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Produced by the NASA Center for Aerospace Information (CASI)
MONTHLY PROGRESS REPORT #5
for
SPACE SHUTTLE PROPULSION PARAMETER ESTIMATION
USING
OPTIMAL ESTIMATION TECHNIQUES

CONTRACT NO NAS8-35324

submitted to
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
MARSHALL SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA

20 September 1983
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REFERENCES
1.0 INTRODUCTION

The fifth monthly progress report includes corrections and additions to the previously submitted reports. The addition of the SRB propellant thickness as a state variable is included with the associated partial derivatives.

During this reporting period, preliminary results of the estimation program checkout was presented to NASA technical personnel.
The system described by equation (1) is observed at discrete times, $t_k$, with not all states being directly measured. Some measurements are non-linear functions of the elements of the state vector $\mathbf{x}(t)$. In general, the measurement process is described as

$$z_k = h_k(x(t_k)) + v_k$$

(2)

where

$z_k$ = m-dimensional observation vector

$h_k$ = functional representation of the measurements in terms of the states

$v_k$ = m-dimensional, zero-mean, white Gaussian noise sequence with covariance

$$E[v_i v_j^T] = R \delta_{i,j}$$

Examples of the elements of the observation vector $z_k$ include radar measurements of range, azimuth, and elevation from the radar site to the vehicle.

It is assumed that the system process noise vector $w(t)$ and the measurement noise vector $v_k$ are uncorrelated. Also, the system state initial condition vector $x_0$ is not correlated with either of these two noise vectors. Therefore

$$E[w(t) v_k^T] = 0, \ E[w(t) x_0^T] = 0, \ E[x_0 v_k^T] = 0$$

where the superscript $T$ denotes transpose. For later reference, the following matrices are defined.
2.0 FILTERING AND SMOOTHING ALGORITHM

The Space Shuttle Parameter Estimation Program utilizes optimal estimation techniques to provide estimates of the propulsion system parameters. The technique selected is the extended Kalman filter and the modified Bryson-Frazier smoother. By modeling the propulsion system parameters as time correlated random variables, improved estimates of these parameters are obtained and are properly time phased by removing the filter induced lag by using the combined filter/smoother. The smoother also provides improved estimates of the initial state estimates.

The system, in state-space notation, is modeled as the continuous dynamical system disturbed by additive Gaussian white noise

$$\dot{x} = f(x(t), t) + G(t) \nu(t) + u(t), \quad x(0) = x_0 \quad (1)$$

where

- $x$ = n-dimensional state vector
- $x_0$ = Gaussian initial condition vector with covariance $P_0$
- $\nu(t)$ = p-dimensional white, zero-mean white Gaussian noise with covariance
  $$E[\nu(t) \nu^T(\tau)] = Q(t) \delta(t - \tau)$$
- $u(t)$ = n-dimensional control vector.

The elements of the vector $x(t)$ represent vehicle position, velocities, attitudes, angular rates, aerodynamic and propulsion parameters, measurement biases, etc. Elements of $u(t)$ include known control inputs such as SSME power level commands.
These matrices are linearizations of the dynamics and measurement models respectively, evaluated about either a nominal or reference value of the state, or about the state estimate.

2.1 Extended Kalman Filter Algorithm

The extended Kalman filter algorithm is in essence a conventional linear Kalman filter algorithm applied to a mathematical model resulting from the linearization of the system model equation (1), and measurement process, equation (2), about a current state estimate. The filter yields optimal estimates if the linearization is accurate, i.e., the state estimate closely approximates the true state. The derivation of the algorithm can be found in reference [1].

The algorithm proceeds as follows. After initialization of the state estimate and covariance, the state estimate and covariance are propagated forward in time until a measurement update is available, by

$$\hat{x} = f(\hat{x}(t), t), \quad t_{k-1} < t \leq t_k$$

(5)

and

$$\dot{P}(t) = F(\hat{x}(t), t) P(t) + P(t) F(\hat{x}(t), t)^T + G(t) Q(t) G(t)^T$$

(6)
At the measurement time, the state estimate and covariance are updated by

\[ \hat{x}_k(+) = \hat{x}_k(-) + K_k(z_k - h_k(\hat{x}_k(-))) \]  

and

\[ P_k(+) = (I - K_k H_k(\hat{x}_k(-))) P_k(-) \]  

where the (-) and (+) represent the appropriate values just before and just after the update. The updated values are used to reinitialize the time propagation equations (3) and (4) for integrating up to the next measurement time. The Kalman gain matrix is computed as

\[ K_k = P_k(-) H_k(\hat{x}_k(-))^T (H_k(\hat{x}_k(-)) P_k(-) H_k(\hat{x}_k(-))^T + R_k)^{-1} \]  

This algorithm is repeated until the last time point, \( t_N \), is processed.

For later use in the smoother algorithm, various combinations of the state estimates (\( \hat{x}_k \)), measurements (\( z_k \)), linearized dynamics matrix (\( F \)) and measurement matrix (\( H \)), measurement noise covariance (\( R \)) and estimation error covariance matrix (\( P \)) must be stored for each time instant to be processed by the smoother algorithm.

2.2 Modified Bryson-Frazier Smoother Algorithm

The operation of the smoother algorithm is similar to the filter algorithm except in reverse time. The derivation of this smoother algorithm is found in reference [2]. This fixed interval smoothing algorithm provides optimal estimates given all the measurements in comparison to the filtering algorithm providing optimal estimates given the previous
measurements processed. Therefore the smoother provides improved estimates in addition to removing the time lag induced by the filter algorithm.

The smoothing algorithm adjoint variables, $\lambda$ and $A$ are "initialized" at the final time processed by the filter, $T$,

$$\lambda(T-) = -H_N^T(H_N P_N H_N^T + R_N)^{-1} (z_N - H_N (\hat{x}_N(-))) \delta_{T, T}$$

and

$$A(T-) = H_N (H_N P_N H_N^T + R_N)^{-1} H_N \delta_{T, T}$$

If $T$ is not an observation time, $\lambda$ and $A$ are zero. The adjoint variables are propagated in reverse time to the next previous measurement time by

$$\dot{\lambda} = -F(\hat{x}(t), t) \lambda \quad , \quad t_k \leq t < t_{k+1}$$

$$\dot{A} = -F(\hat{x}(t), t)^T A - A F(\hat{x}(t), t)$$

At the time of an available measurement, $t_k$, the adjoint variables are updated by

$$\lambda(-) = \lambda(+) - H_k^T(H_k P_k H_k^T + R_k)^{-1} ((z_k - h_k (x_k(-)))

+ (H_k P_k H_k^T + R_k) K_k^T \lambda(+)$$

and

$$A(-) = (I - K_k H_k)^T A(+) (I - K_k H_k) + H_k^T(H_k P_k H_k^T + R_k)^{-1} H_k$$
The smoother state estimate and error covariance are obtained using the filter estimate and covariance and the adjoint variables by

\[ \mathbf{x}^*(t) = \tilde{\mathbf{x}}(t) - \mathbf{P}(t) \mathbf{A}(t) \mathbf{P}(t) \]  

and

\[ \mathbf{P}^*(t) = \mathbf{P}(t) - \mathbf{P}(t) \mathbf{A}(t) \mathbf{P}(t). \]  

Due to the potential number of time points to be processed, smoother estimates may only be computed at the discrete measurement times. For this approach the propagation equations (10) and (11) are replaced by

\[ \lambda_k^+ = \lambda_{k+1}^- \]  

and

\[ \lambda_k^+ = \lambda_{k+1}^- \]  

where \( \phi^T \) is the state transition matrix formed with the linearized dynamics matrix \( F \) to propagate the adjoint variable from time \( t_{k+1} \) to time \( t_k \). The algorithm continues in reverse time until the initial time is reached.

2.3 Iterations with the Filter/Smooher Algorithm

The performance of the filter/smooher algorithm is a direct result of the accuracy of the linearization. Repeated operations of the algorithms with adjustments in initial state estimates and covariance in each cycle can yield improved estimates. This technique is known as global iterated
filtering as defined in reference [3]. Each cycle of operating the algorithms would yield increasing improvements in the state estimates.

This feature of the algorithm operation is of special interest to the propulsion parameter estimation problem using the NASA predictive models. Initial, or nominal, values of the parameters of interest can be used to obtain the necessary partial derivatives indicated earlier. From operating the algorithm improved estimates of those parameters are obtained. Using these improved estimates, more accurate partial derivatives are obtained for use in the algorithms. This process is continued until there is in essence no change in the partial derivatives or quality of the state estimates. If the linearization is accurate, the measurement residual should be a white noise process with known covariance.
3.0 FILTER/Smoother Algorithm System and Measurement Model

The usefulness of the filter/smooother algorithm is to provide estimates of the system states from the observed motion and dynamics while the system is driven by known and unknown elements. These unknown elements are elements of the system state vector to be estimated. The evolution of motion resulting from these known and unknown elements is assumed to be suitably represented for this study by a six degree-of-freedom (6 DOF) rigid body equations of motion. These equations are presented and discussed in section 3.1.

To implement these equations into the filter/smooother algorithm presented in section 2.0, a linearization of the system state and measurement models is required. These linearized equations are presented in section 3.2.

3.1 Equations of Motion and Measurement Equations

3.1.1 Rigid Body Equations of Motion

The rate of change of vehicle velocity in body coordinates, \( \dot{v}_B \), as a result of external forces acting on the vehicle is described by

\[
\dot{v}_B = \frac{\rho A v^2}{2m} c_f + B C I (I) (r(I)) - \omega \times v_B + \frac{f_T}{m} + \frac{f_p}{m}
\]

where

- \( \rho \) = atmospheric density
- \( A \) = aerodynamic coefficient referenced area
- \( v_m \) = magnitude of vehicle velocity relative to the surrounding air mass
m = vehicle mass

c = aerodynamic force coefficient vector

\( \mathbf{g}^{(I)} \) = gravity vector in inertial coordinates

\( \omega \) = angular rotation of the body relative to the inertial frame

\( f^{(B)}_T \) = resultant thrust force vector in body coordinates

\( f^{(B)}_p \) = resultant plasma force vector in body coordinates

The rate of change of vehicle position in inertial coordinates, \( \dot{r}^{(I)} \), is then obtained by

\[
\dot{r}^{(I)} = I_C^B \mathbf{v}^{(B)}
\]  

(21)

where \( I_C^B \) is the transformation matrix from body coordinates to inertial coordinates. The elements of the \( I_C^B \) transformation matrix are obtained from the resulting Euler angles defined by

\[
\begin{bmatrix}
\dot{\psi} \\
\dot{\theta} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

where \( \psi, \theta, \) and \( \phi \) are roll, pitch and yaw attitudes respectively.

The roll, pitch and yaw rates of the body relative to inertial coordinates are \( p, q, \) and \( r \) respectively. Finally, the rate of change of the body rates relative to inertial is given by
\[ \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = (I)^{-1} \left[ \frac{\rho A v^2}{2} \dot{f}_c + \frac{\rho A v^2}{2} (r_A - r_{cg}) \times c_f ight] - \dot{u} \times (\gamma \dot{w}) + \tau_T^{(B)} + \tau_P^{(B)} \]

where

\[ I = \text{vehicle moments of inertia matrix} \]
\[ f_c = \text{aerodynamic coefficient referenced length and moment coefficient vector} \]
\[ r_{cg} = \text{vehicle center-of-gravity vector in body coordinates} \]
\[ r_A^{(B)} = \text{aerodynamic coefficient reference position in body coordinates} \]
\[ \tau_T^{(B)} = \text{resultant thrust torque vector in body coordinates} \]
\[ \tau_P^{(B)} = \text{resultant plane torque vector in body coordinates} \]

The equations of motion represent the first twelve elements of the system state vector. These equations are summarized in Table 3.1.1-1.

The moment of inertia matrix \( I \) in general is given by

\[ I = \begin{bmatrix}
I_x & -I_{xy} & -I_{zx} \\
-I_{xy} & I_y & -I_{yz} \\
-I_{zx} & -I_{yz} & I_z
\end{bmatrix} \]

for the moment axis terms, i.e., \( I_y \), and the product of inertia terms, i.e., \( I_{zx} \).
TABLE 3.1.1-1

Equations of Motion

(first twelve system states)

\[
\ddot{\alpha}^{(I)} = I_C B (B)
\]

\[
\ddot{\omega}^{(B)} = \frac{\rho v^2}{2m} c_\ell + B C q (I_{cg}^{(I)}) - \omega \times \omega^{(B)} + \frac{F_T^{(B)}}{m} + \frac{F_p^{(B)}}{m}
\]

\[
\dot{\vec{r}} = \begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} = \begin{bmatrix}
1 & \sin\theta \tan\theta & \cos\theta \tan\theta \\
0 & \cos\phi & -\sin\phi \\
0 & \sin\phi \sec\theta & \cos\phi \sec\theta
\end{bmatrix} \begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

\[
I_B \ddot{\omega} = [I]^{-1} \left[ \frac{\rho v^2}{2} c_m + \frac{\rho v^2}{2} (r_A - r_{cg})^{(B)} \times c_\ell - \omega \times (I \omega) + T_T + T_p \right]
\]
The aerodynamic force and moment coefficients and plume forces are defined as functions of angle-of-attack, \( \alpha \), and angle-of-sideslip, \( \beta \), as shown in Figure 3.1.1-1. The body referenced relative velocity vector, removing the wind velocity, \( \mathbf{v}_w \), from the vehicle velocity, is given by

\[
\mathbf{v}_r = \mathbf{v}^{(B)} - B_C \mathbf{I} \mathbf{v}_w = \mathbf{v}^{(B)} - B_C^{LL} \mathbf{v}_w^{(LL)}
\]  

(25)

where \( \mathbf{v}_w^{(LL)} \) is the local-level referenced wind velocity vector. The following equations define \( \alpha \) and \( \beta \) in terms of the components of \( \mathbf{v}_r \):

\[
\alpha = \tan^{-1}\left(\frac{v_{r3}}{v_{r1}}\right)
\]

(26)

\[
\beta = \sin^{-1}\left(\frac{v_{r2}}{v_m}\right)
\]

(27)

where

\[
v_m = (v_{r1}^2 + v_{r2}^2 + \ldots)^{1/2}
\]

(28)

The resultant thrust force \( f_T^{(B)} \) is expanded as

\[
\begin{bmatrix}
    f_T \\
    \mathbf{f}_T
\end{bmatrix} = \sum_{i=1}^{n} B_{C_i}^{Q_i} \begin{bmatrix}
    f_{T_i} \\
    \Delta
\end{bmatrix}
\]

(29)

where the transformation matrix \( B_{C_i}^{Q_i} \) transforms the magnitude of thrust for each thrusting device, \( f_{T_i}^{Q_i} \), from its center-line to the body coordinates. The general equation for \( f_{T_i} \) is
FIGURE 3.1.1-1. Body Referenced Wind Axis System
\[ f_{T_{\text{ivac}}} = f_{T_1} - P_{s_1} A_e \]

where

- \( f_{T_{\text{ivac}}} \) = vacuum thrust
- \( P_{s_1} \) = atmospheric pressure at motor exit
- \( A_e \) = motor exit cone area

The matrix \( B_{C_i}^{Q_i} \) is different for the SSME's and SRB's and is given by

\[
B_{C_i}^{Q_i} = \begin{cases} 
B_{C_{\text{MP}}} & \text{SSME} \\
B_{C_{\text{QL}}} & \text{SRB} 
\end{cases} 
\]  \hspace{1cm} (30)

where

- \( B_{C_{\text{MP}}} \) = transformation from the engine mount plane to the body coordinates
- \( MP_{C_{\text{G}}} \) = transformation from the gimbal reference plane to mount plane (structural deformation)
- \( G_{C_{\text{QL}}} \) = transformation from enterline to the gimbal reference plane
- \( B_{C_{\text{QL}}} \) = transformation from SRB nozzle centerline to the body coordinates (gimbal angles).
The resultant thrust torque is the summation of the torque contribution from each thrusting device and is given by

\[
\tau_T = \sum_{i=1}^{n} (\tau_{T_i}^{(B)} - \tau_{cg}^{(B)}) x \begin{bmatrix}
T_i \\
0 \\
0
\end{bmatrix}
\]

where

\[
\tau_{T_i}^{(B)} = \text{body coordinates of the thrust reference point for the } i^{th} \text{ thrusting device.}
\]

3.1.2 Measurement Equations

The measurements assumed available for the filter/smooother algorithm include inertial platform acceleration and attitudes, ground based radar tracking, SRB's head pressure, SSME's chamber pressures, liquid H₂ flow rates, pressurant flow rates. The ET volumetric levels are available; however, due to their limited number (4), they may only be used for alternate checks of the filter/smooother algorithm performance.

The propulsion related measurements will be treated in a separate section. In the following, the inertial platform acceleration measurements, attitude measurements and ground based tracking measurements models will be described for later linearization.
3.1.2.1 Platform Acceleration Measurements

Accelerometers mounted orthogonally on an inertially stabilized platform, not located at the vehicle center of gravity, sense externally applied special forces and accelerations due to body rotation. The accelerometer measurement is modeled by

\[ a_{m}^{(S)} = S_{C}^{P} P_{C}^{P'} P_{C}^{B} \left( \frac{\rho \text{Av}^2}{m} + \frac{\text{f}_{T}}{m} + \frac{\text{f}_{p}}{m} \right) + \omega \times \left( r_{G}^{(B)} - r_{CG}^{(B)} \right) + \omega \times \left( r_{G}^{(B)} - r_{CG}^{(B)} \right) + b_{a}^{(S)} + v_{a}^{(S)} \]  

(32)

where

- \( S_{C}^{P} \) = transformation from platform coordinates to sensing coordinates
- \( P_{C}^{P'} \) = transformation from misaligned platform coordinates to platform coordinates
- \( P_{C}^{B} \) = transformation from body to misaligned platform coordinates
- \( r_{G}^{(B)} \) = body coordinates of the platform center
- \( b_{a}^{(S)} \) = accelerometer bias vector
- \( v_{a}^{(S)} \) = accelerometer measurement noise vector

3.1.2.2 Platform Attitude Measurements

The inertially stabilized platform for the STS is a four axis IMU with a redundant roll axis [4]. Vehicle body attitudes are generated via quaternions [5]. It is assumed that an equivalent representation
can be made to obtain vehicle attitude by a three rotation sequence of roll, pitch, yaw to transform from inertial to body coordinates. This approach has been used in reference [6].

The attitude angle measurement model is given by

$$\theta_m = \theta + b_{\theta} + v_{\theta}(S)$$

where

- $b_{\theta}$ = platform misalignment bias vector (used to formulate $P_{C'P}$)
- $v_{\theta}(S)$ = attitude measurement noise vector.

The transformation matrix used to transform from body to inertial coordinates in terms of the elements of the $\theta$ vector is given by

$$I_{CB} = \begin{bmatrix}
\cos\theta\cos\phi & \sin\phi\sin\theta\cos\phi + \cos\phi\sin\theta & \cos\phi\sin\theta\sin\phi - \sin\phi\cos\theta \\
\sin\theta\sin\phi & -\cos\phi\sin\theta\cos\phi + \sin\phi\cos\theta & \cos\phi\cos\theta \\
\cos\theta\sin\phi & -\sin\phi\sin\theta\sin\phi & \sin\phi\cos\theta \\
\end{bmatrix}$$

3.1.2.3 Ground Based Tracking Measurements

Ground based radar tracking devices can provide measurements of range, azimuth and elevation from the radar sight to the vehicle. Azimuth and elevation are established relative to the sight's local level. If
the tracking device is a passive optical tracker (not laser) then only
azimuth and elevation measurements are available requiring more than one
to establish position information.

Defining \(x, y, \) and \(z\) as the local east, north and up position of the
vehicle relative to the ground based tracking device, the radar measurement
equations are given by

\[
\rho = (x^2 + y^2 + z^2)^{1/2} + b_\rho + v_\rho \tag{35}
\]

\[
A = \tan^{-1}\left(\frac{x}{y}\right) + b_A + v_A \tag{36}
\]

\[
E = \tan^{-1}\left(\frac{z}{\sqrt{x^2 + y^2}}\right) + b_E + \Delta E + v_E \tag{37}
\]

where

\(b_\rho, b_A, b_E = \) range, azimuth, elevation biases

\(\Delta E = \) atmospheric refraction correction

\(v_\rho, v_A, v_E = \) range, azimuth, elevation measurement noise.

The position vector of the vehicle relative to the tracking device is
given by

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \Delta \xi_A^{\text{LL}} = \xi_{\text{LL,ECF}} \left[ \xi_{\text{ECF,ECI}}^{-1}(I) - \xi_{\text{RDR}}^{(ECF)} \right] \tag{38}
\]
where

\[ \text{LL}_C^{\text{ECF}} = \text{transformation from earth center fixed to local level} \]
\[ \text{ECF}_C^{\text{ECI}} = \text{earth centered inertial to earth centered fixed} \]
\[ r_{\text{RDR}}^{(\text{ECF})} = \text{position vector of tracking device in ECF coordinates.} \]

The transformation matrix \( \text{LL}_C^{\text{ECF}} \) is given by

\[
\text{LL}_C^{\text{ECF}} = \begin{bmatrix}
-sin\lambda & -sinL\cos\lambda & cosL\cos\lambda \\
\cos\lambda & -sinL\sin\lambda & cosL\sin\lambda \\
0 & \cosL & \sinL \\
\end{bmatrix}
\]

where \( L \) and \( \lambda \) are the geodetic latitude and east longitude of the device. The transformation matrix \( \text{ECF}_C^{\text{ECI}} \) is given by

\[
\text{ECF}_C^{\text{ECI}} = \begin{bmatrix}
\cos[\omega_E(t - t_{\text{RNP}})] & \sin[\omega_E(t - t_{\text{RNP}})] & 0 \\
-sin[\omega_E(t - t_{\text{RNP}})] & \cos[\omega_E(t - t_{\text{RNP}})] & 0 \\
0 & 0 & 1 \\
\end{bmatrix}_{[\text{RNP}]}
\]

where

\[ \omega_E = \text{earth rotation rate} \]
\[ t_{\text{RNP}} = \text{time tag for RNP matrix} \]

The position vector, \( r_{\text{RDR}}^{(\text{ECF})} \), of the tracking device is given by
$\mathbf{X}_{RDR}^{(ECF)} = \begin{bmatrix}
\frac{R_E}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \cos L \cos \lambda \\
\frac{R_E}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \cos L \sin \lambda \\
\frac{R_E (1 - e)^2}{\sqrt{\cos^2 L + (1 - e)^2 \sin^2 L}} + h \sin L
\end{bmatrix}$

where

$R_E = \text{equatorial radius of Fisher ellipsoid}$

$e = \text{flattening of Fisher ellipsoid}$

$h = \text{altitude of the device above Fisher ellipsoid}$
3.2 Linearized System State and Measurement Equations

The vehicle equations of motion are nonlinear functions of their motion variables and are implicit functions of other elements of the system states. The measurement equations involve similar function relationships. The linearizations for the filter/smooother algorithm require partial derivatives with respect to the motion variables, i.e., $y^{(B)}$ and $\theta$, and with respect to other elements of the state vector, yielding explicit functional relationships for the elements of interest.

For system state equations the partial derivatives will be presented in section 3.2.1 for the state elements in order of occurrence for the first twelve states. Other partial derivatives for candidate state elements will follow in section 3.3.1. The measurement equation partial derivatives for the first twelve states will be presented in section 3.2.2. Partial derivatives of the measurement equations for other candidate states will be presented in section 3.3.2.

The resulting partial derivatives are imbedded into the linearized system state matrix, $F(x(t), t)$, as shown in Figure 3.2-1. A corresponding linearized measurement matrix, $H(x_k)$, is similarly formed with the measurement equations' partial derivatives.

3.2.1 System State Partial Derivatives

Partial derivatives of each of the equations listed in Figure 3.2-1 are developed in their order of occurrence with respect to the order of...
FIGURE 3.2-1. Linearization for Filter/Smooother Model

(Dynamics)

<table>
<thead>
<tr>
<th>$T$ (I)</th>
<th>$V$ (B)</th>
<th>$\theta$</th>
<th>$\omega$</th>
</tr>
</thead>
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<td>$\delta r^*(I)$</td>
<td>$\delta r^*(I)$</td>
</tr>
<tr>
<td>$\delta V(B)$</td>
<td>$\delta r^*(I)$</td>
<td>$\delta V(B)$</td>
<td>$\delta V(B)$</td>
</tr>
<tr>
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<td>$\delta V(B)$</td>
</tr>
<tr>
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<td>0</td>
<td>$\delta \theta$</td>
</tr>
<tr>
<td>$\delta \omega$</td>
<td>$\delta \omega$</td>
<td>$\delta \omega$</td>
<td>$\delta \omega$</td>
</tr>
<tr>
<td>$\delta r^*(I)$</td>
<td>$\delta V(B)$</td>
<td>$\delta V(B)$</td>
<td>$\delta V(B)$</td>
</tr>
</tbody>
</table>

Critical Jakobian 5.5.15: OF POOR QUALITY
the corresponding states. Partial derivatives of thrust terms are presented as though for a single device.

**Inertial Position Rate Equation**

The first nonzero partial derivative of the $\mathbf{r}^{(I)}$ equation is with respect to $\mathbf{v}^{(B)}$:

$$\frac{\partial}{\partial \mathbf{v}^{(B)}} \mathbf{r}^{(I)} = I_{CB}.$$  \hspace{1cm} (42)

The second nonzero partial derivative is with respect to $\theta$. This partial derivative results in a third order tensor and occurs frequently in later developments. The generalized form is presented in Appendix A.

**Body Velocity Rate Equation**

The partial derivative of $\mathbf{v}^{(B)}$ with respect to $\mathbf{r}^{(I)}$ is given for altitude terms approximately as

$$\frac{\partial}{\partial \mathbf{r}^{(I)}} \mathbf{v}^{(B)} = \frac{\partial}{\partial h} \mathbf{r}^{(I)}.$$  \hspace{1cm} (43)

where

$$\frac{\delta \mathbf{v}^{(B)}}{\delta h} = \frac{\delta \mathbf{v}^{(B)}}{\delta \mathbf{r}^{(I)}} \frac{\delta \mathbf{r}^{(I)}}{\delta h} = \frac{\lambda_0^2}{2m} \frac{\delta \mathbf{r}^{(I)}}{\delta h} + \frac{\rho \lambda_0^2}{m} \frac{\delta \mathbf{r}^{(I)}}{\delta h} + \frac{\rho \lambda_0^2}{2m} \frac{\delta \mathbf{r}^{(I)}}{\delta h} + \frac{\rho \lambda_0^2}{2m} \frac{\delta \mathbf{r}^{(I)}}{\delta h}$$

$$+ \frac{1}{m} (B C \frac{\delta \mathbf{r}^{(Q)}}{\delta p_s} + \frac{\delta \mathbf{r}^{(B)}}{\delta p} + \frac{\delta \mathbf{r}^{(B)}}{\delta p} + \frac{\delta \mathbf{r}^{(B)}}{\delta p} + \frac{\delta \mathbf{r}^{(B)}}{\delta p} + \frac{\delta \mathbf{r}^{(B)}}{\delta p})$$  \hspace{1cm} (44)
The partial derivatives of \( \frac{\delta v_m}{\delta v(B)} \), \( \frac{\delta \alpha}{\delta v(B)} \), \( \frac{\delta \beta}{\delta v(B)} \), \( \frac{\delta \alpha}{\delta \alpha} \), \( \frac{\delta \beta}{\delta \alpha} \), and \( \frac{\delta v_m}{\delta \alpha} \) occur frequently and are given in Appendix B.

The gravity vector \( \frac{B}{C} \frac{I}{D} \frac{(1)}{E} \frac{f(I)}{G} \) partial derivative with respect to \( \frac{f(I)}{G} \) is

\[
\frac{B}{C} \frac{I}{D} \frac{\delta g(I)}{\delta f(I)} = \frac{B}{C} \frac{I}{D} \frac{\mu}{|f(I)|^3} \left( \begin{array}{ccc}
\frac{r_1}{|f|} - 1 & \frac{r_1 r_2}{|f|} & \frac{r_1 r_2}{|f|} \\
\frac{r_1 r_2}{|f|} & \frac{r_2}{|f|} & \frac{r_2}{|f|} \\
\frac{r_1 r_3}{|f|} & \frac{r_2 r_3}{|f|} & \frac{r_3}{|f|} - 1
\end{array} \right)
\]

where

\( \mu = \) gravitational constant.

The partial derivative, \( \frac{\delta v}{\delta f(I)} \), is the sum of the matrices in equations 43 and 45.

The partial derivative of \( \frac{v}{g(B)} \) with respect to \( \frac{v}{g(B)} \) is given by

\[
\frac{\delta v}{\delta v(B)} = \frac{\rho A v_m}{m} \frac{\delta v_m}{\delta v(B)} + \frac{\rho A v_2}{2m} \frac{\delta c_2}{\delta v(B)} + \frac{\rho A v_2}{2m} \frac{\delta c_2}{\delta v(B)} + \frac{\rho A v_2}{2m} \frac{\delta c_2}{\delta v(B)}
\]

\[
+ \frac{1}{m} \left[ \frac{\delta f}{\delta \alpha} \right] \left( \frac{\delta v}{\delta v(B)} \right) - \frac{\delta f}{\delta \beta} \left( \frac{\delta v}{\delta v(B)} \right) \]

(46)
\{\omega\} = \text{skew symmetric matrix made from the elements of the vector } \omega \\
\text{and equivalent to the cross product operator } \omega \times ( ).

The partial derivative of \( v^{*}(B) \) with respect to \( \Theta \) is

\[
\frac{\partial v^{*}(B)}{\partial \Theta} = \frac{\rho A v}{m} \frac{\partial v}{\partial \Theta} - \frac{\partial v}{\partial \Theta} \frac{\partial v}{\partial \Theta} + \frac{\rho A v^2}{2m} \frac{\partial c_{m}}{\partial \Theta} \frac{\partial \alpha}{\partial \Theta} \frac{\partial v}{\partial \Theta} \frac{\partial r}{\partial \Theta} \\
+ \frac{\rho A v^2}{2m} \frac{\partial c_{m}}{\partial \Theta} \frac{\partial \beta}{\partial \Theta} \frac{\partial v}{\partial \Theta} \frac{\partial r}{\partial \Theta} + \frac{\partial f_{(B)}}{\partial \Theta} \frac{\partial v}{\partial X} \frac{\partial v}{\partial X} \frac{\partial r}{\partial X} \\
+ \frac{1}{m} \left[ \frac{\partial f_{(B)}}{\partial v} \frac{\partial \alpha}{\partial \Theta} \frac{\partial v}{\partial \Theta} \frac{\partial r}{\partial \Theta} + \frac{\partial f_{(B)}}{\partial v} \frac{\partial \beta}{\partial \Theta} \frac{\partial v}{\partial \Theta} \frac{\partial r}{\partial \Theta} \right].
\] (47)

The partial derivatives of \( \frac{\partial}{\partial X} \) are given in Appendix B and the partial derivative \( \frac{\partial v}{\partial \Theta} \) is given in Appendix A. The last partial derivative is given in Appendix C.

The partial derivative of \( v^{*}(B) \) with respect to \( \omega \) is

\[
\frac{\partial v^{*}(B)}{\partial \omega} = \frac{\rho A v^2}{2m} \frac{\partial c_{m}}{\partial \omega} \frac{\partial \alpha}{\partial \omega} \frac{\partial v}{\partial \omega} + \frac{\partial f_{(B)}}{\partial \omega} \frac{\partial v}{\partial X} \frac{\partial v}{\partial X} \frac{\partial r}{\partial X}.
\] (48)
**Euler Angle Rate Equation**

The Euler angle rate equation is a function of both the Euler angles and the inertial rate. The linearization will yield the two associated matrices.

First with respect to the vector $\theta$, the following matrix results

$$
\frac{\partial \hat{\theta}}{\partial \theta} = \begin{bmatrix}
q \cos \phi \tan \theta - r \sin \phi \tan \theta & q \sin \phi \sec^2 \theta + r \cos \phi \sec^2 \theta & 0 \\
-q \sin \phi - r \cos \phi & 0 & 0 \\
q \cos \phi \sec \theta - r \sin \phi \cos \theta & q \sin \phi \sec \theta \tan \theta + r \cos \phi \sec \theta \tan \theta & 0
\end{bmatrix} (49)
$$

The partial derivative of $\hat{\theta}$ with respect to $\omega$ is

$$
\frac{\partial \hat{\theta}}{\partial \omega} = \begin{bmatrix}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{bmatrix} (50)
$$

**Inertial Angular Acceleration Equation**

The first partial derivative of this equation is with respect to the vector $\Gamma^{(I)}$. Using the approximation indicated in equation 43, this partial derivative is
\[
\frac{\partial \omega^*}{\partial \Gamma(I)} = [I]^{-1} \left\{ \frac{\rho v_m^2 A d c_m}{2} \frac{\partial p}{\partial h} - c_m + \left( \rho v_m A c_m \right) \frac{\partial p}{\partial h} \right\} + \left( \rho v_m A c_m \right) \frac{\partial p}{\partial h}
\]

\[
\left( \frac{\partial \omega^*}{\partial \Gamma(I)} \right)^T \left( \frac{\partial \omega}{\partial \delta \Gamma(B)} \right)
\]

Next, with respect to the vector \( \Gamma(B) \), the partial derivative is

\[
\frac{\partial \omega}{\partial \Gamma(B)} = [I]^{-1} \left\{ \left( \rho v_m A c_m \right) \frac{\partial p}{\partial h} - c_m + \left( \rho v_m A c_m \right) \frac{\partial p}{\partial h} \right\} + \left( \rho v_m A c_m \right) \frac{\partial p}{\partial h}
\]

\[
\left( \frac{\partial \omega}{\partial \Gamma(B)} \right)^T \left( \frac{\partial \omega}{\partial \delta \Gamma(B)} \right)
\]
The partial derivative with respect to the vector $\theta$ is

$$\frac{\partial \omega}{\partial \theta} = [I]^{-1} \left\{ (\rho v_m A) \frac{\partial c_m}{\partial \theta} + \left( r_A^{(B)} - r_{c_g}^{(B)} \right) \times \rho v_m A \frac{\partial c_f}{\partial \theta} \right\} \frac{\partial v_m}{\partial \theta}$$

$$+ \left( \frac{\rho v_m^2 m d c_m}{2} \frac{\partial c_m}{\partial \alpha} + \left( r_A^{(B)} - r_{c_g}^{(B)} \right) \times \frac{\rho v_m^2 A d c_f}{2} \frac{\partial c_f}{\partial \alpha} \right) \frac{\partial \alpha}{\partial \theta}$$

$$+ \left( \frac{\rho v_m^2 m d c_m}{2} \frac{\partial c_m}{\partial \beta} + \left( r_A^{(B)} - r_{c_g}^{(B)} \right) \times \frac{\rho v_m^2 A d c_f}{2} \frac{\partial c_f}{\partial \beta} \right) \frac{\partial \beta}{\partial \theta}$$

$$+ \left( \frac{\partial T}{\partial \alpha} \frac{\partial \alpha}{\partial \theta} + \frac{\partial T}{\partial \beta} \frac{\partial \beta}{\partial \theta} \right).$$

The final partial derivative for the first twelve states is with respect to the vector $\omega$. This operation yields

$$\frac{\partial \omega}{\partial \omega} = [I]^{-1} \left( \frac{\partial c_m}{\partial \omega} \frac{\partial c_m}{\partial \omega} + \left( r_A^{(B)} - r_{c_g}^{(B)} \right) \times \frac{\partial c_f}{\partial \omega} \frac{\partial c_f}{\partial \omega} \right) \frac{\partial v_m}{\partial \omega}$$

$$+ \{ I_{12} \} - \{ \omega \} I.$$
3.2.2 Measurement Partial Derivatives

The measurements assumed to be available, as discussed earlier, include ground based radar tracking, inertially stabilized platform attitudes relative to the vehicle body, and stabilized platform mounted 3 axis orthogonal accelerations. As with the state dynamics matrix, the measurement equations are linearized about the best state estimates.

Radar Track Measurement Equation

Referring to the radar track measurement equations, the required partial derivatives are

\[ \frac{\delta p}{\delta r(I)} = \frac{\delta p}{\delta \Delta r_v(L)} \frac{\delta \Delta r_v(L)}{\delta r(I)} \]  
(55)

\[ \frac{\delta A}{\delta r(I)} = \frac{\delta A}{\delta \Delta r_v(L)} \frac{\delta \Delta r_v(L)}{\delta r(I)} \]  
(56)

\[ \frac{\delta E}{\delta r(I)} = \frac{\delta E}{\delta \Delta r_v(L)} \frac{\delta \Delta r_v(L)}{\delta r(I)} \]  
(57)

The last partial derivative in each of these equations, \( \frac{\delta \Delta r_v(L)}{\delta r(I)} \), is

\[ \frac{\delta \Delta r_v(L)}{\delta r(I)} = LL_{ECF} ECF_{ECI} \]  
(58)
The rest of the required partial derivatives are

\[
\frac{\partial p}{\partial \Delta_r^{(LL)}} = \Delta_r^{(LL)} T / |\Delta_r| \tag{59}
\]

\[
\frac{\partial A}{\partial \Delta_r^{(LL)}} = \left[ \frac{-y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}, 0 \right] \tag{60}
\]

\[
\frac{\partial E}{\partial \Delta_r^{(LL)}} = \left[ \frac{-xz}{\rho^2 \sqrt{x^2 + y^2}}, \frac{-yz}{\rho^2 \sqrt{x^2 + y^2}}, \frac{\sqrt{x^2 + y^2}}{\rho^2} \right] \tag{61}
\]

**Inertially Stabilized Platform Attitude Equation**

The inertial platform is assumed to provide attitude angle measurements of the true attitude plus an attitude bias plus measurement noise. The partial derivative of the measured attitudes with respect to the vector \( \Theta \) yields an identity matrix.

**Accelerometer Measurement Equation**

The accelerometer senses specific body forces excluding gravity along the sensing axes. With reference to the accelerometer equation, the partial derivative with respect to \( r^{(I)} \) is
The partial derivative with respect to $v(B)$ yields

$$\frac{\partial \delta_m(S)}{\partial v(B)} = S C B \left[ \frac{\rho A v m}{2m} \frac{\partial \rho}{\partial h} c_f \frac{\partial v m}{\partial h} + \frac{\partial^2 A v}{2m} c_f \frac{\partial c_f}{\partial h} \right] + \frac{1}{m} \left[ \frac{\partial f_p}{\partial m} \frac{\partial c_f}{\partial v(B)} + \frac{\partial f_p}{\partial \alpha} \frac{\partial c_f}{\partial v(B)} + \frac{\partial f_p}{\partial \beta} \frac{\partial c_f}{\partial v(B)} \right]$$

(63)

For the partial derivative of the accelerometer with respect to the vector $\theta$, the measurement equation is temporarily rewritten as

$$a_m(S) = S C P' P' C B_{(B)} + b_{(S)} + v_{(S)}$$

(64)

where the vector $\mathbf{g}_{(B)}$ represents the sum of the aerodynamic, thrust, plume and rotational coupling terms. The matrix $P' C B$ is the same matrix as $I_C^B$. The required partial derivative results from
The partial derivative on the right hand side is developed in Appendix A with the vector $s^{(B)}$ representing the sum of the terms indicated above.

The final partial derivative for the accelerometer measurement is with respect to the body rotation vector $\omega$. Defining

$$\Delta r_s = \begin{bmatrix} \Delta r_1 \\ \Delta r_2 \\ \Delta r_3 \end{bmatrix} = (r_s^{(B)} - r_{cg}^{(B)})$$

and denoting $\omega_i$ as the $i^{th}$ element of the vector $\omega$, the resulting matrix is

$$\frac{\partial a_m}{\partial \omega} = s_c^B \begin{bmatrix} w_2 \Delta r_2 + w_3 \Delta r_3 & w_1 \Delta r_2 - 2w_2 \Delta r_1 & w_1 \Delta r_3 - 2w_3 \Delta r_1 \\ w_2 \Delta r_1 - 2w_1 \Delta r_2 & w_1 \Delta r_1 + w_3 \Delta r_3 & w_2 \Delta r_3 - 2w_3 \Delta r_2 \\ w_3 \Delta r_1 - 2w_1 \Delta r_3 & w_3 \Delta r_2 - 2w_2 \Delta r_3 & w_1 \Delta r_1 + w_2 \Delta r_2 \end{bmatrix}$$
3.2.3 Additional Parameter Partial Derivatives

The mathematical developments are presented in this section for the partial derivatives of the system and measurement equations to allow for additional candidate parameters to be included in the estimation algorithms. These parameters include center-of-gravity, $r_{cg}$, moments of inertia, $I$, wind velocity, $v_w$, and inertial platform tilt errors. Aerodynamic and plume parameter partial derivatives are also presented.

The computer program is being structured to permit these parameters to be easily incorporated without significant impact on the program code.

3.2.3.1 Center-of-Gravity

From equation 23, the partial derivative of angular acceleration with respect to $r_{cg}$ is

$$\frac{\partial \omega}{\partial r_{cg}} = [I]^{-1} \left[ -\frac{\rho A v^2}{2} \frac{c_f}{c} + \sum_{i=1}^{n} \frac{B_i}{c_i} + \frac{f_{TI}}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

From equation 32, the partial derivative of the measured acceleration with respect to $r_{cg}$ is

$$\frac{\partial a_m}{\partial r_{cg}} = -S C B \frac{\tau w \times w}{w}$$
3.2.3.2 Moments of Inertia

The moments-of-inertia are grouped into "principal" terms, \( i_p \), and cross product terms, \( i_{cp} \). From equation 24, these vectors are defined as

\[
\begin{bmatrix}
I_x \\
I_y \\
I_z
\end{bmatrix}
\]

\( i_p \)  \hspace{1cm} (70)

and

\[
\begin{bmatrix}
I_{xy} \\
I_{zx} \\
I_{yz}
\end{bmatrix}
\]

\( i_{cp} \)  \hspace{1cm} (71)

With these definitions, equation 23 is rewritten as

\[
e_\ast = (I)^{-1} \left[ \Sigma T - \{ \omega \} \right] \begin{bmatrix}
\omega_1 & 0 & 0 \\
0 & \omega_2 & 0 \\
0 & 0 & \omega_3
\end{bmatrix} \begin{bmatrix}
i_p \\
i_{cp}
\end{bmatrix}
\]

\hspace{1cm} (72)
where $\Sigma T$ represents the sum of the nonrotational torques in equation 23.

Defining an intermediate vector $\mathbf{a}$ as

$$\mathbf{a} = \Sigma T - \mathbf{w} \times (I\mathbf{w})$$

the partial derivatives of the angular acceleration with respect to $i_p$ and $i_{cp}$ are

$$\begin{align*}
\frac{\partial \omega_z}{\partial i_p} &= \frac{\partial}{\partial i_p} (I^{-1} \mathbf{a}) \bigg|_{\mathbf{a}-\text{fixed}} - (I^{-1}) \begin{bmatrix}
0 & -\omega_2 \omega_3 & \omega_2 \omega_3 \\
\omega_1 \omega_3 & 0 & -\omega_1 \omega_3 \\
-\omega_1 \omega_2 & \omega_1 \omega_2 & 0
\end{bmatrix} \\
\frac{\partial \omega_z}{\partial i_{cp}} &= \frac{\partial}{\partial i_{cp}} (I^{-1} \mathbf{a}) \bigg|_{\mathbf{a}-\text{fixed}} - (I^{-1}) \begin{bmatrix}
\omega_1 \omega_2 & -\omega_1 \omega_2 & \omega_3 - \omega_2 \\
-\omega_2 \omega_3 & \omega_2 - \omega_3 & \omega_1 \omega_2 \\
\omega_2 - \omega_1 & \omega_3 - \omega_1 & -\omega_1 \omega_3
\end{bmatrix}
\end{align*}$$

where
\[ \frac{\partial}{\partial \zeta_{i,p}} (I^{-1} a) = \frac{1}{\Delta} \begin{bmatrix} \sigma_{z1} + \sigma_{zx3} & \sigma_{y1} + \sigma_{xy2} \\ \sigma_{z2} + \sigma_{yz3} & 0 & \sigma_{x2} + \sigma_{xy1} \\ \sigma_{y3} + \sigma_{yz2} & \sigma_{x3} + \sigma_{zx1} & 0 \end{bmatrix} \]

(76)

\[ \begin{bmatrix} (Iy)_z - (Iy)_z \sigma_{a1} & (Iy)_z - (Iy)_z \sigma_{a1} & (Iy)_z - (Iy)_z \sigma_{a1} \\ (Iy)_z - (Iy)_z \sigma_{a2} & (Iy)_z - (Iy)_z \sigma_{a2} & (Iy)_z - (Iy)_z \sigma_{a2} \\ (Iy)_z - (Iy)_z \sigma_{a3} & (Iy)_z - (Iy)_z \sigma_{a3} & (Iy)_z - (Iy)_z \sigma_{a3} \end{bmatrix} \]

and

\[ \frac{\partial}{\partial \zeta_{i,c,p}} (I^{-1} a) = \]

\[ \begin{bmatrix} \sigma_{z2} + \sigma_{yz3} & \sigma_{y3} + \sigma_{yz2} & -2\sigma_{y2} + \sigma_{zx3} + \sigma_{xy3} \\ \sigma_{z1} + \sigma_{zx3} & \sigma_{y1} + \sigma_{yz2} & \sigma_{x3} + \sigma_{zx1} \\ \sigma_{y1} + \sigma_{zx2} + \sigma_{xy3} & \sigma_{y1} + \sigma_{xy2} & \sigma_{x2} + \sigma_{xy1} \end{bmatrix} \]

(77)
\[
- \frac{1}{\Delta} \begin{bmatrix}
-2(I_{xy} + I_{yz})a_1 & -2(I_{yz} + I_{yx})a_1 & -2(I_{xy} + I_{yx})a_1 \\
-2(I_{yx} + I_{xz})a_2 & -2(I_{yz} + I_{xz})a_2 & -2(I_{xy} + I_{xz})a_2 \\
-2(I_{yx} + I_{yz})a_3 & -2(I_{yz} + I_{xz})a_3 & -2(I_{xy} + I_{xz})a_3
\end{bmatrix}
\]

and

\[
\Delta = I_{xy} I_{yz} I_{xx} I_{yx} I_{xz} I_{zx} I_{xy} I_{yz} I_{xx} I_{zx} I_{xy} I_{xz} I_{yx} I_{zx} I_{zz} I_{xz} I_{zy} I_{yx} I_{yx} I_{zx} I_{zy} I_{xz} I_{zy}
\]  

(78)

3.2.3.3 Wind Velocity

From equations 20 and 25, the partial derivative of the vehicle acceleration with respect to \( V_w \) is

\[
\frac{\partial v(B)}{\partial V_w} = \frac{\partial v}{\partial m} m^2 C_f \frac{\partial v}{\partial \alpha} C_D + \frac{\partial v}{\partial m} m^2 C_f \frac{\partial v}{\partial \delta} C_D + \frac{\partial v}{\partial m} m^2 C_f \frac{\partial v}{\partial \delta} C_D .
\]  

(79)

The first of the partial derivatives in equation 79 can be obtained from the following equation

\[
v_m^2 = v_T v_T = v(B)^T (B) v_w + v_T v_T C_B C_B v_w
\]  

(80)

From equation 80, the following is obtained

\[
\frac{\partial v_m^2}{\partial V_w} = -2 v_T T B C.
\]  

(81)
Denoting the elements of the matrix $B$ as $c_{11}$, $c_{12}$, etc., the following equations are obtained for the partial derivatives of $\alpha$ and $\beta$ with respect to $v_w$:

$$\frac{\partial \alpha}{\partial v_w} = \frac{-1}{v r_1^2 + v r_3^2} \begin{bmatrix} (v r_1 c_{31} - v r_3 c_{11}) \\ (v r_1 c_{32} - v r_3 c_{12}) \\ (v r_1 c_{33} - v r_3 c_{13}) \end{bmatrix}$$

and

$$\frac{\partial \beta}{\partial v_w} = \frac{-1}{v_m \sqrt{v r_2^2 + v r_3^2}} \begin{bmatrix} v r_2 c_{21} - \frac{v r_2}{v_m} (v r_1 c_{11} + v r_2 c_{21} + v r_3 c_{31}) \\ v r_2 c_{22} - \frac{v r_2}{v_m} (v r_1 c_{12} + v r_2 c_{22} + v r_3 c_{32}) \\ v r_2 c_{23} - \frac{v r_2}{v_m} (v r_1 c_{13} + v r_2 c_{23} + v r_3 c_{33}) \end{bmatrix}$$

3.2.3.4 Inertial Platform Tilt

Temporarily rewriting equation (32) as

$$\alpha_m^{(S)} = S^P (I + 6\theta x) \alpha$$

(84)
where

\[ \Delta \theta = \text{vector whose elements are the axes misalignments} \]

\[ s = \text{sum of the bracketed terms in equation 32 multiplied by } P^c B. \]

The following partial derivative of the measured acceleration with respect to \( \Delta \theta \) is obtained

\[
\frac{\partial a}{\partial \Delta \theta} = S^P \left[ s \right], \quad (85)
\]

3.2.3.5 Aerodynamic and Plume Parameters

A linear model for the aerodynamic and plume characteristics is used. This model is expanded as

\[
C_f = C_{fo} + C_{fa} \alpha + C_{f\beta} \beta + \ldots \quad (86)
\]

\[
C_m = C_{mo} + C_{ma} \alpha + C_{mb} \beta + \ldots \quad (87)
\]

\[
f_p = f_{p0} + f_{pa} \alpha + f_{p\beta} \beta + \ldots \quad (88)
\]

where additional terms to represent rates, cross couplings, and controls can be included.

The basic approach of establishing the partial derivatives will be illustrated for a couple of terms, \( C_{fa} \) and \( C_{ma} \). Using these example
illustrations, the rest of the candidate parameters can be similarly obtained. From equation 20, the following partial derivative is obtained

$$\frac{\delta v^{(B)}}{\delta c_f} = \frac{\delta v^{(B)}}{\delta c_f} = \frac{\delta c}{\delta c_f} = \frac{\rho Av^2}{2m} \alpha [U]$$ (89)

where

$$[U] = \text{unit } 3 \times 3 \text{ matrix with one's (1) on the diagonal and zeros off the diagonal}$$

From equation 23, the partial derivative of angular acceleration with respect to \( c_m \) is

$$\frac{\delta \omega^*}{\delta c_m} = \frac{\delta \omega^*}{\delta c_m} = [I]^{-1} \frac{\rho Av^2}{2m} \alpha [U].$$ (90)

The corresponding partial derivative with respect to \( c_f \) is

$$\frac{\delta \omega^*}{\delta c_f} = \frac{\delta \omega^*}{\delta c_f} = [I]^{-1} \frac{\rho Av^2}{2m} \frac{f_{r_{A}}^{(B)}}{r_{c_{g}}^{(B)}} \alpha [U].$$ (91)

The static aerodynamic coefficient model has been obtained by a multiple regression analysis of the current aerodynamic tabular data. This model is presented in Appendix D with the associated regression coefficients.
3.3 Propulsion Parameter States and Measurements

A candidate approach for incorporating the NASA propulsion model's capabilities has been identified. This approach utilizes nominal predicted values of thrust, pressure, propellant and pressurant mass flow rates, and utilizes sensitivities or partial derivatives of these variables with respect to the independent parameters selected for estimation by the algorithm.

The approach is to include deviations from nominal values of measured chamber pressure, power level, propellant and pressurant mass flow rates as states. The models associated for these deviations are time correlated random processes. Then as states, partial derivatives of the first twelve states with respect to these variables will be required.

For the SSME and SRB, this modeling approach is discussed in the following. Additionally, the necessary partial derivatives of the first twelve state variables with respect to the additional states are presented.

3.3.1 SSME Propulsion Parameter Model

For the SSME, the total actual values of vacuum thrust and oxidizer mass flow rates are modeled by

\[ f_T = f_{T, \text{nom}} + \Delta f_T \]  

and

\[ m_{0, 2} = m_{0, 2, \text{nom}} + \Delta m_{0, 2} \]
The measurements of fuel mass flow rate, pressurant mass flow rates and power level are modeled as

\[
\dot{m}_{H_2} = \dot{m}_{H_2}^{\text{nom}} + \Delta m_{H_2} + b_{m_{H_2}} + s_{m_{H_2}} \tag{94}
\]

\[
\dot{m}_{H_2} = \dot{m}_{H_2}^{\text{nom}} + \Delta m_{H_2} + b_{m_{H_2}} + s_{m_{H_2}} \tag{95}
\]

\[
\dot{m}_{O_2} = \dot{m}_{O_2}^{\text{nom}} + \Delta m_{O_2} + b_{m_{O_2}} + s_{m_{O_2}} \tag{96}
\]

and

\[
PL = PL^{\text{nom}} + APL + b_{PL} + s_{PL} \tag{97}
\]

These measured quantities include measurement noise \( s(\) and potential bias states \( b(\) modeled as random constants. In these measurements, the \( \Delta' \)d variables are to be included as states in the estimation algorithm. If the nominal values are zero or unknown, then the \( \Delta' \)d variables absorb the entire estimate. Where required, the estimate for the variables used in the estimation algorithm is formed using the nominal and the estimate of the deviation, etc. In example, thrust and fuel mass flow rate estimates are formed as

\[
\hat{f}_T = f_T^{\text{nom}} + \hat{\Delta f}_T \tag{98}
\]
and

\[ m_{H_2} = m_{H_2}^{\text{nom}} + \Delta m_{H_2} + b m_{H_2} \]  

(99)

The deviation or \( \Delta'd \) measurement variables are modeled as time correlated random variables. This permits these variables to vary within a band of frequencies. The typical model is then given as

\[ \frac{d}{dt} \Delta(\cdot) = -\frac{1}{\tau(\cdot)} \Delta(\cdot) + \frac{1}{\tau(\cdot)} s(\cdot) \]  

(100)

where the parenthesis \( (\cdot) \) would be replaced by the variables, i.e., \( m_{H_2} \).

For the SSME, an additional variable \( \Delta c_{\text{mult}} \) is modeled as in equation 100 and included as a state variable with the \( \Delta'd \) measurement variables.

The thrust deviation is expanded as in the following truncated Taylor series as a function of the independent parameters.

\[ \Delta f = \frac{\partial f}{\partial m_{H_2}} \Delta m_{H_2} + \frac{\partial f}{\partial m_{O_2}} \Delta m_{O_2} + \frac{\partial f}{\partial c} \Delta c + \frac{\partial f}{\partial PL} \Delta PL + \frac{\partial f}{\partial MR} \Delta MR. \]  

(101)

In the \( \nu^{(B)} \) and \( \omega \) equations, with equation 101 replacing \( f_T \),

the partial derivatives of \( f_T \) with respect to the \( \Delta'd \) variables are obtained directly from equation 101.
It is desirable to include vehicle mass bias as a state. The SSME's system contribution to the mass deviation is given by

$$\Delta m_{SSME's} = \sum_i (\Delta m_{H_2}^i + \Delta m_{O_2}^i - \Delta m_{H_2}^i - \Delta m_{O_2}^i).$$  \hspace{1cm} (102)

In equation 101, the $\Delta m_{O_2}$ contribution to the mass deviation is not available from measurements. As with the thrust deviation, this quantity is formed as

$$\Delta m_{O_2} = \frac{\partial m_{O_2}}{\partial H_2} \Delta m_{H_2} + \frac{\partial m_{O_2}}{\partial p} \Delta m_{H_2} + \frac{\partial m_{O_2}}{\partial c} \Delta c^* + \frac{\partial m_{O_2}}{\partial \Delta PL} \Delta PL + \frac{\partial m_{O_2}}{\partial \Delta MR} \Delta MR.$$  \hspace{1cm} (103)

which is in terms of other estimated state variables. In equations 101 and 103 the deviation in mixture ratio, $\Delta MR$, is obtained algebraically from

$$\Delta MR = \frac{\Delta m_{H_2}}{m_{H_2}}.$$  \hspace{1cm} (104)

The partial derivatives for the SSME above have been incorporated into the estimation algorithm as functions of engine power level.
3.3.2 SRB Propulsion Parameter Model

The approach for the SRB modeling follows closely that used for the SSME. Candidate independent parameters include propellant burn rate exponent, $a$, and motor efficiency coefficient, $c_m$. Others can be added using this technique.

The actual value of vacuum thrust is given by equation 92. The only measurement available for the SRB is the total pressure at the forward head end of the motor case and is modeled as

$$ P_{OH} = P_{OH}^{nom} + \Delta P_{OH} + b + s $$

where $b$ and $s$ represents a bias and measurement noise respectively.

The independent parameters, $\Delta a$ and $\Delta c_m$, are included in the model as states. The model assumed can be as given by equation 100 or another suitable dynamical process, i.e., random constant.

The thrust deviation is given by the following truncated Taylor series as a function of the candidate independent parameters.

$$ f_T = \frac{\partial f_T}{\partial a} \Delta a + \frac{\partial f_T}{\partial c_m} \Delta c_m + \ldots $$

The partial derivatives for the $v^B$ and $w$ equations with respect to the independent parameters are obtained directly from equation 106. The mass deviation equation for the SRB is given as

$$ \Delta m_{SRB} = \sum (\Delta m_i) $$
where

\[ \Delta m_i = \frac{\partial m}{\partial a} \Delta a + \ldots \]  

(108)

The head pressure deviation, \( \Delta P_{OH} \), is expanded similarly

\[ \Delta P_{OH} = \frac{\partial P_{OH}}{\partial a} \Delta a + \ldots \]  

(109)

A simplified model for the SRB's thrust, head pressure and mass flow rate has been developed that can be directly incorporated within the filter algorithm for estimating burn rate coefficient, nozzle coefficient, mass flow rate, etc. This model, to be described below, uses apriori specified burn area and port volume as a function of burn depth into the propellant grain. From this simplified model analytical partial derivatives required by the estimation algorithm can be obtained.

The thrust is given by

\[ f_T = c_m c_T c^* \dot{m} \]  

(110)

where

- \( c_m \) = nozzle coefficient
- \( c_T \) = thrust coefficient
- \( c^* \) = characteristic exhaust velocity
- \( \dot{m} \) = mass flow rate

Two of the required partial derivatives with respect to mass flow rate and nozzle coefficient are easily obtained, viz
The partial derivative with respect to burn rate coefficient is

\[
\frac{\delta f_T}{\delta c_T} = c_T \frac{\delta c}{\delta m} + \frac{\delta m}{\delta c_T} c_T c_m
\]  

(113)

where it has been assumed that \( c^* \) is not a function of \( a \). Using the "ideal" expression for \( c_T \) [7]

\[
c_T = \sqrt{\frac{2\gamma^2}{\gamma - 1} \left[ \frac{2}{(\gamma + 1)^2} \left[ 1 - \left( \frac{P_e}{P_o} \right)^{\gamma - 1} \right] + \frac{P_e - P_a}{A_t} \frac{A_e}{P_o} \right]},
\]  

(114)

where

\[
\gamma = \text{ratio of specific heats}
\]

\[
P_e = \text{motor nozzle exit pressure}
\]

\[
P_a = \text{ambient atmospheric pressure at nozzle exit}
\]

\[
A_e = \text{motor nozzle exit area}
\]

\[
A_t = \text{motor nozzle throat area},
\]

the first partial derivative in equation 113 is
To evaluate the second partial derivative in equation 113, the following equation for pressure [8] is used:

\[
P_0 = \left( \frac{c^* \rho_p A_b}{A_t} \right) \frac{1}{1-n}
\]  

(116)

where

\( \rho_p \) = propellant density

\( A_b \) = propellant burn area

\( n \) = propellant burn rate exponent

The following partial derivative is then obtained

\[
\frac{\partial P_0}{\partial a} = \frac{c^* \rho_p A_b}{A_t} \left( \frac{1}{1-n} \right) a^{1-n}
\]  

(117)

The last partial derivative in equation 113 is obtained from

\[
m = \rho_p r_b A_b = \rho_p a \left( \frac{c^* \rho_a A_b}{A_t} \right)^n A_b.
\]  

(118)

The resulting partial derivative is

\[
\frac{\partial m}{\partial a} = \rho_p \left( \frac{c^* \rho_a A_b}{A_t} \right)^{1-n} a^{1-n}
\]  

(119)
To utilize the head pressure measurement and its sensitivity to parameter variations, the following equation [7] is used

\[ P_{OH} = \frac{P_0}{2} \left[ 1 + \sqrt{1 + 4RT \left( \frac{c_r b \rho f}{A_p P_0} \right)} \right] \]  

(120)

where

- \( R \) = gas constant
- \( T \) = gas absolute temperature
- \( c \) = port circumference
- \( A_p \) = port cross section area
- \( r_b \) = propellant burn rate
- \( f \) = distance from motor nozzle to pressure measurement point

This equation assumes a cylindrical port with an approximately constant cross sectional area.

The partial derivatives with respect to burn rate coefficient and mass flow rate are

\[ \frac{\delta P_{OH}}{\delta a} = \frac{1}{4} \left[ 1 + \sqrt{1 + 4RT \left( \frac{A_p}{c \rho f} \right)} \right] \frac{\delta P_0}{\delta a} \]  

(121)
and

\[
\frac{\delta P_{OH}}{\delta \dot{m}} = 1 \left( \frac{1}{\sqrt{1 + 4RT \left( \frac{\dot{m}^2}{v^2P} \right)}} \right) + \frac{\delta P_{O}}{\delta \dot{m}} + \frac{4RT \frac{\dot{m}^2}{v^2P}}{\sqrt{1 + 4RT \left( \frac{\dot{m}^2}{v^2P} \right)}}
\] (122)

In equations 121 and 122 a cylindrical port has been assumed in determining the port volume \( V_p \). Equation 122 was obtained from equation 120 by replacing the term \( c r_b P_{p} \) by \( \dot{m} \). The partial derivative of \( P_{OH} \) with respect to \( c_m \) is obviously zero. In using these analytical partial derivatives, the basic performance measures of thrust, mass flow rate, head and nozzle pressures, etc. are matched between this model and the NASA SOBER internal ballistics routine results. The burn area and port volume are adjusted in the simple model to obtain the agreement. Then using the adjusted area and volume as a function of burn depth, the partial derivatives are evaluated.

The inclusion of solid propellant thickness, \( t \), as a state variable necessitates the development of the partial derivatives of \( t \) with respect to the solid propulsion parameters and the partial derivative of the measurement \( P_{OH} \) with respect to \( t \). These partial derivatives are given as

\[
\frac{\delta t}{\delta a} = \frac{c^* \rho_d a}{1 - n} \left( \frac{\dot{m}^2}{v^2P} \right)^{1-n}
\] (123)


\[
\frac{\delta P_O}{\delta \tau} = \frac{1}{4} \left( 1 + 16 \frac{\pi RT}{V_p} \frac{a^3}{V_p c^* a_b} \right) \frac{\delta P_O}{\delta \tau} \]

\[
= \frac{8 \pi RT}{V_p} \frac{\delta}{\delta \tau} \left( \frac{a^3}{V_p} \right) \frac{\delta a_b}{\delta \tau} \frac{2}{c^* a_b} + \frac{16 \pi RT}{V_p} \frac{\delta}{\delta \tau} \left( \frac{a^3}{V_p} \right) \frac{a_t^2}{c^* a_b} \]  
\text{(124)}

where

\[
\frac{\delta P_O}{\delta \tau} = \left[ \frac{\rho a_b}{a_t} \frac{\delta c^*}{\delta \tau} + \frac{c^* a_b}{a_t} \frac{\delta a_b}{\delta \tau} + c^* \rho a_b \frac{\delta a_t}{\delta \tau} \right] . \]
\text{(125)}

Here, the partial derivatives of \( c^* \), \( a_b \), and \( a_t \) with respect to \( \tau \) are evaluated numerically.

Finally, the partial derivative of thrust with respect to \( \tau \) is given by

\[
\frac{\delta \Delta f_T}{\delta \tau} = \frac{\delta P_O}{\delta \tau} a_t + \frac{\rho P_O}{\delta \tau} \frac{\delta a_t}{\delta \tau} \]
\text{(126)}
3.3.3 Vehicle Mass

The total rate of change of vehicle mass is given by

$$\frac{d}{dt}(m) = m_{SSME_{nom}} + m_{SRB_{nom}} + \Delta m_{SSME} + \Delta m_{SRB} + m_{NON-CONSUME} \quad (127)$$

The first two terms in this equation are the apriori assumed nominal values. The third and fourth terms were discussed earlier. The last term should be zero; however it can include a mass bias uncertainty \( \Delta m_b \).

The equations, state and measurement, in which mass occurs are the \( v(B) \) and \( a_m \) equations. Assuming equation 123 can be summarized as \( \dot{m} + \dot{\Delta m}_b \) then the mass can be written as \( m + \Delta m_b \). Replacing this expression for the mass in the two indicated equations yields the following partial derivatives with respect to the \( \Delta m_b \).

$$\frac{\partial v^{(B)}}{\partial \Delta m_b} = -\frac{1}{(m + \Delta m_b)^2} \left( \frac{p v_m^2 a}{2} \right) \frac{c_f}{f_p^{(B)}} + 3 \frac{c_\rho}{c_\tau} f_{T_1}(Q) \quad (128)$$

and

$$\frac{\partial a^{(S)}}{\partial \Delta m_b} = -\frac{1}{(m + \Delta m_b)^2} \left( \frac{p v_m^2 a}{2} \right) \frac{c_f}{f_p^{(B)}} + B \frac{c_\rho}{c_\tau} f_{T_1}(Q) \quad (129)$$
4.0 PROJECTED ACTIVITIES DURING UPCOMING MONTH

During the upcoming month the computer programs will be exercised on the NASA computers, and documentation of the computer routines will be developed.

Due to minor delays, SDI is requesting a one month extension at no cost to the government to maximize the capabilities of the computer routines developed.
REFERENCES


APPENDIX A

PARTIAL DERIVATIVE OF THE VECTOR $I_B^C \mathbf{v}$ wrt $\theta$

This partial derivative is one of several that occurs frequently in the formulation of the linearized system state and measurement equations. The desired partial derivative is

$$
\begin{align*}
\frac{\partial }{\partial \theta} & \left[
\begin{array}{c}
(c \cos \theta \cos \phi) v_1 + (\sin \theta \sin \phi) v_2 + (\cos \theta \sin \phi) v_3 \\
(c \sin \theta \sin \phi) v_1 + (-\cos \phi) v_2 + (\sin \phi) v_3 \\
(-\sin \theta) v_1 + (\sin \phi \cos \theta) v_2 + (\cos \phi \cos \theta) v_3
\end{array}
\right] \\
\end{align*}
$$

The resulting matrix is given in Table A-1.
<table>
<thead>
<tr>
<th>( C^B )</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta \cos \theta \mathbf{v}_1 )</td>
<td>( -\cos \theta \sin \theta \mathbf{v}_1 )</td>
</tr>
<tr>
<td>( (\cos \theta \sin \theta \cos \psi + \sin \theta \sin \psi) \mathbf{v}_2 )</td>
<td>( +\sin \theta \cos \theta \cos \psi \mathbf{v}_2 )</td>
</tr>
<tr>
<td>( -(\sin \theta \sin \theta \cos \psi + \cos \theta \sin \psi) \mathbf{v}_3 )</td>
<td>( -\cos \theta \sin \theta \mathbf{v}_3 )</td>
</tr>
<tr>
<td>( \cos \theta \mathbf{v}_2 )</td>
<td>0</td>
</tr>
<tr>
<td>( -\sin \theta \cos \theta \mathbf{v}_3 )</td>
<td>( -\sin \theta \sin \theta \mathbf{v}_3 )</td>
</tr>
</tbody>
</table>
APPENDIX B

\[
\frac{\delta v_m}{\delta v(B)}, \quad \frac{\delta \alpha}{\delta v(B)}, \quad \frac{\delta \beta}{\delta v(B)}, \quad \frac{\delta \alpha}{\delta h}, \quad \frac{\delta \beta}{\delta h} \quad \text{and} \quad \frac{\delta v_m}{\delta h} \quad \text{Expressions}
\]

These partial derivatives occur frequently and will be developed in this appendix. The equation for \( v_r \) is

\[
v_r = \frac{v(B)}{v^{(LL)}} - B_{LL} \frac{v^{(LL)}}{v_w}
\]

Since the wind velocity, \( v^{(LL)} \), is only a function of altitude then

\[
\frac{\partial}{\partial v} = \frac{\partial}{\partial v(B)}. \tag{B-2}
\]

The first partial derivative, \( \frac{\delta v_m}{\delta v(B)} \), is

\[
\frac{\delta v_m}{\delta v(B)} = \begin{bmatrix}
\begin{array}{c}
\frac{v_r_1}{v_m} \\
\frac{v_r_2}{v_m} \\
\frac{v_r_3}{v_m}
\end{array}
\end{bmatrix}^T. \tag{B-3}
\]

The second, \( \frac{\delta \alpha}{\delta v(B)} \), is given by
\[
\frac{\partial \alpha}{\partial v(B)} = \begin{bmatrix}
-v_r_3 \\
-v_r_1^2 + v_r_3^2 \\
v_r_1 \\
v_r_1^2 + v_r_3^2
\end{bmatrix}
\]

The equation for \( \frac{\partial \beta}{\partial v(B)} \) is

\[
\frac{\partial \beta}{\partial v(B)} = \begin{bmatrix}
-v_r_1 v_r_2 \\
v_m \sqrt{v_r_1^2 + v_r_3^2} \\
\sqrt{v_r_1^2 + v_r_3^2} \\
v_m \\
-v_r_2 v_r_3 \\
v_m \sqrt{v_r_1^2 + v_r_3^2}
\end{bmatrix}
\]
The following equations define the last three required partial derivatives

\[ \frac{\partial \alpha}{\partial h} = - \frac{\partial \alpha}{\partial v(B)} B_{C,LL} \frac{\partial v^{(LL)}}{\partial \omega} \]

B-6

\[ \frac{\partial \beta}{\partial h} = - \frac{\partial \beta}{\partial v(B)} B_{C,LL} \frac{\partial v^{(LL)}}{\partial \omega} \]

B-7

\[ \frac{\partial v_m}{\partial h} = - \frac{\partial \beta}{\partial v(B)} B_{C,LL} \frac{\partial v^{(LL)}}{\partial \omega} \]

B-8
APPENDIX C

PARTIAL DERIVATIVE OF THE VECTOR $B_C^I v$ wrt $\theta$

The third of the frequently occurring required partial derivatives is

$$\frac{\partial}{\partial \theta} \begin{bmatrix} \cos \theta \cos \psi v_1 + \cos \theta \sin \psi v_2 - \sin \theta v_3 \\ -\cos \phi \sin \psi v_1 + \sin \phi \sin \theta \sin \psi v_2 + \sin \phi \cos \theta v_3 \\ \sin \phi \sin \theta \cos \psi v_1 + \cos \phi \sin \theta \sin \psi v_2 + \cos \phi \cos \theta v_3 \end{bmatrix}$$

The resulting matrix is given in Table C-1.
TABLE C-1. Partial Derivative of $B^C_v$ wrt $\theta$

\[
\begin{align*}
0 & \quad -\sin\theta \cos\phi \ v_1 \\
& \quad -\sin\theta \sin\phi \ v_2 \\
& \quad -\cos\theta \ v_3 \\
(cos\phi \sin\theta \cos\phi + \sin\theta \sin\phi) v_1 & \quad \sin\phi \cos\theta \cos\phi \ v_1 \\
-(cos\phi \sin\theta \sin\phi - \sin\phi \cos\theta) v_2 & \quad +\sin\phi \cos\theta \sin\phi \ v_2 \\
+ \cos\phi \cos\theta \ v_3 & \quad -\sin\phi \sin\theta \ v_3 \\
-(\sin\phi \sin\theta \cos\phi - \cos\phi \cos\phi) v_1 & \quad \cos\phi \cos\theta \cos\phi \ v_1 \\
-(\sin\phi \sin\theta \sin\phi + \cos\phi \cos\phi) v_2 & \quad +\cos\phi \cos\theta \sin\phi \ v_2 \\
- \sin\phi \cos\theta \ v_3 & \quad -\cos\phi \sin\theta \ v_3 \\
\end{align*}
\]
APPENDIX D
AERODYNAMIC MODELING REGRESSION ANALYSIS AND RESULTS

The aerodynamic data tables provided as IVBC3 data has been incorporated into an aerodynamic coefficient polynomial model. This modeling effort reduces the dimensionality of the numerical tables to one and reduces the storage requirements for the aerodynamic model.

The coefficient model used for the two stages differ slightly as a result of the available data. The regression analysis led in the selection of the form of the aerodynamic model. Terms with insignificant correlation were eliminated from the model.

In equation form, the first stage static coefficients of axial force, $C_A$; normal force, $C_N$; pitching moment, $C_m$; rolling moment, $C_f$; side force, $C_Y$; and yawing moment, $C_n$; are given below

\[
C_A = C_{A_o} + C_{A_\alpha} \alpha + C_{A_\alpha^2} \alpha^2 + C_{A_{\alpha B}} \alpha B + C_{A_B^2} B^2
\]

\[
C_N = C_{N_o} + C_{N_\alpha} \alpha + C_{N_{\alpha B^2}} \alpha B^2
\]

\[
C_m = C_{m_o} + C_{m_\alpha} \alpha + C_{m_{\alpha B^2}} \alpha B^2
\]

\[
C_f = C_{f_o} + C_{f_\beta} \beta + C_{f_{\alpha B}} \alpha B + C_{f_{\alpha^2 \beta}} \alpha^2 B
\]

\[
C_n = C_{n_o} + C_{n_\alpha} \alpha + C_{n_{\alpha B^2}} \alpha B^2
\]
The corresponding second stage model is given by the following equations

\[
C_y = C_{y0} + C_{y\beta} \beta + C_{y\alpha \beta} \alpha \beta + C_{y2} \alpha^2 \beta
\]

\[
C_n = C_{n0} + C_{n\beta} \beta + C_{n\alpha \beta} \alpha \beta + C_{n2} \alpha^2 \beta
\]

\[
C_A = C_{A0} + C_{A\alpha} \alpha + C_{A2} \alpha^2
\]

\[
C_N = C_{N0} + C_{N\alpha} \alpha + C_{N2} \alpha^2
\]

\[
C_m = C_{m0} + C_{m\alpha} \alpha + C_{m2} \alpha^2
\]

\[
C_I = C_{I0} + C_{I\beta} \beta + C_{I\alpha \beta} \alpha \beta + C_{I2} \alpha^2 \beta
\]

\[
C_Y = C_{Y0} + C_{Y\beta} \beta + C_{Y\alpha \beta} \alpha \beta + C_{Y2} \alpha^2 \beta
\]

\[
C_n = C_{n0} + C_{n\beta} \beta + C_{n\alpha \beta} \alpha \beta + C_{n2} \alpha^2 \beta
\]

For the first stage, data from an angle-of-attack range of -6 to +6 degrees was used in the regression analysis. Data from a range of -8 to +4 degrees was used for the second stage. The results, TCXX..., from the regression analysis is presented below for each of the coefficients, C\text{XX}..., above.
### Table D-1. First Stage Axial Force Coefficient

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*Original page of poor quality*
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TABLE D-2. First Stage Pitching Moment Coefficient

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### TABLE D-6. First Stage Yawing Moment Coefficient

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### TABLE D-7. Second Stage Longitudinal Coefficients

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#### PROBLEM SET: CNF2

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### TABLE D-8. Second Stage Lateral Coefficients

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