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Canonical Forms for Nonlinear Systems

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CANONICAL FORMS FOR NONLINEAR SYSTEMS

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SUMMARY

Necessary and sufficient conditions for transforming a nonlinear system to a controllable linear system have been established, and this theory has been applied to the automatic flight control of aircraft. These transformation results show that the nonlinearities in a system are often not intrinsic, but are the result of unfortunate choices of coordinates in both state and control variables. Given a nonlinear system (that may not be transformable to a linear system), we construct a canonical form in which much of the nonlinearity is removed from the system. If a system is not transformable to a linear one, then the obstructions to the transformation are obvious in the canonical form. If the system can be transformed (it is called a linear equivalent), then the canonical form is a usual one for a controllable linear system. Thus our theory of canonical forms generalizes the earlier transformation (to linear systems) results. Our canonical form is not unique, except up to solutions of certain partial differential equations we discuss. In fact, the important aspect of this paper is the constructive procedure we introduce to reach the canonical form. As is the case in many areas of mathematics, it is often easier to work with the canonical forms than in arbitrary coordinate variables.

I. INTRODUCTION

Suppose we have a nonlinear system

$$\dot{x}(t) = f(x(t)) + u(t)g(x(t)) \tag{1}$$

where f and g are real-analytic vector fields on $\mathbb{R}^n$ and $f(0) = 0$. If $f(x) = Ax$ and $g(x) = b$, where A is $n\times n$ and b is $n\times 1$, and $b, Ab, \ldots, A^{n-1}b$ are linearly independent, then we can always find new coordinates $y_1, y_2, \ldots, y_n, v$, where the y_i are functions of x, and v is a function of (x,u), such that the system becomes a "string of integrators"

$$\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
&\vdots \\
\dot{y}_{n-1} &= y_n \\
\dot{y}_n &= v
\end{aligned} \tag{2}$$

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This is a canonical form for the controllable linear system.

We let $[f,g],(ad^2f,g),\ldots,(ad^kf,g),\ldots$ denote successive Lie brackets (definitions are given in the second section) of the vector fields f and g from equation (1). If $g,[f,g],\ldots,(ad^{n-1}f,g)$ are linearly independent and $g,[f,g],\ldots,(ad^{n-2}f,g)$ are involutive, then from reference 1 we know there exists a neighborhood of the origin in x-space and new coordinates $y_1,y_2,\ldots,y_n$ and control v so that we have equation (2). Thus the system (1) appears as a nonlinear system only because of an unfortunate choice of coordinates, and system (2) is a canonical form for system (1).

If we want to build a controller for system (1), then a controller can be designed for system (2) and applied to system (1) through the transformation (from x-space to y-space) and its inverse. This is the basic design technique for the automatic flight control of aircraft applied in references 2-5. The general multi-input version of transformations of nonlinear to linear systems is given in reference 6.

Suppose we retain the assumption that $g,[f,g],\ldots,(ad^{n-1}f,g)$ span $\mathbb{R}^n$ for points near the origin, but remove the assumption about involutivity. What canonical form can we then derive for system (1)? We certainly want this canonical form to show "a certain amount of linearity." Under what Lie bracket conditions does this canonical form exist, and can we actually present the form in such a way that it accentuates the important Lie brackets?

For example, the system

$$\begin{aligned}\dot{x}_1 &= x_2 + x_3^2 \\ \dot{x}_2 &= x_3 + x_2^3\,, \\ \dot{x}_3 &= u\end{aligned} \qquad f = \begin{bmatrix} x_2 + x_3^2 \\ x_3 + x_2^3 \\ 0 \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

can be transformed by $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3 + x_2^3$, $v = u + 3x_2^2(x_3 + x_2^3)$ to

$$\left.\begin{aligned}\dot{y}_1 &= y_2 + (y_3 - y_2^3)^2 = y_2 + y_3^2 - 2y_2^3y_3 + y_2^6 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= v\end{aligned}\right\} \tag{4}$$

Letting $\tilde{y}_1 = y_1 + (y_2^4)/2$, $\tilde{y}_2 = y_2$, $\tilde{y}_3 = y_3$, $\tilde{v} = v$, we have (deleting the $\sim$ notation)

$$\left.\begin{aligned}\dot{y}_1 &= y_2 + y_3^2 + y_2^6 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= v\end{aligned}\right\} \tag{5}$$

If we write system (5) as $\dot{y} = \bar{f}(y) + v\bar{g}(y)$, system (5) cannot be reduced to the linear system (2) since the set $\{\bar{g},[\bar{f},\bar{g}]\}$ is not involutive (see ref. 1). However, because

$$\bar{g} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\bar{f},\bar{g}] = \begin{bmatrix} 2y_3 \\ 1 \\ 0 \end{bmatrix}, \quad \Big[\bar{g},[\bar{f},\bar{g}]\Big] = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad (ad^2\bar{f},\bar{g}) = \begin{bmatrix} 1 + 6y_2^5 \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

we can write the form (5) as

$$\left.\begin{aligned} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ y_3 \\ 0 \end{bmatrix} + \begin{bmatrix} y_3^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2^6 \\ 0 \\ 0 \end{bmatrix} \\ &= \bar{g}v + [\bar{f},\bar{g}]\Big|_{y_3=0} y_3 + \frac{1}{2!}\Big[[\bar{f},\bar{g}],\bar{g}\Big]\Big|_{y_3=0} y_3^2 \\ &\quad + (ad^2\bar{f},\bar{g})\Big|_{y_2=y_3=0} y_2 \\ &\quad + \frac{1}{6!}\Big(-(ad^5[\bar{f},\bar{g}],(ad^2\bar{f},\bar{g})\Big)\Big|_{y_2=y_3=0} y_2^6 \end{aligned}\right\} \tag{7}$$

In equation (5) we point out the two dimensional linear subsystem which is a string of integrators

$$\left.\begin{aligned} \dot{y}_2 &= y_3 \\ \dot{y}_3 &= v \end{aligned}\right\} \tag{8}$$

Equation (5) (or (7)) is the canonical form of interest to use for this example. We derive such forms for n dimensional systems, and give conditions under which a form shows a particular type of linear subsystem (as in eq. (8)). We provide a constructive procedure to move a system from the original coordinates to the canonical form, and this procedure is the main contribution of this paper. Also, the canonical form is unique only up to solutions of partial differential equations we introduce.

These canonical forms are important because they show the intrinsic nonlinearities of a system and the obstructions to having a transformation to the linear system (2). Also, as in all areas of mathematics, it is often much easier to prove theorems if we assume some canonical form which exhibits the basic mathematical properties (e.g., Jordan form, rational canonical form, companion form, Brunovsky (ref. 7) form).

As noted before, one interesting problem is characterizing those nonlinear systems which can be transformed to controllable linear systems in canonical form (this research contains multi-input as well as single-input results). We refer here to the work of Krener (ref. 8), Brockett (ref. 9), Jakubcyzk and Respondek (ref. 10), Hermann (ref. 11), and the authors (refs. 1, 6, and 12-15). Hermann applies the theory of equivalence of exterior differential systems to study canonical forms under feedback for nonlinear systems.

The results of this paper can easily be generalized to multi-input systems. We consider this problem and examine a two-input control system as an example at the end of the paper.

II. DEFINITIONS AND PRELIMINARIES

For vector fields f and g on $\mathbb{R}^n$ we define the Lie bracket of f and g

$$[f,g] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$$

where $\partial f/\partial x$ and $\partial g/\partial x$ denote Jacobian matrices. This is the negative of the standard definition, but it allows for easier notation in our present study. We can also define $[f,[f,g]]$, $[g,[f,g]]$, $[f,[f,[f,g]]]$, $[g,[f,[f,g]]]$, etc. In fact, we let

$$(ad^0 f,g) = g$$
$$(ad^1 f,g) = [f,g]$$
$$(ad^2 f,g) = \Big[f,[f,g]\Big]$$
$$\vdots$$
$$(ad^k f,g) = [f,(ad^{k-1} f,g)]$$

A set of $\mathscr{C}^\infty$ (smooth) vector fields $\{X_1,X_2,\ldots,X_r\}$ on $\mathbb{R}^n$ is <u>involutive</u> if there exist $\mathscr{C}^\infty$ functions $\gamma_{ijk}(x)$ so that

$$[X_i,X_j](x) = \sum_{k=1}^{r} \gamma_{ijk}(x) X_k(x) , \quad 1 \le i,\, j \le r,\; i \ne j$$

If f is a vector field on $\mathbb{R}^n$ and $h(x)$ is a $\mathscr{C}^\infty$ function, then

$$L_f(h) = \langle dh,f\rangle = \frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \cdots + \frac{\partial h}{\partial x_n} f_n$$

where dh is the gradient of h. Similarly, for a $\mathscr{C}^\infty$ one form $\omega = \omega_1\, dx_1 + \omega_2\, dx_2 + \cdots + \omega_n\, dx_n$ we define

$$\langle \omega,f\rangle = \omega_1 f_1 + \omega_2 f_2 + \cdots + \omega_n f_n$$

and the Lie derivative of ω with respect to f

$$L_f(\omega) = \left(\frac{\partial \omega^*}{\partial x} f\right)^* + \omega \frac{\partial f}{\partial x}$$

where $*$ denotes the transpose and $\partial\omega^*/\partial x$ and $\partial f/\partial x$ are Jacobian matrices. For g a vector field on $\mathbb{R}^n$ and f, h, and ω as before we have the formulas (see ref. 12)

$$L_f\langle\omega,g\rangle = \langle L_f(\omega),g\rangle - \langle\omega,[f,g]\rangle \tag{9}$$

$$dL_f(h) = L_f(dh) \tag{10}$$

The following lemma is implied by results in reference 1, but for the sake of completeness, we present a proof. Let f and g be as in system (1) (the assumption of real analytic can be relaxed).

Lemma 1. Suppose the set of vector fields $\{g,[f,g], \ldots,(ad^{n-1}f,g)\}$ are linearly independent and $\{g,[f,g], \ldots,(ad^q f,g)\}$ is involutive for some integer q, $1 \le q \le n-2$. Then the sets of vector fields

$$\{g,[f,g], \ldots,(ad^{q-1}f,g)\}$$

$$\{g,[f,g], \ldots,(ad^{q-2}f,g)\}$$

$$\vdots$$

$$\{g,[f,g]\}$$

are also involutive.

Proof. Since $\{g,[f,g], \ldots,(ad^q f,g)\}$ is involutive, by the classical Frobenius Theorem there are functions $T_1,T_2, \ldots,T_{n-q-1}$ so that $dT_1,dT_2, \ldots,dT_{n-q-1}$ are linearly independent and $\langle dT_i,(ad^j f,g)\rangle = 0$, $i = 1,2, \ldots,n-q-1$ and $j = 0,1, \ldots,q$. There is at least one T_i, say T_{n-q-1}, so that

$$\langle dT_{n-q-1},(ad^{q+1}f,g)\rangle \neq 0$$

Define $T_{n-q} = L_f(T_{n-q-1})$

By formula (9)

$$L_f\langle dT_{n-q-1},(ad^{j-1}f,g)\rangle = \langle L_f(dT_{n-q-1}),(ad^{j-1}f,g)\rangle - \langle dT_{n-q-1},(ad^j f,g)\rangle$$

for $j = 1,2, \ldots,q$

Thus

$$\langle L_f(dT_{n-q-1}),(ad^{j-1}f,g)\rangle = 0$$

and by equation (10)

$$\langle dT_{n-q},(ad^{j-1}f,g)\rangle = 0 , \quad j = 1,2, \ldots,q$$

Hence

$$\langle dT_i,(ad^j f,g)\rangle = 0 , \quad i = 1,2, \ldots,n-q \text{ and } j = 0,1, \ldots,q-1$$

Formulas (9) and (10) are applied again to show

$$\langle dT_{n-q}, (ad^q f, g)\rangle = \langle dT_{n-q-1}, (ad^{q+1} f, g)\rangle \neq 0$$

We claim that the vectors $dT_1, dT_2, \ldots, dT_{n-q}$ are linearly independent, knowing that $dT_1, dT_2, \ldots, dT_{n-q-1}$ are independent by assumption. Take constants $c_1, c_2, \ldots, c_{n-q}$ so that

$$c_1\, dT_1 + c_2\, dT_2 + \ldots + c_{n-q}\, dT_{n-q} = 0$$

We dual product this with $(ad^q f, g)$

$$c_1\langle dT_1, (ad^q f, g)\rangle + c_2\langle dT_2, (ad^q f, g)\rangle + \ldots + c_{n-q}\langle dT_{n-q}, (ad^q f, g)\rangle = 0$$

and find that $c_{n-q} = 0$. Hence $c_1 = c_2 = \ldots = c_{n-q-1} = 0$ and the desired gradients are linearly independent.

By the Frobenius Theorem, linearly independent gradients satisfying

$$\langle dT_i, (ad^j f, g)\rangle = 0 , \quad i = 1,2, \ldots, n - q \quad \text{and} \quad j = 0,1, \ldots, q - 1$$

imply that the set $\{g, [f,g], \ldots, (ad^{q-1} f, g)\}$ is involutive. Repeating this process $q - 2$ more times completes the proof.

A $\mathscr{C}^\infty$ <u>distribution</u> Δ on $\mathbb{R}^n$ is an assignment $\Delta(x)$ of a linear subspace of $\mathbb{R}^n$ at each point x of $\mathbb{R}^n$. We assume that Δ is of positive constant dimension k and identify Δ with the set of vector fields in it. We also let Δ be involutive and regular (in ref. 16 this means that the quotient set $\mathbb{R}^n/\Delta$ is a $\mathscr{C}^\infty$ manifold). The distribution Δ is (f,g) <u>invariant</u> if there exist $\alpha(x)$ and $\beta(x)$ so that

$$[\tilde{f}, \Delta] \subseteq \Delta$$

$$[\tilde{g}, \Delta] \subseteq \Delta$$

where $\tilde{f} = f + g\alpha$ and $\tilde{g} = g\beta$. Here f and g are as in equation (1).

The following lemma is found in reference 16.

<u>Lemma 2</u>. Let Δ be an (f,g) invariant regular distribution. Then Δ induces a regular equivalence relation on $\mathbb{R}^n$ such that the dynamics in system (1) passes to the quotient denoted by $\mathbb{R}^n/\Delta$, whose dimension is $n - k$.

In our later application of this result, since our theory is local, $\mathbb{R}^n$ is replaced by an open neighborhood of the origin.

III. CANONICAL FORMS

We consider system (1) and assume the following conditions hold on an open neighborhood of the origin in $\mathbb{R}^n$. The vector fields $g, [f,g], \ldots, (ad^{n-1} f, g)$ are linearly independent and the set $\{g, [f,g], \ldots, (ad^q f, g)\}$ is involutive for some fixed integer q, $0 \le q \le n-2$. All arguments and results are local and hold in a neighborhood of the origin. Some coordinate changes used are similar to those in reference 9.

Since g is nonvanishing there is a well known coordinate change so that g becomes

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

We thus assume that equation (1) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_{n-1} \\ f_n \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \tag{11}$$

Using feedback (new u = old $u + f_n$) we can assume $f_n = 0$.

Let

$$\dot{x} = Ax + bu = \frac{\partial f(0)}{\partial x} x + gu \tag{12}$$

be the linearization of system (11) about the origin. Since at the origin $g = b, [f,g] = Ab, \ldots, (ad^{n-1}f,g) = A^{n-1}b$, we have that this linearization is controllable. It is well known that coordinate changes and feedback can be applied to take system (12) to the string of integrators (2). Hence equation (11) can be put in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + \ldots \\ x_3 + \ldots \\ \cdot \\ \cdot \\ \cdot \\ x_n + \ldots \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} = f + ug \tag{13}$$

where $+ \ldots$ denotes higher order terms.

Replace $x_n + \ldots$ by a new x_n, compute $\dot{x}_n$, and apply feedback to return the last entry in f to 0. For equation (13) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + \ldots \\ x_3 + \ldots \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} + \ldots \\ x_n \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix} = f + ug \tag{14}$$

We consider the terms

$$\begin{bmatrix} a_1(x_1,x_2,\ldots,x_{n-1})x_n \\ a_2(x_1,x_2,\ldots,x_{n-1})x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{n-2}(x_1,x_2,\ldots,x_{n-1})x_n \\ x_n \\ 0 \end{bmatrix}$$

in the f vector field. Using coordinate changes on $x_1,x_2,\ldots,x_{n-1}$ space we take

$$\begin{bmatrix} a_1(x_1,x_2,\ldots,x_{n-1})x_n \\ a_2(x_1,x_2,\ldots,x_{n-1})x_n \\ \cdot \\ \cdot \\ \cdot \\ a_{n-2}(x_1,x_2,\ldots,x_{n-1})x_n \\ x_n \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \end{bmatrix} x_n$$

Thus we have form (14) where $a_1 = a_2 = \ldots = a_{n-2} = 0$.

We have not used the fact that the set $\{g,[f,g],\ldots,(ad^q f,g)\}$ is involutive if $q > 0$ and the implications of Lemma 1. We take (14) and compute certain Lie brackets.

We find

$$g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad [f,g] = \begin{bmatrix} \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_n} \\ \vdots \\ \frac{\partial f_{n-1}}{\partial x_n} \\ 0 \end{bmatrix}, \quad \big[[f,g],g\big] = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_n^2} \\ \frac{\partial^2 f_2}{\partial x_n^2} \\ \vdots \\ \frac{\partial^2 f_{n-1}}{\partial x_n^2} \\ 0 \end{bmatrix}$$

with $\partial f_{n-1}/\partial x_n = 1$ and $\partial^2 f_{n-1}/\partial x_n^2 = 0$. By Lemma 1 we must have $\gamma_1(x)$ and $\gamma_2(x)$ such that

$$\big[[f,g],g\big] = \gamma_2(x)[f,g] + \gamma_1(x)g$$

This implies that $[[f,g],g] = 0$ vector and $\partial^2 f_1/\partial x_n^2 = \partial^2 f_2/\partial x_n^2 = \ldots = \partial^2 f_{n-1}/\partial x_n^2 = 0$ (and hence $\partial f_1/\partial x_n = 0$, $\partial f_2/\partial x_n = 0, \ldots, \partial f_{n-2}/\partial x_n = 0$ since $a_1 = a_2 = \ldots = a_{n-2} = 0$).

Replace $x_{n-1} + \ldots$ in (14) by a new x_{n-1}, compute $\dot{x}_{n-1}$, make coordinate changes for x_n, compute $\dot{x}_n$, and apply feedback to return the last entry in f to 0. For equation (14) we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-3} \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 + \ldots \\ x_3 + \ldots \\ \vdots \\ x_{n-2} + \ldots \\ x_{n-1} \\ x_n \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = f + ug \tag{15}$$

We examine the terms

$$\begin{bmatrix} b_1(x_1,x_2,\ldots,x_{n-2})x_{n-1} \\ b_2(x_1,x_2,\ldots,x_{n-2})x_{n-1} \\ \vdots \\ b_{n-3}(x_1,x_2,\ldots,x_{n-2})x_{n-1} \\ x_{n-1} \\ 0 \\ 0 \end{bmatrix}$$

in the f vector field from equation (15). Coordinate changes on $x_1,x_2,\ldots,x_{n-2}$ space move

$$\begin{bmatrix} b_1(x_1,x_2,\ldots,x_{n-2}) \\ b_2(x_1,x_2,\ldots,x_{n-2}) \\ \vdots \\ b_{n-3}(x_1,x_2,\ldots,x_{n-2}) \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Hence in equation (15) we assume $b_1 = b_2 = \ldots = b_{n-3} = 0$.

If $q > 1$, then $\{g,[f,g],(ad^2f,g)\}$ is involutive by Lemma 1. Computing again,

$$g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [f,g] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad (ad^2f,g) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{n-1}} \\ \frac{\partial f_2}{\partial x_{n-1}} \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \left[(ad^2f,g),[f,g]\right] = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_{n-1}^2} \\ \frac{\partial^2 f_2}{\partial x_{n-1}^2} \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

There are $\gamma_1(x)$, $\gamma_2(x)$, and $\gamma_3(x)$ such that

$$\left[(ad^2 f, g), [f,g]\right] = \gamma_3(x)(ad^2 f, g) + \gamma_2(x)[f,g] + \gamma_1(x) g$$

But this implies that $\partial^2 f_i / \partial x_{n-1}^2 = 0$ for $i = 1,2, \ldots, n-3$, and hence $\partial f_1/\partial x_{n-1} = 0$, $\partial f_2/\partial x_{n-1} = 0, \ldots, \partial f_{n-3}/\partial x_{n-1} = 0$ since $b_1 = b_2 = \ldots = b_{n-3} = 0$.

We then repeat the above arguments until our equation (15) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-q-2} \\ \dot{x}_{n-q-1} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-q-2} \\ f_{n-q-1} \\ \vdots \\ f_{n-1} \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with

$$\begin{aligned}
f_1 &= x_2 + \sum_{i=2}^{\infty} h_{1,1,i} x_1^i + \sum_{j=2}^{n-q} \sum_{i=1}^{\infty} h_{1,j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
f_2 &= x_3 + \sum_{i=2}^{\infty} h_{2,1,i} x_1^i + \sum_{j=2}^{n-q} \sum_{i=1}^{\infty} h_{2,j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
&\vdots \\
f_{n-q-2} &= x_{n-q-1} + \sum_{i=2}^{\infty} h_{(n-q-2),1,i} x_1^i + \sum_{j=2}^{n-q} \sum_{i=1}^{\infty} h_{(n-q-2),j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
f_{n-q-1} &= x_{n-q} \\
&\vdots \\
f_{n-1} &= x_n \\
f_n &= 0
\end{aligned} \tag{17}$$

the definitions and functional dependences of the h's being obvious.

We apply coordinate changes on $x_1, x_2, \ldots, x_{n-q-1}$ space to take

$$\begin{bmatrix} h_{1,n-q,1}(x_1,x_2,\ldots,x_{n-q-1}) \\ h_{2,n-q,1}(x_1,x_2,\ldots,x_{n-q-1}) \\ \vdots \\ h_{(n-q-2),n-q,1}(x_1,x_2,\ldots,x_{n-q-1}) \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Also in new coordinates on $x_1, x_2, \ldots, x_{n-q-2}$ space

$$\begin{bmatrix} h_{1,n-q-1,1}(x_1,x_2,\ldots,x_{n-q-2}) \\ h_{2,n-q-1,1}(x_1,x_2,\ldots,x_{n-q-2}) \\ \vdots \\ h_{(n-q-3),n-q-1,1}(x_1,x_2,\ldots,x_{n-q-2}) \\ 1 + h_{(n-q-2),n-q-1,1}(x_1,x_2,\ldots,x_{n-q-2}) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{appears as} \quad \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Similar arguments allow us to arrive at

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-q-2} \\ \dot{x}_{n-q-1} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-q-2} \\ f_{n-q-1} \\ \vdots \\ f_{n-1} \\ f_n \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{18}
$$

where

$$
\left.
\begin{aligned}
f_1 &= x_2 + \sum_{j=1}^{n-q} \sum_{i=2}^{\infty} h_{1,j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
f_2 &= x_3 + \sum_{i=2}^{\infty} h_{2,1,i} x_1^i + \sum_{i=1}^{\infty} h_{2,2,i}(x_1) x_2^i + \sum_{j=3}^{n-q} \sum_{i=2}^{\infty} h_{2,j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
&\vdots \\
f_{n-q-2} &= x_{n-q-1} + \sum_{i=2}^{\infty} h_{(n-q-2),1,i} x_1^i + \sum_{j=2}^{n-q-2} \sum_{i=1}^{\infty} h_{(n-q-2),j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
&\quad + \sum_{j=n-q-1}^{n-q} \sum_{i=2}^{\infty} h_{(n-q-2),j,i}(x_1, x_2, \ldots, x_{j-1}) x_j^i \\
f_{n-q-1} &= x_{n-q} \\
&\vdots \\
f_{n-1} &= x_n \\
f_n &= 0
\end{aligned}
\right\} \tag{19}
$$

with new h functions.

Equation (18) is our canonical form for the nonlinear system (1). We have proved the following result.

Theorem 1. Suppose f and g in system (1) satisfy, in a neighborhood of the origin,

1. $g, [f,g], \ldots, (ad^{n-1}f, g)$ are linearly independent,
2. $g, [f,g], \ldots, (ad^{q}f, g)$ are involutive.

Then there are new state space coordinates, which are also called $x_1, x_2, \ldots, x_n$, and a new control u so that system (1) becomes system (18).

If $q = n - 2$ then equation (18) is a linear system, and we have the results of reference 1.

Suppose we are given two systems, (18) and

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_{n-q-2} \\ \dot{y}_{n-q-1} \\ \vdots \\ \dot{y}_{n-1} \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_{n-q-2} \\ \bar{f}_{n-q-1} \\ \vdots \\ \bar{f}_{n-1} \\ 0 \end{bmatrix} + v \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

with $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{n-1}$ as in equation (19) except that $x_1, x_2, \ldots, x_n, u$ are replaced by $y_1, y_2, \ldots, y_n, v$ and h by $\bar{h}$. We wish to know if they are equivalent in the sense that system (20) is simply system (18) in new coordinates $y_1(x), y_2(x), \ldots, y_n(x), v(x,u)$. If this is true, there is a transformation from system (18) to system (20). Since involutivity is invariant under these transformations we assume that the integer q is the same for both systems.

If (18) is equivalent to (20), then we have a transformation as above which satisfies the following partial differential equations.

$$\left.\begin{aligned}
\sum_{i=1}^{n-1} \frac{\partial y_1}{\partial x_i} f_i &= \bar{f}_1 \\
\sum_{i=1}^{n} \frac{\partial y_2}{\partial x_i} f_i &= \bar{f}_2 \\
&\vdots \\
\sum_{i=1}^{n} \frac{\partial y_{n-1}}{\partial x_i} f_i &= \bar{f}_{n-1}
\end{aligned}\right\} \tag{21}$$

Here we need $y_1, y_2, \ldots, y_{n-1}$ as functions of $x_1, x_2, \ldots, x_{n-1}$ only and y_n a function of $x_1, x_2, \ldots, x_n$. Writing these equations out we obtain

$$\left.\begin{aligned}
&\frac{\partial y_1}{\partial x_1}(x_2 + \ldots) + \frac{\partial y_1}{\partial x_2}(x_3 + \ldots) + \ldots + \frac{\partial y_1}{\partial x_{n-1}} x_n \\
&\qquad = y_2 + \sum_{j=1}^{n-q} \sum_{i=2}^{\infty} \bar{h}_{1,j,i}(y_1, y_2, \ldots, y_{j-1}) y_j^i \\
&\frac{\partial y_2}{\partial x_1}(x_2 + \ldots) + \frac{\partial y_2}{\partial x_2}(x_3 + \ldots) + \ldots + \frac{\partial y_2}{\partial x_{n-1}} x_n \\
&\qquad = y_3 + \sum_{i=2}^{\infty} \bar{h}_{2,1,i} y_1^i + \sum_{i=1}^{\infty} \bar{h}_{2,2,i}(y_1) y_2^i + \sum_{j=3}^{n-q} \sum_{i=2}^{\infty} \bar{h}_{2,j,i}(y_1, y_2, \ldots, y_{j-1}) y_j^i \\
&\qquad \vdots \\
&\frac{\partial y_{n-1}}{\partial x_1}(x_2 + \ldots) + \frac{\partial y_{n-1}}{\partial x_2}(x_3 + \ldots) + \ldots + \frac{\partial y_{n-1}}{\partial x_{n-1}} x_n \\
&\qquad = y_n
\end{aligned}\right\} \tag{22}$$

In the left-hand side of the first (n - 2) equations, the terms $(\partial y_1/\partial x_{n-1})x_n$, $(\partial y_2/\partial x_{n-1})x_n, \ldots, (\partial y_{n-2}/\partial x_{n-1})x_n$ consist of functions of $x_1, x_2, \ldots, x_{n-1}$ times the variable x_n. In the right-hand side of these equations, the only possible terms of this type are those that contain a y_n (and this can occur only if $q = 0$). But such terms are raised to powers of two or greater. Hence $y_1, y_2, \ldots, y_{n-2}$ are functions of $x_1, x_2, \ldots, x_{n-2}$ only.

We examine the terms $(\partial y_1/\partial x_{n-2})x_{n-1}$, $(\partial y_2/\partial x_{n-2})x_{n-1}, \ldots, (\partial y_{n-3}/\partial x_{n-2})x_{n-1}$ from the left-hand side of the first (n - 3) equations. The only possible way such terms can appear on the right-hand side of these equations is through y_{n-1} and y_n

(if $q \leq 1$), but again these are raised to powers greater than one. We repeat this argument to achieve

$$
\begin{gathered}
y_1(x_1, x_2, \ldots, x_{n-q-2}) \\
y_2(x_1, x_2, \ldots, x_{n-q-2}) \\
\vdots \\
y_{n-q-2}(x_1, x_2, \ldots, x_{n-q-2}) \\
y_{n-q-1}(x_1, x_2, \ldots, x_{n-q-1}) \\
\vdots \\
y_{n-1}(x_1, x_2, \ldots, x_{n-1}) \\
y_n(x_1, x_2, \ldots, x_n)
\end{gathered}
\tag{23}
$$

The partial differential equations (22) now become

$$
\left.
\begin{aligned}
&\sum_{i=1}^{n-q-2} \frac{\partial y_1}{\partial x_i}(x_{i+1} + \ldots) \\
&\qquad = y_2 + \sum_{j=1}^{n-q} \sum_{i=2}^{\infty} \bar{h}_{1,j,i}(y_1, y_2, \ldots, y_{j-1}) y_j^i \\
&\sum_{i=1}^{n-q-2} \frac{\partial y_2}{\partial x_i}(x_{i+1} + \ldots) \\
&\qquad = y_3 + \sum_{i=2}^{\infty} \bar{h}_{2,1,i} y_1^i + \sum_{i=1}^{\infty} \bar{h}_{2,2,i}(y_1) y_2^i + \sum_{j=3}^{n-q} \sum_{i=2}^{\infty} \bar{h}_{2,j,i}(y_1, y_2, \ldots, y_{j-1}) y_j^i \\
&\qquad \vdots \\
&\sum_{i=1}^{n-q-2} \frac{\partial y_{n-q-2}}{\partial x_i}(x_{i+1} + \ldots) \\
&\qquad = y_{n-q-1} + \sum_{i=2}^{\infty} \bar{h}_{(n-q-2),1,i} y_1^i + \sum_{j=2}^{n-q-2} \sum_{i=1}^{\infty} \bar{h}_{(n-q-2),j,i}(y_1, y_2, \ldots, y_{j-1}) y_j^i \\
&\qquad\quad + \sum_{j=n-q-1}^{n-q} \sum_{i=2}^{\infty} \bar{h}(y_1, y_2, \ldots, y_{j-1}) y_j^i
\end{aligned}
\right\}
\tag{24}
$$

$$\left.\begin{aligned}
&\sum_{i=1}^{n-q-2} \frac{\partial y_{n-q-1}}{\partial x_i}(x_{i+1} + \ldots) + \frac{\partial y_{n-q-1}}{\partial x_{n-q-1}} x_{n-q} = y_{n-q} \\
&\sum_{i=1}^{n-q-2} \frac{\partial y_{n-q}}{\partial x_i}(x_{i+1} + \ldots) + \sum_{i=n-q-1}^{n-q} \frac{\partial y_{n-q}}{\partial x_i} x_{i+1} = y_{n-q+1} \\
&\qquad\vdots \\
&\sum_{i=1}^{n-q-2} \frac{\partial y_{n-1}}{\partial x_i}(x_{i+1} + \ldots) + \sum_{i=n-q-1}^{n-1} \frac{\partial y_{n-1}}{\partial x_i} x_{i+1} = y_n
\end{aligned}\right\} \quad \begin{array}{c}(24)\\(\text{Cont})\end{array}$$

Thus we have reduced the equivalence problem of two systems to that of finding a transformation of the form (23) which satisfies equations (24). We present examples in this direction.

<u>Example 1</u>. We take two systems which are already in the canonical form.

$$\left.\begin{aligned}
\dot{x}_1 &= x_2 + \frac{1}{2} x_1 x_3^2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
\dot{y}_1 &= y_2 + y_1 y_3^2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= v
\end{aligned}\right\} \tag{26}$$

Now by the results of reference 8, there is no state space coordinate changes which takes one system to the other. However, if we allow transformations also involving controls, this is possible.

By equation (23) we have $y_1(x_1)$, $y_2(x_1,x_2)$ and $y_3(x_1,x_2,x_3)$. Substituting into equation (24) we find

$$\left.\begin{aligned}
\frac{\partial y_1}{\partial x_1}\left(x_2 + \frac{1}{2} x_1 x_3^2\right) &= y_2 + y_1 y_3^2 \\
\frac{\partial y_2}{\partial x_1}\left(x_2 + \frac{1}{2} x_1 x_3^2\right) + \frac{\partial y_2}{\partial x_2} x_3 &= y_3
\end{aligned}\right\} \tag{27}$$

Since we are looking for real-analytic solutions we expand

$$\begin{aligned}
y_2(x_1,x_2) &= r_{20}(x_1) + r_{21}(x_1)x_2 + r_{22}(x_1)x_2^2 + \ldots \\
y_3(x_1,x_2,x_3) &= r_{30}(x_1,x_2) + r_{31}(x_1,x_2)x_3 + r_{32}(x_1,x_2)x_3^2 + \ldots
\end{aligned} \tag{28}$$

Thus the first equation in (27) gives

$$\frac{\partial y_1}{\partial x_1} x_2 + \frac{\partial y_1}{\partial x_1}\left(\frac{1}{2} x_1 x_3^2\right) = r_{20}(x_1) + r_{21}(x_1)x_2 + r_{22}(x_1)x_2^2 + \ldots$$
$$+ y_1(x_1)\left(r_{30}(x_1,x_2) + r_{31}(x_1,x_2)x_3 + r_{32}(x_1,x_2)x_3^2 + \ldots\right)^2$$

Computing we have

$$\left.\begin{aligned} y_2(x_1,x_2) &= r_{21}(x_1)x_2 \\ y_3(x_1,x_2,x_3) &= r_{31}(x_1)x_3 \end{aligned}\right\} \tag{29}$$

and from equation (27)

$$\left.\begin{aligned} \frac{\partial y_1}{\partial x_1} x_2 + \frac{\partial y_1}{\partial x_1}\frac{1}{2} x_1 x_3^2 &= r_{21}(x_1)x_2 + y_1(x_1)\left(r_{31}(x_1)x_3\right)^2 \\ \frac{\partial y_2}{\partial x_1}\left(x_2 + \frac{1}{2} x_1 x_3^2\right) + \frac{\partial y_2}{\partial x_2} x_3 &= r_{31}(x_1)x_3 \end{aligned}\right\} \tag{30}$$

Hence $\partial y_2/\partial x_1 = 0$, $y_2(x_1,x_2) = y_2(x_2) = r_{21}x_2$, $\partial y_2/\partial x_2 = r_{31}(x_1) = r_{31}$, and $\partial y_1/\partial x_1 = r_{21}$ where r_{21} and r_{31} are constants. Integrating we have

$$y_1 = r_{21}x_1$$
$$y_2 = r_{31}x_2$$
$$y_3 = r_{31}x_3$$

Since $y_2 = r_{31}x_2$, $r_{31} = r_{21}$ and there is a constant $r \neq 0$ such that

$$y_1 = rx_1$$
$$y_2 = rx_2$$
$$y_3 = rx_3$$

To find r we substitute into the first equation in (30).

$$rx_2 + \frac{1}{2} rx_1x_3^2 = rx_2 + rx_1r^2x_3^2$$

or

$$\frac{1}{2} r = r^3$$

with solution $r = 1/(2)^{1/2}$.

Hence the transformation

$$y_1 = \frac{1}{\sqrt{2}} x_1$$

$$y_2 = \frac{1}{\sqrt{2}} x_2$$

$$y_3 = \frac{1}{\sqrt{2}} x_3$$

$$v = \frac{1}{\sqrt{2}} u$$

takes system (25) to system (26).

Example 2. Again our systems are in canonical form

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_2 x_3^2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned}\right\} \tag{31}$$

$$\left.\begin{aligned} \dot{y}_1 &= y_2 + y_1 y_3^2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= v \end{aligned}\right\} \tag{32}$$

The partial differential equations (24) in this example are

$$\left.\begin{aligned} \frac{\partial y_1}{\partial x_1}(x_2 + x_2 x_3^2) &= y_2 + y_1 y_3^2 \\ \frac{\partial y_2}{\partial x_1}(x_2 + x_2 x_3^2) + \frac{\partial y_2}{\partial x_2} x_3 &= y_3 \end{aligned}\right\} \tag{33}$$

We take the expansions (28) for $y_2(x_1,x_2)$ and $y_3(x_1,x_2,x_3)$. The first equation in (33) is

$$\frac{\partial y_1}{\partial x_1} x_2 + \frac{\partial y_1}{\partial x_1} x_2 x_3^2 = r_{20}(x_1) + r_{21}(x_1)x_2 + r_{22}(x_1)x_2^2 + \ldots$$
$$+ y_1(x_1)\Big(r_{30}(x_1,x_2) + r_{31}(x_1,x_2)x_3 + r_{32}(x_1,x_2)x_3^2 + \ldots\Big)^2$$

This implies

$$
\begin{aligned}
y_2(x_1,x_2) &= r_{21}(x_1)x_2 \\
y_3(x_1,x_2,x_3) &= r_{31}(x_1,x_2)x_3
\end{aligned}
\tag{34}
$$

and from equation (33)

$$
\left.
\begin{aligned}
\frac{\partial y_1}{\partial x_1} x_2 + \frac{\partial y_1}{\partial x_2} x_2 x_3^2 &= r_{21}(x_1)x_2 + y_1(x_1)r_{31}^2(x_1,x_2)x_3^2 \\
\frac{\partial y_2}{\partial x_1}(x_2 + x_2x_3^2) + \frac{\partial y_2}{\partial x_2} x_3 &= r_{31}(x_1,x_2)x_3
\end{aligned}
\right\}
\tag{35}
$$

Therefore, $\partial y_1/\partial x_1 = r_{21}(x_1)$, $(\partial y_1/\partial x_1)x_2 = y_1 r_{31}^2(x_1 x_2)$, $\partial y_2/\partial x_1 = 0$, $y_2(x_2) = r_{21}x_2$, $\partial y_2/\partial x_2 = r_{31}(x_2)$, where r_{21} is now a constant. Also

$$
\left.
\begin{aligned}
r_{21}(x_1)x_2 &= y_1 r_{31}^2(x_2) \\
r_{21} &= r_{31}(x_2)
\end{aligned}
\right\}
\tag{36}
$$

Then r_{31} is a constant and $r_{31} = r_{21}$. The first equation in (36)

$$
r_{21}x_2 = r_{31}^2 y_1(x_1)
$$

is impossible to satisfy, and our systems are not equivalent.

We now discuss the linear subsystem contained in equations (18) and (19). It is clear that $\{g,[f,g], \ldots,(ad^q f,g)\}$ being involutive implies the existence of the linear subsystem

$$
\begin{aligned}
\dot{x}_{n-q-1} &= x_{n-q} \\
\dot{x}_{n-q} &= x_{n-q-1} \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= u
\end{aligned}
\tag{37}
$$

However, there may be a "larger" linear subsystem as illustrated by the following example.

Example 3. Consider on $\mathbb{R}^4$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_4^2 \\ \dot{x}_2 &= x_3 + x_2^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{aligned}\right\} \tag{38}$$

Now the set $\{g,[f,g]\}$ is not involutive, implying $q = 0$. But the coordinate changes

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= x_3 + x_2^2 \\ y_4 &= x_4 + 2x_2x_3 + 2x_2^3 \\ v &= u + 2x_2x_4 + 2(x_3 + x_2^2)x_3 + 6x_2(x_3 + x_2^2) \end{aligned}$$

yield the system

$$\left.\begin{aligned} \dot{y}_1 &= y_2 + y_4^2 - 4y_2y_3y_4 + 4y_2^2y_3^2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_4 \\ \dot{y}_4 &= v \end{aligned}\right\} \tag{39}$$

Thus we have a "larger" linear system than indicated by the involutive assumption. However, the coordinate changes just used are contained in those applied in the proof of Theorem 1. Hence the process we introduced in Theorem 1 will provide the linearity beyond that given to us by the integer q. In our system (39) above we can also use coordinate changes on y_1, y_2, y_3 space to send

$$\begin{bmatrix} -4y_2y_3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ to } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

For system (39) let Δ be the distribution consisting of the vector field

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then $[f,\Delta] \subseteq \Delta$ and $[g,\Delta] \subseteq \Delta$ and Δ is an (f,g) invariant distribution with $\alpha = 0$ and $\beta = 1$. By Lemma 2, the dynamics on the quotient manifold $\mathbb{R}^4/\Delta$ is linear.

We return to system (18) and let Δ be the distribution spanned by the vector fields

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \ldots \quad \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

where the 1 in the last vector field is in the $(n - q - 2)$ position. Since the Jacobian matrix of f in equation (18) has only zero elements on and below the diagonal,

$$[f,\Delta] \subseteq \Delta$$

Trivially,

$$[g,\Delta] \subseteq \Delta$$

and Δ is an (f,g) invariant distribution with $\alpha = 0$ and $\beta = 1$.

Hence the dynamics on the quotient manifold $\mathbb{R}^n/\Delta$, where $\mathbb{R}^n$ is actually an open neighborhood of the origin, is linear. This dynamics is actually the linear part of equations (18) and (19), and linear design techniques can be applied on $\mathbb{R}^n/\Delta$. As Example 3 shows, if more linear dynamics exists, this procedure can still be applied.

Equations (18) and (19) emphasize the essential nonlinearities of the nonlinear system (1). Since a part of the system actually appears in linear form, a design technique based on equation (18) should be simpler than on the original system.

If the integer q is equal to $n - 2$, then equation (18) is a linear system, and we have a transformation from a nonlinear to a linear system. The design of an automatic flight controller involving a multi-input generalization of this transformation is presented in references 2-5, as mentioned before.

Suppose we have a multi-input system

$$\dot{x}(t) = f\bigl(x(t)\bigr) + \sum_{i=1}^{m} u_i(t) g_i\bigl(x(t)\bigr) \tag{40}$$

where $f, g_1, \ldots, g_m$ are real analytic. It is not difficult to extend the results of this paper to develop a canonical form for equation (40). We consider the case $n = 7$, $m = 2$, and $g_1, [f,g_1], (ad^2 f,g_1), (ad^3 f,g_1), g_2, [f,g_2], (ad^2 f,g_2)$ span $\mathbb{R}^7$.

We also assume the set $\{g_1, g_2, [f,g_1], [f,g_2]\}$ is involutive, and a parallel result to Lemma 1 implies the set $\{g_1, g_2\}$ is involutive. Thus we take (possibly renaming controls)

$$g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

After applying feedback we have as our system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ 0 \\ f_5 \\ f_6 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{41}$$

We use linear feedback and linear coordinate changes on $\mathbb{R}^7$ to take the linearization of (41) about the origin to Brunovsky (ref. 7) form. Hence equation (41) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} x_2 + \cdots \\ x_3 + \cdots \\ x_4 + \cdots \\ 0 \\ x_6 + \cdots \\ x_7 + \cdots \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{42}$$

where $+ \ldots$ denotes higher order terms.

Replace $x_7 + \ldots$ by x_7, $x_4 + \ldots$ by x_4, compute $\dot{x}_7$ and $\dot{x}_4$, and apply feedback so that equation (42) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} x_2 + \ldots \\ x_3 + \ldots \\ x_4 \\ 0 \\ x_6 + \ldots \\ x_7 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = f + u_1 g_1 + u_2 g_2 \qquad (43)$$

Next we compute

$$[f,g_1] = \begin{bmatrix} \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_4} \\ 1 \\ 0 \\ \frac{\partial f_5}{\partial x_4} \\ 0 \\ 0 \end{bmatrix}, \qquad [f,g_2] = \begin{bmatrix} \frac{\partial f_1}{\partial x_7} \\ \frac{\partial f_2}{\partial x_7} \\ 0 \\ 0 \\ \frac{\partial f_5}{\partial x_7} \\ 1 \\ 0 \end{bmatrix}, \qquad \big[[f,g_2],g_2\big] = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_7^2} \\ \frac{\partial^2 f_2}{\partial x_7^2} \\ 0 \\ 0 \\ \frac{\partial^2 f_5}{\partial x_7^2} \\ 0 \\ 0 \end{bmatrix},$$

$$[f,g_1],g_2 = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_4 \partial x_7} \\ \frac{\partial^2 f_2}{\partial x_4 \partial x_7} \\ 0 \\ 0 \\ \frac{\partial^2 f_5}{\partial x_4 \partial x_7} \\ 0 \\ 0 \end{bmatrix}, \qquad \big[[f,g_1],g_1\big] = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_4^2} \\ \frac{\partial^2 f_2}{\partial x_4^4} \\ 0 \\ 0 \\ \frac{\partial^2 f_5}{\partial x_4^2} \\ 0 \\ 0 \end{bmatrix}$$

The set $\{g_1, g_2, [f,g_1], [f,g_2]\}$ being involutive implies that $\partial^2 f_i/\partial x_4^2$, $\partial^2 f_i/\partial x_7^2$, $\partial^2 f_i/\partial x_4 \partial x_7$ vanish for $i = 1,2,5$. Also, coordinate changes can be made on x_1, x_2, x_3, x_5, x_6 space to convert

$$[f,g_1] \text{ to } \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } [f,g_2] \text{ to } \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We assume (43) is in these new coordinates.

Replace $x_3 + \ldots$ by a new x_3, $x_6 + \ldots$ by a new x_6, compute $\dot{x}_3$ and $\dot{x}_6$, make coordinate changes for x_4 and x_7, compute $\dot{x}_4$ and $\dot{x}_7$, and apply feedback to return the fourth and last entries in f to zero. Thus we find

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{44}$$

with

$$\left.\begin{aligned} f_1 &= x_2 + \tilde{f}_1 \\ f_2 &= x_3 \\ f_3 &= x_4 \\ f_4 &= 0 \\ f_5 &= x_6 \\ f_6 &= x_7 \\ f_7 &= 0 \end{aligned}\right\} \tag{45}$$

$\tilde{f}_1$ being a function of x_1, x_2, x_3, x_5, x_6 only and containing no linear terms.

It is interesting to note the various equations (e.g., eq. (44)) that result if we take different involutive assumptions (in the above we took $\{g_1, g_2, [f, g_1], [f, g_2]\}$ as our involutive set). For example, if the sets $\{g_1, g_2, [f, g_1], [f, g_2]\}$ and $\{g_1, g_2, [f, g_1], [f, g_2], (ad^2 f, g_1), (ad^2 f, g_2)\}$ are both involutive, we have a linear system as shown in reference 6.

IV. CONCLUDING REMARKS

We have introduced a canonical form for the nonlinear system

$$\dot{x}(t) = f\Big(x(t)\Big) + u(t)g\Big(x(t)\Big)$$

which emphasizes the intrinsic nonlinearities of the system. This form is derived by proceeding through a series of coordinate changes in state and control variables. Applications of this canonical form theory to the problem of system equivalence and an extension to multi-input systems are also discussed. If a system is transformable to a controllable linear system, then its canonical form is the Brunovsky one.

REFERENCES

1. Su, R.: On the Linear Equivalents of Nonlinear Systems. System and Control Letters, vol. 2, no. 1, 1982, pp. 48-52.

2. Meyer, G.; and Cicolani, L.: A Formal Structure for Advanced Flight Control Systems. NASA TN D-7940, 1975.

3. Meyer, G.; and Cicolani, L.: Applications of Nonlinear System Inverses to Automatic Flight Control Design-System Concepts and Flight Evaluations. AGARDograph on Theory and Applications of Optimal Control in Aerospace Systems, P. Kant, ed., 1980, pp. 10-1, 10-29.

4. Smith, G. A.; and Meyer, G.: Application of the Concept of Dynamic Trim Control to Automatic Landing of Carrier Aircraft. NASA TP-1512.

5. Meyer, G.: The Design of Exact Nonlinear Model Followers. 1981 Joint Automatic Control Conference, vol. II, FA3A.

6. Hunt, L. R.; Su, R.; and Meyer, G.: Design for Multi-Input Nonlinear Systems. Differential Geometric Control Theory Conference, Birkhäuser, Boston, Cambridge, to appear.

7. Brunovsky, P.: A Classification of Linear Controllable Systems. Kibernetika (Praha), vol. 6, 1970, pp. 173-188.

8. Krener, A. J.: On the Equivalence of Control Systems and Linearization of Nonlinear Systems. SIAM J. Control, vol. 11, 1973, pp. 670-676.

9. Brockett, R. W.: Feedback Invariants for Nonlinear Systems. IFAC Congress, Helsinki, 1978.

10. Jakubcyzk, B.; and Respondek, W.: On Linearization of Control Systems. Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., vol. 28, 1980, pp. 517-522.

11. Hermann, R.: The Theory of Equivalence of Pfaffian Systems and Input Systems Under Feedback, preprint.

12. Hunt, L. R.; Su, R.; and Meyer, G.: Global Transformation of Nonlinear Systems. IEEE Trans. Autom. Contr., vol. 28, no. 1, 1983, to appear.

13. Hunt, L. R.; Su, R.; and Meyer, G.: Robustness in Nonlinear Control. Differential Geometric Control Theory Conference, Birkhäuser, Boston, Cambridge, to appear.

14. Su, R.; Hunt, L. R.; and Meyer, G.: Theory of Design Using Nonlinear Transformations. 1982 Automatic Control Conference, pp. 247-251.

15. Meyer, G.; Hunt, L. R.; and Su, R.: Design of a Helicopter Autopilot by Means of Linearizing Transformations. AGARD 35th GCA Symposium, Lisbon, Portugal, 1982.

16. Isidori, A.; Krener, A. J.; Gori-Giorgi, C.; and Monaco, S.: Nonlinear Decoupling Via Feedback: A Differential Geometric Approach. IEEE Trans. Autom. Contr., vol. 26, 1981, pp. 331-345.

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16. Abstract

Necessary and sufficient conditions for transforming a nonlinear system to a controllable linear system have been established, and this theory has been applied to the automatic flight control of aircraft. These transformations show that the nonlinearities in a system are often not intrinsic, but are the result of unfortunate choices of coordinates in both state and control variables. Given a nonlinear system (that may not be transformable to a linear system), we construct a canonical form in which much of the nonlinearity is removed from the system. If a system is not transformable to a linear one, then the obstructions to the transformation are obvious in canonical form. If the system can be transformed (it is called a linear equivalent), then the canonical form is a usual one for a controllable linear system. Thus our theory of canonical forms generalizes the earlier transformation (to linear systems) results. Our canoncial form is not unique, except up to solutions of certain partial differential equations we discuss. In fact, the important aspect of this paper is the constructive procedure we introduce to reach the canonical form. As is the case in many areas of mathematics, it is often easier to work with the canonical form than in arbitrary coordinate variables.

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