CAD OF CONTROL SYSTEMS: APPLICATION OF NONLINEAR PROGRAMMING TO A LINEAR–QUADRATIC FORMULATION

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CAD OF CONTROL SYSTEMS: APPLICATION OF NONLINEAR PROGRAMMING TO A LINEAR-
QUADRATIC FORMULATION

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ABSTRACT. The familiar suboptimal regulator design approach is recast as a
constrained optimization problem and incorporated in a CAD package where
both design objective and constraints are quadratic cost functions. This
formulation permits the separate consideration of, for example, model-
following errors, sensitivity measures and control energy as objectives to
be minimized or limits to be observed. Efficient techniques for computing
the interrelated cost functions and their gradients are utilized in
conjunction with a nonlinear programming algorithm. The effectiveness of
the approach and the degree of insight into the problem which it affords is
illustrated in a helicopter regulation design example.

Keywords. Computer-aided design; control engineering computer
applications; multivariable control systems; nonlinear programming; optimal
control; optimization.

INTRODUCTION
Since the interest in control system optimization generated by the linear
quadratic regulator (LQR) approach there have been a number of quite separate
developments. Some workers followed the
pattern set by Levine and Athans (1970) who
imposed constraints on controller structure but retained the scalar quadratic cost
function adopted in LQR design. Various
features can be incorporated within this
context such as sensitivity reduction and
model-following and an efficient solution
technique typically employs an unconstrained
gradient minimization algorithm (see Fleming
1979). Computationally this is an
attractive approach but from the design
point of view it is dogged by the
difficulty of using a scalar quadratic
measure to describe all the desired facets
of system performance.

Another line of attack was prompted by
Zakian's Method of Inequalities (Zakian,
1973) in which system constraints and
specifications are represented by a set of
simultaneous algebraic inequalities.
Control engineers found this approach
attractive because the problem description
could include such basic control design
parameters as overshoot, damping, settling
time, etc. Polak and Payne advanced this
idea by positing semi-infinite programming
problems in which design objectives are
realized by minimizing a function subject to
a set of inequalities where these objectives
can be expressed as infinite dimensional
constraints (Payne and co-workers, 1982).

Such problems require special mathematical
programming algorithms to handle the
infinite dimensional constraints and,
sometimes, nondifferentiable functions.

The method described here strikes a
compromise between those two approaches in
that the quadratic measures of the
suboptimal linear regulator approach are
retained but the problem is recast as a
constrained minimization problem in which
the constraints may represent bounds on
control energy, model-following errors,
sensitivity, etc. The minimizing controller
gain matrix is obtained efficiently using
readily available software tools and a CAD
package has been mounted on a 32-bit
minicomputer. What appears at first sight
to be a relatively straightforward extension
of suboptimal regulator design yields a
numerical solution technique which, from
the designer's viewpoint, is much more efficient
than the original design approach and
provides results which reveal useful
insights into the design problem. The
method is exercised within the CAD context
by means of a design example on helicopter
regulation in which the identification of
active constraints at the minimum plays a
leading role in understanding the nature of
the problem.

PROBLEM STATEMENT
For clarity a simplified statement of the
problem follows:
Given the linear time-invariant plant
description
\[ \dot{x}_p = A x_p + B u_p \quad x_p(0) = x_{p0} \]  
\[ y_p = C_x x_p \]

and feedback control law

\[ y_p = K x_p \]

find the control, \( u_p(t) \), which minimizes the quadratic cost function

\[ J_0 = \int \left( x_p^T Q x_p + u_p^T R u_p \right) dt = \int x_p^T Q x_p dt, \]

where \( Q_p = Q + C_x^T R C_x \), subject to the inequality constraints

\[ J_1 \leq k_1, \quad i = 1, 2, \ldots, q, \]

where \( J_1 \) is a quadratic cost function of the form

\[ J_1 = \int \left( x_d^T Q x_d + u_d^T R u_d \right) dt \]

and \( Q_d = Q + C_x^T R C_x \).

The inequality bounds, \( z_i \), are designer-specified and the weighting matrices \( Q_i, R_i \), \( i = 0, 1, \ldots, q \), which are usually diagonal, are selected by the designer.

For the full problem formulation, as incorporated in the CAD package, the plant state vector, \( x_p \), may be augmented by a state compensator vector, \( x_s \), a trajectory sensitivity vector, \( x_m \), and a model state vector, \( x_m \), to accommodate a dynamic compensator option for the controller and the inclusion of sensitivity reduction and model-following terms in the cost functions (see Fleming, 1979).

One application, for example, might seek a controller gain matrix, \( K \), to achieve a certain degree of model-following within some control energy constraints, i.e.,

\[ \text{minimize} \quad J = \int (y_r - y_m)^T Q_0 (y_r - y_m) dt \]

subject to

\[ \int u_p^2 dt < k_1, \quad i = 1, 2, \ldots, m, \]

where \( y_m \) represents the desired model output response. This example is typical of one which would be solved using the straightforward suboptimal linear regulator approach by successive minimization of

\[ J = \int \left( (y_r - y_m)^2 Q_0 (y_r - y_m)+ R u_p^2 \right) dt, \]

where the designer strives to find the appropriate choice of \( Q_0 \) and \( R \) to satisfy the control energy constraints. Thus what was previously solved in an approximate way in a number of unconstrained minimizations is now solved in a single constrained minimization.

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**FORMULATION OF THE NONLINEAR PROGRAMMING PROBLEM**

If we let \( k \) represent the minimizing variables, i.e., the variable elements of gain matrix, \( K \), then we have the nonlinear programming problem:

\[
\text{minimize} \quad J_0,
\]

subject to the inequality constraints

\[ J_1 \leq z_i, \quad i = 1, 2, \ldots, q. \]

The cost functions, \( J_j, i = 0, 1, \ldots, q \), Eqs. (4) and (5), can be computed from

\[ J_1 = \text{tr}(P_1 k), \quad i = 0, 1, \ldots, q, \]

where \( X_0 = X_0 x_0 \) and \( P_1 \) satisfies the Lyapunov matrix equation

\[ P_1 A + A^T P_1 = -Q_1, \quad i = 0, 1, \ldots, q, \]

where \( A = A_1 + R_1 K C_1 \) (see Eqs. (1) - (3)).

Gradient information may be obtained from the gradient matrices

\[ \frac{\partial J_1}{\partial K} = 2 \left( B^T P_1 R_1 K + A^T C_1 K \right) A_1 C_1^T, \quad i = 0, 1, \ldots, q \]

where \( A \) satisfies the Lyapunov matrix equation

\[ A C_1 + A_1 C_1 = -Q_1. \]

For the case where initial conditions are not known matrix \( X_0 \) may be modified so that either the expected values, the average values or the worst-case values of \( X_0 \) in \( J_j, i = 0, 1, \ldots, q \) are measured (see Fleming, 1979).

There are a variety of constrained optimization algorithms employing gradient information which will solve this problem. In the implementation (Vandervorst, 1973) of Zoutendijk's Method of Feasible Directions (Zoutendijk, 1960) which was used, the algorithm initially finds a feasible point and the proceeds by iteratively searching along feasible directions, employing the conjuate direction method of Fletcher and Reeves (1964) if no constraints are violated. At each iteration gradient information is required only for the active or violated constraints.

**IMPLEMENTATION OF THE METHOD**

To compute the objective function, Eq. (4) and the constraint functions, Eq. (5), together with their associated gradients, Eq. (8), matrices \( P_i, i = 0, 1, \ldots, q \) and \( A \) must be obtained from the set of Lyapunov
matrix equations (7) and (9) at each iteration of the minimization algorithm; also the set of equations (7) must be solved several times within each iteration to compute the objective and constraint functions at each step of the linear search. The method of Barlete and Stewart (1972), generally accepted as the most efficient Lyapunov matrix equation solver, is a transformation method which reduces \( \bar{A} \), Eq. (7), to its real Schur form and then obtains the solution by solving a set of low order (4) linear equations. It is well suited for this application where only one Schur reduction of \( \bar{A} \) per estimate of \( K \) is necessary for the solution of the (4x4) equations (7) and (9). The solution of these equations is further speeded up by exploiting certain structural and sparseness properties of \( \bar{A} \) and \( \bar{P} \).

The adoption of the Barlete-Stewart method has the added advantage that system eigenvalues may be determined for each new estimate of \( K \) from the diagonal and principal subdiagonal elements of the real Schur form of \( \bar{A} \) at little extra computational cost. Monitoring of eigenvalues is necessary since Eq. (7) is only valid for stable values of \( \bar{A} \). Although the problem is such that, in general, the minimization routine will tend to generate values of \( K \) with unstable excursions of \( \bar{A} \), clearly the minimization routine also requires that the initial estimate of \( K \) stabilizes \( \bar{A} \), and, when the open-loop plant is unstable, it should be necessary for the program to search for a stabilizing value of \( K \) using a steepest descent technique similar to that of Koenigsberg and Frederick (1970).

The CAD program operates in conversational mode permitting easy modification of design parameters \( Q_1, Q_2, \) and \( z_1 \) as well as specification of the gain parameters in matrix \( K \) to be optimized. Solution times are dependent on the number of constraints but typically the numerical minimization takes 2 - 4 times longer to compute than the corresponding suboptimal regulator approach. However this method is much more efficient in achieving the design goal thereby reducing design time as well as affording the designer increased understanding of the problem.

**DESIGN EXAMPLE - HELICOPTER REGULATION**

The plant describing helicopter longitudinal dynamics (Michael and Farrar, 1973) has four states and two controllers:

\[
\begin{bmatrix}
\begin{bmatrix}
    z_{x_1} & z_{x_2} & z_{x_3} & z_{x_4}
\end{bmatrix}
\end{bmatrix}^T = \begin{bmatrix}
    [u_{x_1} u_{x_2} 0 0]
\end{bmatrix}^T, \quad u_p = \begin{bmatrix}
    [u_{x_1} u_{x_2}]
\end{bmatrix}^T.
\]

Prior to this design a standard LQR design had been carried out and a satisfactory result obtained which gave good regulation for \( u_x \) and \( u_z \) (forward and vertical velocities) observing control magnitude constraints on \( u_1 \) and \( u_2 \) (longitudinal cyclic pitch and main rotor collective pitch). These controllers utilize feedback from all four plant states, i.e. \( \mu = K_{opt} x \), and the resulting responses are characterized by

\[
\begin{bmatrix}
    x_p
\end{bmatrix} = (\bar{A} - K_{opt} \bar{P})^{-1} x_0 \bar{A} x_p,
\]

where \( \bar{A}_{opt} \) is defined to be the "model" matrix. Since feedback from \( u_x \) and \( u_z \) requires strapdown sensors which are undesirable, the objective here is to examine whether similar regulation responses can be achieved without sensing \( u_x \) and \( u_z \) and without violating control magnitude constraints.

We therefore wish to minimize an objective function which measures the error between the new plant response and the model response:

\[
J_0 = \int_0^\infty (x_p - x_m)^T Q_0 (x_p - x_m) dt,
\]

where \( Q_0 = \text{diag}(1,1,0,0) \), subject to the constraints

\[
J_1 < x_{z_i}^2, \quad i = 1, 2,
\]

where

\[
J_1 = \int_0^\infty u_1^2 dt, \quad x_1 = 4.06, \quad 0,
\]

\[
J_2 = \int_0^\infty u_2^2 dt, \quad x_2 = 6.57, \quad 0.2
\]

and \( x_{z_0} = (1 1 1 1)^T \).

The values of \( x_1 \) and \( x_2 \) correspond to the values of \( J_1 \) and \( J_2 \) obtained for \( \mu = K_{opt} x \) with the above initial conditions. The constraints therefore place limits on control energy and indirectly limit control magnitudes.

Various controllers structures were considered:

**Controller 1**

\[
\begin{bmatrix}
    u_{x_1} \\
    u_{x_2}
\end{bmatrix} = \begin{bmatrix}
    k_{11} & k_{12} \\
    k_{21} & k_{22}
\end{bmatrix} \begin{bmatrix}
    x_{c_1} \\
    x_{c_2}
\end{bmatrix}
\]

**Controller 2**

\[
\begin{bmatrix}
    u_{x_1} \\
    u_{x_2}
\end{bmatrix} = \begin{bmatrix}
    k_{11} & k_{12} \\
    k_{21} & x_{c_2}
\end{bmatrix} \begin{bmatrix}
    x_{c_1} \\
    1 & k_{32}
\end{bmatrix}
\]

**Controller 3**

\[
\begin{bmatrix}
    u_{x_1} \\
    u_{x_2}
\end{bmatrix} = \begin{bmatrix}
    k_{11} & k_{12} \\
    k_{21} & k_{22}
\end{bmatrix} \begin{bmatrix}
    x_{c_1} \\
    x_{c_2}
\end{bmatrix}
\]
Design results are tabulated above. Note that for dynamic compensator, Controller 3, in order to obtain a minimal parameter form element $k_{31}$ is nonvarying and set to unity.

Excellent model-following is achieved with Controller 1, pitch angle and pitch rate feedback, although there is some deterioration when Controller 2, pitch angle feedback alone, is used. This situation is remedied by using a first-order dynamic compensator together with pitch angle feedback, Controller 3. Graphical results, Figs. 1 and 2, illustrate the model-following capabilities of these three controllers. Controllers 4 and 5 explore the possibility of using $u_1$ or $u_2$ alone. Poor model-following is realized by Controller 4 and it was found to be impossible to stabilize the system using Controller 5 within the constraint bounds. The algorithm failed to find a feasible solution for this case in ten iterations; the results at this point are presented.

The last two columns of Table 1 and Fig. 3 indicate that the control energy constraints are effective in keeping the control magnitudes within bounds. The LQR controller has an initial value of 2.04 (bracketed in the table) which swiftly falls from this level. It is assumed, therefore, that the level $u_2(t) = -1.66$ is the more important magnitude to be observed since control effort is around this level for a significant period. Figure 3 illustrates how well this bound is observed. If, however, $u_2$ magnitudes up to 2.04 can be accommodated then constraint 2 should be relaxed by increasing $z_2$ by an appropriate factor.

From the results we learn that the magnitude of $u_2$ is the limiting component which prevents further improvement in model-following capability with Controller Structures 1 and 2. With the dynamic compensator structure, Controller 3, its best performance is achieved within the constraints. Controller Structures 4 and 5 have proved themselves incapable of producing good results.

The implications, then, are clear:

1) Both controls $u_1$ and $u_2$ are necessary,
2) It is desirable to feedback $\theta$ and $\dot{\theta}$ however, if $\dot{\theta}$ is unavailable, satisfactory results can be achieved with a first-order compensator,
3) A relaxation of the magnitude constraint on $u_2$ will lead to improved results.

It is in the nature of the problem formulation that designs for each controller structure were obtained in one optimization process enabling the designer to effectively experiment with different controller structures. Further, using the gain-fixing facility of the program, system integrity can be taken into account in the manner of Fleming (1981).

CONCLUDING REMARKS

A new optimization approach based on the linear-quadratic formulation has been proposed for control system design.
Exploiting the mathematical tractability of the formulation an efficient CAD program has evolved for solving linear regulator problems. The use of constraint functions in the problem description frees the designer to a large extent from the chore of weighting matrix selection and presents additional information concerning the nature of the design problem.

Additional features may be incorporated within the program. For example, the inclusion of side constraints on gain values will be particularly useful in dynamic compensator design and the handling of large parameter variations may be implemented in a similar way to that of Vinkler and co-workers (1979). Also, since plant eigenvalues and eigenvectors are easily obtained from the Schur form of $A$, arising from the approach these may also be incorporated in the design objective and constraints.

Although the package performs well current work is directed at further improving its effectiveness by identifying the best optimization algorithm in terms of speed of convergence and robustness for this particular nonlinear programming problem.

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**Fig. 1.** Forward velocity responses, $\mu_x$, illustrating model-following capabilities of a) Controller 1, b) Controller 2 and c) Controller 3.
Fig. 2 Vertical velocity responses, $\mu_z$, illustrating model-following capabilities of a) Controller 1, b) Controller 2 and c) Controller 3.

Fig. 3. Comparison of $u_2$ responses.
Application to a Linear-Quadratic Formulation

REFERENCES


The familiar suboptimal regulator design approach is recast as a constrained optimization problem and incorporated in a CAD package where both design objective and constraints are quadratic cost functions. This formulation permits the separate consideration of, for example, model-following errors, sensitivity measures and control energy as objectives to be minimized or limits to be observed. Efficient techniques for computing the interrelated cost functions and their gradients are utilized in conjunction with a nonlinear programming algorithm. The effectiveness of the approach and the degree of insight into the problem which it affords is illustrated in a helicopter regulation design example.
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