

NASA CR-172,141

NASA Contractor Report 172141

ICASE

FOR REFERENCE

NOT TO BE TAKEN FROM THIS ROOM

NASA-CR-172141
19830020667

NUMERICAL VISCOSITY AND THE ENTROPY CONDITION
FOR CONSERVATIVE DIFFERENCE SCHEMES

Eitan Tadmor

Contract No. NAS1-17070
June 1983

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

NASA
National Aeronautics and
Space Administration
Langley Research Center
Hampton, Virginia 23665



LIBRARY COPY

JUN 11 1983

LANGLEY RESEARCH CENTER
LIBRARY, NASA
HAMPTON, VIRGINIA



NUMERICAL VISCOSITY AND THE ENTROPY CONDITION
FOR CONSERVATIVE DIFFERENCE SCHEMES

Eitan Tadmor
Institute for Computer Applications in Science and Engineering

Abstract

Consider a scalar, nonlinear conservative difference scheme satisfying the entropy condition. It is shown that difference schemes containing more numerical viscosity will necessarily converge to the unique, physically relevant weak solution of the approximated conservative equation. In particular, entropy satisfying convergence follows for E schemes — those containing more numerical viscosity than Godunov's scheme.

1980 Mathematics Subject Classification 65P05, 35L65.

Research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-17070 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Introduction

There is a close relation between the concepts of entropy and viscosity, associated with systems of conservation laws. It is well known, for example, that vanishing viscosity weak solutions for such systems must satisfy the entropy inequality across their discontinuity, and that the converse holds, at least in the small - in the large for scalar problems; both are being used to identify the so-called "physically relevant" solution of such systems, e.g., [7].

In this paper we amplify a certain aspect of this relation, with regard to conservative difference schemes

(1.1)

$$v_v(t+k) = v_v(t) - \lambda [h(v_{v-p+1}(t), \dots, v_{v+p}(t)) - h(v_{v-p}(t), \dots, v_{v+p-1}(t))]$$

serving as consistent approximations to the scalar conservation law

$$(1.2) \quad \frac{\partial u}{\partial t}(x,t) + \frac{\partial f}{\partial x}(u(x,t)) = 0.$$

To make our point, consider a difference scheme which is known to satisfy the entropy inequality; roughly speaking, this should indicate according to the above, the existence of certain amount of numerical viscosity present in such a scheme. It is therefore plausible to assert that other schemes, containing more numerical viscosity, will also have to satisfy the entropy inequality. After putting these terms in a more precise framework, we show the validity of the above assertion subject to the technical assumptions listed below. Thus, we prove the entropy inequality by means of comparison.

In [12], Osher introduced, for the method of lines, a class of E schemes which were shown to converge to the physically relevant solution. Making use of the terminology just introduced, the so-called E schemes can be identified as exactly those having no less numerical viscosity than that of Godunov. Since the latter is known to satisfy the entropy inequality, we are able to extend Osher's ideas to the fully discrete case, as a special case of the above assertion. This is carried out in Section 5, paving the way for the proof of the more general assertion in Section 6. Prior to that, we give in Sections 3 and 4, a brief discussion on the entropy inequality in relation to the all important Godunov and Lax-Friedrichs schemes.

2. Preliminaries

We consider difference schemes

$$(2.1a) \quad v_v(t+k) = H(v_{v-p}(t), \dots, v_{v+p}(t); f, \lambda)$$

which admit a conservative form

(2.1b)

$$H(v_{v-p}, \dots, v_{v+p}; f, \lambda) = v_v - \lambda [h(v_{v-p+1}, \dots, v_{v+p}) - h(v_{v-p}, \dots, v_{v+p-1})],$$

and are serving as consistent approximations to the scalar conservation law

$$(2.2) \quad \frac{\partial u}{\partial t}(x, t) + \frac{\partial f}{\partial x}(u(x, t)) = 0.$$

Here, $v_v(t) \equiv v(x_v, t)$ is denoting the approximation value at the gridpoint

$(x_v \equiv v\Delta x, t)$, k and Δx are respectively, the temporal and spatial meshsize such that the mesh ratio $\lambda \equiv k/\Delta x$ is being kept fixed, and p , a natural number. Finally, $h_{v+1/2} \equiv h(v_{v-p+1}, \dots, v_{v+p})$ is the Lipschitz continuous numerical flux consistent with the differential one, $h(w, w, \dots, w) = f(w)$; for the sake of simplifying the notations, its possible dependence on f and λ is being suppressed.

We begin by putting the scheme (2.1) in an increment form: using the difference operator $\Delta w_{v+1/2} \equiv w_{v+1} - w_v$, we set for $v_{v+1} \neq v_v$

$$(2.3a) \quad C_{v+1/2}^+ = \lambda \frac{f_v - h_{v+1/2}}{\Delta v_{v+1/2}}$$

$$(2.3b) \quad C_{v+1/2}^- = \lambda \frac{f_{v+1} - h_{v+1/2}}{\Delta v_{v+1/2}} ;$$

adding and subtracting f_v to the RHS of (2.1b) and making use of (2.3), (2.1a) reads

$$(2.4) \quad v_v(t+k) = v_v(t) + C_{v+1/2}^+ \Delta v_{v+1/2}(t) - C_{v-1/2}^- \Delta v_{v-1/2}(t).$$

Next, we denote

$$(2.5) \quad Q_{v+1/2} \equiv C_{v+1/2}^+ + C_{v+1/2}^- = \lambda \frac{f_v + f_{v+1} - 2h_{v+1/2}}{\Delta v_{v+1/2}} .$$

Noting the identity $C_{v+1/2}^- - C_{v+1/2}^+ = \lambda \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}}$, the incremental coefficients $C_{v+1/2}^\pm$ equal

$$(2.6) \quad C_{v+1/2}^\pm = \frac{1}{2} \left(Q_{v+1/2} \mp \lambda \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right).$$

Inserted into (2.4), our scheme then recast into the form

(2.7)

$$v_v(t+k) = v_v(t) - \frac{\lambda}{2} [f(v_{v+1}(t)) - f(v_{v-1}(t))] + \frac{1}{2} [\Delta(Q_{v-1/2} \Delta v_{v-1/2}(t))],$$

which reveals the role Q plays as the numerical viscosity coefficient. We will therefore use Q as a measurement of the amount of viscosity present in such a scheme.

REMARK: In the case of 3-point schemes, $p = 1$, this measure of viscosity is rather general in the sense that such schemes are completely determined by their coefficient of numerical viscosity, e.g., [10]. We do not claim the generality for $(2p+1)$ -point schemes $p > 1$: this definition of numerical viscosity is in fact 3-point oriented, as we shall see in a more precise form, later on.

Let $TV[v(t)] \equiv \sum_v |v_{v+1}(t) - v_v(t)|$ denote the total variation of the computed solution at time t ; we then have the following

Lemma 2.1

The scheme (2.1) has a total variation non-increasing provided its numerical viscosity coefficient, $Q_{v+1/2}$, satisfies

$$(2.8) \quad \lambda \left| \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right| < Q_{v+1/2} < 1.$$

Proof: The inequalities (2.8), expressed in terms of the incremental coefficients in (2.6), are translated into

$$(2.9) \quad c_{v+1/2}^+ > 0, \quad c_{v+1/2}^- > 0, \quad 1 - c_{v+1/2}^+ - c_{v+1/2}^- > 0.$$

A straightforward calculation, based on the incremental form (2.4) and the inequalities (2.9), shows the non-increase in total variation, $TV[v(t+k)] < TV[v(t)]$, see [5].

Lemma 2.1 implies, in particular, the convergence of the scheme (2.1), provided its numerical viscosity coefficient meets the requirement (2.8): one can select a boundedly a.e. converging subsequence, $v_v(t; \Delta x')$, such that its limit $v(x,t) = \lim_{x=v\Delta x', \Delta x' \rightarrow 0} v_v(t; \Delta x')$ satisfies (2.2) in the weak sense, e.g., [1], [3], [9]⁽¹⁾. Weak solutions of (2.2) however, are not unique. The lower-bound on the LHS of (2.8) requiring that much of viscosity for convergence to a limit weak solution, does not guarantee this weak solution to be the physically relevant one: it is well-known for example, that the 3-point Courant-Isaacson-Rees scheme where $Q_{v+1/2} = \lambda \left| \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right|$, may admit limit weak solutions violating the physically relevant entropy condition, e.g., [5], [12]; thus, a greater amount of viscosity is required for the entropy condition to hold. In the next section we discuss Godunov's scheme which turns out to play a central role in determining that additional required amount.

We note in passing, the fundamentally different role played by the upper bound on the numerical viscosity, appearing on the RHS of (2.8). It is related to the hyperbolic nature of the approximated equation (2.2), as it amounts to the CFL-like condition, see (2.5),

(1) We consider compactly supported initial data; a further L^∞ bound, derived below, is required for the more general initial data in $L^1 \cap L^\infty \cap BV$.

$$(2.10) \quad \lambda \left| (f(v_v) - h_{v+1/2}) + (f(v_{v+1}) - h_{v+1/2}) \right| < |v_{v+1} - v_v|,$$

which usually results in placing a limitation on the mesh ratio, λ , being used (recall that $h(\dots)$ may depend on λ as well). A stricter CFL condition of this type was introduced in [9]. In particular, the numerical flux of a difference scheme satisfying (2.10) admits the consistency relation

$$(2.11) \quad h(v_{v-p+1}, \dots, v_{v-1}, w, w, v_{v+2}, \dots, v_{v+p}) = f(w).$$

Such essentially 3-point schemes include, beside the standard 3-point schemes, several of the recently constructed second-order accurate converging schemes, e.g., [5], [8].

Finally, we would like to point out that by halving the CFL number, one obtains a maximum principle; that is

Lemma 2.2

Consider the scheme (2.1) with a numerical viscosity coefficient, $Q_{v+1/2}$, satisfying

$$(2.12) \quad \lambda \left| \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right| < Q_{v+1/2} < \frac{1}{2}.$$

Then, the following maximum principle

$$(2.13) \quad \inf_{\mu} [v_{\mu}(t)] < v_v(t+k) < \sup_{\mu} [v_{\mu}(t)]$$

holds.

Proof: The incremental coefficients in (2.6) do not exceed a value of

$$0 < c_{v+1/2}^{\pm} = \frac{1}{2} \left(Q_{v+1/2} \mp \lambda \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right) < \frac{1}{2} \cdot 2Q_{v+1/2} < \frac{1}{2} .$$

Making use of the incremental form of the scheme, see (2.4),

$$v_v(t+k) = c_{v+1/2}^+ v_{v+1}(t) + (1 - c_{v+1/2}^+ - c_{v-1/2}^-) v_v(t) + c_{v-1/2}^- v_{v-1}(t)$$

and noting the convexity of the combination on the RHS, (2.13) follows.

3. The Entropy Condition and Godunov's Scheme

The building block in Godunov's scheme, [4], is the solution of the Riemann problem. Let $u^R(x/t; u_{\text{left}}, u_{\text{right}})$ denote the similarity solution of the Riemann problem (2.2) subject to initial conditioning

$u(x, t=0) = \frac{1 - \text{sgn}(x)}{2} u_{\text{left}} + \frac{1 + \text{sgn}(x)}{2} u_{\text{right}}$. Godunov's scheme is determined by

$$(3.1a) \quad v_v(t+k) = H^G(v_{v-1}, v_v, v_{v+1}) \equiv \frac{v_{v-1/2}^- + v_{v+1/2}^+}{2} ,$$

where

$$(3.1b) \quad v_{v-1/2}^- \equiv v^-(v_{v-1}, v_v) = \frac{1}{\Delta x/2} \int_0^{\Delta x/2} u^R(x/k; v_{v-1}, v_v) dx$$

$$(3.1c) \quad v_{v+1/2}^+ \equiv v^+(v_v, v_{v+1}) = \frac{1}{\Delta x/2} \int_0^{\Delta x/2} u^R(x/k; v_v, v_{v+1}) dx$$

Assume the CFL condition

$$(3.2a) \quad \lambda \cdot \max_u |a(u)| < \frac{1}{2}, \quad a(u) \equiv \dot{f}(u)$$

holds. The RHS of (3.1b-c) can be evaluated from the integral form of (2.2), see Figure 3-1, giving

$$(3.2b) \quad v_{v-1/2} = v_v - 2\lambda (f_v - h_{v-1/2}^G),$$

$$(3.2c) \quad v_{v+1/2}^+ = v_v + 2\lambda (f_v - h_{v+1/2}^G),$$

where

$$(3.3) \quad h_{v+1/2}^G = h^G(v_v, v_{v+1}) = f(u^R(0^+; v_v, v_{v+1}))$$

stands for the numerical flux of Godunov's scheme: indeed, by averaging (3.2b-c), (3.1a) takes the desired conservative form

$$(3.4) \quad v_v(t+k) = v_v - \lambda (h_{v+1/2}^G - h_{v-1/2}^G).$$

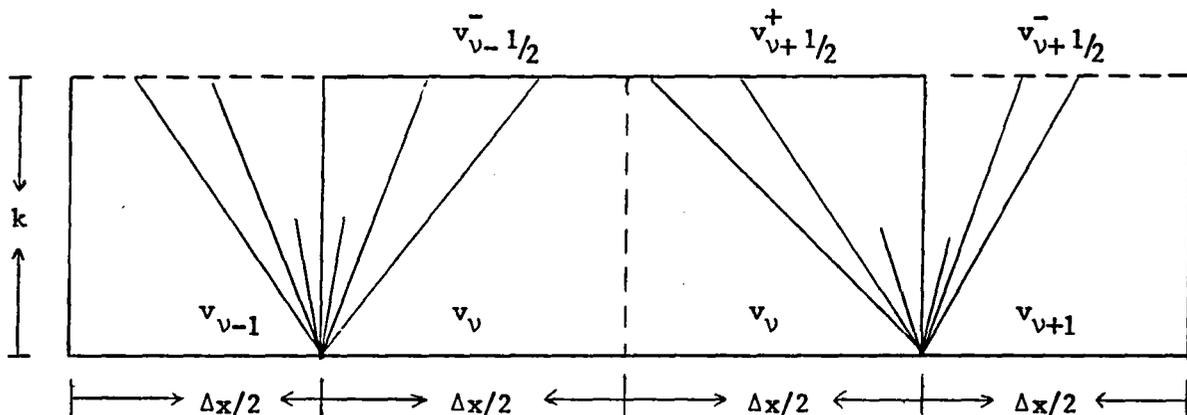


Figure 3-1

Consider a pair of scalar functions $(U(w), F(w))$ such that

$$(3.5a) \quad \dot{U}(w)\dot{f}(w) = \dot{F}(w), \quad \ddot{U}(w) > 0;$$

the entropy condition for a physically relevant solution of (2.2), $u \equiv u(x,t)$, requires the following entropy inequality

$$(3.5b) \quad \frac{\partial}{\partial t} U(u) + \frac{\partial}{\partial x} F(u) < 0 \text{ (weakly)}$$

to hold for all entropy pairs related through (3.5a). Recalling (3.1b-c), Jensen's inequality and the integral form of (3.5b) yield

$$(3.6a) \quad U(v_{v+1/2}^-) < \frac{1}{\Delta x/2} \int_0^{\Delta x/2} U(u^R(x/k; v_{v-1}, v_v)) dx < U(v_v) - 2\lambda(F(v_v) - F_{v-1/2}^G)$$

$$(3.6b) \quad U(v_{v+1/2}^+) < \frac{1}{\Delta x/2} \int_{-\Delta x/2}^0 U(u^R(x/k; v_v, v_{v+1})) dx < U(v_v) + 2\lambda(F(v_v) - F_{v+1/2}^G)$$

where

$$(3.7) \quad F_{v+1/2}^G \equiv F^G(v_v, v_{v+1}) = F(u^R(0^+; v_v, v_{v+1}))$$

is Godunov's numerical entropy flux, consistent with the differential one, $F^G(w,w) = F(w)$. Averaging (3.6a-b) we find on account of (3.1a), that Godunov's scheme is consistent with the differential entropy inequality (3.5b),

$$(3.8) \quad U(v_v(t+k)) < \frac{U(v_{v-1/2}^-) + U(v_{v+1/2}^+)}{2} < U(v_v) - \lambda(F_{v+1/2}^G - F_{v-1/2}^G).$$

We now summarize what we have shown in the following

Lemma 3.1.

Assume the CFL condition

$$(3.9) \quad \lambda \cdot \max_u |a(u)| < \frac{1}{2}$$

holds. For $v_{v\pm 1/2}^\pm$ given by

$$(3.10a) \quad v_{v-1/2}^- = v_v - \lambda(f_v - f_{v-1}) - Q_{v-1/2}^G \Delta v_{v-1/2}$$

$$(3.10b) \quad v_{v+1/2}^+ = v_v - \lambda(f_{v+1} - f_v) + Q_{v+1/2}^G \Delta v_{v+1/2},$$

we have the following entropy inequalities

$$(3.11a) \quad U(v_{v-1/2}^-) < U(v_v) - 2\lambda(F(v_v) - F_{v-1/2}^G)$$

$$(3.11b) \quad U(v_{v+1/2}^+) < U(v_v) + 2\lambda(F(v_v) - F_{v+1/2}^G).$$

Proof: Inserting the definition of the numerical viscosity coefficient in (2.5), one reads (3.10a-b) from (3.2b-c). The conclusion appears in (3.6a-b).

REMARK: We have shown that Godunov's scheme satisfies the entropy inequality (3.8) by averaging (3.11a-b), while assuming the CFL condition (3.2a), $\lambda \cdot \max_u |a(u)| < \frac{1}{2}$; the latter was required in order to guarantee that waves issued from the two opposite faces of the v_v -cell, do not interact. In the scalar case, an entropy solution is known to exist whether or not these waves

interact. Hence, (3.8) follows from the integral form of (3.5b) applied over the whole v_v -cell (rather than — as we have done — over its left and right halves), provided the relaxed CFL condition $\lambda \cdot \max_u |a(u)| < 1$ holds, thus preventing these waves from reaching the cell's other faces. The reason for our introduction of $v_v(t+k)$ as the average of $v_{v-1/2}^-$ and $v_{v+1/2}^+$, each of which satisfies the entropy inequality (3.11a-b), will prove itself essential, however, in studying E schemes in Section 5 below. We note that the so introduced averaging is nothing else but a restatement of the following identity whose verification is left to the reader

$$(3.12) \quad H^G(v_{v-1}, v_v, v_{v+1}; \lambda) \equiv \frac{H^G(v_{v-1}, v_v, v_v; 2\lambda) + H^G(v_v, v_v, v_{v+1}; 2\lambda)}{2} .$$

In closing this section, we would like to point out the following geometric interpretation of the numerical viscosity coefficient associated with Godunov's scheme, $Q_{v+1/2}^G$: integrating (2.2) over the left half of v_{v+1} -cell, see Figure 3-1, we find

$$v_{v+1/2}^- = v_{v+1} - 2\lambda(f_{v+1} - h_{v+1/2}^G)$$

while integration over the right half of v_v -cell yields, as before,

$$v_{v+1/2}^+ = v_v + 2\lambda(f_v - h_{v+1/2}^G) .$$

Subtracting the second from the first, we have

$$v_{v+1/2}^- - v_{v+1/2}^+ = v_{v+1} - v_v - 2\lambda(f_v + f_{v+1} - 2h_{v+1/2}^G) \equiv (1 - 2Q_{v+1/2}^G) \cdot (v_{v+1} - v_v) .$$

Thus, $1 - 2Q_{v+1/2}^G$ gives us the compression ratio
 $(v_{v+1/2}^- - v_{v+1/2}^+) / (v_{v+1} - v_v)$.

4. Lax-Friedrichs Scheme and its Entropy Satisfying Modification

Lax-Friedrichs scheme [2], [6], given by

(4.1)

$$v_v(t+k) = H^{LF}(v_{v-1}, v_v, v_{v+1}; \lambda) \equiv \frac{v_{v+1}(t) + v_{v-1}(t)}{2} - \frac{\lambda}{2} (f(v_{v+1}) - f(v_{v-1})),$$

has the most allowable numerical viscosity under the total variation non-increasing requirement (2.8), $Q_{v+1/2}^{LF} \equiv 1$. A. Harten has observed [private communication] that the scheme coincides with that of Godunov, when the latter is applied over a staggered grid, see Figure 4-1,

$$v_v(t+k) = \frac{1}{2\Delta x} \int_{-\Delta x}^{\Delta x} u^R(x/k; v_{v-1}, v_{v+1}) dx \equiv H^{LF}(v_{v-1}, v_v, v_{v+1}; \lambda),$$

provided the CFL condition

$$(4.2) \quad \lambda \cdot \max_u |a(u)| < 1$$

is met.

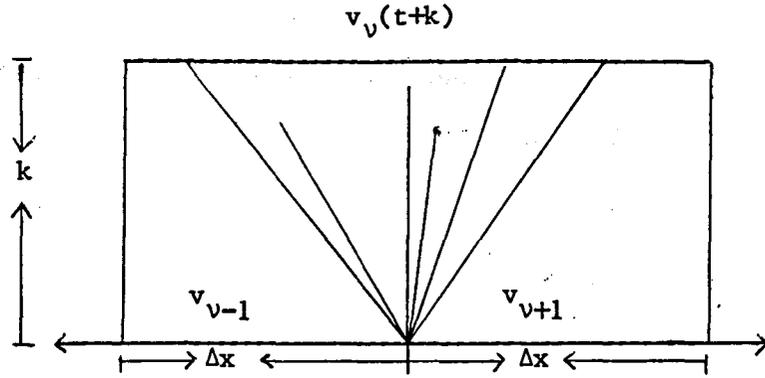


Figure 4-1

Integrating the differential entropy inequality (3.5b) over the same domain, we end up with its discrete version

$$U(v_v(t+k)) < \frac{U(v_{v+1}(t)) + U(v_{v-1}(t))}{2} - \frac{\lambda}{2} (F(v_{v+1}) - F(v_{v-1}));$$

after little rearrangement, it can be put into the more standard form, compare (3.8),

$$(4.3a) \quad U(v_v(t+k)) < U(v_v) - \lambda (F_{v+1/2}^{LF} - F_{v-1/2}^{LF}),$$

where

$$(4.3b) \quad F_{v+1/2}^{LF} \equiv F^{LF}(v_v, v_{v+1}) = \frac{F(v_{v+1}) + F(v_v)}{2} - \frac{1}{2\lambda} (U(v_{v+1}) - U(v_v))$$

is LF numerical entropy flux, consistent with the differential one

$$F^{LF}(w, w) = F(w).$$

We note that LF scheme does not admit a simple averaging of the type introduced above for Godunov's scheme. Instead, one might consider the following modification

(4.4)

$$v_v(t+k) = H^M(v_{v-1}, v_v, v_{v+1}; \lambda) \equiv \frac{v_{v+1} + 2v_v + v_{v-1}}{4} - \frac{\lambda}{2} (f(v_{v+1}) - f(v_{v-1})).$$

The so-modified scheme has half the numerical viscosity of LF-scheme, $Q_{v+1/2}^M \equiv 1/2$, and can be rewritten in the desired averaged form

$$(4.5a) \quad v_v(t+k) = H^M(v_{v-1}, v_v, v_{v+1}) \equiv \frac{v_{v-1/2}^- + v_{v+1/2}^+}{2}$$

where

$$(4.5b) \quad v_{v+1/2}^- \equiv v^-(v_{v-1}, v_v) = \frac{v_v + v_{v-1}}{2} - \lambda(f_v - f_{v-1})$$

$$(4.5c) \quad v_{v+1/2}^+ \equiv v^+(v_v, v_{v+1}) = \frac{v_{v+1} + v_v}{2} - \lambda(f_{v+1} - f_v).$$

The new scheme introduced, (4.4), can also be viewed as a two-cell averaging of two non-interacting Riemann problems, see Figure 3-1,

$$v_v(t+k) = H^M(v_{v-1}, v_v, v_{v+1}) \equiv \frac{1}{2\Delta x} \left[\int_{-\Delta x/2}^{\Delta x/2} u^R(x/k; v_{v-1}, v_v) dx \right. \\ \left. + \int_{-\Delta x/2}^{\Delta x/2} u^R(x/k; v_v, v_{v+1}) dx \right]$$

provided the CFL limitation

$$(4.6a) \quad \lambda \cdot \max_u |a(u)| < \frac{1}{2}$$

is met. Integrating the entropy inequality (3.5b) over the same two-cell domain, the scheme is found to satisfy that entropy inequality is its standard discrete version

$$(4.6b) \quad U(v_v(t+k)) < U(v_v(t)) - \lambda(F_{v+1/2}^M - F_{v-1/2}^M)$$

with a numerical entropy flux

$$(4.6c) \quad F_{v+1/2}^M \equiv F_v^M(v_v, v_{v+1}) = \frac{F(v_{v+1}) + F(v_v)}{2} - \frac{1}{4\lambda} (U(v_{v+1}) - U(v_v))$$

consistent with the differential one, $F^M(w, w) = F(w)$.

In analogy with Lemma 3, we are now ready to state

Lemma 4.1

Assume the CFL condition

$$(4.7) \quad \lambda \cdot \max_u |a(u)| < \frac{1}{2}$$

holds. For $v_{v\pm 1/2}^\pm$, given by

$$(4.8a) \quad v_{v-1/2}^- = v_v - \lambda(f_v - f_{v-1}) - Q_{v-1/2}^M \Delta v_{v-1/2}$$

$$(4.8b) \quad v_{v+1/2}^+ = v_v - \lambda(f_{v+1} - f_v) + Q_{v+1/2}^M \Delta v_{v+1/2},$$

we have the following entropy inequalities

$$(4.9a) \quad U(v_{v-1/2}^-) < U(v_v) - 2\lambda(F(v_v) - F_{v-1/2}^M)$$

$$(4.9b) \quad U(v_{v+1/2}^+) < U(v_v) + 2\lambda(F(v_v) - F_{v+1/2}^M).$$

Proof: Recalling that $Q_{v+1/2}^M \equiv \frac{1}{2}$, the RHS of (4.8a) and (4.8b) coincide with $H^{LF}(v_{v-1}, v_v, v_v; 2\lambda)$ and $H^{LF}(v_v, v_{v+1}, v_{v+1}; 2\lambda)$ respectively; applying for the latter the LF entropy inequality as quoted in (2.3a-b), one reads the conclusion (4.9a-b).

REMARK: In [7], P. D. Lax gave a direct proof of the entropy inequality (4.3), for LF scheme approximating arbitrary system of conservation laws. (In comparison, the arguments used in the above scalar analysis, requires the existence of an entropy satisfying Riemann solution in the large.) Since the modified scheme is nothing else but an average of two LF-solvers, Lax's result goes over in this case; that is, for arbitrary system of conservation laws, both LF and the modified scheme, satisfy the entropy inequality for all entropy pairs associated with the differential system⁽²⁾. As much as we are aware, these are the only two known examples satisfying the entropy condition in such generality.

5. The Entropy Condition and E Schemes

In this section we study difference schemes containing no less numerical viscosity than that of Godunov, $Q > Q^G$. Such E schemes — after Osher [12]

(2) For the exact CFL limitation in this case, see [7].

— are shown to converge to the unique physically relevant solution of (2.2), provided the CFL limitation

$$(5.1) \quad \lambda \left| (f(v_v) - h_{v+1/2}) + (f(v_{v+1}) - h_{v+1/2}) \right| < \frac{1}{2} |v_{v+1} - v_v|$$

is met. Ideally, one would like to allow the relaxed CFL limitation (2.10) to be used; the reason for introducing the stricter (5.1) — half the usual CFL number — stems from the fact that we were unable to rewrite LF scheme in the desirable averaged form as discussed in Section 4. We note that (5.1) takes the equivalent form

$$(5.1') \quad |Q_{v+1/2}| < \frac{1}{2},$$

which, in the case of Godunov's scheme, amounts to preventing waves interaction. As before, such a CFL limitation yields, in particular, the consistency relation (2.11), characterizing essentially 3-point schemes.

Theorem 5.1

An E scheme converges to the physically relevant solution of (2.2), under the CFL restriction (5.1).

Proof: Convergence was established in Lemma 2.1, since an E scheme is necessarily total-variation non-increasing

$$\lambda \left| \frac{\Delta f_{v+1/2}}{\Delta v_{v+1/2}} \right| < Q_{v+1/2}^G < Q_{v+1/2} < \frac{1}{2} < 1.$$

We turn to examine the entropy inequality. We attach the superscript G, M,

and E to distinguish between Godunov's scheme (3.1), the modified scheme (4.4) and the E scheme under consideration (2.7)

$$v_v(t+k) = v_v(t) - \frac{\lambda}{2} (f(v_{v+1}(t)) - f(v_{v-1}(t))) + \frac{1}{2} (\Delta(Q_{v-1/2} \Delta v_{v-1/2}(t))).$$

We rewrite the latter in the averaged form

$$(5.2) \quad v_v(t+k) = \frac{v_{v-1/2}^{E-} + v_{v+1/2}^{E+}}{2}$$

where

$$(5.3a) \quad v_{v-1/2}^{E-} = v_v - \lambda(f_v - f_{v-1}) - Q_{v-1/2} \Delta v_{v-1/2}$$

$$(5.3b) \quad v_{v+1/2}^{E+} = v_v - \lambda(f_{v+1} - f_v) + Q_{v+1/2} \Delta v_{v+1/2}.$$

Recall the corresponding averaging forms for Godunov's scheme, see (3.10),

$$(5.4a) \quad v_{v-1/2}^{G-} = v_v - \lambda(f_v - f_{v-1}) - Q_{v-1/2}^G \Delta v_{v-1/2}$$

$$(5.4b) \quad v_{v+1/2}^{G+} = v_v - \lambda(f_{v+1} - f_v) + Q_{v+1/2}^G \Delta v_{v+1/2}$$

and that for the modified scheme, see (4.8),

$$(5.5a) \quad v_{v-1/2}^{M-} = v_v - \lambda(f_v - f_{v-1}) - \frac{1}{2} \Delta v_{v-1/2}$$

$$(5.5b) \quad v_{v+1/2}^{M+} = v_v - \lambda(f_{v+1} - f_v) + \frac{1}{2} \Delta v_{v+1/2}.$$

According to our assumption

$$(5.6) \quad Q_{v_{\pm}1/2} = \theta_{v_{\pm}1/2} Q_{v_{\pm}1/2}^G + (1 - \theta_{v_{\pm}1/2}) \frac{1}{2}, \quad 0 < \theta_{v_{\pm}1/2} < 1.$$

Multiply (5.4a) by $\theta_{v_-1/2}$, (5.5a) by $(1 - \theta_{v_-1/2})$ and add to find that (5.3a) amounts to

$$v_{v_-1/2}^{E-} = \theta_{v_-1/2} v_{v_-1/2}^{G-} + (1 - \theta_{v_-1/2}) v_{v_-1/2}^{M-};$$

similarly, multiplying (5.4b) by $\theta_{v_+1/2}$, (5.5b) by $(1 - \theta_{v_+1/2})$ and adding, we end up with (5.3b) having the form

$$v_{v_+1/2}^{E+} = \theta_{v_+1/2} v_{v_+1/2}^{G+} + (1 - \theta_{v_+1/2}) v_{v_+1/2}^{M+}.$$

Averaging the last two equalities, (5.2) reads

$$(5.7) \quad v_v(t+k) = \frac{\theta_{v_-1/2}}{2} v_{v_-1/2}^{G-} + \frac{(1 - \theta_{v_-1/2})}{2} v_{v_-1/2}^{M-} \\ + \frac{\theta_{v_+1/2}}{2} v_{v_+1/2}^{G+} + \frac{(1 - \theta_{v_+1/2})}{2} v_{v_+1/2}^{M+}.$$

Thus, we see that every E scheme can be written as a convex combination of one-sided averaged Riemann solutions.

Let $(U(w), F(w))$ be an entropy pair associated with (2.2). By the convexity of U , (5.7) implies

$$(5.8) \quad U(v_v(t+k)) \leq \frac{\theta_{v_-1/2}}{2} U(v_{v_-1/2}^{G-}) + \frac{(1 - \theta_{v_-1/2})}{2} U(v_{v_-1/2}^{M-}) \\ + \frac{\theta_{v_+1/2}}{2} U(v_{v_+1/2}^{G+}) + \frac{(1 - \theta_{v_+1/2})}{2} U(v_{v_+1/2}^{M+}).$$

Next, we invoke the entropy inequalities concluded in Lemmas 3.1 and 4.1

$$U(v_{v-1/2}^{G-}) < U(v_v) - 2\lambda(F(v_v) - F_{v-1/2}^G)$$

$$U(v_{v+1/2}^{G+}) < U(v_v) + 2\lambda(F(v_v) - F_{v+1/2}^G)$$

$$U(v_{v-1/2}^{M-}) < U(v_v) - 2\lambda(F(v_v) - F_{v-1/2}^M)$$

$$U(v_{v+1/2}^{M+}) < U(v_v) + 2\lambda(F(v_v) - F_{v+1/2}^M).$$

When inserted into (5.8), we end up with the desired entropy inequality

$$(5.9a) \quad U(v_v(t+k)) < U(v_v) - \lambda(F_{v+1/2}^E - F_{v-1/2}^E)$$

with a numerical entropy flux

$$(5.9b) \quad F_{v+1/2}^E = \theta_{v+1/2} F_{v+1/2}^G + (1 - \theta_{v+1/2}) F_{v+1/2}^M$$

consistent with the differential one, $F^E(\dots, w, w, \dots) = F(w)$.

REMARKS:

(i) An explicit formula for Godunov's numerical flux, $h^G(v_v, v_{v+1}) = \text{Min}[\text{sgn}(v_{v+1} - v_v) f(v)]$, $v_{v+1/2}^{\min} < v < v_{v+1/2}^{\max}$ was given in [12]; here $v_{v+1/2}^{\min/\max} = \text{Min/Max}(v_v, v_{v+1})$. Hence, an equivalent characterization for E schemes, requiring

$$\text{sgn}(v_{v+1} - v_v) [h_{v+1/2} - f(v)] < 0, \quad v_{v+1/2}^{\min} < v < v_{v+1/2}^{\max},$$

shows that a 3-point monotone scheme is an E scheme. Unfortunately, E schemes like monotone ones, are at most first-order accurate [12].

(ii) We have seen that E schemes satisfy the entropy inequality (5.9) for all entropy functions, $U(\cdot)$; their corresponding numerical entropy fluxes are given as convex combinations of two numerical fluxes associated with monotone schemes -- Godunov and the modified LF scheme (4.4). Hence, an L_1 -convergence rate estimate of order $(\Delta x)^{1/2}$ follows along the lines of [9, Theorem IV]

$$\|v(\cdot, t) - u(\cdot, t)\|_{L^1} < \|v(\cdot, t=0) - u(\cdot, t=0)\|_{L^1} + K \cdot (t\Delta x)^{1/2}.$$

Considerations of the constant coefficients case shows this L_1 -estimate to be sharp, e.g., [11, Sections 9 and 10].

(iii) As an immediate corollary from Theorem 5.1 we obtain verification of the following "folklore" result.

Corollary 5.2

A conservative difference scheme with a non-vanishing numerical viscosity, $0 < Q_{\min} < Q_{v+1/2}(\lambda) < \frac{1}{2}$, is converging to the unique entropy solution for sufficiently small mesh ratio, λ .

Such non-vanishing viscosity schemes were specifically "tailored", for example, in [5, Section 5]. Here we note, that the CFL-like restriction on the mesh ratio, λ , depends heavily on the behavior of the flux, f , near the sonic points.

6. Numerical Viscosity and the Entropy Condition

In this section we would like to systemize the kind of arguments introduced above, emphasizing those essential ingredients which prevail in the more general context.

We consider a general conservative scheme which we rewrite in the averaged form, compare (2.7),

$$(6.1) \quad v_v(t+k) = \frac{[v_v(t) - \lambda(f_v - f_{v-1}) - Q_{v-1/2} \Delta v_{v-1/2}] + [v_v(t) - \lambda(f_{v+1} - f_v) + Q_{v+1/2} \Delta v_{v+1/2}]}{2};$$

the entropy condition follows by constructing a consistent discrete entropy inequality for each of the averaged terms on the RHS (6.1), thus opening the door for showing the former by means of comparison. For that purpose, pick a 3-point entropy condition satisfying scheme

$$(6.2) \quad \tilde{v}_v(t+k) = H(\tilde{v}_{v-1}, \tilde{v}_v, \tilde{v}_{v+1}; f, \lambda) \equiv v_v - \lambda(h_{\tilde{v}_{v+1/2}} - h_{\tilde{v}_{v-1/2}}),$$

such that the following holds:

Assumption: The numerical flux $h_{\tilde{v}_{v+1/2}}$ is independent of the mesh ratio λ .

The plausibility of the above assumption stems from the fact that the Riemann problem admits similarity solution $u^R(x, t; u_{\text{left}}, u_{\text{right}}) \equiv u^R(x/t; u_{\text{left}}, u_{\text{right}})$, and hence all difference approximations based on Riemann solvers must satisfy such a requirement; this is not the case, for example, with LF scheme (4.1), where we were forced to consider instead its modification (4.4).

The reason for introducing the last assumption is becoming clear upon writing

(6.3)

$$\begin{aligned} & \mathbb{H}(v_{v-1}, v_v, v_{v+1}; \lambda) \\ &= \frac{[v_v - \lambda(f_v - f_{v-1}) - Q_{v-1/2} \Delta v_{v-1/2}] + [v_v - \lambda(f_{v+1} - f_v) + Q_{v+1/2} \Delta v_{v+1/2}]}{2} \end{aligned}$$

where, see (2.5),

$$(6.4) \quad Q_{v+1/2} \equiv \lambda \frac{f_v + f_{v+1} - 2h_{v+1/2}}{\Delta v_{v+1/2}}$$

depends linearly on λ ; hence, the two averaged terms on the RHS of (6.3) -- abbreviated as before by $\bar{x}_{v-1/2}$ and $x_{v+1/2}^+$ -- can be equivalently expressed as

$$\bar{x}_{v-1/2} \equiv \mathbb{H}(v_{v-1}, v_v, v_v; 2\lambda)$$

$$x_{v+1/2}^+ \equiv \mathbb{H}(v_v, v_v, v_{v+1}; 2\lambda),$$

each of which satisfies the entropy inequality, provided the CFL limitation is being halved. Termwise comparison of the averaged forms, (6.1) and (6.3), shows their difference only in the numerical viscosity coefficients; assuming $Q_{v+1/2}$ to vary between two coefficients of numerical viscosity associated with entropy satisfying schemes, we are able to represent (6.1) as a convex combination of the latter. The discrete entropy inequality follows for the corresponding convex combination of entropy fluxes.

We have shown

Theorem 6.1

Consider the difference scheme (6.1) and assume that the CFL condition

$$|Q_{v+1/2}| \equiv \left| \lambda \frac{f_v + f_{v+1} - 2h_{v+1/2}}{\Delta v_{v+1/2}} \right| < \frac{1}{2}$$

holds⁽³⁾. Then, the scheme satisfies the entropy condition, provided we can find another entropy satisfying difference approximation with less numerical dissipation, $\tilde{Q}_{v+1/2}$,

$$Q_{v+1/2} < \tilde{Q}_{v+1/2}.$$

The corresponding numerical entropy flux is given by

$$F_{v+1/2} = \left[\frac{1/2 - Q_{v+1/2}}{1/2 - \tilde{Q}_{v+1/2}} \right] \cdot \tilde{F}_{v+1/2} + \left[\frac{Q_{v+1/2} - \tilde{Q}_{v+1/2}}{1/2 - \tilde{Q}_{v+1/2}} \right] \cdot F_{v+1/2}^M.$$

⁽³⁾One may assume, instead, $|Q_{v+1/2}| \leq |\tilde{Q}_{v+1/2}|$, $\tilde{Q}_{v+1/2}$ denoting the numerical viscosity coefficient of a difference scheme admitting the desired entropy satisfying averaged form.

References

- [1] M. CRANDALL and A. MAJDA, "Monotone difference approximations for scalar conservation laws", Math. Comp., v. 34, 1980, pp. 1-21.

- [2] K. O. FRIEDRICHS, "Symmetric hyperbolic linear differential equations", Comm. Pure Appl. Math., v. 7, 1954, pp. 345-392.

- [3] J. GLIMM, "Solution in the large for nonlinear hyperbolic systems of equations", Comm. Pure Appl. Math., v. 18, 1965, pp. 697-715.

- [4] S. K. GODUNOV, "A finite difference method for the numerical computation of discontinuous solutions of the equations of flow dynamics", Mat. Sb., v. 47, 1959, pp. 271-290.

- [5] A. HARTEN, "High resolution schemes for hyperbolic conservation laws", J. Comp. Phys., v. 49, 1983, pp. 357-393.

- [6] P. D. LAX, "Weak solutions of nonlinear hyperbolic equations and their numerical computation", Comm. Pure Appl. Math., v. 7, 1954, pp. 159-193.

- [7] P. D. LAX, "Shock waves and entropy", in Contributions to Nonlinear Functional Analysis, ed. E. A. Zarantonello, pp. 603-634, Academic Press, New York, NY, 1971.

- [8] A. MAJDA and S. OSHER, "Numerical viscosity and the entropy condition", Comm. Pure Appl. Math., v. 32, 1979, pp. 797-838.

- [9] R. SANDERS, "On convergence of monotone finite difference schemes with variable spatial efficiency", Math. Comp., v. 40, 1983, pp. 91-106.

- [10] E. TADMOR, "The large-time behavior of the scalar, genuinely nonlinear Lax-Friedrichs scheme", NASA CR-172138, 1983.

- [11] V. THOMÉE, "Stability theory for partial difference operators", SIAM Review, v. 11, 1969, pp. 152-195.

- [12] S. OSHER, "Riemann solvers, the entropy condition and difference approximations", preprint, Department of Mathematics, University of California-Los Angeles, Los Angeles, California, 1983.

1. Report No. NASA CR-172141		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle NUMERICAL VISCOSITY AND THE ENTROPY CONDITION FOR CONSERVATIVE DIFFERENCE SCHEMES				5. Report Date June 1983	
				6. Performing Organization Code	
7. Author(s) Eitan Tadmor				8. Performing Organization Report No. 83-20	
				10. Work Unit No.	
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665				11. Contract or Grant No. NAS1-17070	
				13. Type of Report and Period Covered Contractor Report	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546				14. Sponsoring Agency Code	
15. Supplementary Notes Langley Technical Monitor: Robert H. Tolson Final Report					
16. Abstract Consider a scalar, nonlinear conservative difference scheme satisfying the entropy condition. It is shown that difference schemes containing <u>more</u> numerical viscosity will necessarily converge to the unique, physically relevant weak solution of the approximated conservative equation. In particular, entropy satisfying convergence follows for E schemes -- those containing more numerical viscosity than Godunov's scheme.					
17. Key Words (Suggested by Author(s)) numerical viscosity entropy condition convergence			18. Distribution Statement Unclassified-Unlimited Subject Category 64		
19. Security Classif. (of this report) Unclassified		20. Security Classif. (of this page) Unclassified		21. No. of Pages 28	22. Price A03

End of Document