OBSERVABILITY FOR TWO DIMENSIONAL SYSTEMS

L.R. Hunt and Renjeng Su

ABSTRACT

Sufficient conditions that a 2 dimensional system with output is locally observable are presented. Known results depend on time derivatives of the output and the inverse function theorem. In some cases, no information is provided by these theories, and one must study observability by other methods. We dualize the observability problem to the controllability problem, and apply the deep results of Hermes on local controllability to prove a theorem concerning local observability.

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I. Introduction

Suppose we take a nonlinear system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \\ y &= h(x) \end{aligned} \tag{1}$$

where f is a real analytic vector field on $\mathbb{R}^2$ (or on a 2 dimensional manifold in general) and h is a real analytic output function on $\mathbb{R}^2$. Given a point $x_0 \in \mathbb{R}^2$, under what conditions on f and h can we guarantee that there is an open neighborhood U of x_0 so that knowledge of the observed output y of the trajectories of $\dot{x} = f(x)$ starting at points in U allow us to distinguish between x_0 and any other point in U? We also want to distinguish between any two points x_1 and x_2 in U where $h(x_1) = h(x_2) = h(x_0)$ when $t = 0$.

The known results in the literature (e.g. [1] and [2]) give sufficient conditions which involve the time derivatives of the output (or equivalently, the Lie derivatives of the output function h with respect to the vector field f) and the inverse function theorem. The results of Kou, Elliott and Tarn [1] can be applied for n dimensional C^∞ systems with several outputs and those of Hermann and Krener [2] also involve a system with inputs, whereas in this paper,

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we consider the two dimensional system (1).

Easy examples like the following one are of interest for this problem. Take

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} {x_2}^3 \\ \\ 0 \end{bmatrix} = f(x(t))$$
$$y = h(x) = x_1 \tag{2}$$

Computing the time derivatives of output we find

$$\begin{aligned} y &= x \\ \dot{y} &= {x_2}^3 \\ \ddot{y} &= 0 \\ \dddot{y} &= 0 \\ &\vdots \end{aligned}$$

Thus the inverse function theory provides an answer only if $x_2 \neq 0$. No information is provided by this method at $x_2 = 0$, and if one is constructing a state estimator based on the above time derivatives, then one obtains bad results near $x_2 = 0$. However, if we draw a phase plane portrait of the trajectories of $\dot{x} = f(x)$, we realize that those trajectories with initial values in a particular level set of the output function (i.e. $\{x: h(x) = \text{constant}\}$) and above the line $x_2 = 0$ move to the right, and those below, to the left. Moreover, the trajectories starting at any two points in a level set move to different level sets in a given time $t > 0$. Hence, there should be some calculation (involving Lie derivatives at a point x_0 where $x_2 = 0$) that should let us know this is occurring, and also imply the ability to distinguish between x_0 and the other points in some open neighborhood of x_0 in $\mathbb{R}^2$. In addition, we also want to differentiate any two points in the level of h through x_0 by watching the output in tim

As emphasized in the paper of Hermann and Krener [2], the duality between controllability and observability is simply that between vector fields and differential forms. Since the gradient of the output $y = h(x)$ in (2) is nonzero, we can find a nonvanishing vector field, say $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so that the dual product of dy and g is zero. Then we consider the control problem

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 \\ \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix} = f + ug. \tag{3}$$

The local controllability along a reference trajectory results of Hermes [3] present a way to compute the precise Lie brackets involving f and g at a point where $x_2 = 0$ that provide the needed information about the movement of the flow of $\dot{x} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix}$ on the level sets of $y = x_1$ in (2). This is true because Hermes studies the attainable set from a point x_0 at a time t.

If we compute Lie brackets for system (3) at a point x_0 where $x_2 = 0$ we find

$$g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [f,g] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [g,[f,g]] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [g,[g,[f,g]]] = \begin{bmatrix} -6 \\ 0 \end{bmatrix}.$$

The fact that $[g,[g,[f,g]]$ and g are linearly independent at x_0, and this is the first Lie bracket with this property, implies that the trajectories of $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 \\ 0 \end{bmatrix}$ perform as previously described along an integral curve of g (i.e. a level set of the output $y = h$). From this we can deduce the existence of an open neighborhood U of x_0 in which we can distinguish points in $U-\{x_0\}$ from x_0, and any two points in the level set of h through x_0 intersected with U can be distinguished.

The purpose of this paper is to present conditions under which system (1) has for a given point x_0 such a neighborhood U. One assumption is that the gradient of h(x) at x_0 is nonvanishing, implying the level set through x_0 is a 1 dimensional manifold. In our example, f(x) vanishes at a point x_0 where $x_2 = 0$. We will also provide a theory concerning observability in the case that f and g are linearly independent at x_0.

For other research into the problem of nonlinear observability we refer to [4], [5], [6], [7], [8], [9], [1], [10], [2], [11], [12], [13], and [14]. Open problems concerning observability are generated by this paper. Can the assumption that the gradient of $y = h(x)$ at x_0 is nonvanishing be reduced? What theory exists in n dimensions, and how does one handle several outputs and the introduction of inputs?

II. Definitions and Results

First we motivate the notion of observability which is appropriate for our theory.

Suppose $x_0 \in \mathbb{R}^2$ and the gradient of $y = h(x)$ in (1) is nonzero at x_0. Then there is a neighborhood V of x_0 in $\mathbb{R}^2$ so that the level sets of $y = h(x)$ form a 1 parameter family of 1 dimensional manifolds which folitate V as the parameter varies. Restricting to this set V, it is clear we can certainly distinguish between 2 points which are in distinct level sets. The problem is to differentiate between 2 points that start in the same level set by watching the movement under $\dot{x} = f(x)$ as time advances. If for arbitrarily fixed small positive time, all points in the level set of h(x) through x_0 are carried to distinct level sets of $h(x)$, then we can distinguish between them.

Let C_{x_0} be the level set of $h(x)$ in (1) through x_0.

Definition. The system (1) is locally level set observable at x_0 if there is an open neighborhood U of x_0 in $\mathbb{R}^2$ and a one-to-one correspondence between the set $U \cap C_{x_0}$ and the set of trajectories of the observed output $y(t)$ for arbitrarily small time $t > 0$. Equivalently, for arbitrary small time $t > 0$ the trajectories of $\dot{x} = f(x(t))$ starting at any two distinct points in C_{x_0} lie in different level sets of $h(x)$.

Of course if (1) is locally level set observable at x_0, it is easy to distinguish x_0 from all points in $U-\{x_0\}$ for U sufficiently small.

Suppose we have C^∞ vector fields f and g on $\mathbb{R}^2$. The Lie bracket of f and g is

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g,$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are Jacobian matrices. We can then define $[f,[f,g]], [g,[f,g]], [f,f[f,g]]]$, etc. For notation we take

$$\begin{aligned}
(ad^1 f,g) &= [f,g] \\
(ad^2 f,g) &= [f,[f,g]] \\
&\vdots \\
(ad^k f,g) &= [f,(ad^{k-1} f,g)]
\end{aligned}$$

and similarly for $(ad^k g,f)$.

For h a C^∞ function on $\mathbb{R}^2$ and f a C^∞ vector field we let

$$L_f(h) = \langle dh,f \rangle,$$

with $<.,.>$ denoting the duality between one forms and vector fields, and we can inductively take $L^2_f(h), L^3_f(h)$, etc.
If w is a C^∞ one form on $\mathbb{R}^2$

$$L_f(w) = (\frac{\partial w^*}{\partial x} f)^* + w \frac{\partial f}{\partial x},$$

where * denotes transpose. Note that $L_f(dh) = dL_f(h)$. The three "Lie derivatives" $[f,g], L_f(h)$ and $L_f(w)$ are related by the rule

$$L_f\langle w,g\rangle = \langle L_f(w),g\rangle + \langle w,[f,g]\rangle \tag{4}$$

In system (1) we assume as before that the gradient of h is nonzero at x_0. Choose a real analytic vector field g such that $\langle dh,g\rangle = 0$ and g is nonvanishing at x_0. Consider the 2 dimensional control system

$$\dot{x}(t) = f(x) + ug(x). \tag{5}$$

Using formula (4) we find

$$L_f\langle dh,g\rangle = \langle L_f(dh,g\rangle + \langle dh,[f,g]\rangle.$$

Since $\langle dh,g\rangle = 0$, we have $\langle dh,[f,g]\rangle = -\langle L_f(dh),g\rangle = -\langle dL_f(h),g\rangle$. Thus g and $[f,g]$ are linearly independent at x_0 if and only if dh and $L_f(h)$ are linearly independent there. Similarly, if g and $[f,g]$ are dependent at x_0 (or equivalently, dh and $dL_f(h)$ are), then by applying formula (4) again we find that g and $(ad^2 f,g)$ are linearly independent at x_0 if and only if dh and $dL^2_f(h)$ are. This process can be continued indefinitely (and in some cases like systems (2) and (3) we obtain no linear independence). On the one hand if there is some $L_f^k(h)$ satisfying the condition dh and $dL_f^k(h)$ are independent at x_0, there is an open neighborhood U of x_0 so system (1) is observable on U in the sense of [1] and [2]. Equivalently, if there is an $(ad^k f,g)$ with g and $(ad^k f,g)$ independent at x_0, then the results of Hermes [15] imply local controllability along a reference trajectory (starting at x_0) at time t (to be defined momentarily). Thus the duality between observability and controllability is easily realized in this way. However, Hermes [3],[16],[17], [18], and [19] has results on controllability that are much more general than those that depend on the vector fields $g,[f,g],(ad^2 f,g),\ldots$ being linearly independent.

Let $\varphi(t,x_0)$ be the solution of $\dot{x} = f(x(t))$ in (1) or (5) at time t with $\varphi(0,x_0) = x_0$. We say that the system (5) is <u>locally controllable along φ at time $t > 0$</u> if all points in some 2-dimensional open neighborhood of $\varphi(t,x_0)$ can be reached at time t by solutions of (5) initiating from x_0.

Now we define the following sets (see [16])

$$S^1 = \{g,[f,g],(ad^2 f,g),(ad^3 f,g),\ldots\}$$
$$S^2 = \{g,(ad^2 g,f),[f,(ad^2 f,g)],(ad^2 f,(ad^2 f,g)),\ldots\}$$
$$S^3 = \{g,(ad^3 g,f),[f,(ad^3 f,g)],(ad^2 f,(ad^3 f,g)),\ldots\}$$
$$\vdots$$

Let dim span $S^k_{x_0}$ = the dimension of the span of S^k at x_0.

The following result is of interest in our study of 2 dimensional observability. Assume the gradient of h in system (1) is nonvanishing at $x_0 \in \mathbb{R}^2$ and let g be defined as in section (5).

<u>Theorem</u>. If either of the following conditions hold, then system (1) is locally level set observable at x_0 in $\mathbb{R}^2$:

1) f and g are linearly independent at x_0 and the smallest

integer m so that dim span $S_{x_0}^m = 2$ is odd.

2) $f(x_0) = 0$ and the smallest integer m so that g and $(ad^m g, f)$ are linearly independent at x_0 is odd.

<u>Remark</u>. The corresponding results in [1] and [2], if dualized as we have done, consider only the set S^1 in statement 1) and the Lie bracket $[f,g]$ [illegible] $[g,$[illegible]$)$ in statement 2).

<u>Proof</u>. As stated before, since the gradient of h is nonvanishing at x_0, there is an open neighborhood V of x_0 on which this gradient is nonzero, and V consists of a 1-parameter foliation of level sets of h(i.e. integral curves of g).

For $x \in V$, we denote by $(\exp t f)(x)$ (or $\varphi(t,x)$) the integral curve (or solution curve) of f with initial value x. For fixed t, $(\exp t f)(x)$ also denotes the value of the solution at that time. For any $t \geq 0$, let L_t denote the integral curve of g through the point $(\exp t f)(x_0)$. Choose a point $x \in L_0$ close to x_0, travel from x_0 to x instantaneously along L_0 (assuming unbounded controls) and then travel along $(\exp t f)(x)$ for t units of time. If f and g are linearly independent at x_0 and the integral curves of g and $[f,g]$ through x_0 cross at x_0, then Hermes shows in [19] that our final destination is a point in some L_τ with $\tau \neq t$. If $\tau = t$, instantaneous movement along L_t to x_0 contradicts the fact that $\tau < t$ (or $\tau > t$) as Hermes has indicated. In fact, we have $\tau < t$ for those points x in L_0 close to x_0 and on one side of x_0 in L_0 and $\tau > t$ for those x in L_0 on the other side. By continuity arguments, for each x in L_0 sufficiently close to x_0, we arrive in time t at a distinct L_{τ_x}. Thus, by shrinking V to an open set U, if necessary, all points in

$L_0 \cap U$ move in time $t > 0$ (and t sufficiently small) to different level sets of h. Hence, system (1) is locally level set observable at x_0. Hermes proves in [16] that the integral curves of g and [f,g] cross at x_0 if statement 1) holds. We remark that Hermes' labeling of the sets S^i is different from ours.

If condition 2) holds, then the work of Hermes in [3] applies. In this case x_0 is an equilibrium point of f. Hermes shows that as we move along the integral curve of g at x_0, the vector field f "changes sides" as we pass through the point x_0, as in the example in the introduction. Thus if V and $t > 0$ are sufficiently small, the trajectories of $\dot{x} = f(x(t))$ starting at points in the integral curve of g through x_0 (and contained in V) move to different integral curves of g in the time t. Note that if we begin at x_0 we stay there for all time t. We have the desired observability in an open neighborhood U of x_0. $\square$

The example in the introduction has the property that $f(x_0) = 0$ if x_0 is a point where $x_2 = 0$. We now provide an example, similar to one in [16], where statement 1) of the Theorem applies.

Let $f(x) = \begin{bmatrix} 4 + x_1 x_2^3 \\ 0 \end{bmatrix}$,

$y = h(x) = x_1$,

and x_0 = the origin. In this case $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The smallest integer m so that dim span $S^m_{x_0} = 2$ is 3, and we have the local level set observability.

The problems of trying to extend the Theorem to $n > 2$ dimensions are quite interesting. Let us consider the system

$$\dot{x}(t) = f(x(t))$$
$$y = h(x) \tag{6}$$

with f and h being real analytic on $\mathbb{R}^n$. Here h is real-valued and has a nonvanishing gradient at $x_0 \in \mathbb{R}^n$. The level sets of h are real analytic (n-1) dimensional submanifolds of $\mathbb{R}^n$ near x_0. Thus the dual system is

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-1} u_i(t) g_i(x(t)), \tag{7}$$

where $g_1, g_2, \ldots, g_{n-1}$ are real analytic vector fields forming an involutive set with integral manifolds being the level sets of h. The theory of Hermes in [17] can be applied to give conditions under which we can distinguish between certain points with initial values in the same level set of the output.

If h is a p-vector valued function in (6), then $h = (h_1, h_2, \ldots, h_p)$ and we assume their gradients are linearly independent at x_0. In this case the dual system becomes

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n-p} u_i(t) g_i(x(t)) \tag{8}$$

where the set $\{g_1, g_2, \ldots, g_{n-p}\}$ is involutive near x_0. If the results of Hermes [15] using linearization are not applicable, then the problems concerning observability of (6) appear to be very difficult.

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