

SOME RECENT DEVELOPMENTS IN THE ENDOCHRONIC THEORY  
WITH APPLICATION TO CYCLIC HISTORIES

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ABSTRACT

Constitutive equations with only two easily determined material constants can predict with computational ease the stress (strain) response of normalized mild steel to a variety of general strain (stress) histories, without a need for special unloading-reloading rules that are otherwise so evident in the literature.

These equations are derived from the endochronic theory of plasticity of isotropic materials with an intrinsic time scale defined in the plastic strain space. Agreement between theoretical predictions and experiments are excellent quantitatively in cases of various uniaxial constant strain amplitude histories, variable uniaxial strain amplitude histories and cyclic relaxation. The cyclic ratcheting phenomenon is also predicted by the present theory, in routine fashion.

## INTRODUCTION

In recent years, cyclic plasticity, which deals with the rate-independent inelastic response of materials to cyclic stress or strain histories, has become an important subject of research in applied mechanics and engineering design. Past experimental work, theoretical studies and engineering analysis are well documented in the literature. For details see, typically, References [1-16].

On the basis of existing experimental results, one concludes that generally, when subjected to symmetric stress or strain cycles, annealed or soft materials will harden and will tend to a stable response, while cold-worked or hard materials will soften. When a stable response is reached, hysteresis loops in the stress-strain space become stable, closed and symmetric. Also stable loops at various strain (or stress) amplitudes are similar in shape. This has led to the definition of a cyclic stress-strain curve which is the locus of the tips of stable hysteresis loops. It is found that some metals, e.g. 7075-T6 aluminum, follow the Masing rule. However, some metals, e.g. normalized mild steel, do not follow this rule at all [8].

In the presence of a history of unsymmetric stress cycles, the material response involves a progressive increase of plastic (or total) strain in the

direction of mean stress. This is called cyclic "creep" or "ratcheting". On the other hand, a history of unsymmetric strain cycles, will result in "cyclic stress relaxation" toward zero mean stress. Both phenomena occur whether the material response is stable or not.

Under variable amplitude cycling, metals have a strong memory of their most recent point of reversal.

If the number of cycles is large enough, then effects of prior plastic history tend to disappear. More precisely a material has a "fading" memory, in terms of the intrinsic time scale  $\zeta$ , of the history of plastic deformation that preceded the cyclic history  $\bar{\epsilon}_n$ , as the latter progresses.

Attempts to describe the above phenomena analytically in terms of constitutive laws have been tried. However, so far, an elegant, simple but realistic constitutive law is still not at hand.

In this paper, we use a recent model of endochronic theory in the study of cyclic plasticity of stable materials. This model, proposed by Valanis  $\bar{[13]}$ , has been applied to metals by authors  $\bar{[14]}$ . In the case of normalized mild steel, it is shown that the constitutive equations derived from the theory can predict quantitatively stable hysteresis loops pertaining to various strain amplitudes. The broader capability of the theory is critically tested by demonstrated agreement with the observed cyclic response of normalized mild steel to variable uniaxial strain amplitude histories. In the final section we show that cyclic ratcheting and cyclic relaxation are phenomena which are readily predicted by the present theory.

## 1. A BRIEF REVIEW OF THE ENDOCHRONIC THEORY

In the late 1960's, the formulation of constitutive theories of visco-elastic materials from concepts of irreversible thermodynamics and internal state variables reached an advanced level of development. It was natural to inquire if a similar approach could be used to establish a theory of plasticity, and the attempt by Valanis to explore this question led to the development of the endochronic theory in 1971 [15].

In its early stages of development, the theory rested on the notion that the stress response of dissipative materials is a function of the deformation (strain) path. When the material behavior considered is rate-independent, the path in question must also be rate independent. The early version of the endochronic theory was constructed in terms of a path in the strain space  $\epsilon_{ij}$ . In this space, every point represents a deformation (strain) state. A sequence of strain states traces a path in this space (Figure 1). The distance along the path between the two strain states P and P' is denoted by  $d\zeta$ . If  $P$ , a fourth order positive definite tensor, is the metric of the space, then

$$d\zeta^2 = P_{ijkl} d\epsilon_{ij} d\epsilon_{kl} \quad (1.1)$$

The tensor  $P$  is a material property in the sense that in general it will vary from material to material. Since successive strain states on a strain path are distinct and  $d\zeta$  is always positive, the latter can serve as a time measure which is a property of the material at hand, since  $P$  is such. The length of the path  $\zeta$  is then an intrinsic time scale where "time" is used here in a very general sense. The stress at point P is not determined simply by the strain

at P, but by the history of the strain along the path OP. Materials for which the stress is a function of the history of strain with respect to an intrinsic time scale, have been called "endochronic" by the first author and the theory of the mechanical response of such materials is called "endochronic theory".

In the applications, it was found that it is appropriate to define an intrinsic time scale  $z$  which is related to the intrinsic time measure  $\zeta$ , by the relation:

$$dz = \frac{d\zeta}{f} \quad (1.2)$$

where  $f$  is a function of the history of strain. The function  $f$ , generally considered to be a function of  $\zeta$ , is of thermodynamic origin and is related proportionally to the degree of internal friction in a material. If a material hardens,  $f(\zeta)$  increases with  $\zeta$ ; if it softens,  $f$  decreases with  $\zeta$  and is constant otherwise.

The power of the thermodynamic development that follows lies in the fact that it does not depend on an explicit definition of  $z$ . Thus one can envision a thermodynamic framework, applicable to a large class of materials, from which an explicit constitutive equation, pertaining to a sub-class, can be obtained by simply choosing the appropriate form of  $z$ .

The intrinsic time defined by equation (1.2) leads to a so-called simple endochronic theory. In the case of linear isotropic theory the constitutive equations so derived can be decomposed into deviatoric and hydrostatic parts. The deviatoric stress  $\underline{s}$  is related to the history of the deviatoric strain  $\underline{e}$  by the linear functional relation:

$$\underline{s} = 2 \int_0^z \mu(z - z') \frac{\partial e}{\partial z'} dz' \quad (1.3)$$

where in the reference configuration,  $\underline{s}$  is zero,  $z = 0$ , and the shear modulus,  $\mu(z)$ , is given by a Dirichlet series, i.e.,

$$\mu(z) = \lambda_\infty + \sum_{r=1}^n \lambda_r e^{-\beta_r z} \quad (1.4)$$

where  $\lambda_\infty$ ,  $\lambda_r$  and  $\beta_r$  are positive constants. The hydrostatic stress,  $\sigma_H$ , is related to the history of volumetric strain,  $\theta$ , in a similar fashion by the linear functional relation:

$$\sigma_H = \int_0^z K(z - z') \frac{\partial \theta}{\partial z'} dz' \quad (1.5)$$

where  $\sigma_H = \sigma_{kk}/3$  and  $\theta = \epsilon_{kk}$ , in the usual notation where the summation convention is employed. The bulk modulus,  $K(z)$ , is given by a Dirichlet series of the form of equation (1.4). Note again that  $\sigma_H = 0$  in the reference configuration.

For further details of the derivation of equations above see [15], where it is shown that both  $\mu(z)$  and  $K(z)$  are composed of finite sums of positive exponentially decaying terms. In particular,  $\mu(0)$  and  $K(0)$  are the shear and bulk elastic moduli, respectively.

The simple endochronic theory has been applied with success to a number of problems of practical interest [7,15,16].

Despite this fact, it failed to predict closed hysteresis loops for "small" unloading-reloading processes in one-dimensional conditions. For

such deformation histories, the theory predicted a slope at the reloading point that was smaller than the unloading slope at the same point. This feature of the theory is at odds with the observed behavior of most metals.

It was shown that the openness of the hysteresis loops is thermodynamic in nature and has to do with the fact that the intrinsic time rate of dissipation at the onset of unloading is equal to the intrinsic time rate of dissipation upon continuation of loading. However, from experience, most rate-insensitive materials initially unload in an elastic manner and, therefore, with essentially zero rate of dissipation. In view of this, the discrepancy between prediction and observation was bound to arise [17].

It was subsequently demonstrated, however, that if the measure of intrinsic time is redefined in terms of the increment of plastic strain, the rate of dissipation at the onset of unloading and reloading is, in fact, zero. Therefore, it was appropriate to adopt the plastic strain increment as the measure of intrinsic time. Moreover, the constitutive equations (1.3) and (1.5) are recast in a form whereby the stress is related to the history of plastic strain. This was done by the first author recently [17]. This model was used to prove mathematically the existence of yield surface and that the kinematic hardening rule is a consequence of the theory. Of greater theoretical and practical consequence, however, is the fact that new measure of intrinsic time makes feasible the complete elimination of the yield surface by shrinking its size to zero and thereby reducing the surface to a point. This is done by introducing weakly singular kernel functions in the linear functional representation of stresses in terms of history of plastic strain by allowing the kernel functions to possess an integrable singularity at the origin (i.e.  $z = 0$ ). On the basis of the above considerations, endochronic constitutive equations of

isotropic materials, which exhibit yielding immediately upon application of loading, are as follows

$$\underline{s} = 2 \int_0^{z_D} \rho (z_D - z'_D) \frac{\partial \underline{e}^P}{\partial z'_D} dz'_D, \quad \rho(0) = \infty \quad (1.6)$$

$$\sigma_{kk} = 3 \int_0^{z_H} \kappa (z_H - z'_H) \frac{\partial \epsilon_{kk}^P}{\partial z'_H} dz'_H, \quad \kappa(0) = \infty \quad (1.7)$$

and

$$\int_0^{z_H} \kappa(z'_H) dz'_H < \infty; \quad \int_0^{z_D} \rho(z'_D) dz'_D < \infty, \quad \text{for all finite } z_H \text{ and } z_D,$$

where D and H denote the deviatoric and hydrostatic state, respectively. Also

$$d\underline{e}^P = d\underline{e} - \frac{d\underline{s}}{2\mu_1} \quad (1.8)$$

$$d\epsilon_{kk}^P = d\epsilon_{kk} - \frac{d\sigma_{kk}}{3K_1} \quad (1.9)$$

where  $\mu_1$  and  $K_1$  are the appropriate elastic moduli. The intrinsic time scale increments  $dz_H$  and  $dz_D$  are related to the intrinsic time measures by the equations:

$$dz_D = d\zeta_D / f_D(\zeta_D) \quad (1.10)$$

$$dz_H = d\zeta_H / f_H(\zeta_H) \quad (1.11)$$

where

$$d\zeta_D = |de_{ij}^P de_{ij}^P|^{\frac{1}{2}} \quad (1.12)$$

$$d\zeta_H = |d\epsilon_{kk}^P| \quad (1.13)$$

Here  $|\cdot|$  denotes the absolute value. Other more general definitions are possible, see reference [13]. The kernels  $\rho$  and  $\kappa$  are given by the series

$$\rho(z_D) = \sum_{r=1}^{\infty} \rho_r e^{-\alpha_r z_D} \quad (1.14)$$

$$\kappa(z_H) = \sum_{r=1}^{\infty} \kappa_r e^{-\omega_r z_H} \quad (1.15)$$

which must be convergent for all values of  $z > 0$ , but should diverge at  $z = 0$ . The above equations summarize the new model of the isotropic endochronic theory.

In conclusion, two significant results are accomplished: (1) The slope of the deviatoric (or hydrostatic) stress-strain curve at points of unloading and reloading or strain rate reversal is always elastic, i.e., equal to the slope at the origin of the appropriate stress-strain curve. (2) The hysteresis loops in the first quadrant of the stress-strain space are always closed. For details see reference [13].

#### Constitutive Relations in Tension-Torsion

The constitutive equations that apply in this specific case are found from equations (1.6) and (1.7) and are given below.

$$\tau = 2 \int_0^{z_D} \rho(z_D - z'_D) \frac{\partial \eta^P}{\partial z'_D} dz'_D \quad (1.16)$$

$$\sigma_1 = 2 \int_0^{z_D} \kappa(z_D - z'_D) \frac{\partial}{\partial z'_D} (\epsilon_1^P - \epsilon_2^P) dz'_D \quad (1.17)$$

$$\sigma_1 = 3 \int_0^{z_D} \kappa(z_H - z'_H) \frac{\partial}{\partial z'_H} (\epsilon_1^P + 2\epsilon_2^P) dz'_H \quad (1.18)$$

where  $\epsilon_i^P$  and  $\sigma_i$  are the axial plastic strains and stresses, respectively, along the axes  $x_i$  and  $\epsilon_2^P = \epsilon_3^P$  to satisfy the condition of isotropy. Also  $\tau$  and  $\eta^P$  stand for  $s_{12}$  and  $e_{12}^P$ , respectively, in the notation of equation (1.6).

Because in the experiments to be investigated the hydrostatic strain was not measured we shall proceed to make the usual (approximate) assumption of elastic hydrostatic response, in which case equation (1.7) does not apply, but instead the plastic incompressibility condition

$$\epsilon_1^P + 2\epsilon_2^P = 0 \quad (1.19)$$

is used. In the following, we will omit the subscripts D and H.

In light of the above hypotheses and in view of equations (1.16) and (1.19) the appropriate constitutive equations in tension-torsion are the following:

$$\tau = 2 \int_0^z \rho(z - z') \frac{\partial \eta^P}{\partial z'} dz' \quad (1.20a)$$

$$\sigma_1 = \int_0^z E(z - z') \frac{\partial \epsilon_1^P}{\partial z'} dz' \quad (1.20b)$$

$$\epsilon_1 + 2\epsilon_2 = \frac{\sigma_1}{3K_1} \quad (1.20c)$$

where

$$E(z) = 3 \rho(z) \quad (1.21)$$

$$dz = dz_D = \frac{d\zeta}{f(\zeta)} \quad (1.22)$$

$$d\zeta = d\zeta_D = \left| \left[ \frac{2}{3} (d\epsilon_1^P - d\epsilon_2^P)^2 + 2(d\eta^P)^2 \right]^{\frac{1}{2}} \right| \quad (1.23a)$$

Alternatively,  $d\zeta$  can be expressed in terms of the engineering shear strain  $\gamma^P = 2\eta^P$ , in which case, upon using equation (1.19),

$$d\zeta = \left| \left[ \frac{3}{2} (d\epsilon^P)^2 + \frac{1}{2} (d\gamma^P)^2 \right]^{\frac{1}{2}} \right| \quad (1.23b)$$

Here  $\epsilon^P = \epsilon_1^P$ .

In the applications that follow we will use the above equations in the study of cyclic response to a variety of test conditions.

## 2. APPLICATIONS TO STEADY CYCLIC RESPONSE

In subsequent applications, it is expedient to rescale  $d\zeta$  by a constant  $\sqrt{2}$  so that

$$d\zeta = \left| \left[ 3(d\epsilon^P)^2 + (d\gamma^P)^2 \right]^{\frac{1}{2}} \right| \quad (2.1)$$

The values of  $\rho(z)$  and  $E(z)$  are rescaled by the same constant.

### Cyclic Shear Response

It follows from equation (2.1) that in pure shear

$$d\zeta = |d\gamma^P| \quad (2.2a)$$

In addition, if the cyclic response is steady, then  $f(\zeta)$  is a constant, which we set equal to 1. Thus equation (1.22) becomes

$$dz = |d\gamma^P| \quad (2.2b)$$

In reference [13], we let  $\rho(z)$  be a function of the form

$$\rho(z) = \rho_0 z^{-\alpha} \quad (2.3)$$

where  $\rho_0$  and  $\alpha$  are material constants and  $0 < \alpha < 1$ . This type of kernel satisfies the constraint imposed by equation (1.6) and leads to the Ramberg-Osgood equation for the tensile response. In view of these remarks, we use equation (2.3) for the present study.

Upon substitution of equation (2.3) in equation (1.20a), the shear stress is expressed as a function of the history of plastic strain as follows:

$$\tau = \int_0^z \frac{\rho_0}{(z - z')^\alpha} \frac{d\gamma^P}{dz'} dz' \quad (2.4)$$

At the completion of  $n$  reversals and by virtue of equations (2.2a, b) and (2.4), the following relation applies,

$$\tau = \sum_{i=1}^n \int_{z_{i-1}}^{z_i} (-1)^{i-1} \frac{\rho_0}{(z - z')^\alpha} dz' + (-1)^n \int_{z_n}^z \frac{\rho_0}{(z - z')^\alpha} dz' \quad (2.5a)$$

where  $z_i$  denotes the value of  $z$  at the point where the  $i$ th reversal has been completed and  $z_0 \stackrel{\text{def}}{=} 0$ . By simple analysis, the above equation leads to the result

$$\tau = \frac{\rho_0}{1-\alpha} \left[ z^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (z - z_i)^{1-\alpha} \right] \quad (2.5b)$$

Equation (2.5b) is suitable for the prediction of the stress response, once the functional relationship between  $z$  and the (plastic) shear strain history is known.

#### Cyclic Uniaxial Response --

In this case, we use equations (2.1) and (2.2a, b) to obtain the essential relation.

$$dz = d\zeta = \sqrt{3} |d\epsilon^P| \quad (2.6)$$

In the fashion outlined above, the steady cyclic uniaxial response is found from equations (1.20b), (1.21), (2.3) and (2.6) and is given by the relations,

$$\sigma = \frac{\sqrt{3} \rho_o}{1-\alpha} \left[ z^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (z - z_i)^{1-\alpha} \right] \quad (2.7)$$

If, instead of using equation (2.6), we use

$$dz = |d\epsilon^P| \quad (2.8)$$

then equation (2.7) becomes

$$\sigma = \frac{3\rho_o}{1-\alpha} \frac{1}{3^{\alpha/2}} \left[ z^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (z - z_i)^{1-\alpha} \right] \quad (2.9)$$

The scaling of the intrinsic time by a constant is a matter of convenience and may be done at will, without interference with the theory. We observe that equations (2.5b) and (2.9) obey the linear homogeneous transformation between indicated stresses and strains given below:

$$\tau = \sigma/\sqrt{3} , \quad \gamma^P = \sqrt{3} \epsilon^P \quad (2.10a,b)$$

To test the validity of the theory, we appeal to the experimental results on normalized mild steel obtained by Jhansal and Topper [6].

Constant Uniaxial Strain Amplitude --

We consider the class of metals whose asymptotic stress response to sustained cyclic strain excitation at constant strain amplitude is a periodic

stress history with constant amplitude. Specifically in a uniaxial test of this type, the axial stress amplitude  $\Delta\sigma$  is constant and therefore the axial plastic strain amplitude  $\Delta\epsilon^P$  is also constant, following equation (1.8).

Thus

$$\Delta\epsilon^P = \Delta\epsilon - \frac{\Delta\sigma}{E_1} \quad (2.11)$$

where  $\Delta\epsilon$  is the axial strain amplitude and  $E_1$  is Young's modulus. As a result, the value of  $z$  during cyclic tension and compression can be found by integrating equation (2.8). After an odd number  $n$  of reversals has been completed, the value of  $z - z_n$  can be calculated by integrating the relation  $dz = -d\epsilon^P$  with  $\Delta\epsilon^P$  as the lower limit of integration. If  $n$  is even, then the relation  $dz = d\epsilon^P$  applies with  $-\Delta\epsilon^P$  as the lower limit of integration. The results are as follows:

$$z = 2n\Delta\epsilon^P \mp \epsilon^P \quad (2.12a)$$

and

$$z_n = (2n - 1) \Delta\epsilon^P \quad (2.12b)$$

where in equation (2.12a) "-" is used for  $n = \text{odd}$  and "+" for  $n = \text{even}$ .

Equation (2.12b) applies to both cases.

Upon substitution of equations (2.12a, b) in equation (2.9), one obtains the result

$$\sigma(\epsilon^P) = \frac{3\rho_0}{1-\alpha} \frac{1}{3^{\alpha/2}} (\Delta\epsilon^P)^{1-\alpha} F_n(\alpha, x) \quad (2.13)$$

$$x = \epsilon^P / \Delta\epsilon^P \quad (2.14a)$$

$$F_n(\alpha, x) = (2n \pm x)^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (2n - 2i + 1 \pm x)^{1-\alpha} \quad (2.14b)$$

where the "+" and "-" signs correspond to n even and n odd, respectively.

The algebraic value of the peak stress (i.e., stress amplitude) is found from equation (2.14b) by choosing n odd and setting x = 1 in equation (2.14b), i.e.,

$$F_n(\alpha) = (2n - 1)^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (2n - 2i)^{1-\alpha} \quad (2.14c)$$

where n = 1, 3, 5, . . . . The peak stress at n = even is given by the same equation, i.e., equation (2.14c). Thus equation (2.14c) is applicable for all n. It can be shown that, in the limit of n → ∞, F<sub>n</sub> converges to a constant F<sub>∞</sub>(α), where F<sub>∞</sub> varies with α but is essentially close to unity. For instance, for α = 0.864, F<sub>∞</sub> is equal to 1.03  $\sqrt[14]{}$ . Thus the asymptotic value of Δσ as n tends to infinity is given by the equation

$$\Delta\sigma = \frac{3\rho_0}{1-\alpha} \frac{1}{3^{\alpha/2}} (\Delta\epsilon^P)^{1-\alpha} F_{\infty}(\alpha) \quad (2.15)$$

This is the equation of the cyclic stress-(plastic) strain curve.

Cyclic steady response in shear can be found in a similar fashion or by using equations (2.10a, b).

To test the validity of the theory, we use experimental data on normalized mild steel [6]. In reference [6], a set of stable uniaxial hysteresis loops corresponding to various constant strain amplitudes was presented in the stress-strain space. A propos of the ensuing theoretical predictions we note that the geometric shape of the loops is given by equation (2.13), whereas the peak stresses are given by equation (2.15). We also note that there are only two undetermined parameters in these equations,  $\alpha$  and  $\rho_0$ . The form of equation (2.15) was corroborated in reference [14] where a semi-logarithmic plot of the experimental values of  $\Delta\sigma$  vs  $\Delta\epsilon^p$  gave rise to a linear relation. The plot also determines  $\alpha$  and  $\rho_0$  which were found to be 0.864 (a pure number) and 48.4 MPa (7.02 ksi), respectively. These values are then used in equation (2.13), and the shape of the hysteresis loops is thereby calculated. Agreement between theory and experiment is excellent as shown in Figure 2.

We wish to devote a few lines to these powerful results. The reader will note that two constants are sufficient to define the cyclic stress- (plastic) strain response as well as the hysteretic behavior of normalized mild steel. It is also pertinent to mention that the analytical expressions involved (equations (2.13) and (2.15)) are not empirical formulae but closed forms derived from a general constitutive equation pertaining to three-dimensional histories. Also of importance is that the prediction of unloading and reloading behavior did not necessitate special rules or special treatment but was dealt with routinely, as part of the total experimental history of interest. Specifically, the celebrated Bauschinger effect is predicted quantitatively and correctly from one and the same constitutive equation.

We make in passing, an observation of historical interest. Equation (2.15) agrees with the empirical relationship proposed by Landgraff et al. [2]

for steels, i.e.,

$$\Delta\sigma \sim (\Delta\varepsilon^p)^{1-\alpha}$$

where  $1-\alpha$  ranges from 0.12 to 0.17. In our case,  $1-\alpha = .136$ .

#### Variable Uniaxial Strain Amplitudes --

To extend the experimentally verified domain of validity of the theory and to broaden our view of its capabilities, we test it under conditions of variable uniaxial strain amplitude histories. The stress response to such histories is found by using equations (2.13) and (2.14b). The analytical results are compared with the experimental data on normalized mild steel [6]. The experiment consists of a constant uniaxial strain amplitude cyclic test (until stable hysteresis loops are reached) followed by a variable uniaxial strain amplitude test. The experimental data are shown in Figure 3. Despite the complexity of the history, agreement between theory and experiment is obtained and shown in Figure 3. Again the theory predicts the stress history routinely without the use of special rules present in other theories [3,5,6,10,14]. At this point, we may reasonably conclude that the theory as expressed by equation (2.5b) (or equation (2.9)) is suitable for the prediction of the stress response to cyclic straining, in the case of normalized mild steel.

#### Cyclic Relaxation --

Here we address the case where the plastic shear strain is increased monotonically to a value  $\gamma_+^p$ , and is followed by a cyclic shear strain history with amplitude  $\Delta\gamma^p$  about a mean value  $\gamma_0^p$ .

To calculate the stress response we use equation (2.5b). The cyclic shear strain history is shown in Figure 4. With reference to Figure 4, to make the following definitions

$$\gamma_+^p = \gamma_0^p + \Delta\gamma^p \quad (2.16a)$$

$$\gamma_-^p = \gamma_0^p - \Delta\gamma^p \quad (2.16b)$$

The value  $z_i$  of  $z$  at  $i$ th reversal, is found from equation (2.2b). Thus

$$z_i = \gamma_0^p + (2i - 1)\Delta\gamma^p, \quad i = 1, 2, \dots, n. \quad (2.17)$$

After  $n$  reversals have been completed, the value of  $z$  at the current shear strain  $\gamma^p$  is

$$z = 2n \Delta\gamma^p + \gamma_0^p \mp \bar{\gamma}^p \quad (2.18)$$

where

$$\bar{\gamma}^p = \gamma^p - \gamma_0^p \quad (2.19)$$

and the minus and plus signs correspond to  $n$  odd and even respectively. The shear response, after  $n$  reversals is found upon using equations (2.5b), (2.17) and (2.18). Specifically,

$$\tau = \frac{\rho_0}{1-\alpha} (\Delta\gamma^p)^{1-\alpha} F_n(\alpha, x_0, x) \quad (2.20)$$

where

$$F_n(\alpha, x_0, x) = (2n + x_0 \bar{\gamma} x)^{1-\alpha} + 2 \sum_{i=1}^n (-1)^i (2n - 2i + 1 \bar{\gamma} x)^{1-\alpha} \quad (2.21)$$

and

$$x_0 = \gamma_0^p / \Delta\gamma^p \quad (2.22a)$$

$$x = \bar{\gamma}^p / \Delta\gamma^p \quad (2.22b)$$

If  $n = \text{odd}$ , then  $x$  varies from 1 to -1; while if  $n = \text{even}$ , then  $x$  varies from -1 to 1.

Equations (2.21) and (2.14b) differ only in the first term on their right-hand side. It is  $x_0$  which allows cyclic relaxation to take place. The results are shown in Figure 4 where the material constants, found previously, were used.

We notice that as  $n$  is very large, the effect of  $x_0$  in equation (2.21) disappears as a result of the relation  $\lim_{n \rightarrow \infty} F_n(\alpha, x_0, x) = F_\infty(\alpha, x)$ . The hysteresis loops then become stable and symmetric with respect to  $\gamma_0^p$  and have exactly the same shape as those with zero mean shear strain.

#### Other Complex Histories

A strain history of practical importance is shown in Figure 5, where a cyclic strain history at a fixed strain amplitude is followed by another at a lower strain amplitude. The experimental results are shown in Figure 5. In

order to predict the stress response, we use the numerical scheme developed in the section on variable uniaxial strain amplitudes. The theoretical results obtained are also shown in Figure 5. Again agreement between theory and experiment is demonstrated.

It is important to observe that the decreasing effect of the previous history on the stress response to a periodic strain history (cyclic test at constant strain amplitude) is the natural consequence of the monotonically decaying kernel function used in the present theory, i.e., in equation (2.3). This type of kernel does indeed impart to the material a fading memory with respect to the endochronic time scale.

### 3. CYCLIC RATCHETING

In this case the cyclic stress history is given. The numerical scheme developed in the previous sections is still useful. In addition, an iterative method is used to ensure the correct value of the stress at the point of reversal. Such schemes are easy to implement in the computer program. For purposes of theoretical study, the constitutive equations for shear under symmetric and unsymmetric stress cycles were used. Specifically, equation (2.5b) with material constants of normalized mild steel found previously, predicted the cyclic ratcheting phenomena shown in Figures 6(A) - 6(D). It is clear that, under unsymmetric stress cycles, the increment of plastic shear strain per cycle  $\delta\gamma_N^p$  is positive and decreasing but not equal to zero, as shown in Figure 6(D). This indicates that, whether the material response is stable or not, the direction of progressive (plastic) shear strain is in the "direction" of mean shear stress. However, in the case of symmetric stress

cycles, the first stress cycle gives rise to a hysteresis loop which lies toward the right-hand side in the stress-strain space. The subsequent cycles will then cause the hysteresis loops to move toward the left-hand side until a stable symmetric hysteresis loop is reached. Due to the effects of the first stress cycle, the center of stably symmetric hysteresis loops does not lie at the origin of the stress-strain space. We find that the sign of the "off-center" value of the strain is the same as the sign of the strain at the point of first reversal. This phenomenon is essentially the counterpart of the cyclic relaxation after initial loading as indicated in Figure 4.

Comparisons between theoretical predictions and experiments must await further experimental information.

#### 4. CONCLUSIONS

On the basis of the results presented in this paper, we conclude that the constitutive equations derived from the endochronic theory are very suitable for the analytical prediction cyclic response of stable materials under a variety of conditions. Moreover, the theory has its origins in irreversible thermodynamics of internal variables shown to be a powerful tool in the derivation of constitutive theories for several classes of materials (e.g. viscoelastic, plastic and viscoplastic materials).

Also noteworthy is the fact that a constitutive equation with two material constants, which are easily determined, can predict with computational ease the stress (strain) response of a material to a variety of general strain (stress) histories, without a need for special rules that are otherwise so evident in the literature.

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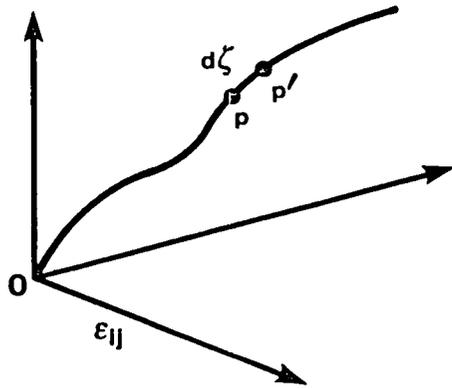


Fig. 1 Path in strain space.

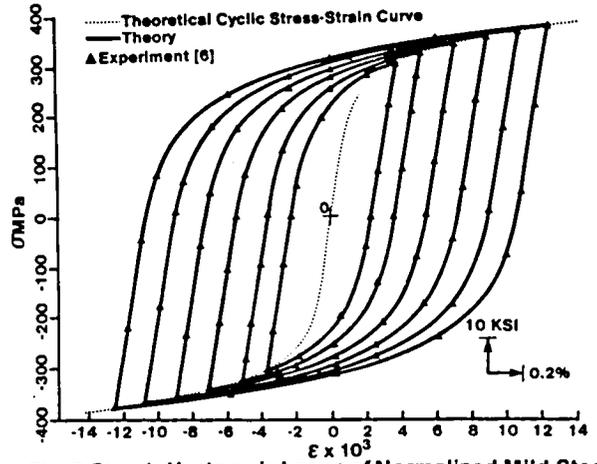


Fig. 2 Steady Hysteresis Loops of Normalized Mild Steel

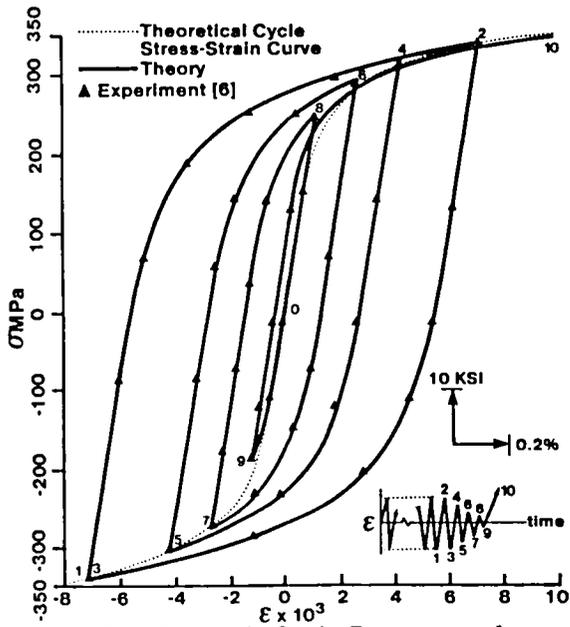


Fig. 3 Steady Cycle Response of Normalized Mild Steel

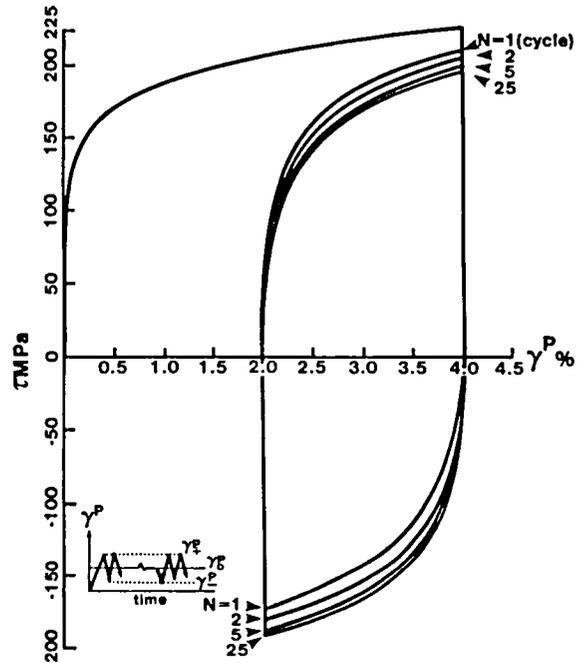


Fig. 4 Shear Cyclic Relaxation of Normalized Mild Steel

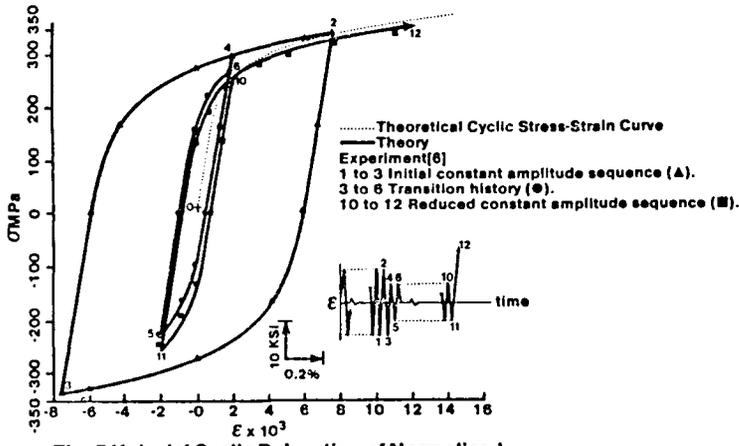
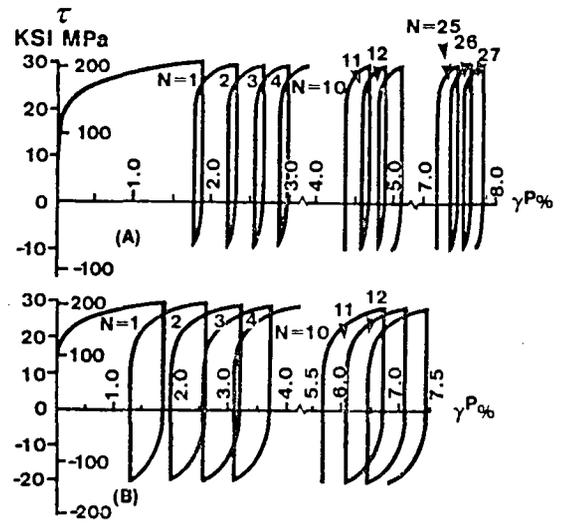


Fig. 5 Uniaxial Cyclic Relaxation of Normalized Mild Steel



Figs. 6(A)&(B) Shear Ratcheting of Normalized Mild Steel Under Unsymmetric Stress Cycles

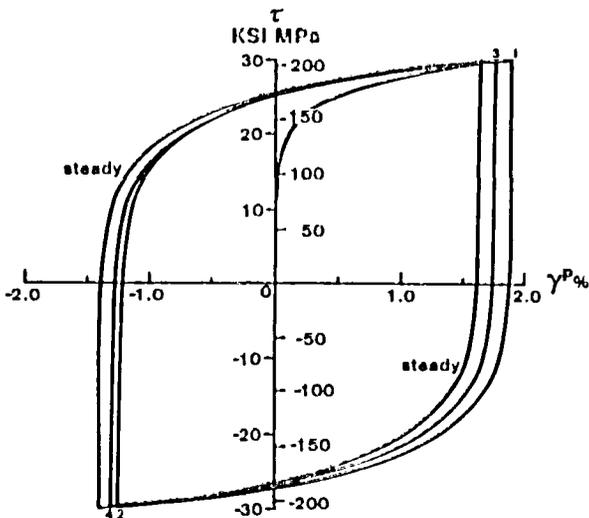


Fig. 6(C) Shear Ratcheting of Normalized Mild Steel Under Symmetric Stress Cycles

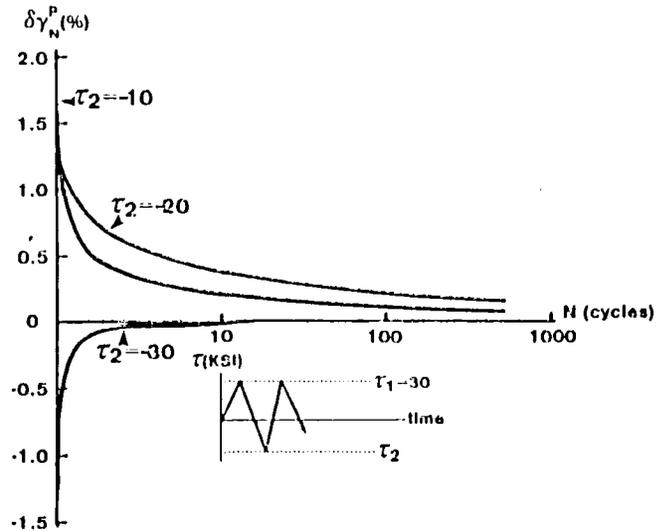


Fig. 6(D) Plastic Shear Strain Increment Per cycle v.s. Number of Cycles