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An Optimal Diagnostic Strategy for Finding Malfunctioning Components in Systems

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INTRODUCTION

It is often difficult and time consuming, if not computationally impossible, to locate a failed component in a large complex system. Recently, the U.S. Army Research and Technology Laboratories at Moffett Field, California, have established a theory stating that the minimum number of test points required for conclusive detection of system failure is equal to the total number of terminal test points; this set of points constitutes the optimal choice. In this report we have developed an optimal diagnostic strategy for finding a failed component in a malfunctioning system.

Four problems intrinsically related to this strategy are given as an introduction. The first problem (ref. 2) deals with finding a selected object from a given set, each object of which has a known probability of being chosen. The problem is solved using a series of yes-no questions, arbitrarily dividing the set of objects into two groups and asking in which group the selected object is contained. The yes-no questioning process continues until the selected object is known. The second problem is the gold-coin-in-the-box problem (ref. 2). Briefly, a gold coin is concealed in one of m boxes of copper coins, where each box i, $i = 1, 2, 3, \ldots, m$. Assuming a known probability of finding the gold coin and the associated cost involved, one must choose a strategy to maximize the probability of finding the gold coin when a budget ceiling is imposed, incorporating a repeated search through the set of boxes until the coin is found. The third problem (ref. 3) does not allow for the repeated search of the gold-coin-in-the-box problem. This problem addresses how to find the state of every component in a k/n system (an n-component system having the property that the system is functioning if at least k out of n components are functioning) in the minimum optimal time, given that the system is not functioning. Finally, the last problem (ref. 4) is stated as follows: Given an n-component functional system in which numerous components are functionally interdependent, determine a diagnostic strategy for conclusive detection of the operational status of the system with the least expenditure.

The present paper concerns itself with the problem of optimally finding a failed component in a malfunctioning system. A concise statement of the problem, together with the main result, will be given in the third section, entitled Admissible Diagnostic Strategies, after some relevant definitions and assumptions have been introduced in the second section. The fourth section contains illustrative examples for determining the optimal strategies based on the theorem obtained in the third section.

DEFINITIONS AND ASSUMPTIONS

In this section we shall state explicitly the definitions and assumptions upon which our strategy is based. These definitions and assumptions are generally accepted in the mathematical sciences and engineering communities; see, for example, references 5-7.

Definitions

Failure: A condition characterized by the inability of a material, structure, or system to fulfill its intended purpose or task.

Malfunction: Failure to operate in the normal or expected manner, or level of performance.

Test: An observation or measurement procedure that provides sufficient information to determine whether or not all members of a particular subset of elements are functioning properly.

Coherent systems: Those systems for which the replacement of a failed component by a functioning one will not induce a functioning system to fail. (A rigorous mathematical definition is given in appendix A.)

Strategy: An ordering of the components which are tested sequentially in the predetermined order until a failed component can be found.

Admissible strategy: A strategy whose expected expenditure is minimum.

Optimal strategy: An admissible strategy which obeys Bellman's Principle of Optimality.

Bellman's Principle of Optimality: Whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Assumptions

It is hypothesized that

1. At any instant or stage, the system or equipment under consideration may be in only one of two states: functioning or faulty.

2. The system can be schematically decomposed into a finite number of components (or modules), each of which, at any instant, is in one of the two possible states.

3. The state of the system depends solely on the states of its components.

Hypothesis (1) is a realistic assumption because if the performance level of a given component is degraded to an "unsatisfactory" level (or beyond the tolerance as cited in the specifications of the equipment), then the component is in the malfunctioning state. Hypothesis (2) demands only the feasibility of schematic system decomposition, not necessarily a physical decomposition. Hypothesis (3) tacitly assumes that a proper environment exists for the system under question.

ADMISSIBLE DIAGNOSTIC STRATEGIES

Before we discuss the main result of this paper, it is necessary to restate concisely the problem under consideration: Suppose that we are given an n-component, coherent, malfunctioning system for which the component reliability and the associated test time are known. A malfunctioning component must be found such that the expected test time is optimal in the sense of Bellman's Principle of Optimality.

The following lemma plays an important role in the subsequent deductive process. (The mathematical proof is provided in appendix B.)

Lemma. Given that an n-component, coherent system is not functioning, and p_i and t_i, $i = 1, 2, 3, \ldots, n$, are the reliability and test time of component i, respectively, then the expected time $T(a)$ of a strategy a for finding a malfunctioning component of the system is given by

$$T(a) = \sum_{i=1}^{n-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i$$

where

$$P(0) \equiv 1$$

and the conditional probability

$$P(k) \equiv P\left(k \,\middle|\, \bar{S} \bigwedge_{m=1}^{k-1} m \right)$$

when the kth component of a is functioning, given that the system is down, and all the first $k - 1$ components are functioning.

With this lemma at our disposal, it is a trivial exercise to deduce the following theorem (thus, the proof is omitted).

Theorem. Given that an n-component, coherent system is down, the corresponding reliability and test time are known, and it is necessary to find a malfunctioning component of the system, then the following statements are equivalent:

1. Strategy s is admissible

2. $T(s) \leq T(a), \forall a \in \mathscr{A}$

where $\mathscr{A}$ is the set of all ordering of n components.

The main drawbacks of the theorem include the amount of algebraic computation and enumeration, and the bookkeeping of $\mathscr{A}$ (e.g., for $n = 15$, $\mathscr{A}$ contains more than 1.3×10^{12} strategies). Therefore, even for a "moderate" value of n, the computation would defy the capability of a present-day electronic computer. Since the number of strategies grows factorially as a function of n, an obvious remedy to overcome this difficulty is to partition the system into a sequence of nested levels (or subsystems) in which the number of components involved in each level is computationally manageable. Unfortunately, the resulting strategy is not necessarily admissible or optimal in the global sense, even though it is admissible and optimal locally. A better way to solve this combinatorial problem is to reduce the computational complexity of statement 2 of the above theorem by finding its equivalence. This would simplify the computation immensely while preserving the global admissibility and optimality. (This result will be published in a subsequent paper.)

EXAMPLES FOR FINDING OPTIMAL STRATEGIES

The main result of the preceding section is to provide a means to determine optimal diagnostic strategies for finding a malfunctioning component in a failed coherent system.

We begin with an intuitively simple example. Suppose that a four-component coherent system is not functioning. Let α, β, γ, and δ be the components whose reliabilities p and test times t are as follows:

Component	p	t
α	0.9	2
β	0.9	4
γ	0.9	2
δ	0.9	3

Then for this example the set $\mathscr{A}$ contains 24 strategies:

$A = \langle\alpha, \beta, \gamma, \delta\rangle$
$B = \langle\delta, \alpha, \beta, \gamma\rangle$
$C = \langle\gamma, \delta, \alpha, \beta\rangle$
$D = \langle\beta, \gamma, \delta, \alpha\rangle$
$E = \langle\beta, \delta, \gamma, \alpha\rangle$
$F = \langle\delta, \gamma, \beta, \alpha\rangle$
$G = \langle\gamma, \delta, \beta, \alpha\rangle$
$H = \langle\delta, \beta, \gamma, \alpha\rangle$
$I = \langle\gamma, \beta, \delta, \alpha\rangle$
$J = \langle\delta, \gamma, \alpha, \beta\rangle$
$K = \langle\delta, \beta, \alpha, \gamma\rangle$
$L = \langle\gamma, \beta, \alpha, \delta\rangle$
$M = \langle\beta, \gamma, \alpha, \delta\rangle$
$N = \langle\beta, \delta, \alpha, \gamma\rangle$
$O = \langle\gamma, \alpha, \beta, \delta\rangle$
$P = \langle\gamma, \alpha, \delta, \beta\rangle$
$Q = \langle\delta, \alpha, \gamma, \beta\rangle$
$R = \langle\beta, \alpha, \delta, \gamma\rangle$
$S = \langle\beta, \alpha, \gamma, \delta\rangle$
$U = \langle\alpha, \beta, \delta, \gamma\rangle$
$V = \langle\alpha, \gamma, \delta, \beta\rangle$
$W = \langle\alpha, \delta, \gamma, \beta\rangle$
$X = \langle\alpha, \gamma, \beta, \delta\rangle$
$Y = \langle\alpha, \delta, \beta, \gamma\rangle$

Also, the conditional probabilities are: (In what follows all numbers have been rounded off to four decimal places.)

$P(\alpha|\bar{S}) = 0.7092$
$P(\beta|\bar{S}) = .7092$
$P(\gamma|\bar{S}) = .7092$
$P(\delta|\bar{S}) = .7092$
$P(\alpha|\bar{S}\wedge\beta) = 0.6310$
$P(\alpha|\bar{S}\wedge\gamma) = .6310$
$P(\alpha|\bar{S}\wedge\delta) = .6310$
$P(\beta|\bar{S}\wedge\alpha) = .6310$
$P(\beta|\bar{S}\wedge\gamma) = 0.6310$
$P(\beta|\bar{S}\wedge\delta) = .6310$
$P(\gamma|\bar{S}\wedge\alpha) = .6310$
$P(\gamma|\bar{S}\wedge\beta) = .6310$
$P(\gamma|\bar{S}\wedge\delta) = .6310$
$P(\delta|\bar{S}\wedge\alpha) = .6310$
$P(\delta|\bar{S}\wedge\beta) = .6310$
$P(\delta|\bar{S}\wedge\gamma) = .6310$

and the expected times of the strategies are:

$T(A) = 5.7318$
$T(B) = 6.2084$
$T(C) = 5.0226$
$T(D) = 6.7609$
$T(E) = 7.0226$
$T(F) = 6.2084$
$T(G) = 5.9176$
$T(H) = 6.7318$
$T(I) = 6.1793$
$T(J) = 5.3134$
$T(K) = 6.7318$
$T(L) = 5.7318$
$T(M) = 6.3134$
$T(N) = 7.0226$
$T(O) = 5.2084$
$T(P) = 4.7609$

T(Q) = 5.3134	T(V) = 4.7609
T(R) = 6.7609	T(W) = 5.0226
T(S) = 6.3134	T(X) = 5.2084
T(U) = 6.1793	T(Y) = 5.9176

The minimum value is attained by strategies P and V. By definition, both P and V are admissible. To determine optimality, we compute for each admissible strategy the expected time at each of the three subsequences of tests.

Admissible strategy	Expected time of first test	Expected time of first two tests	Expected time of first three tests
P	2.0000	3.4184	4.7609
V	2.0000	3.4184	4.7609

In this case, the corresponding expected times at each stage are equal. Thus, P and V are also optimal. This is not surprising for two reasons: (1) the component test times are the deciding parameters for the system whose entropy is maximum, and (2) any admissible strategy with nondecreasing, termwise-smallest sequence of component test times is optimal.

For the next example, let us reconsider the last problem using the following reliability data:

Component	P
α	0.9
β	0.8
γ	0.6
δ	0.8

For this example the conditional probabilities are:

$P(\alpha \mid \bar{S}) = 0.8472$	$P(\beta \mid \bar{S}\wedge\gamma) = 0.5284$
$P(\beta \mid \bar{S}) = .6944$	$P(\beta \mid \bar{S}\wedge\delta) = .6479$
$P(\gamma \mid \bar{S}) = .3887$	$P(\gamma \mid \bar{S}\wedge\alpha) = .3506$
$P(\delta \mid \bar{S}) = .6944$	$P(\gamma \mid \bar{S}\wedge\beta) = .2958$
$P(\alpha \mid \bar{S}\wedge\beta) = 0.8239$	$P(\gamma \mid \bar{S}\wedge\delta) = .2958$
$P(\alpha \mid \bar{S}\wedge\gamma) = .7643$	$P(\delta \mid \bar{S}\wedge\alpha) = .6753$
$P(\alpha \mid \bar{S}\wedge\delta) = .8239$	$P(\delta \mid \bar{S}\wedge\beta) = .6479$
$P(\beta \mid \bar{S}\wedge\alpha) = .6753$	$P(\delta \mid \bar{S}\wedge\gamma) = .5284$

and the expected times of the strategies are:

T(A) = 6.5330	T(L) = 3.9656
T(B) = 6.6773	T(M) = 5.7996
T(C) = 3.5769	T(N) = 6.9830
T(D) = 6.0050	T(O) = 3.9657
T(E) = 6.9830	T(P) = 3.6687
T(F) = 5.2104	T(Q) = 5.5330
T(G) = 3.9877	T(R) = 7.1051
T(H) = 6.6774	T(S) = 6.5330
T(I) = 4.1710	T(U) = 7.1051
T(J) = 4.7996	T(V) = 4.5855
T(K) = 6.6774	T(W) = 5.6858

$T(X) = 4.8825$ $T(Y) = 6.8301$

Obviously, strategy C is the only admissible strategy, and hence it is optimal also.

It is interesting to note that the optimal strategy, $C = \langle\gamma, \delta, \alpha, \beta\rangle$ proceeds with the most unreliable and the least test time component γ as the first component to be tested; next in the test sequence is δ, which is the next most unreliable and costly component. Between the remaining components α and β, C yields α for the third position; yet β is less reliable than α. This apparent contradiction to intuition can be explained by reexamining the preceding lemma, which asserts that the expected test time of the last component is zero. That is to say, equivalently, between the last two components, an optimal strategy always chooses the one having a smaller test time regardless of their reliability data.

CONCLUSIONS

An optimal diagnostic strategy for finding a failed component in an n-component, coherent, malfunctioning system is presented. It was found that even for a moderate value of n the amount of algebraic computation and enumeration of the set of all possible strategies becomes computationally infeasible because of its factorial-growth character. An obvious solution to this problem is to partition the system into a sequence of nested subsystems, in which the number of components involved in each subsystem is computationally manageable. Unfortunately, this strategy is not necessarily optimal or even admissible in the global sense.

APPENDIX A

A MATHEMATICAL DEFINITION OF COHERENT SYSTEMS

Let S be the set of all n components of an n-component system. Then a state function X on S onto the set $\{0,1\}$ is defined as follows:

$$\forall\, c \in S,\ X(c) = \begin{cases} 1, \text{ if component } c \text{ is functioning} \\ 0, \text{ if component } c \text{ is faulty} \end{cases}$$

A structure function ϕ on the set of all n-tuple, $X(S)$, onto $\{0,1\}$ is defined by

$$\forall\, t \in X(S),\ \phi(t) = \begin{cases} 1, \text{ if the system is functioning} \\ 0, \text{ if the system is faulty} \end{cases}$$

A structure function ϕ is monotone if it has the following property:

$$\forall\, a,b \in X(S),\ a \leq b \Rightarrow \phi(a) \leq \phi(b)$$

A coherent system is a system whose structure function is monotone.

APPENDIX B

A MATHEMATICAL PROOF FOR $T(s_n) = \sum_{i=1}^{n-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i$

Lemma. Given that an n-component, coherent system is not functioning, and p_i and t_i, $i = 1, 2, 3, \ldots, n$ are the component reliability and test time of i, respectively, then the expected time $T(s_n)$ of a strategy s_n for finding a malfunctioning component of the system is given by

$$(*) \qquad T(s_n) = \sum_{i=1}^{n-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i$$

where

$$P(0) \equiv 1$$

and the conditional probability

$$P(k) \equiv P\left(k \,\middle|\, \bar{S} \bigwedge_{m=1}^{k-1} m\right)$$

when the kth component of s_n is functioning, given that the system is not functioning, and all the first $k-1$ components are functioning.

Proof: We prove the expression (*) by mathematical induction on n. To show the basis of induction, let $n = 1$ (i.e., a one-component system). In this case, $T(s_1) = 0$, since no test is needed. Also, for $n = 1$, the right-hand side of (*) is zero because a null sum is defined to be zero. Next, by inductive hypothesis, (*) is valid for an n-component system. It remains to be shown that the inductive step is valid whenever the inductive hypothesis is valid; that is, we must prove that for an $(n + 1)$-component system

$$T(s_{n+1}) = \sum_{i=1}^{(n+1)-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i$$

Note that

$$\sum_{i=1}^{(n+1)-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i = \sum_{i=1}^{n-1} \left[\prod_{k=0}^{i-1} P(k) \right] t_i + P(1)P(2) \ldots P(n-1)t_n$$

$$= T(s_n) + P(1)P(2) \ldots P(n-1)t_n$$

$$= T(s_n) + P(1)P(2) \ldots P(n-1)\{t_n + P(n)\tau_0(s_{n+1})$$

$$+ [1 - P(n)]\tau_1(s_{n+1})\}$$

The next to the last equality holds by the inductive hypothesis, and, given that the system is not functioning and the first n components are functioning, the last equality follows from the fact that the expected remaining test time (denoted by $\tau_0(s_{n+1})$), is zero. Likewise the expected remaining test time, $\tau_1(s_{n+1})$ is zero if the first $n - 1$ components are functioning and the nth component is not functioning. The quantity in the braces in the last term is precisely the expected time for testing the nth component in s_{n+1}. Therefore, together with the first term, $T(s_n)$, it constitutes the expected test time of s_{n+1}; that is,

$$T(s_n) + P(1)P(2) \ldots P(n-1)\{t_n + P(n)\tau_0(s_{n+1}) + [1 - P(n)]t_1(s_{n+1})\} = T(s_{n+1})$$

Hence, by the principle of mathematical induction, the assertion follows.

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16. Abstract

In this paper, a solution to the following problem is presented: Given that an n-component functional system is down, it is required to find a malfunctioning component of the system such that the expected expenditure is minimum.

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