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H. Boduroglu and F. Erdogan

LEHIGH UNIVERSITY
Bethlehem, Pennsylvania 18015

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INTERNAL AND EDGE CRACKS IN A PLATE OF FINITE WIDTH UNDER BENDING*

by

H. Boduroglu** and F. Erdogan
Lehigh University, Bethlehem, PA 18015

ABSTRACT

In this paper the title problem is studied by using Reissner's transverse shear theory. The main purpose of the paper is to investigate the effect of stress-free boundaries on the stress intensity factors in plates under bending. Among the results found, particularly interesting are those relating to the limiting cases of the crack geometries. The numerical results are given for a single internal crack, two collinear cracks, and two edge cracks. Also studied is the effect of Poisson's ratio on the stress intensity factors.

1. Introduction

In many relatively thin-walled plate and shell structures through cracks may develop as a result of cyclic loading. To analyze this fatigue crack propagation process the stress intensity factor calculated from the elastic analysis of the structure appears to be the most widely used correlation parameter representing the severity of part-flaw geometry and the intensity of applied loads. In plates containing through cracks and subjected to membrane loading only, usually the solution obtained by ignoring local three-dimensional effects and by assuming the validity of conditions of the generalized plane stress seems to be quite adequate. Partly because of the practical importance of the problem of plates under membrane loading and partly because of the relative simplicity of the related elasticity problems, the two-dimensional crack problems have been studied very extensively. Even though in many applications the bending components of the external loads are also present, as in,

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(**) Permanent address: Dept. of Civil Engineering, Istanbul Technical University, Istanbul, Turkey.

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for example, transversely loaded plates and structures undergoing flow-induced vibrations, the solution of the plate bending problem seems to have been carried out only for an infinite plate [1]-[5]. These studies have demonstrated the importance of transverse shear effects on the stress intensity factors and have shown that the bending results are sufficiently different from the plane stress results. It is, therefore, worthwhile to investigate the influence of finite in-plane dimensions, particularly that of stress-free edges on the stress intensity factors in plates undergoing bending.

The problem considered in this paper is a relatively long rectangular plate containing collinear cracks perpendicular to its long sides. Of particular interest is the investigation of the edge cracks and crack-free boundary interaction. As in [1]-[4] the external loads are assumed to by symmetric with respect to the plane of the crack and a transverse shear theory [6], [7] is used to formulate the problem.

2. The Formulation of Bending Problem

Consider a relatively long flat plate of finite width which contains symmetrically located collinear cracks perpendicular to its sides (Figure 1). It is assumed that $x_2 = 0$ is a plane of symmetry with respect to loading and geometry and the problem in the absence of cracks has been solved under the given applied loads. Thus, through a proper superposition the crack problem may be reduced to a stress perturbation problem in which the self-equilibrating crack surface tractions are the only external loads. Also, it is assumed that the plate is acted upon by a sufficiently large tensile membrane load so that there is no crack surface interference (on the compression side) in the bending problem. Thus, the results given in this paper should be considered together with the solution given in [8] where the corresponding generalized plane stress problem was studied for the same crack geometry as Figure 1.

By using the Reissner's transverse shear theory, the basic equations for elastic plates under bending may be expressed as follows (see, for example, [9] for the general case):

$$\nabla^4 w = 0 \,, \tag{1}$$

$$\frac{1-\nu}{2}\ \kappa\ \nabla^2\Omega - \Omega = 0\ , \tag{2}$$

$$\kappa\ \nabla^2\ \psi - \psi - w = 0\ , \tag{3}$$

$$\beta_x = \frac{\partial\psi}{\partial x} + \frac{1-\nu}{2}\ \kappa\ \frac{\partial\Omega}{\partial y}\ ,\quad \beta_y = \frac{\partial\psi}{\partial y} - \frac{1-\nu}{2}\ \kappa\ \frac{\partial\Omega}{\partial x}\ , \tag{4}$$

$$M_{xx} = \frac{a^*}{h\lambda^4}\left[\frac{\partial^2\psi}{\partial x^2} + \nu\ \frac{\partial^2\psi}{\partial y^2} + \frac{\kappa}{2}\ (1-\nu)^2\ \frac{\partial^2\Omega}{\partial x\partial y}\right]\ , \tag{5}$$

$$M_{yy} = \frac{a^*}{h\lambda^4}\left[\frac{\partial^2\psi}{\partial y^2} + \nu\ \frac{\partial^2\psi}{\partial x^2} - \frac{\kappa}{2}\ (1-\nu)^2\ \frac{\partial^2\Omega}{\partial x\partial y}\right]\ , \tag{6}$$

$$M_{xy} = \frac{a^*(1-\nu)}{2h\lambda^4}\left[2\ \frac{\partial^2\psi}{\partial x\partial y} + \frac{\kappa}{2}\ (1-\nu)\left(\frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial x^2}\right)\right]\ , \tag{7}$$

$$V_x = \frac{\partial w}{\partial x} + \frac{\kappa}{2}\ (1-\nu)\ \frac{\partial\Omega}{\partial y} + \frac{\partial\psi}{\partial x}\ , \tag{8}$$

$$V_y = \frac{\partial w}{\partial y} - \frac{\kappa}{2}\ (1-\nu)\ \frac{\partial\Omega}{\partial x} + \frac{\partial\psi}{\partial y}\ . \tag{9}$$

The dimensionless quantities which appear in (1)-(9) are defined in Appendix A. The dimensions are given in Figure 1. In the usual notation M_{ij} and V_i $(i,j = 1,2)$ are the bending and the transverse shear resultants β_1 and β_2 are the components of the rotation vector, u_1, u_2, and u_3 are the components of the displacement vector, and a^* is a length parameter representing the crack size ($a^* = a$ for $c > 0$, $d \leq b$, $a^* = d$ for $c = 0$, $d < b$, Figure 1).

As in the corresponding plane stress problem [8], here it is assumed that $x_1 = 0$ is a plane symmetry. Thus, in the perturbation problem under consideration the solution of the differential equations (1)-(3) may be expressed as

$$w(x,y) = \frac{2}{\pi}\int_0^\infty (A_1 + yA_2)e^{-\alpha y}\cos\alpha x\ d\alpha$$
$$+ \frac{2}{\pi}\int_0^\infty (C_1\cosh\beta x + C_2 x\ \sinh\beta x)\cos\beta y\ d\beta\ , \tag{10}$$

$$\Omega(x,y) = \frac{2}{\pi}\int_0^\infty B_1 e^{-r_1 y} \sin\alpha x \, d\alpha + \frac{2}{\pi}\int_0^\infty B_2 \sinh r_2 x \sin\beta y \, d\beta \, , \tag{11}$$

$$\psi(x,y) = \frac{2}{\pi}\int_0^\infty [-A_1+(2\kappa\alpha - y)A_2]e^{-\alpha y} \cos\alpha x \quad d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [-(C_1+2\kappa\beta C_2)\cosh\beta x - C_2 x \sinh\beta x]\cos\beta y \, d\beta \, , \tag{12}$$

where

$$r_1 = [\alpha^2 + \frac{2}{\kappa(1-\nu)}]^{\frac{1}{2}} \quad , \quad r_2 = [\beta^2 + \frac{2}{\kappa(1-\nu)}]^{\frac{1}{2}} \quad , \tag{13}$$

and the unknowns A_1, A_2 and B_1 are functions of α, and C_1, C_2 and B_2 are functions of β.

By substituting from (10)-(12) into (4)-(9) the components of rotation, the moment resultants and transverse shear resultants are found to be

$$\beta_x(x,y) = \frac{2}{\pi}\int_0^\infty (-[-A_1+(2\kappa\alpha-y)A_2]e^{-\alpha y} - \frac{r_1}{\gamma^2} B_1 \, e^{-r_1 y})\sin\alpha x \, d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [-(C_1+2\kappa\beta C_2)\beta\sinh\beta x - C_2 x\beta\cosh\beta x - C_2 \sinh\beta x$$

$$- \frac{\beta}{\gamma^2} B_2 \sinh r_2 x]\cos\beta y \, d\beta \, , \tag{14}$$

$$\beta_y(x,y) = \frac{2}{\pi}\int_0^\infty [-(-A_1+(2\kappa\alpha-y)A_2)\alpha e^{-\alpha y} - \frac{\alpha}{\gamma^2} B_1 \, e^{-r_1 y}]\cos\alpha x \, d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [[(C_1+2\kappa\beta C_2)\cosh\beta x + C_2 x \sinh\beta x]\beta$$

$$- \frac{r_2}{\gamma^2} B_2 \cosh r_2 x]\sin\beta y \, d\beta \, , \tag{15}$$

$$M_{xx}(x,y) = \frac{a^*}{h\lambda^4}\frac{2}{\pi}\{\int_0^\infty([(1-\nu)\alpha^2A_1 - (\alpha^2(1-\nu)(2\kappa\alpha-y) - 2\nu\alpha)A_2]e^{-\alpha y}$$

$$- \frac{1-\nu}{\gamma^2} r_1B_1e^{-r_1y})\cos\alpha x d\alpha + \int_0^\infty(\frac{1-\nu}{\gamma^2}\beta\, r_2B_2\cosh r_2x$$

$$- (1-\nu)\beta^2C_1\cosh\beta x - [(1-\nu)x\beta^2\sinh\beta x$$

$$+ 2\beta(1+(1-\nu)\kappa\beta^2)\cosh\beta x]C_2)\cos\beta y\, d\beta\}, \qquad (16)$$

$$M_{yy}(x,y) = \frac{a^*}{h\lambda^4}\frac{2}{\pi}\{\int_0^\infty([-(1-\nu)\alpha^2A_1 + (\alpha^2(1-\nu)(2\kappa\alpha-y) + 2\alpha)A_2]e^{-\alpha y}$$

$$+ \frac{(1-\nu)}{\gamma^2}\alpha r_1\, B_1e^{-r_1y})\cos\alpha x\, d\alpha$$

$$+ \int_0^\infty(- \frac{(1-\nu)}{\gamma^2}\beta r_2B_2\cosh r_2x + \beta^2(1-\nu)C_1\cosh\beta x$$

$$+ [(\frac{4\beta^3}{\gamma^2} - 2\nu\beta)\cosh\beta x + (1-\nu)\beta^2x\sinh\beta x]C_2)\cos\beta y\, d\beta\}, \qquad (17)$$

$$M_{xy}(x,y) = \frac{a^*}{h\lambda^4}\frac{2}{\pi}\{\int_0^\infty([2\alpha^2A_1 + (2\alpha^2y-2\alpha-4\kappa\alpha^3)A_2]e^{-\alpha y}$$

$$- \frac{1}{\gamma^2}(r_1{}^2+\alpha^2)B_1e^{-r_1y})\sin\alpha x d\alpha + \int_0^\infty[\frac{1}{\gamma^2}(\beta^2+r_2{}^2)B_2\sinh r_2x$$

$$+ 2\beta^2C_1\sinh\beta x + (2\beta^3x\cosh\beta x + (2\beta+4\kappa\beta^3)\sinh\beta x)C_2]\sin\beta y\, d\beta\}\ , \qquad (18)$$

$$V_x(x,y) = -\frac{2}{\pi}\int_0^\infty [2\kappa\alpha^2 A_2 e^{-\alpha y} + \frac{r_1}{\gamma^2} B_1 e^{-r_1 y}]\sin\alpha x \, d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [\frac{1}{\gamma^2} B_2 \sinh r_2 x + (1-2\kappa\beta^2)C_2 \sinh\beta x]\cos\beta y \, d\beta \ , \tag{19}$$

$$V_y(x,y) = -\frac{2}{\pi}\int_0^\infty [2\kappa\alpha^2 A_2 e^{-\alpha y} + \frac{\alpha}{\gamma^2} B_1 e^{-r_1 y}]\cos\alpha x \, d\alpha$$

$$+ \frac{2}{\pi}\int_0^\infty [-\frac{r_2}{\gamma^2} B_2 \cosh r_2 x + \beta C_1 \cosh\beta x$$

$$+ (2\kappa\beta^2\cosh\beta x + \beta\sinh\beta x)C_2]\sin\beta y \, d\beta \ , \tag{20}$$

where

$$\gamma^2 = \frac{2}{\kappa(1-\nu)} = \frac{10(a^*)^2}{h^2} \ . \tag{21}$$

Because of the assumed symmetry, it is sufficient to consider the problem for $0 \leq x_1 < b$, $0 \leq x_2 < \infty$ only. Thus, referring to Figure 1, the boundary and symmetry conditions of the problem may be expressed as follows:

$$M_{xx}(b',y) = 0, \ M_{xy}(b',y) = 0, \ V_x(b',y) = 0, \ 0 \leq y < \infty \ , \tag{22}$$

$$M_{xy}(0,y) = 0, \ V_x(0,y) = 0, \ \beta_x(0,y) = 0, \ 0 \leq y < \infty \ , \tag{23}$$

$$M_{xy}(x,0) = 0, \ V_y(x,0) = 0, \ 0 \leq x < b' \ , \tag{24}$$

$$M_{yy}(x,0) = m(x) \ , \ c' < x < d' \ , \tag{25}$$

$$\beta_y(x,0) = 0, \ 0 \leq x < c' \ , \ d' < x < b' \ , \tag{26}$$

where the normalized length parameters are defined by

$$b' = b/a^* \ , \ c' = c/a^* \ , \ d' = d/a^* \ . \tag{27}$$

From the expressions (14), (18) and (19) it may be seen that the (symmetry) conditions (15) are identically satisfied. By using the five homogeneous conditions (22) and (24), five of the unknowns A_i, B_i, C_i, $(i = 1,2)$ may be eliminated. The sixth unknown may then be determined from the mixed boundary conditions (25) and (26). By substituting from (18) and (20) into the homogeneous conditions (24) we find

$$A_1 = \frac{1+\nu}{4\alpha^2} B_1 \quad , \quad A_2 = -\frac{1-\nu}{4\alpha} B_1 \quad . \tag{28}$$

If we now define

$$\frac{\partial}{\partial x} \beta_y(x,+0) = f(x) \ , \ 0 \leq x < b' \quad , \tag{29}$$

from (15), (26), and (28) it can be shown that (Figure 1)

$$B_1(\alpha) = -2 \int_{c'}^{d'} f(t)\sin\alpha t \, dt \ , \tag{30}$$

and

$$A_1(\alpha) = -\frac{1+\nu}{2\alpha^2} \int_{c'}^{d'} f(t)\sin\alpha t dt \ , \ A_2(\alpha) = \frac{1-\nu}{2\alpha} \int_{c'}^{d'} f(t)\sin\alpha t \, dt. \tag{31}$$

By using the expressions (16), (18) and (19) the boundary conditions (22) may be reduced to

$$B_2 \frac{\beta}{\gamma^2} \sinh r_2 b' - C_2 \, 2\beta^2\kappa \sinh\beta b' = \frac{2}{\pi} \int_0^\infty$$

$$\frac{\beta^2}{(r_1{}^2+\beta^2)(\alpha^2+\beta^2)} B_1 \sin b'\alpha \, d\alpha \ , \tag{32}$$

$$B_2 \frac{\beta^2+r_2^2}{\gamma^2} \sinh r_2 b' - 2C_1\beta^2 \sinh\beta b' - 2C_2\beta[\beta b' \cosh\beta b'$$

$$+ (1+2\beta^2\kappa)\sinh\beta b'] = \frac{2}{\pi}\int_0^\infty [\frac{2\alpha^2\beta}{\kappa\gamma^2(\alpha^2+\beta^2)^2}$$

$$- \frac{\beta\gamma^2+2\alpha^2\beta}{(\alpha^2+\beta^2)(r_1^2+\beta^2)}] B_1 \sin b'\alpha \, d\alpha , \tag{33}$$

$$B_2 \frac{1-\nu}{\gamma^2} \beta r_2 \cosh r_2 b' - C_1(1-\nu)\beta^2\cosh\beta b' - C_2[(1-\nu)\beta^2 b' \sinh\beta b'$$

$$+ 2\beta(1+(1-\nu)\beta^2\kappa)\cosh\beta b'] = -\frac{(1-\nu)^2}{2\pi}\int_0^\infty [-\frac{2\alpha\beta^2\gamma^2\kappa}{(\alpha^2+\beta^2)(r_1^2+\beta^2)}$$

$$- \frac{\alpha(\alpha^2-\beta^2)}{(\alpha^2+\beta^2)^2}] B_1 \cos b'\alpha \, d\alpha . \tag{34}$$

Equations (30)-(34) indicate that all of the unknowns in the problem can be expressed in terms of the new unknown function f(t). It is also seen that all of the boundary conditions (22)-(26) except (25) are satisfied. The equation to determine f(t) may, therefore, be obtained by substituting from (30)-(34) and (17) into (25). From the formulation of the problem one may observe that the unknown functions A_1, A_2 and B_1 refer to the "infinite" plate and should give the kernel found in [4]. Indeed, after some simple manipulations it may be shown that

$$\frac{a^*(1-\nu^2)}{2\pi h\lambda^4}\int_{c'}^{d'} f(t)\{\frac{3+\nu}{1+\nu}(\frac{1}{1-x}+\frac{1}{1+x}) - \frac{4\kappa(1-\nu)}{1+\nu}[\frac{1}{(t-x)^3}+\frac{1}{(t+x)^3}]$$

$$+ \frac{4}{1+\nu}[\frac{1}{t-x}K_2(\gamma|t-x|) + \frac{1}{t+x}K_2(\gamma|t+x|)]\}dt$$

$$+ \lim_{y\to+0}\int_0^\infty [\frac{4\beta^2}{1+\nu}C_1\cosh\beta x + \frac{8}{1+\nu}(\kappa\beta^3\cosh\beta x + \frac{\beta^2}{2}x \sinh\beta x$$

$$- \frac{\nu}{1-\nu}\beta\cosh\beta x)C_2 - \frac{2\kappa(1-\nu)}{1+\nu} r_2\beta B_2\cosh r_2 x]\cos\beta y \, d\beta$$

$$= m(x) , \quad c' < x < d' , \tag{35}$$

where K_2 is the modified Bessel function of the second kind. By solving (32)-(34) for C_1, C_2 and B_2 and by substituting into (35) we find

$$\frac{a^*(1-\nu^2)}{2\pi h\lambda^4}\int_{c'}^{d'}\{(\frac{3+\nu}{1+\nu}(\frac{1}{1-x}+\frac{1}{1+x})-\frac{4\kappa(1-\nu)}{1+\nu}(\frac{1}{(t-x)^3}+\frac{1}{(t+x)^3})$$

$$+\frac{4}{1+\nu}[\frac{1}{t-x}K_2(\gamma|t-x|)+\frac{1}{t+x}K_2(\gamma|t+x|)])+k(x,t)$$

$$-k(x,-t)\}f(t)dt=m(x),\quad c'<x<d', \tag{36}$$

where the Fredholm kernel $k(x,t)$ is given by

$$k(x,t)=\int_0^\infty\{[-\frac{3+\nu}{1+\nu}-\frac{1-\nu}{1+\nu}\beta(b'-t)]\frac{1+e^{-2\beta x}}{1-e^{-2\beta b'}}e^{-(2b'-t-x)\beta}$$

$$-\frac{2\kappa(1-\nu)}{1+\nu}\frac{1+e^{-2r_2x}}{1-e^{-2r_2b'}}(\beta^2e^{-(b'-t)r_2}-\beta r_2e^{-(b'-t)\beta})e^{-(b'-x)r_2}$$

$$+[(\frac{2\beta}{1-\nu}-\frac{2b'\beta^2}{1+\nu}\frac{1+e^{-2b'\beta}}{1-e^{-2b'\beta}})(1+e^{-2\beta x})+\frac{4}{1+\nu}\{\kappa\beta^3(1+e^{-2\beta x})$$

$$+\frac{\beta^2}{2}x(1-e^{-2\beta x})-\frac{\nu}{1-\nu}\beta(1+e^{-2\beta x})\}]\frac{1}{D}[D_1e^{-(2b'-t-x)\beta}$$

$$+D_2e^{-(b'-x)\beta}e^{-(b'-t)r_2}]-\frac{4\kappa}{1+\nu}\beta^2r_2(1+e^{-2r_2x})\frac{1}{D}[D_1e^{-(b'-t)\beta}$$

$$+D_2e^{-(b'-t)r_2}]\frac{1-e^{-2b'\beta}}{1-e^{-2b'r_2}}e^{-(b'-x)r_2}\}d\beta, \tag{37}$$

$$D_1=\frac{2\beta}{\gamma^2}r_2(1-e^{-2b'\beta})\frac{1+e^{-2b'r_2}}{1-e^{-2b'r_2}}-2(1+e^{-2b'\beta})$$

$$+\frac{1+e^{-2b'\beta}}{\kappa\gamma^2}[1-(b'-t)\beta]-(1-\nu)[\frac{\beta}{2}(b'-t)-\kappa\beta^2](1-e^{-2b'\beta}), \tag{38}$$

$$D_2 = -\frac{2\beta^2}{\gamma^2}\frac{1+e^{-2b'r_2}}{1-e^{-2b'r_2}}(1-e^{-2b'\beta}) - \kappa\beta^2(1-\nu)(1-e^{-2b'\beta}) , \tag{39}$$

$$D = 4b'\beta^2 e^{-2b'\beta} - (\frac{3+\nu}{1-\nu}\beta + 2\kappa\beta^3)(1-e^{-4b'\beta})$$

$$+ 2\beta^2\kappa r_2 \frac{1+e^{-2b'r_2}}{1-e^{-2b'r_2}}(1-e^{-2b'\beta})^2 . \tag{40}$$

If the cracks are internal cracks as shown in Figure 1, then from (26) and (29) it follows that

$$\int_{c'}^{d'} f(t)dt = 0 . \tag{41}$$

Thus, the integral equation (36) must be solved under the additional condition (41).

From the following asymptotic behavior of $K_2(z)$ for small values of z

$$K_2(z) = \frac{2}{z^2} - \frac{1}{2} + O(z^2 \log z) , \tag{42}$$

it may be shown that the kernel of the integral equation (36) has only a Cauchy type singularity. Hence, the solution is of the form

$$f(t) = \frac{F(t)}{(t-c')^{\frac{1}{2}}(d'-t)^{\frac{1}{2}}} , \quad c' < t < d' , \tag{43}$$

and the bounded function F(t) may be obtained by using the numerical method described, for example, in [10].

If the plate contains a single symmetrically located crack, i.e., for c=0, d<b (see Figures 1 and 3), by using the symmetry of the problem and by observing that f(t) = -f(-t), (36),(41), and (43) may be epxressed as

$$\frac{d(1-\nu^2)}{2\pi h\lambda^4}\int_{-d'}^{d'}\{[\frac{3+\nu}{1+\nu}\frac{1}{t-x} - \frac{4\kappa(1-\nu)}{1+\nu}\frac{1}{(t-x)^3} + \frac{4}{1+\nu}\frac{1}{t-x}K_2(\gamma|t-x|)]$$

$$+ k(x,t)\}f(t)dt = m(x), \quad -d' < x < d' , \tag{44}$$

$$\int_{-d'}^{d'} f(t)dt = 0 \quad , \tag{45}$$

$$f(t) = \frac{F(t)}{\sqrt{d^2-t^2}} \quad , \quad -d < t < d \quad , \tag{46}$$

where $a^* = d$ is used for the normalizing length parameter (Appendix A).

3. The Stress Intensity Factors

In the symmetric plate problem under consideration the bending component of the Mode I stress intensity factor at the crack tips is defined by (Figure 1)

$$k_{1c}(x_3) = \lim_{x_1 \to c} [2(c-x_1)]^{\frac{1}{2}} \sigma_{22}(x_1,0,x_3) \quad , \tag{47}$$

$$k_{1d}(x_3) = \lim_{x_1 \to d} [2(x_1-d)]^{\frac{1}{2}} \sigma_{22}(x_1,0,x_3) \quad . \tag{48}$$

Let σ_b be a stress amplitude calculated on the plate surface and used for normalizing the stress intensity factors. For example, in a plate subjected to uniform bending $M_{22} = M_o$ away from the crack region

$$\sigma_b = 6M_o/h^2 \quad . \tag{49}$$

The stress intensity factors are then normalized with respect to $\sigma_b\sqrt{a^*}$. If the stress intensity factors on the plate surface are defined by

$$k(c) = k_{1c}(h/2) \ , \ k(d) = k_{1d}(h/2) \ , \tag{50}$$

it is sufficient to calculate k(c) and k(d) in terms of which we have

$$k_{1c}(x_3) = \frac{x_3}{h/2} k(c) \ , \ k_{1d}(x_3) = \frac{x_3}{h/2} k(d) \quad . \tag{51}$$

We now note that (36) gives the normalized bending resultant m(x) on $y = 0$ outside as well as inside the crack. Thus, a relatively straightforward asymptotic analysis would show that [11]

$$k_{1c}(x_3) = \frac{x_3}{h/2}\frac{hE}{4}\lim_{x_1\to c}\sqrt{2(x_1-c)}\;\frac{\partial\beta_2}{\partial x_1}, \tag{52}$$

$$k_{1d}(x_3) = -\frac{x_3}{h/2}\frac{hE}{4}\lim_{x_1\to d}\sqrt{2(d-x_1)}\;\frac{\partial\beta_2}{\partial x_1}\;. \tag{53}$$

From (43) and (51)-(53) it then follows that

$$k(c) = \frac{hE}{4a^*\sigma_b}F(c')\;,\; k(d) = -\frac{hE}{4a^*\sigma_b}F(d')\;. \tag{54}$$

In plane elasticity problems for cracks it is known that in the close neighborhood of a crack tip $x_1 = d$, $x_2 = 0$ we have

$$\sigma_{22}(x_1,0) = \frac{k}{\sqrt{2(x_1-d)}} + O(1)\;, \tag{55}$$

$$k_1 = -\frac{2\mu}{1+\kappa_0}\lim_{x_1\to d}\sqrt{2(d-x_1)}\;\frac{\partial}{\partial x_1}[u_2(x_1,+0) - u_2(x_1,-0)]\;, \tag{56}$$

where k_1 is the Mode I stress intensity factor, μ is the shear modulus, $\kappa_0 = 4-3\nu$ for plane strain and $\kappa_0 = (3-\nu)/(1+\nu)$ for the generalized plane stress. In the symmetric bending problem under consideration the crack surface displacement is given by

$$u_2(x_1,+0,x_3) = x_3\beta_2(x_1,+0)\;. \tag{57}$$

From (48), (53) and (55)-(57) it may be observed that the results found from the solution of the plane elasticity and the bending problems given by (53) and (55) are identical provided κ_0 is selected as $(3-\nu)/(1+\nu)$, (i.e., if the value for plane stress rather than for plane strain is used for κ_0). Also,

as shown in [9] and [11] the transverse shear theory used in the present analysis gives an angular distribution for the asymptotic stress state around the crack tip which is identical to that found for the plane elasticity problem.

4. The Edge Cracks

An important special case of the problem described in Figure 1 is the edge cracks for which d=b and c>0. In this case as x and t approach the end point d'=b' simultaneously the kernel k(x,t) given by (37) becomes unbounded and consequently influences the singular nature of the solution. Since the integrand in (37) is bounded in any finite interval in $0 \leq \beta < \infty$, the unbounded terms in k(x,t) will be due to the asymptotic behavior of the integrand. Thus, by separating the asymptotic part of the integrand, (37) may be expressed as

$$k(x,t) = \int_0^\infty [K(x,t,\beta) - K_\infty(x,t,\beta)]d\beta + \int_0^\infty K_\infty(x,t,\beta)d\beta \ . \tag{58}$$

The first integral in (58) is bounded and the second may be evaluated in closed form. After a somewhat lengthy analysis similar to that described in [8] we obtain

$$\int_0^\infty K_\infty(x,t,\beta)d\beta = k_s(x,t) = \frac{1}{2b'-x-t} - \frac{6(b'-x)}{(2b'-x-t)^2} + \frac{4(b'-x)^2}{(2b'-x-t)^2} \ . \tag{59}$$

One may note that $k_s(x,t)$ given by (59) is identical to that found for the plane edge crack problem given in [8] and, together with the Cauchy kernel 1/(t-x), constitutes a generalized Cauchy kernel.

Referring to the definition of f(t) given by (29) and the boundary condition (26), it is seen that the condition (41) is not valid for the edge crack and, as pointed out in [8], is not needed for the solution of the integral equation (36). In this case the generalized Cauchy kernel $k_g(x,t) = 1/(t-x) + k_s(x,t)$ has the property that $k_g(x,b') = 0$, $k_g(b',t) = 0$, and f(t) is nonsingular at t=b'. Thus, the numerical solution of the problem is obtained by letting

$$f(t) = \frac{F(t)}{\sqrt{t-c'}} \quad , \quad c' < t < b' \quad , \tag{60}$$

5. Results and Discussion

The problem is solved for three crack geometries shown in Figures 1, 3 and 4 and for a uniform bending moment $M_{22} = M_0$ (per unit plate width) away from the crack region. The results for a symmetrically located internal crack of length 2d are given in Tables 1-3 (see the insert in Figure 3). Since the problem has three length parameters, namely h, b and d, the results depend on two dimensionless length constants. Table 1 shows the normalized stress intensity factor as a function of b/d for fixed values of b/h. One may note that, as expected for $(b/d) \to 1$ the stress intensity factor becomes unbounded. Also, for fixed plate dimensions b and h in the other limiting case of $d \to 0$ the stress intensity ratio is seen to approach unity, which is the result given by the plane elasticity solution for an infinite medium with a through crack. These trends are clearly observed in Figure 2 where the asymptotes are indicated by b/h = constant lines. The behavior of the solution for the limiting case of $(d/h) \to 0$ may also be shown analytically. Referring to (44), in this case the problem is one of an infinite plate for which the Fredholm kernel k(x,t) is zero. Noting that for $d \to 0$ $\gamma \to 0$, by using (42) it may easily be shown that in limit the kernel of the integral equation (44) would reduce to $1/(t-x)$. Also, by observing that the crack opening displacement on the plate surface is $u_2 = \beta_y h/2$, $m(x) = M_{22}/(h^2E) = M_0/(h^2E) = \sigma_b/(6E)$, and $\lambda^4 = 12(1-\nu^2)d^2/h^2$, if we replace f(x) by

$$f(x) = \partial\beta_y/\partial x = (2d/h)\partial v/\partial x \ , \ v = u_2(x_1,+0,h/2) \ , \tag{61}$$

equation (44) becomes

$$\frac{E}{2}\frac{1}{\pi}\int_{-d'}^{d'} \frac{1}{t-x}\frac{\partial v}{\partial t}\,dt = -\sigma_b \quad , \quad -d' < x < d' \quad , \tag{62}$$

which is the integral equation for an infinite plane under uniform stress σ_b and for which the stress intensity factor is $k(d) = \sigma_b\sqrt{d}$.

Table 1. Stress intensity factor in a plate of finite width containing a symmetrically located single crack and subjected to uniform bending (M_0) away from the crack region, $\nu = 0.3$, $\sigma_b = 6M_0/h^2$, (Fig. 1, c=0).

b = 2h		b = 4h		b = 6h		b = 8h		b = 10h	
b/d	$k(d)/\sigma_b\sqrt{d}$	b/d	$k(d)/\sigma_b\sqrt{d}$	b/d	$k(d)/\sigma_b\sqrt{d}$	b/d	$k(d)/\sigma_b\sqrt{d}$	b/d	$k(d)/\sigma_b\sqrt{d}$
$\to\infty$	$\to 1.0$	$\to\infty$	$\to 1.0$	$\to\infty$	$\to 1.0$	$\to\infty$	$\to 1.0$	$\to\infty$	$\to 1.0$
40	0.9887	40	0.9680	60	0.9676	80	0.9676	100	0.9675
20	0.9697	20	0.9231	30	0.9218	40	0.9213	50	0.9210
10	0.9296	10	0.8530	15	0.8491	20	0.8476	25	0.8469
5	0.8747	6.666	0.8125	10	0.8045	13.333	0.8019	12.5	0.7716
2.5	0.8694	5	0.7910	7.5	0.7780	10	0.7737	10	0.7526
2	0.9094	4	0.7812	6.0	0.7620	8	0.7556	7.5	0.7332
1.5	1.0664	2	0.8320	3.0	0.7502	4	0.7267	5	0.7166
1.04	3.7426	1.04	3.4252	1.5	0.9282	2	0.7817	2.5	0.7347
$\to 1.0$	$\to\infty$	1.02	5.1526	1.04	3.2040	1.04	3.0449	2	0.7702
		1.01	7.4209	1.01	7.1933	1.01	6.9928	1.04	2.9256
		$\to 1.0$	$\to\infty$	$\to 1.0$	$\to\infty$	$\to 1.0$	$\to\infty$	1.01	6.8141
								$\to 1.0$	$\to\infty$

In bending problem the kernel of the integral equation is a function of the Poisson's ratio ν. Therefore, unlike the plane elasticity problems the stress intensity factors in bending are dependent on ν. Most of the results given in this paper have been calculated for $\nu = 0.3$. However, to show the influence of ν on the stress intensity factors, for a central crack and for two edge cracks the results are also given for $\nu = 0$, 0.2, and 0.5. Table 2 shows the results for the internal crack. It is seen that the stress intensity factor slightly increases with increasing Poisson's ratio.

Table 2. The effect of Poisson's ratio on the stress intensity factor in a plate of finite width containing a single crack and subjected to uniform bending, ($\sigma_b = 6M_0/h^2$, b/h = 10, Fig. 1, c = 0).

b/d	$k(d)/\sigma_b\sqrt{d}$			
	$\nu = 0$	$\nu = 0.2$	$\nu = 0.3$	$\nu = 0.5$
$\to \infty$	$\to 1.0$	$\to 1.0$	$\to 1.0$	$\to 1.0$
100	0.9583	0.9650	0.9675	0.9717
50	0.9002	0.9151	0.9210	0.9307
25	0.8119	0.8368	0.8469	0.8638
12.5	0.7278	0.7587	0.7716	0.7938
10	0.7074	0.7392	0.7526	0.7758
7.5	0.6868	0.7194	0.7332	0.7573
5	0.6689	0.7023	0.7166	0.7416
2.5	0.6850	0.7197	0.7347	0.7612
2	0.7192	0.7547	0.7707	0.7975
1.04	2.8152	2.8916	2.9258	2.9881
1.01	6.6721	6.7715	6.8141	6.8886
$\to 1.0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$

Table 3. Stress intensity factor vs. width-to-crack length ratio in a plate containing a single crack and subjected to uniform bending, (ν=0.3, d/h = 1, $\sigma_b = 6M_0/h^2$, Fig. 1, c=0).

b/d	$\to 1$	2	4	8	10	20	$\to \infty$
$k(d)/\sigma_b\sqrt{d}$	$\to \infty$	0.9094	0.7812	0.7556	0.7526	0.7488	0.7475

Table 3 shows the effect of b/d ratio on k(d) for d/h = 1. Again, as $b \to d$ k(d) becomes unbounded, and as $b \to \infty$ k(d) is seen to approach the infinite plate result given in [4].

To give some idea about the distribution of the stresses in the plate, the bending moment $M_{22}(x_1,0) = M(x)$ is given in Figure 3. The result is obtained from (44) which shows that the moment is $-M_0$ for $0 \leq x_1 < d$, and has

a singularity at $x_1 = d + 0$. The figure indicates that M_{22} is a monotonically decreasing function of x_1.

Some results for collinear cracks shown in Figure 1 are given in Table 4. One set of results shows the stress intensity factors for fixed crack and plate dimensions and for varying crack location. The other set of results shows the effect of crack length for a fixed crack location (as determined by its midpoint). As $c \to 0$ or $2a/(c+d) \to 1$ it is seen that $k(c)$ becomes unbounded which is expected. The somewhat unusual result in this case is the steep rise of $k(d)$ to the single central crack value as $c \to 0$. As pointed out in [4], a smooth continuation of $k(d)$ as $c \to 0$ would correspond to the "pinched" crack solution - the steep rise in $k(d)$ being the result of the relaxation of the crack surface rotation β_2 at $x_1 = 0$ from zero to the single crack value. Even though it does not seem to be possible to analyze this phenomenon in the bending problem, it can be done for the collinear cracks in plane elasticity. For example, from the expression given for $k(d)$ [12]

$$\frac{k(d)}{\sigma_0\sqrt{d}} = \frac{1}{k}\left[1 - \frac{E(k)}{K(k)}\right] , \quad k = \sqrt{1-c^2/d^2} \tag{63}$$

in an infinite plane containing two collinear cracks along $x_2 = 0$, $c < |x_1| < d$ and subjected to uniform tension σ_0 away from the crack region, if we consider $k(d)$ a function of c, it may be shown that for fixed d

$$\frac{k(d)}{\sigma_0\sqrt{d}} \to 1 \quad \text{and} \quad \frac{d}{dc}k(d) \to -\infty \quad \text{for} \quad c \to 0 \quad , \tag{64}$$

meaning that for $c \to 0$ the approach of $k(d)$ to the single crack value is very steep. In (63) K and E are the complete elliptic integrals of first and second kind, respectively.

The results for the edge cracks are shown in Tables 5 and 6 and in Figure 4. Table 5 and Figure 4 also show the results obtained from the plane elasticity problem for the identical crack geometry. The effect of the Poisson's ratio on the stress intensity factor in the plate under bending is shown in Table 6.

Table 4. Stress intensity factors in a plate containing two symmetrically located collinear cracks and subjected to uniform bending, (Fig. 1, ν=0.3, a=(d-c)/2, b/h=10, $\sigma_b=6M_o/h^2$).

b/h = 10			c+d = b		
$\frac{2a}{c+d}$	$k(c)/\sigma_b\sqrt{a}$	$k(d)/\sigma_b\sqrt{a}$	b/a	$k(c)/\sigma_b\sqrt{a}$	$k(d)/\sigma_b\sqrt{a}$
→1/9	→1.0170		→∞	→ 1.0	→ 1.0
0.14	0.7758	0.7835	10	0.76846	0.76847
0.15	0.7720	0.7765	8	0.76058	0.76060
0.2	0.7685	0.7685	7.5	0.75969	0.75971
0.3	0.7764	0.7720	6	0.76299	0.76292
0.4	0.7923	0.7803	5	0.77550	0.77552
0.5	0.8167	0.7916	4	0.81218	0.81212
0.6	0.8519	0.8058	3	0.9442	0.9432
0.7	0.9014	0.8232	2.5	1.2083	1.2041
0.8	0.9751	0.8447	2.08	3.1343	3.2639
0.9	1.1172	0.8734	→ 2	→ ∞	→ ∞
0.95	1.3018	0.8944			
0.96	1.3743	0.9000			
0.97	1.4791	0.9066			
0.98	1.6516	0.9148			
0.99	2.0268	0.9265			
→1.0	→ ∞	→1.0134			

Table 5. Stress intensity factors in a plate of finite width containing symmetric edge cracks and subjected to uniform bending or membrane loading away from the crack region. $\sigma_b=6M_o/h^2$, $\sigma_m=N_o/h$, ν=0.3 (see insert in Fig. 3).

c/b	Bending: $k(c)/\sigma_b\sqrt{a}$			Tension
	b=10h	b=6h	b=2h	$k(c)/\sigma_m\sqrt{a}$
→ 0	→ ∞	→ ∞	→ ∞	→ ∞
0.01	8.7889			
0.05	3.4726			
0.1	2.2893	2.4567	2.7957	2.9467
0.2	1.5754	1.6656	1.9359	2.1769
0.3	1.3082	1.3690	1.5802	1.8744
0.4	1.1693	1.2171	1.3922	1.7136
0.5	1.0878	1.1130	1.2849	1.6328
0.6	1.0396	1.0800	1.2266	1.6080
0.7	1.0157	1.0583	1.2069	1.5970
0.8	1.0170	1.0665	1.2320	1.5915
0.9	1.0694	1.1369	1.3501	1.5883
0.95	1.1666	1.2645	1.4798	
0.98	1.3466	1.4383	1.5452	
0.99	1.4589			
→1.0	→1.5869	→1.5869	→1.5869	→1.5869

Table 6. Effect of Poisson's ratio on the stress intensity factors in a plate containing symmetric edge cracks which is subjected to uniform bending away from the crack region. $b/h = 10$, $\sigma_b = 6M_o/h^2$ (Fig. 1, d=b, 2a = b-c).

c/b	$k(c)/\sigma_b\sqrt{a}$			
	$\nu = 0$	$\nu = 0.2$	$\nu = 0.3$	$\nu = 0.5$
$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$
0.05	3.3451	3.4352	3.4726	3.5360
0.1	2.1681	2.2533	2.2893	2.3514
0.2	1.4744	1.5452	1.5754	1.6281
0.3	1.2197	1.2816	1.3082	1.3548
0.4	1.0889	1.1450	1.1693	1.2118
0.5	1.0132	1.0653	1.0878	1.1274
0.6	0.9695	1.0184	1.0396	1.0769
0.7	0.9495	0.9956	1.0157	1.0510
0.8	0.9548	0.9981	1.0170	1.0502
0.9	1.0127	1.0522	1.0694	1.1000
0.95	1.1159	1.1151	1.1666	1.1943
0.98	1.3003	1.3293	1.3466	1.3631
$\to 1$	$\to 1.5869$	$\to 1.5869$	$\to 1.5869$	$\to 1.5869$

The results given in the tables are self-explanatory. It should, however, be emphasized that (a) the stress intensity factor for the plate under bending increases with decreasing b/h ratio, (b) bending values are always smaller than those of the plane elasticity, (c) as $c \to b$ (or as the crack length 2a approaches zero) bending as well as the plane elasticity results approach to that of a semi-infinite plane containing an edge crack (i.e., $k(c) \to 1.5869\sigma\sqrt{a}$, $\sigma = \sigma_b$ or $\sigma = \sigma_m$) (see [8]), and (d) if the results are normalized with respect to $\sigma'\sqrt{c}$ (rather than $\sigma\sqrt{a}$), it is seen that as $c \to 0$ in both cases $k(c)/\sigma'\sqrt{c}$ approach $2/\pi$ which is the value obtained from the closed form elasticity solution of an infinite plane containing two semi-infinite edge cracks and subjected to tension equivalent to an average net section stress σ_m', where $\sigma_m' = \sigma_m\, b/c$, $\sigma_b' = \sigma_b\, b/c$ (see Figure 4).

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APPENDIX A

The normalized quantities for the plate bending problem,

$$x = x_1/a^* \ , \ y = x_2/a^* \ , \ z = x_3/a^* \ ;$$

$$u = u_1/a^* \ , \ v = u_2/a^* \ , \ w = u_3/a^* \ ;$$

$$\beta_x = \beta_1 \ , \ \beta_y = \beta_2 \ ;$$

$$M_{xx} = \frac{M_{11}}{h^2E} \ , \ M_{yy} = \frac{M_{22}}{h^2E} \ , \ M_{xy} = \frac{M_{12}}{h^2E} \ ;$$

$$\sigma_{xx} = \frac{\sigma_{11}}{E} \ , \ \sigma_{yy} = \frac{\sigma_{22}}{E} \ , \ \sigma_{xy} = \frac{\sigma_{12}}{E} \ ;$$

$$V_x = \frac{V_1}{hB} \ , \ V_y = \frac{V_2}{hB} \ ;$$

$$B = \frac{5E}{12(1+\nu)} \ , \ \kappa = \frac{E}{B\lambda^4} \ , \ \lambda^4 = \frac{12(1-\nu^2)a^{*2}}{h^2} \quad .$$

In the problem described by Figure 1 $a^*=a$ for $0 < c < d \leq b$ and $a^*=d$ for $c = 0$, $d < b$.

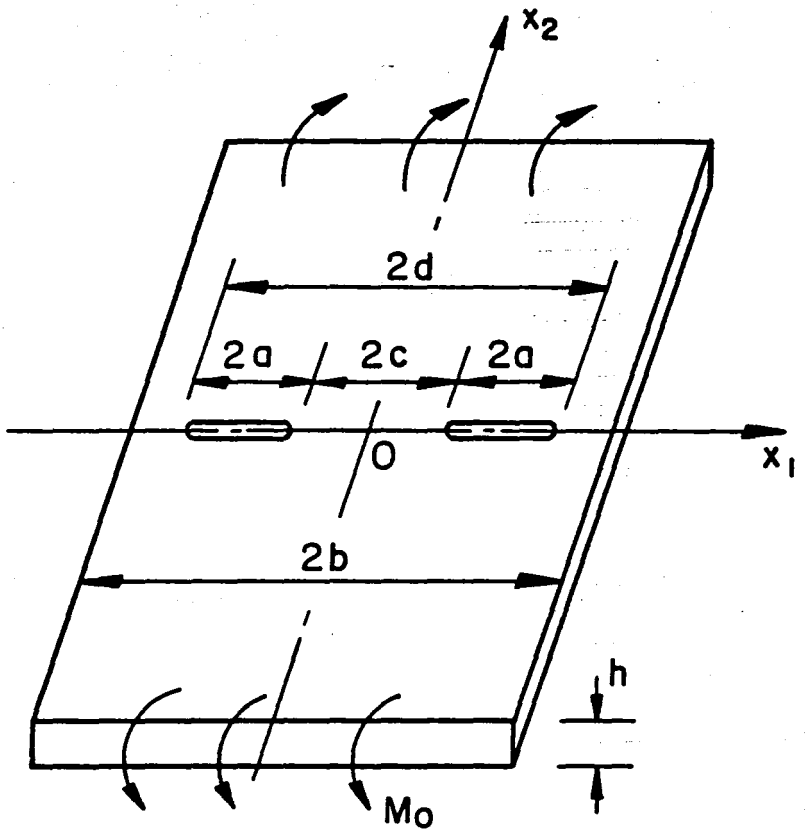

Figure 1. The geometry of the plate.

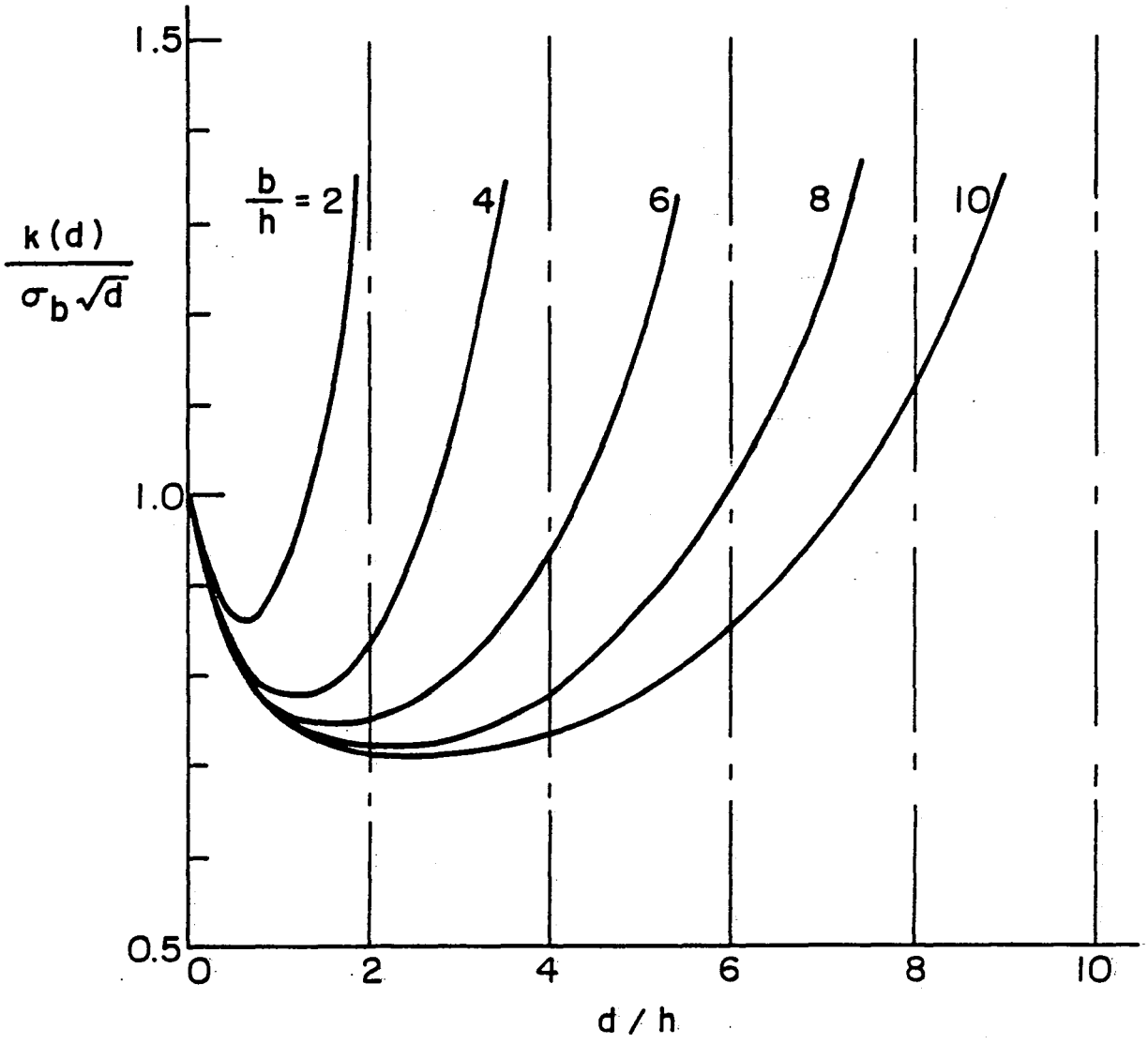

Figure 2. The stress intensity factor in a plate of finite width containing a symmetrically located single internal through crack which is subjected to uniform bending moment $M_{22} = M_0$ away from the crack region (see insert in Figure 3); $\nu = 0.3$, $\sigma_b = 6M_0/h^2$.

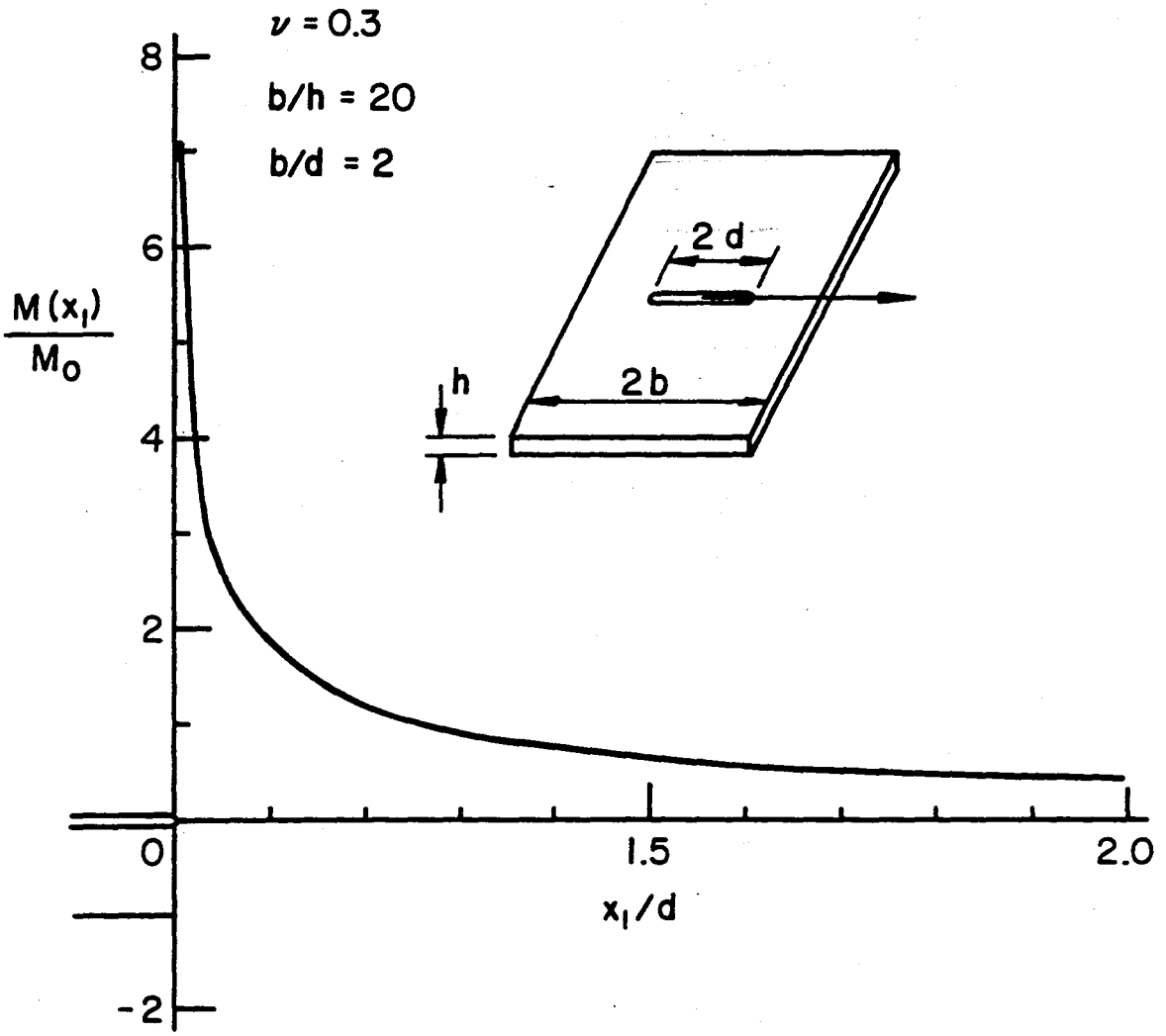

Figure 3. Distribution of the bending moment $M_{22}(x_1,0) = M(x_1)$ in the plane of the crack for a plate containing a single symmetric crack and subjected to $M_{22}(x_1,0) = -M_0$ on the crack surface $-d < x_1 < d$.

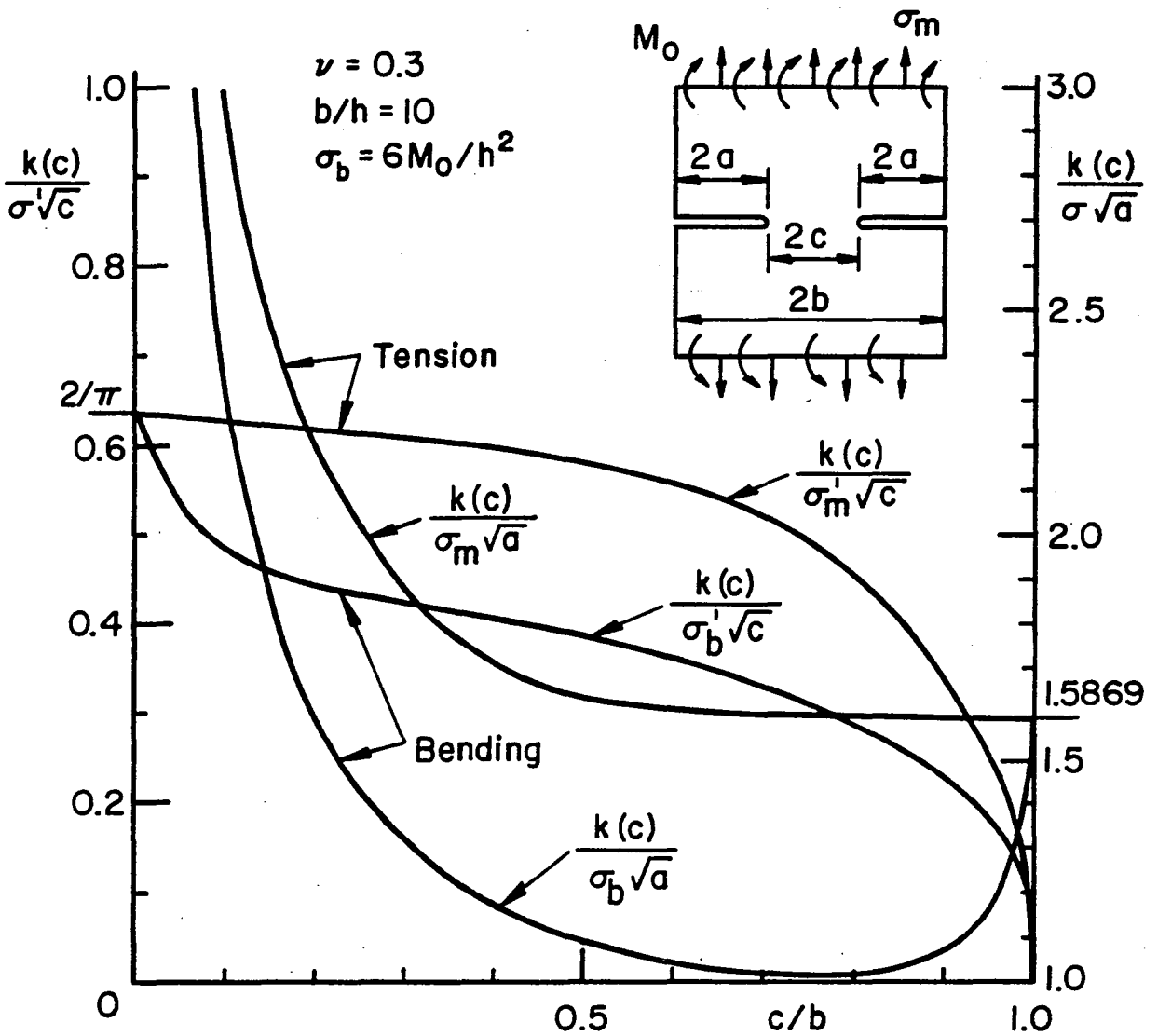

Figure 4. Stress intensity factor in a plate containing two symmetric edge cracks which is subjected to uniform bending moment M_o or uniform tensile stress σ_m away from the crack region; $\nu = 0.3$, $b/h = 10$, $\sigma_b = 6M_o/h^2$, $\sigma_b' = \sigma_b\, b/c$, $\sigma_m' = \sigma_m\, b/c$ where σ_b' and σ_m' are the average net section stresses.

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15. Supplementary Notes

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16. Abstract

In this paper the title problem is studied by using Reissner's transverse shear theory. The main purpose of the paper is to investigate the effect of stress-free boundaries on the stress-intensity factors in plates under bending. Among the results found, particularly interesting are those relating to the limiting cases of the crack geometries. The numerical results are given for a single internal crack, two collinear cracks, and two edge cracks. Also studied is the effect of Poisson's ratio on the stress-intensity factors.

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Plate	Reissner's theory	Unclassified - Unlimited
Bending	Finite width	
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Collinear cracks		
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