	0	0	−205
$\nu_4 - \nu_6 - \theta_3$	0	0	0	−416
$\nu_4 - \nu_6 - \theta_4$	0	0	0	537
$\nu_4 - \nu_6 - \theta_6$	0	0	0	−145
$\nu_4 - \nu_6 + \theta_4$	0	0	0	608
$\nu_4 - \nu_6 + \theta_3$	0	0	0	−251
$\nu_3 - \nu_4 - \theta_1$	−233	0	0	0
$\nu_3 - \nu_4 - \theta_3$	0	280	−499	7947
$\nu_3 - \nu_4 - \theta_4$	0	0	0	3757
$\nu_3 - \nu_4 + \theta_4$	320	−2029	1741	1136
$\nu_3 - \nu_4 + \theta_3$	−206	1392	−1192	0
$\nu_3 - \nu_4 + \theta_1$	−207	0	0	0
$\nu_3 - \nu_5 + \theta_3$	−221	0	0	0
$\nu_3 - \nu_6 - \theta_3$	0	0	0	−119
$\nu_3 - \nu_6 - \theta_4$	0	0	0	198
$\nu_3 - \nu_6 + \theta_4$	0	0	0	214
$\nu_2 - \nu_4 - \theta_3$	−114	0	0	0
$\nu_2 - \nu_5 - \theta_1$	170	0	0	0
$\nu_2 - \nu_5 + \theta_1$	117	0	0	0
$\nu_1 - \nu_2 - \theta_1$	−3777	0	0	0
$\nu_1 - \nu_2 - \theta_2$	3170	176	124	0
$\nu_1 - \nu_2 - \theta_3$	117	0	0	0
$\nu_1 - \nu_2 + \theta_7$	−232	0	0	0
$\nu_1 - \nu_2 + \theta_2$	787	0	0	0
$\nu_1 - \nu_2 + \theta_1$	−7950	1062	903	206
$\nu_1 - \nu_3 - \theta_1$	117	0	0	0
$\nu_1 - \nu_3 - \theta_3$	−413	0	0	0
$\nu_1 - \nu_3 - \theta_4$	−229	0	0	0
$\nu_1 - \nu_3 + \theta_1$	126	0	0	0
$\nu_1 - \nu_4 - \theta_1$	−115	0	0	0
$\nu_1 - \nu_4 - \theta_3$	277	0	0	0
$\nu_1 - \nu_4 - \theta_4$	−1021	0	0	0
$\nu_1 - \nu_4 + \theta_1$	−134	0	0	−606
$\nu_1 - \nu_5 - \theta_1$	−3353	477	401	0
$\nu_1 - \nu_5 - \theta_2$	282	0	0	0
$\nu_1 - \nu_5 + \theta_2$	1387	0	0	0
$\nu_1 - \nu_5 + \theta_1$	−1388	0	0	0
$\nu_1 + \nu_5 - \theta_1$	396	0	0	0
$\nu_1 + \nu_2 - \theta_1$	−589	0	0	0
$\nu_1 + \nu_2 - \theta_2$	138	0	0	0
$2\nu_1 - \theta_1$	1746	0	0	0
$2\nu_1 - \theta_2$	−401	0	0	0
$2\nu_1 - \theta_3$	140	0	0	0

Argument	Jupiter	Saturn	Uranus	Neptune
θ_8	−2	−2	12	0
θ_7	0	2	0	−12
$2\theta_7 - \theta_8$	0	0	1	0
θ_6	0	0	−9	0
θ_1	−3	−4	3	0
$\nu_7 - \nu_8 - \theta_7$	0	0	2	1
$\nu_7 - \nu_8 - \theta_8$	−1	−1	33	−2
$\nu_7 - \nu_8 + \theta_8$	0	0	−1	1
$\nu_7 - \nu_8 + \theta_7$	−3	−3	5	31
$2\nu_7 - \theta_7$	0	0	3	−2
$2\nu_7 - \theta_8$	0	0	−1	1
$\nu_6 - \nu_7 - \theta_6$	0	0	−1	0
$\nu_6 - \nu_7 - \theta_7$	−1	3	3	0
$\nu_6 - \nu_7 + \theta_7$	0	0	3	0
$\nu_6 - \nu_7 + \theta_6$	−1	0	17	−1
$\nu_6 - \nu_8 + \theta_6$	0	0	2	0
$\nu_6 + \nu_7 - \theta_6$	0	−1	1	0
$\nu_6 + \nu_7 - \theta_7$	0	0	−1	0
$2\nu_6 - \theta_6$	−10	26	−1	0
$2\nu_6 - \theta_7$	0	0	2	0
$\nu_5 - \nu_6 - \theta_6$	1	−22	46	0
$\nu_5 - \nu_6 - \theta_7$	0	−1	9	0
$\nu_5 - \nu_6 + \theta_8$	0	0	1	0
$\nu_5 - \nu_6 + \theta_7$	−3	7	9	0
$\nu_5 - \nu_6 + \theta_6$	8	−11	−4	0
$\nu_5 - \nu_7 - \theta_6$	5	−13	−3	0
$\nu_5 - \nu_7 - \theta_7$	−11	−8	225	−30
$\nu_5 - \nu_7 - \theta_8$	−2	−2	12	16
$\nu_5 - \nu_7 + \theta_8$	−1	−1	−2	18
$\nu_5 - \nu_7 + \theta_7$	−13	−11	220	−15
$\nu_5 - \nu_7 + \theta_6$	5	−14	5	0
$\nu_5 - \nu_8 - \theta_7$	0	0	−1	1
$\nu_5 - \nu_8 - \theta_8$	0	0	8	−1
$\nu_5 - \nu_8 + \theta_8$	0	0	1	0
$\nu_5 - \nu_8 + \theta_7$	0	0	0	8
$\nu_5 + \nu_7 - \theta_6$	0	1	1	0
$\nu_5 + \nu_7 - \theta_7$	0	0	−13	5
$\nu_5 + \nu_7 - \theta_8$	0	0	3	−3
$\nu_5 + \nu_6 - \theta_6$	10	−23	−5	0
$\nu_5 + \nu_6 - \theta_7$	0	−1	8	0
$2\nu_5 - \theta_6$	−5	14	−2	0
$2\nu_5 - \theta_7$	0	−1	21	−3
$2\nu_5 - \theta_8$	0	0	0	3
$\nu_3 - \nu_4 - \theta_3$	0	−2	0	0
$\nu_3 - \nu_4 - \theta_4$	0	−1	0	0
$\nu_3 - \nu_4 + \theta_4$	0	−1	0	0
$\nu_1 - \nu_2 - \theta_1$	0	0	−1	0
$\nu_1 - \nu_2 + \theta_1$	1	−1	0	0

CONCLUSION

In this study of the long period variations of the planetary elements, we added to the Lagrangian solution the terms of third order excentricity and inclination arising from the long period portion of the perturbation function calculated for the planets as a whole. We also took into consideration the influence of short period terms of second order mass. We particularly concentrated on determining the integration constants that make the solutions agree with the mean elements when t = 1850.0 is used as time zero.

The terms calculated with these constants are grouped together in Tables 8 to 13. Notice in these results the very strong coupling that exists, for a long period problem, in the planetary system. The magnitude of the terms arising from the short periods shows that there is no point to extending a theory to the fifth order on the basis of the perturbation function's long periods if the short periods are not taken into account.

This work's essential task was therefore the comparison of the various effects according to their origin so as to have an overall view of this problem and to be able to embark on the complete construction of a long period theory. Our solution is in fact still incomplete. Even so, we should take into account the direct terms of fifth order that must have an influence, especially for Mercury, Venus, Earth, and Mars. We have yet to calculate the influence of short periods of higher orders of excentricity and inclination, and maybe even part of the third order with respect to masses in the case of the resonant argument $2\lambda_j - 5\lambda_s$ between Jupiter and Saturn. Such an investigation would be very important. However, since the largest contributions have already been considered, it would no longer present any great difficulties for the determination of integration constants.

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