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DIGITAL MULTI-SHAKER MODAL TESTING

by

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DIGITAL MULTI-SHAKER MODAL TESTING

A Report to
NASA Marshall Space Flight Center
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by

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Introduction

In the design of a structure it is important to know how the structure will respond to forces that are expected to be present in its operating environment. Much time is spent developing analytical models that predict responses, and when the structure is finally built it needs to be physically tested before being placed into operation. This modal testing has become very important in the last few decades with the structures becoming more complex and being placed in more extreme operating environments.

In Chapter 1 of this thesis, a review of several modal testing techniques will be made, along with brief discussions of their advantages and limitations. In Chapter 2, a new technique will be presented which overcomes many of the previous limitations. Several simulated experiments are included in Chapter 3 to verify the validity and accuracy of the new method. Finally, in Chapter 4 conclusions are drawn from the simulation studies and recommendations for further work are presented.

The complete computer code configured for the simulation study is presented in an Appendix.
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Chapter 1

1.1 Overview

The need for modal testing (sometimes referred to as modal analysis) of structures is well known and documented. The problem is that there exists no fool-proof and totally efficient method to extract modal coefficients from structures. Over the past 40 years several different types of analysis have been developed and practiced with varying degrees of success. The analyses range from time-domain to frequency-domain, and from single-excitation to multiple-excitation. To employ each of these methods properly, it is necessary to examine them to find their respective advantages and disadvantages.

1.2 Multi-Shaker Sine Dwell Testing

The oldest and most widely used method is the Multi-Shaker Sine Dwell (MSSD) test, first described by Lewis and Wrisley (1). This method uses shakers attached at several points on the structure. The shakers are tuned in amplitude and phase simultaneously at a given frequency such that the acceleration outputs are 90 degrees out of phase with the corresponding force inputs (this is indicated by a circle on a Lissajous plot). The amplitudes and phases are then recorded giving the mode shape of the structure at that given frequency. The damping can also be obtained by simultaneously shutting off all the applied forces and analyzing the resultant free-decay rate.
The advantages of this method are that a very accurate mode shape can be obtained due to the high signal-to-noise ratio, and input levels can be varied to permit identification of pseudo-modal parameters for non-linear structures. The disadvantages, though, usually outweigh these benefits. To measure the mode shapes accurately, accessibility to several shakers is required, and in some instances the attachment of a shaker at a given location is physically impossible. A second drawback is the time involved in stepping through the frequency range one mode at a time and tuning and re-tuning each shaker to obtain a 90 degree phase lag (or lead).

A revision of the MSSD method by Anderson (2) speeds up the tuning procedure. In this method, interest is placed on maximizing the quadrature part of the output to input relationship. Several closely-spaced modes are tuned in conjunction with one another with the number of shakers used being dependent upon the number of modes being tuned at a time. Though the time required by this method is less, it is still an iterative procedure, which, in itself, makes this method time consuming.

1.3 Time-Domain Analysis

As far as time-domain analyses go, there are basically two types, the Least-Squares Curve-Fitting Time-Domain (LSTD) Method (3) and the eigenvalue Ibrahim Time Domain (ITD) Method (4). The LSTD method is an older and more widely known method than ITD, though ITD is more sophisticated and has fewer limitations.

The LSTD method uses the time history of a free-decay response and upon an initial approximation of the modal parameters, it performs an iterative solution routine, based on the Newton-Raphson procedure, to
produce accurate estimates. The problems with this method are the computer time involved for the iterations (up to 50), the need for user-supplied initial approximations, and the reported high inaccuracies after the first two or three modes (3).

A revision of the LSTD method has recently been developed by Smith (3). In this revision a new type of Newton-Raphson iterative procedure is used. This method is also semi-automatic, with the computer analyzing an FFT of the free-decay response to gather the initial natural frequencies (the damping estimates are still user-supplied). This revision, though, also has difficulty in separating closely-coupled modes. If two peaks cannot be resolved in the initial FFT, then one of the two modes will be omitted from the analysis. Also, to properly identify two closely-spaced modes, the time history must include one or two neat frequency cycles, thus causing a significant increase in the length of the required data record.

The ITD method uses time histories of free-decay responses and time-delayed free-decay responses of several degrees-of-freedom (DOF) to set up an eigenvalue problem that solves for the modal coefficients. To solve for more modes than there are measured DOF, ITD creates pseudo-DOF by using time-delayed responses based on the original responses. This also provides the means for determining the validity of each acquired mode by a comparison of mode shape coefficients.

The ITD method has proven to be an effective tool in modal analysis. There is a problem, though, in the use of this method: it is fairly complicated. For each test there are seven analysis parameters to choose. Also, when there is a low signal-to-noise ratio, ITD produces some systematic errors that must be compensated for by several additional analyses (5).
1.4 Frequency-Domain Analysis

Frequency-domain analysis encompasses several different styles of curve-fitting or eigenvalue analysis. The common features among all the methods are their similarity of excitation and their use of curve-fitting of frequency response functions (FRFs). The structure is commonly excited by one shaker with an input signal in the form of a swept sine wave or some type of random noise. Sometimes, though, the structure may be excited by a single impulse, usually administered by a hammer with a force cell attached.

FRFs are usually acquired by a single-input, single-output procedure. The system is either excited by a hammer or a single point random (SPR) shaker set-up (6). By dividing the cross-spectrum of the input and output by the auto-spectrum of the input an accurate estimate of the FRF can be obtained.

Another procedure for gathering transfer functions has been described by Allemang (7). In this method the structure is excited simultaneously by several random-noise-driven shakers. The excitations and responses are measured and then manipulated.

\[
\{x\} = [H] \{f\}
\]

\[
[GXF] = [H]^{-1} [GFF]
\]

\[
[H] = [GXF] [GFF]^{-1}
\]

where,

- \([GXF]\) = m x n frequency dependent matrix
- \([GXF]\) = Input-output cross-spectrum matrix
- \([GFF]\) = n x n frequency dependent matrix
- \([GFF]\) = Input cross-spectrum matrix
- \(GFF(i,j)\) = Cross-spectrum of input i with respect to input j
This method has the advantage over SPR in that several columns of the transfer matrix can be computed at one time. This method has the disadvantage, though, of a high computational 'cost,' since the $[GFF]$ matrix must be inverted at each frequency.

The simplest curve-fitting technique is the 'peak-picking' method. A plot of imaginary response vs. frequency is used to find the points of maximum magnitude in the imaginary component of the response. If the structure can be represented as a single DOF (SDOF) system in the vicinity of the natural frequency, the point of maximum magnitude corresponds to the natural frequency, and its value is the mode shape component. A half-power scheme can be employed to determine the damping value.

If another mode is close to the chosen mode, errors in the modal parameters will become significant. One way to compensate for this is the SDOF circle-fit technique. With this method a plot of imaginary vs. real values of the FRF is analyzed. When a natural frequency occurs, a circle will be formed in the Argand plane. If two or more modes interact, the circles will not be complete; one circle will be only partially formed when the next circle starts (8). An estimate for each complete circle can be made, and from this the amplitude of the mode shape (diameter of the circle), natural frequency (mid-point on the circle), and the damping (the circle is bisected and the corresponding magnitudes and frequencies are used) can be obtained. This procedure was originally suggested by Kennedy and Pancu (9).

This SDOF circle-fitting method will also break down if the modes become too closely spaced such that the individual circles interfere significantly with each other.
One method that can help produce better estimates is a routine described by Richardson and Kniskern (10). With this method FRFs are taken from the structure using various reference exciter locations, so that components from more than one column of the transfer matrix, \( H \), are included. After initial analysis, the natural frequencies and residues are input to an algorithm that weights and averages the residues to produce modal vectors.

The problem with this method is that it is not self-sufficient nor analytically rigorous. Since a preliminary analysis must be performed beforehand, the time factor increases significantly. The added accuracy is not always sufficient to warrant the use of this algorithm either. A numerical example cited in the paper by Richardson and Kniskern showed only a slight improvement as a result of using this method.

There are, though, several curve-fitting routines that can properly handle multiple DOF (MDOF) situations. Among them is one called the Least-Squares Complex Exponential (LSCE) technique (11). In this method the FRFs are obtained and then transformed back into the time domain, forming the unit impulse response functions. The analytical form of the response function can be written

\[
X(t_k) = \sum_{r=1}^{2N} A_r U_r^T \quad (1-1)
\]

where,

- \( A_r \) = residue of mode \( r \)
- \( A_{r+1} \) = complex conjugate of \( A_r \) (\( r=1,3,5,... \))
- \( U_r = e^{s_r T} \)
sr = \pm \xi_r \nu_r + j \nu_r (1 - \xi_r^2)^{1/2}

sr+1 = \text{complex conjugate of } s_r (r=1,3,5,\ldots)

T = \text{value of time subinterval}

k = \text{time point index } (k=0,1,2,\ldots,2n)

2n = \text{number of time steps}

N = \text{number of modes}

If \( N = n \), then there is a polynomial of order \( 2N \) that must satisfy the equation

\[ U^{2N} + a_1 U^{2N-1} + a_2 U^{2N-1} + \ldots + a_{2N-1} U_1 + a_{2N} = 0 \]  \hspace{1cm} (1-2)

whose roots are \( U_1, U_2, \ldots, U_{2N} \).

Pre-multiplying Eq. (1) by \( a_{2N-k} \) and summing yields

\[ \sum_{k=0}^{2N} a_{2N-k} X(t_k) = \sum_{r=1}^{2N} A_r \left( \sum_{k=0}^{2N} a_{2N-k} U_r^k \right) \]

The following equality holds

\[ \sum_{k=0}^{2N} a_{2N-k} U_r^k = 0 \]

since \( U_r \) is a root of Eq. (1-2).

Thus

\[ \sum_{k=0}^{2N} a_{2N-k} X(t_k) = 0 \text{ , } a_0 = 1 \]  \hspace{1cm} (1-3)

Equation (1-3) can be expanded into a series of equations by beginning each successive equation with a different response time, i.e.

\[ \sum_{i=0}^{m} \left( \sum_{k=0}^{2N} a_{2N-k} X(t_{k+i}) = 0 \right) \text{ , } m > 2N \]  \hspace{1cm} (1-4)

Since \( a_0 \) is known, the other \( a \)'s may be obtained from the equation
and the sum square error $E$ is

$$E = R^T R$$  \hspace{1cm} (1-8)$$

Once the $a$'s are known they can be substituted back into Eq. (1-2) and a polynomial root solver can be used to find the roots, $U_r$. The values for $w_r$ and $\xi_r$ can be found directly from $U_r$. The residues, $A_r$, for each $U_r$ can be found by creating $2N$ equations in the form of Eq. (1-1), each with a different initial time.

Though this method was just described for one unit impulse response record, it can be easily extended to several records. Since the $a$'s are system parameters, they should remain constant for each response record. Therefore, the summation on $i$ can be extended to include any number of response records. This also means that the excitation location can be changed so that more than one column of the $H$ matrix can be used simultaneously in the analysis.

To establish whether the estimate for $N$, the number of modes, is sufficiently large, a simple procedure is used involving a series of progressively larger estimates for the number of modes. Equation (1-5) is first rewritten in matrix form

$$Aa = B$$  \hspace{1cm} (1-6)$$

and the least-squares error is found by computing

$$R = Aa - B$$  \hspace{1cm} (1-7)$$

and the sum square error $E$ is

$$E = R^T R$$  \hspace{1cm} (1-8)$$
A plot of the error vs. the number of modes assumed will show a significant drop-off when \( N \) is sufficiently large.

The LSCE method has been incorporated into commercial software packages and has demonstrated great success in extracting modal parameters. LSCE does have a problem, though. It must transform frequency-domain data back to the time domain. When this additional transform is performed, information can be lost or contaminated.

A second method which uses FRFs from SPR testing and which can use multiple columns of the transfer matrix, \( H \), is an eigenvalue solver developed by Coppolino, called the Simultaneous Frequency Domain (SFD) Method (12). The idea of 'simultaneous' is that single-input FRFs from several response locations and reference locations can be used simultaneously in the analysis. In this method the matrix equation of motion for a structure may be transformed to the frequency domain and written as

\[
[M] \{\tilde{X}(w)\} + [B] \{\tilde{X}(w)\} + [K] \{X(w)\} = \{D\} f(w) \tag{1-9}
\]

where

- \([M]\) = mass matrix
- \([B]\) = damping matrix
- \([K]\) = stiffness matrix
- \(\{X(w)\}\) = array of displacements of points on the structure
- \(\{D\}\) = external force distribution array
- \(f(w)\) = external force amplitude
- \(f(w)\) = 1, since the transfer function for a unit impulse is 1.

Pre-multiplying Eq. (1-9) by \([M]^{-1}\) and solving for \(\{\tilde{X}(w)\}\) gives
\[
\{ \ddot{x}(w) \} = -[\{M\}^{-1} [B] | [M]^{-1} [K] | [M]^{-1} [D]] \{ \ddot{x}(w) \}
\]

(1-10)

The response arrays can be expanded into matrices such that each column corresponds to a frequency point. A least-squares solution can then be obtained for the quantities \([M]^{-1} [B]\) and \([M]^{-1} [K]\). Once these two dynamic matrices are known, it is a simple procedure to use them in an eigenvalue problem

\[
\lambda \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} [M]^{-1}[B] & [M]^{-1}[K] \\ -[I] & [0] \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(1-11)

to solve for the frequencies and modes.

The only disadvantage of SFD compared to the LSCE single-input MDOF parameter solver is that the order of the matrices, thus the number of modes obtained at a time, is directly dependent upon how many response locations are available for the analysis.

A third method that utilizes multiple rows and columns is the force appropriation, or Asher, method (13-15). The Asher method is based on the same idea as the MSSD test; that at a natural frequency an appropriate force vector will produce responses 90 degrees out of phase with the excitation. Assuming that the applied forces are real, this means

\[
\{x\} = [H] \{f\} = ([H_R] + i[H_I]) \{f\}
\]

(1-12)

For the response to be in quadrature with the excitation

\[
[H_R] \{f\} = 0
\]

(1-13)

Thus, the condition for natural frequencies is
\[
\text{Det} [H_R (w_r)] = 0 \quad r = 1, 2, \ldots, n \quad (1-14)
\]

Once the frequencies have been found from Eq. (1-14), a solution for the required force vector, \( \mathbf{F}_r \), can be obtained by using Eq. (1-13). Finally, the mode shape can be obtained from

\[
\{X(w_r)\} = i[H_I (w_r)] \{F\} \quad (1-15)
\]

The Asher method is usually accurate, but the condition of Eq. (1-13) is only a necessary, but not sufficient, requirement for a natural mode. If the order of the determinant is much less than the order of the system, then artificial or 'spurious' modes can appear.

The Asher method requires that the number of excitation and response points be equal, whereas the number of response points usually far exceeds the number of excitation points used in a test. To avoid this Asher method limitation, another method has been developed which uses non-square matrices, and thus permits more response points. This method, termed the minimum coincident response (MCR) method (15-16), starts with basically the same equations

\[
\{X\} = ([H_R] + i[H_I]) \{F\} \quad p < n \quad m < n \quad (1-16)
\]

\[
\{X_R\} = [H_R] \{F\} \quad (1-17)
\]

Instead of setting Eq. (1-17) to zero, a coincident response error norm given by

\[
e = \{X_R\}^T \{X_R\}
\]

is minimized. The force vector \( \mathbf{F} \) is appropriated such that \( e \) is minimized subject to an amplitude constraint on \( X_{11} \), in quadrature.
response at the ith response point. This leads to a force vector

\[ \{F\} = \frac{1}{\{H_{II}\}} \left( [H_R]^T [H_R] \right)^{-1} \{H_{II}\} \quad (1-18) \]

where \( H_{II} \) is the ith row of the imaginary part of the transfer matrix. The mode shape can then be obtained from

\[ \{X_I\} = i[H_I (w_r)] \{F\} \quad (1-19) \]

The MCR method has been shown to give favorable results (15), although it can still produce 'spurious' modes. This problem can be diagnosed by various methods. One is to check the slope of the \( \varepsilon \) vs. frequency curve. Shallow slopes seem to indicate 'spurious' modes, while steep slopes correlate with actual modes (15). Another difficulty with the MCR method is that, while the Asher method interpolates to find zero crossings, MCR searches for minima.
Chapter 2

2.1 Development of a New Multi-Shaker Algorithm

For several years, since the advent of digital computers and the development of the Fast Fourier Transform (FFT), considerable time has gone into performing and perfecting single point random (SPR) excitation structural dynamics measurement techniques. The SPR method has proven to be reasonably fast and simple to perform. There are, though, some inherent problems with an SPR test. Two of the chief problems are energy dissipation within the structure and the possibility of locating a shaker near a node line.

A technique to avoid these problems is to apply forces at multiple locations. The method to be developed in this chapter is an expansion of the eigenvalue SFD method described by Coppolino (12). Since the standard modal analysis methods are applicable to single-shaker testing only, this alternative approach involves a reformulation of the equations of motion to accommodate the use of multiple forces.

The Multi-Shaker Modal Analysis (MSMA) method presented here assumes that a linear structure can be accurately described over a limited frequency range by a finite number of degrees of freedom (DOF). This assumption yields the standard frequency-domain matrix equation relating the output accelerations at given points on the structure to the input forces

\[ \{\ddot{x}(w)\} = [H(w)] \{F(w)\} \]  

(2-1)
where,

\[
\{\ddot{x}(w)\} = \text{output accelerations }, \quad (n \times 1)
\]
\[
\{F(w)\} = \text{input forces }, \quad (n \times 1)
\]
\[
[H(w)] = \text{complex frequency response function }, \quad (n \times n)
\]
\[
[H(w)] = [\phi] [\phi]^T
\]
\[
[\phi] = \text{modal matrix }, \quad (n \times n)
\]
\[
[\phi]^T = \text{diagonal modal admittance matrix }, \quad (n \times n)
\]
\[
[\phi] = \begin{bmatrix}
\bar{w}_r^2 - \omega^2 & -\bar{w}_r \\
\bar{w}_r & 1
\end{bmatrix}^{\rho_i, n}
\]

\[w_r = \text{natural circular frequency of mode } r\]
\[\xi_r = \text{critical damping ratio of mode } r\]
\[w = \text{circular frequency}\]

The force vector is rewritten in a form that includes only the DOF where forces are applied. A distribution matrix relates this new force vector, \(\{f(w)\}\), to the original force vector, \(\{F(w)\}\).

\[
\{F(w)\} = [D] \{f(w)\} \quad (2-2)
\]

For example, if a 4DOF system has exciters at points 1 and 3, the equation would look like

\[
\{F(w)\} = \begin{bmatrix}
f_1 \\
of_1 \\
f_3 \\
of_3
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_3
\end{bmatrix}
\]

So, Eq. (2-1) is now

\[
\{\ddot{x}(w)\} = [\phi] [\phi]^T [D] \{f(w)\} \quad (2-3)
\]
It is assumed that there are \( n \) points where the responses are measured, and if it is assumed that the structure can be accurately described over the frequency range of interest by \( m \) DOF \((m < n)\), then Eq. (2-3) can be partitioned into sets of 'independent' DOF and 'dependent' DOF equations. Thus,

\[
\begin{bmatrix}
\dot{X}_i(w) \\
\dot{X}_d(w)
\end{bmatrix} = \begin{bmatrix}
\mathcal{O}_i[1] \\
\mathcal{O}_d
\end{bmatrix} /A(w)/ \begin{bmatrix}
\mathcal{O}_i^T \mathcal{O}_d^T
\end{bmatrix} \begin{bmatrix}
[D_i] \\
[D_d]
\end{bmatrix} \{f(w)\} \quad (2-4)
\]

or

\[
\begin{bmatrix}
\dot{X}_i(w) \\
\dot{X}_d(w)
\end{bmatrix} = \begin{bmatrix}
[I] \\
\mathcal{O}_d \mathcal{O}_d^{-1}
\end{bmatrix} \begin{bmatrix}
\mathcal{O}_i[1] /A(w)/ \mathcal{O}_i^T
\end{bmatrix} \begin{bmatrix}
[D_i] \\
[D_d]
\end{bmatrix} \{f(w)\} \quad (2-5)
\]

where

\[
[D_{id}] = [D_i] + \mathcal{O}_i^{-1} \mathcal{O}_d^T [D_d]
\]

From Eq. (2-5) a relationship between the independent and dependent DOF accelerations is

\[
\{\ddot{X}_d(w)\} = \mathcal{O}_d \mathcal{O}_d^{-1} \{\ddot{X}_i(w)\} \quad (2-6a)
\]

This equation will be employed later to determine the modal components of the dependent DOF for the determined natural frequencies. The equation relating the independent accelerations to the forces is

\[
\{\ddot{X}_i\} = \mathcal{O}_i[1] /A(w)/ \mathcal{O}_i^T [D_{id}] \{f(w)\} \quad (2-6b)
\]

Pre-multiplying both sides of Eq. (2-6b) by \((\mathcal{O}_i[1] /A(w)/ \mathcal{O}_i^T)^{-1}\) and expanding, yields
\[
\begin{align*}
([\bar{\theta}_i]^{-1}/A(w)/)[\bar{\theta}_i]^T \{\ddot{x}_i\} &= [D_{1d}] \{f(w)\} \\
[\bar{\theta}_i]^T \{(1/A(w)/)[\bar{\theta}_i]^{-1} \{\ddot{x}_i\} &= [D_{1d}] \{f(w)\} \\
[\bar{\theta}_i]^T \{(1 - 12\xi_r (w_r/w) - (w_r/w)^2)/[\bar{\theta}_i]^{-1} \{\ddot{x}_i\} &= [D_{1d}] \{f(w)\} \\
[\bar{\theta}_i]^T \{\bar{\theta}_i\}^{-1}\{\ddot{x}_i\} + [\bar{\theta}_i]^T \{2\xi_r w_r/[\bar{\theta}_i]^{-1} \{-1/w\} \{\ddot{x}_i\} \\
 + [\bar{\theta}_i]^T \{w_r^2/[\bar{\theta}_i]^{-1} \{-1/w^2\} \{\ddot{x}_i\} &= [D_{1d}] \{f(w)\}
\end{align*}
\]

or, in summary,

\[
[M_i] \{\ddot{x}_i\} + [C_i] \{\dot{x}_i\} + [K_i] \{x_i\} = [D_{1d}] \{f(w)\}
\] (2-7)

where

\[
[M_i] = [\bar{\theta}_i]^T [\bar{\theta}_i]^{-1}
\]

\[
[C_i] = [\bar{\theta}_i]^T \{2\xi_r w_r/[\bar{\theta}_i]^{-1}
\]

\[
[K_i] = [\bar{\theta}_i]^T \{w_r^2/[\bar{\theta}_i]^{-1}
\]

\[
\{\ddot{x}_i\} = \{-1/w\} \{x_i\}
\]

\[
\{\dot{x}_i\} = \{-1/w^2\} \{x_i\}
\]

Least-squares averaging may be employed to estimate the matrices \([M_i]\), \([C_i]\), and \([K_i]\). Therefore, the accelerations, velocities, displacements and forces in Eq. (2-7) are expanded into matrices, where each column corresponds to a discrete frequency point. That is,

\[
[X_i] = [\ddot{x}_i(w_1) | \ddot{x}_i(w_2) | ... | \ddot{x}_i(w_k)]
\] (2-8)

where \(k > m\). Multiplication of Eq. (2-7) by \([M_i]^{-1}\) yields

\[
-[\ddot{x}_i] = \begin{bmatrix} C_i \end{bmatrix} \begin{bmatrix} K_i \end{bmatrix} \begin{bmatrix} D_{1d} \end{bmatrix} \begin{bmatrix} X_i \end{bmatrix} \begin{bmatrix} f(w) \end{bmatrix}
\] (2-9)

where,
\[
\begin{align*}
\mathbf{C}_1 &= \mathbf{M}_i^{-1} \mathbf{C}_i \\
\tilde{\mathbf{K}}_i &= \mathbf{M}_i^{-1} \mathbf{K}_i \\
\mathbf{B}_1 &= -\mathbf{M}_i^{-1} \mathbf{D}_1
\end{align*}
\] (2-10a, 2-10b, 2-10c)

Since \([\mathbf{C}_1], [\tilde{\mathbf{K}}_i], \) and \([\mathbf{B}_1])\) are real matrices, Eq. (2-9) can be rewritten for the real and imaginary parts of the responses and inputs.

\[
\begin{bmatrix}
-\mathbf{C}_1 & \tilde{\mathbf{K}}_i \\
\mathbf{C}_1 & \mathbf{B}_1
\end{bmatrix}
\begin{bmatrix}
\dot{\mathbf{x}}_i \\
\mathbf{x}_i
\end{bmatrix}
= \begin{bmatrix}
\dot{\mathbf{f}}_{\text{Re}} \\
\mathbf{f}_{\text{Im}}
\end{bmatrix}
\] (2-11)

Since this equation is over-determined (\(k > m\)), a least-squares solution is used. Equation (2-11) may be rewritten symbolically

\[
\mathbf{Y} = \mathbf{T} \mathbf{Z}
\] (2-12)

The least-squares solution for \([\mathbf{T}]\) is

\[
\mathbf{T} = \mathbf{Y} \mathbf{Z}^T \left( \mathbf{Z} \mathbf{Z}^T \right)^{-1}
\] (2-13)

The \([\mathbf{C}_1], \) and \([\tilde{\mathbf{K}}_i]\) that were solved for in Eq. (2-13) can now be used to determine the eigenvalues and eigenvectors for the system in the following manner. From Eq. (2-9) the free vibration equation

\[
\ddot{\mathbf{x}}_i + [\mathbf{C}_1] \dot{\mathbf{x}}_i + [\tilde{\mathbf{K}}_i] \mathbf{x}_i = 0
\] (2-14)

may be written. Since the matrices \([\mathbf{C}_1]\) and \([\tilde{\mathbf{K}}_i]\) are constant, \(X\) and its derivatives in Eq. (2-14) may be written in the time-domain, rather than in the frequency-domain as originally expressed in Eq. (2-7). Thus, let
\[ x_i(t) = x_0 e^{\lambda t} \] (2-15a)
\[ \dot{x}_i = \lambda x_i \] (2-15b)
\[ \ddot{x}_i = \lambda \dot{x}_i \] (2-15c)

From Eq. (2-15b)
\[ \lambda x_i - \dot{x}_i = 0 \] (2-16d)

Thus, combining Eqs. (2-14) and (2-15) yields

\[
\lambda \begin{bmatrix} \dot{x}_i \\ x_i \end{bmatrix} + \begin{bmatrix} \bar{C} & \bar{K} \\ -I & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_i \\ x_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \] (2-17)

which is a standard linear eigenvalue problem. The eigenvectors will produce mode shape components for the 'independent' DOF only. To include the 'dependent' DOF, Eq. (2-6a) is expanded to matrix form for the \( k \) frequency points and then manipulated, via the least-squares technique, to produce a relationship between the 'dependent' and 'independent' modal coefficients.

\[
\begin{bmatrix} \ddot{x}_d \\ \ddot{x}\end{bmatrix} = \begin{bmatrix} \varphi_d \\ \varphi \end{bmatrix} \begin{bmatrix} \varphi_d \end{bmatrix}^{-1} \begin{bmatrix} \ddot{x}_i \\ \ddot{x}\end{bmatrix} \]
\[
\begin{bmatrix} \varphi_d \\ \varphi \end{bmatrix} = \begin{bmatrix} \ddot{x}_d \\ \ddot{x}_i \end{bmatrix} \begin{bmatrix} \ddot{x}_i \end{bmatrix}^T \begin{bmatrix} \ddot{x}_i \end{bmatrix} \begin{bmatrix} \ddot{x}_i \end{bmatrix}^{-1} \begin{bmatrix} \varphi_i \end{bmatrix} \] (2-18)

2.2 Forcing Functions

As indicated by Eqs. (2-4) and (2-5), the forces may be applied at the dependent DOF as well as the independent DOF. This feature of the equations allows for forces to be applied where either accelerations could not be measured or where an accelerometer produced bad readings and was subsequently discarded. It also allows the user more freedom in
determining which locations are to be identified as independent DOF. If, after data acquisition is completed and a general analysis done, it is determined that a particular DOF is not important, its responses can be reduced out of further calculations, even though it was forced.

In addition to the freedom of location of forces, there is also a freedom in the type of forces which can be applied, subject to one constraint. From Eq. (2-13) it is seen that, if two forces are fully coherent the inverse of the matrix will be singular. However, the restriction is not too severe. If two forcing functions are identical, only one forcing function needs to be included in the force array. The force distribution matrix will handle the duplication in the applied force spectrums. Since, in reality, a structure will have different input impedances at each location, similar signals into the shakers will produce different forces. The best way to avoid any potential problems, however, is to have a low coherence between forces. This can be achieved by using either of two forcing methods, independent random signals at each shaker or simultaneous sine sweeps with each shaker having a different phase shift.

Another type of excitation is a free-decay response. A minor reformulation of the equations is involved, with Eq. (2-11) being modified to the form

\[
- [\mathbf{C}_i] [\mathbf{X}_i] - [\mathbf{R}_i] [\mathbf{X}_i] = [\mathbf{C}_i] [\mathbf{X}_i] - [\mathbf{R}_i] [\mathbf{X}_i] 
\]

This formulation eliminates the need for measurement of the forces, thus allowing a more general type of excitation. This can also be used in
conjunction with the Randomdec method where a free-decay response is generated from a structure that is being randomly excited.

2.3 Averaging

There are two possible methods to eliminate noise contamination in the eigenvalue results. Both methods involve a type of signal averaging. The first method involves averaging FRFs from several runs to form a single acceleration and force FRF spectrum at each location before insertion into Eq. (2-13).

There is one serious drawback to this method. With random excitations, excessive averaging can produce an FRF spectrum with a uniform value, effectively giving fewer independent columns in Eq. (2-13) with which to do a least-squares fit. For a swept-sine test, this averaging technique should suffice, since the FRF will not tend to become uniform, especially if the signal generator has a phase modulator.

The second method is to string the multiple runs together in the same manner as the individual frequencies were in Eq. (2-8). That is

\[
[\ddot{x}(w)] = [\ddot{x}_1(w)] | [\ddot{x}_2(w)] | \ldots | [\ddot{x}_p(w)]
\]

This increases the number of columns, therefore the accuracy of the least-squares fit, but the required storage space for the matrices is also increased.

2.4 Pseudo-Degrees-of-Freedom

If the structure being tested has more modes in the frequency range of interest than there are measurement locations, there is a technique to
increase the number of DOF based on existing measurement DOF. This method involves sampling extra sets of data and placing each new set into the equations as if they were new DOF. With this approach is an alternate method in which the responses of the originally measured data are partitioned into subsets and correspondingly labeled as different sets of data. In this way the number of DOF is increased without the need for more measurements. This, however, gives fewer columns of data for the least-squares process, since this procedure takes some of the columns in Eq. (2-8) and makes new rows out of them.

2.5 Mode Selection

Proper selection of the true modes of vibration from the eigen-solution of Eq. (2-16) is just as important as any other part of the modal analysis. When the number of modes is overspecified, either through oversight or to compensate for noise contamination, a systematic method is needed to identify the spurious modes. Usually the spurious modes can be eliminated immediately by a quick examination of the eigenvalues. The damping values are often excessively high, low, or even negative. The natural frequencies might have values that are out of the measured frequency range. At other times, however, the spurious mode eigenvalues are within the range of tolerance. Then, an inspection of the eigenvectors is needed. One procedure involves the use of pseudo-DOF. A comparison of the portion of the eigenvector representing the pseudo-DOF with the portion of the eigenvector which corresponds to the independent DOF from which the pseudo DOF were produced can readily determine the validity of the mode. A favorable relationship would be one where there is a high correlation of
eigenvector values at corresponding DOF. Another method would be to analyze results of analyses from several data sets; modes that repeatedly show up would indicate valid modes.

2.6 Time-Domain Analysis

The MSMA procedure just described may also be performed directly in the time domain instead of first transforming the signal into the frequency-domain, although this raises some new problems. The relationship between acceleration, velocity, and displacement is not as simple. To acquire the three quantities there are three possible methods. The first is to measure all three quantities directly. The second is to measure the acceleration and then integrate to obtain the velocity and displacement. In this case, the initial velocity and displacement are needed. The third method is to measure the displacement and then differentiate the signal to obtain the velocity and acceleration. Since none of these procedures is easily accomplished, the frequency-domain approach remains the preferred approach.
Chapter 3

3.1 Simulation Studies

To verify the validity and accuracy of the MSMA technique described in Chapter 2, several computer simulations were developed and examined. The primary concerns evaluated in the tests were: (1) resolution of the modes in high noise, (2) identification of modes less than $\xi \omega_n$ apart, (3) accuracy of dependent DOF, and (4) application of forces at dependent DOF.

A computer program was designed such that the system natural frequencies, damping, and mode shapes, and a noise-to-signal ratio are input. The noise level is calibrated by using the root-mean-square of the response signal over the entire frequency range. Also input are the frequency range, number of independent DOF and their corresponding locations, number of dependent DOF and their locations, number of forced DOF and their locations, and the method of excitation. The program creates the frequency responses for the accelerations and forces and then proceeds to manipulate them in accordance with the equations of Chapter 2.

The first series of tests was on a system with nine natural frequencies, seven of which were below 10 Hz. Two of the frequencies were selected less than $\xi \omega_n$ apart, so that the effects of closely-coupled modes could be observed. The noise-to-signal ratio was selected at 20% in conjunction with an applied random force signal. Changed during the test series were the number of forced DOF: 1, 4, and 8 forces were used. As indicated in Table 1, the mode shapes had magnitudes at each station of...
1.0, except for one DOF in each mode whose magnitude was set at 0.01 to simulate a point close to a node line.

A sampling run was first made using a single shaker to obtain an estimate of the number of modes in the 0-10 Hz. range. A check of the FRFs showed that six modes were fairly apparent, so an estimate of eight modes (six for the observed ones and two more for any missed modes or any pseudo or spurious modes created by the noise) did not seem unreasonable.

Three runs were made for each of the forced-DOF conditions plus one additional run utilizing the averaging technique of placing several runs into one large array. The averaged analysis used three individual runs which differed from the previous three runs.

To check the accuracy of the results an error analysis was used that looked at the percentage of error in the natural frequency, damping, and real part of the modal vector compared to the input system characteristics. The natural frequency and damping errors were computed directly. The modal vector, though, was a bit more complicated. The mode shapes all had magnitudes of 1 at each DOF, except for one with .01, so the estimated mode shapes were each scaled such that the largest component had an amplitude of 1 and a phase angle of 0 degrees. The real part of each component was then squared, summed (except for the one component of .01 amplitude, which was omitted from the error analysis), divided by 7 (the corresponding magnitude of the original vector); and then the square root was taken. The final value, which ranged from 0 to 1, was then used directly for the error analysis.

To highlight the results, the errors for three modes were investigated. The first was the 5.9 Hz. mode, which was relatively uncoupled.
The other two modes were the coupled modes, 5.5 Hz. and 5.55 Hz. Referring to the diagrams in Fig. 1, it can be seen that with 20% noise the natural frequency estimates are highly accurate, even for the closely-coupled modes. Also evident in the diagrams is the overall higher accuracy of the parameters corresponding to the 5.9 Hz. mode compared to the 5.5 and 5.55 Hz. modes. Although the errors for the damping and modal vectors are large for the 20% noise single-shaker case, the diagrams show a steady trend towards lower error levels as the number of shakers is increased. In fact, the error levels for the closely-coupled modes are comparable to the error level for the 5.9 Hz. mode, when eight shakers were employed.

Also tested with this simulation model was a set of runs in which the noise-to-signal ratio was changed from 0% to 10% and finally to 20%. To illustrate the effect of noise on the results and the effect of additional shakers, the modal parameters corresponding to the 5.0, 5.5, and 5.55 Hz. modes are tabulated (see Tables 2 through 10). An analysis of the tables shows, once again, a clear trend toward higher levels of accuracy at a given noise level for an increase in the number of shakers. A dramatic result from this set of runs was the absence of the 5.0 Hz mode from the single-shaker analysis when noise was present. As the number of shakers increased though, the 5.0 Hz mode appeared and then increased in accuracy. Overall, by a comparison of the tables, it is seen that the algorithm does not become unstable with increasing noise levels. On the contrary, good results are obtained even at relatively high noise levels when several shakers are used.

The second series of tests analyzed goals (3) and (4). To do this, an 8 DOF system was used, with four modes in the 0-10 Hz. range. In this
case the noise-to-signal ratio was also 20%, but the number of forces was set at 1, 5, and 8 (an initial sampling run indicated three, possibly four, modes present in the 0-10 Hz. range, so five independent DOF were selected).

The previous series of tests was again performed, but since the only substantial errors occurred in the coupled modes, their parameters were the only ones analyzed. In this analysis, only the normalized independent modal vectors and a ratio of the normalized dependent to independent modal vectors were calculated (see Fig. (2)).

Again a convergence toward higher accuracy is seen in these results as the number of applied forces is increased. An important feature of the results is that the accuracy of the dependent DOF, calculated from Eq. (2-17), follows closely the same trends as the independent DOF. Also to be noted in the diagrams is the error levels for five shakers and for eight shakers. Although three shakers are exciting dependent DOF in the eight shaker case, the accuracy is either maintained or improved in almost every instance. Even if there happens to be a higher error indication for an increase in shakers, there is a definite lessening in the error spread.

Two more series of tests were run. The first was a set of runs identical to the first test case with the exception that the forcing functions were swept-sines instead of random excitation. The results of this set were basically identical to the random excitation set. The final series also used the same set-up as the first test, except that no noise was introduced and the test range was from 0-1024 Hz. with 256 data points (a delta frequency of over 4 Hz.). The ninth mode corresponding to 25 Hz. was also deleted. All modes were identified to extremely high accuracy.
(the error was less than one part per billion) in spite of the fact that there were seven natural frequencies between two of the data points.

Some preliminary testing of the algorithm utilizing the concept of pseudo-DOF was performed with mixed results. When a large percentage of the needed independent DOF were actually pseudo-DOF, there was a significant degradation in the accuracy of the results. However, if the pseudo-DOF were used sparingly to obtain the number of needed DOF, the algorithm performed on the same level as in the previously cited tests. In either case, the use of pseudo-DOF in spurious mode detection was satisfactory. An increase in the number of pseudo-DOF beyond the number of needed DOF had no ill effect on the algorithm's performance. In fact it helped facilitate the spurious mode identification.
### Table 1. System Frequencies, Damping Factors, and Mode Shapes (defined at 8 DOF only); Forces and Locations

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Table 1. System Frequencies, Damping Factors, and Mode Shapes (defined at 8 DOF only); Forces and Locations
### Table 2

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**Damping** = .100156E-01

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**Noise Ratio** = 0.0
Undamped Natural Frequency = 0.549815E+01
Damping = 0.922684E-02

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00000E+01</td>
</tr>
<tr>
<td>2</td>
<td>0.961693E+00</td>
</tr>
<tr>
<td>3</td>
<td>-0.111589E+00</td>
</tr>
<tr>
<td>4</td>
<td>-0.953094E+00</td>
</tr>
<tr>
<td>5</td>
<td>-0.866331E+00</td>
</tr>
<tr>
<td>6</td>
<td>-0.874222E+00</td>
</tr>
<tr>
<td>7</td>
<td>0.852086E+00</td>
</tr>
<tr>
<td>8</td>
<td>0.852671E+00</td>
</tr>
</tbody>
</table>

Undamped Natural Frequency = 0.556574E+01
Damping = 0.125684E-01

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
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<td>0.969798E+00</td>
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<tr>
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<td>1.00000E+01</td>
</tr>
<tr>
<td>3</td>
<td>-0.601954E+00</td>
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<tr>
<td>4</td>
<td>-0.306331E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.663191E+00</td>
</tr>
<tr>
<td>6</td>
<td>0.655221E+00</td>
</tr>
<tr>
<td>7</td>
<td>-0.574999E+00</td>
</tr>
<tr>
<td>8</td>
<td>-0.641525E+00</td>
</tr>
</tbody>
</table>

No. of Forces = 1  Noise Ratio = 0.1

No mode corresponding to 5 Hz was identified

Table 3
Undamped Natural Frequency = 550027E+01
Damping = 979364E-02

<table>
<thead>
<tr>
<th>Coordinate</th>
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<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.897034E+00</td>
<td>-0.161891E-01</td>
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<tr>
<td>2</td>
<td>1.00000E+01</td>
<td>0</td>
</tr>
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<td>3</td>
<td>-0.263962E+00</td>
<td>-0.827695E-01</td>
</tr>
<tr>
<td>4</td>
<td>-0.717776E+00</td>
<td>0.560193E-01</td>
</tr>
<tr>
<td>5</td>
<td>-0.923468E-01</td>
<td>0.215160E+00</td>
</tr>
<tr>
<td>6</td>
<td>-0.173389E+00</td>
<td>0.196507E+00</td>
</tr>
<tr>
<td>7</td>
<td>0.551836E+00</td>
<td>-0.126410E+00</td>
</tr>
<tr>
<td>8</td>
<td>0.510949E+00</td>
<td>-0.138541E+00</td>
</tr>
</tbody>
</table>

Undamped Natural Frequency = 558794E+01
Damping = 110378E-01

<table>
<thead>
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<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.721310E+00</td>
<td>0.649048E-01</td>
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<tr>
<td>2</td>
<td>-0.560135E+00</td>
<td>0.146199E-01</td>
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<td>3</td>
<td>0.833390E+00</td>
<td>-0.662602E-02</td>
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<td>4</td>
<td>-0.769025E-01</td>
<td>-0.250925E-01</td>
</tr>
<tr>
<td>5</td>
<td>-0.414973E+00</td>
<td>-0.122354E-01</td>
</tr>
<tr>
<td>6</td>
<td>-0.508665E+00</td>
<td>-0.241696E-01</td>
</tr>
<tr>
<td>7</td>
<td>1.00000E+01</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.947173E+00</td>
<td>0.470355E-02</td>
</tr>
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</table>

No. of Forces = 1 \hspace{1cm} \text{Noise Ratio} = 0.2

No mode corresponding to 5 Hz was identified

Table 4
**Undamped Natural Frequency = 0.499677E+01**  
**Damping = 0.995318E-02**

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>-0.289103E-03</td>
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<tr>
<td>2</td>
<td>0.100000E+01</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.998304E+00</td>
<td>0.122927E-03</td>
</tr>
<tr>
<td>4</td>
<td>0.999471E+00</td>
<td>0.385209E-04</td>
</tr>
<tr>
<td>5</td>
<td>0.997609E+00</td>
<td>0.171764E-03</td>
</tr>
<tr>
<td>6</td>
<td>0.997915E+00</td>
<td>0.149431E-03</td>
</tr>
<tr>
<td>7</td>
<td>0.996893E+00</td>
<td>0.223324E-03</td>
</tr>
<tr>
<td>8</td>
<td>0.997265E+00</td>
<td>0.196417E-03</td>
</tr>
</tbody>
</table>

**Undamped Natural Frequency = 0.549998E+01**  
**Damping = 0.100134E-01**

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.100000E+01</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.999560E+00</td>
<td>0.154268E-02</td>
</tr>
<tr>
<td>3</td>
<td>-0.112414E-01</td>
<td>0.434173E-02</td>
</tr>
<tr>
<td>4</td>
<td>-0.999586E+00</td>
<td>-0.142288E-02</td>
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<td>5</td>
<td>-0.998737E+00</td>
<td>-0.438455E-02</td>
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<td>6</td>
<td>-0.998722E+00</td>
<td>-0.443570E-02</td>
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<tr>
<td>7</td>
<td>0.998114E+00</td>
<td>0.657796E-02</td>
</tr>
<tr>
<td>8</td>
<td>0.998166E+00</td>
<td>0.639820E-02</td>
</tr>
</tbody>
</table>

**Undamped Natural Frequency = 0.554972E+01**  
**Damping = 0.999856E-02**

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.992035E+00</td>
<td>-0.217490E-03</td>
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<tr>
<td>2</td>
<td>-0.988883E+00</td>
<td>-0.304124E-03</td>
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<tr>
<td>3</td>
<td>0.997943E+00</td>
<td>0.577118E-04</td>
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<td>4</td>
<td>0.927480E-02</td>
<td>0.239258E-04</td>
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<td>5</td>
<td>-0.995724E+00</td>
<td>-0.113431E-03</td>
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<td>6</td>
<td>-0.995813E+00</td>
<td>-0.110535E-03</td>
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<tr>
<td>7</td>
<td>0.100000E+01</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.999567E+00</td>
<td>0.122911E-04</td>
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</table>

**No. of Forces = 4**  
**Noise Ratio = 0.0**

Table 5
Undamped Natural Frequency = .499588E+01
Damping = .101836E-01

<table>
<thead>
<tr>
<th>Coordinate</th>
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<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.180826E-01</td>
<td>.648566E-02</td>
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<tr>
<td>2</td>
<td>.989374E+00</td>
<td>.422064E-03</td>
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<td>3</td>
<td>.984864E+00</td>
<td>.601076E-02</td>
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<tr>
<td>4</td>
<td>.100000E+01</td>
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<td>.975333E+00</td>
<td>.364940E-02</td>
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<td>6</td>
<td>.987170E+00</td>
<td>-.791028E-03</td>
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<td>7</td>
<td>.977551E+00</td>
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<tr>
<td>8</td>
<td>.975623E+00</td>
<td>-.161876E-03</td>
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</tbody>
</table>

Undamped Natural Frequency = .549835E+01
Damping = .979900E-02

<table>
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<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>-.995591E+00</td>
<td>.353937E-01</td>
</tr>
<tr>
<td>2</td>
<td>-.963411E+00</td>
<td>.140834E-01</td>
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<td>3</td>
<td>-.193424E-01</td>
<td>-.112736E-01</td>
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<tr>
<td>4</td>
<td>.987221E+00</td>
<td>.125764E-01</td>
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<td>.952534E+00</td>
<td>-.241061E-01</td>
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<td>.100000E+01</td>
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<td>7</td>
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<td>-.490844E-01</td>
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<tr>
<td>8</td>
<td>-.960618E+00</td>
<td>-.491432E-01</td>
</tr>
</tbody>
</table>

Undamped Natural Frequency = .554825E+01
Damping = .943557E-02

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
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<tbody>
<tr>
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<tr>
<td>2</td>
<td>.100000E+01</td>
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<td>3</td>
<td>-.973952E+00</td>
<td>.261893E-02</td>
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<td>4</td>
<td>-.554699E-01</td>
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<td>5</td>
<td>.851456E+00</td>
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<td>6</td>
<td>.903672E+00</td>
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<td>7</td>
<td>-.925752E+00</td>
<td>.291869E-01</td>
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<tr>
<td>8</td>
<td>-.905322E+00</td>
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No. of Forces = 4
Noise Ratio = 0.1

Table 6
Undamped Natural Frequency = 5.00344E+01
Damping = .949072E-02

<table>
<thead>
<tr>
<th>Coordinate</th>
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<th>Coefficient</th>
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<tbody>
<tr>
<td>1</td>
<td>-.431481E-02</td>
<td>-.383221E-02</td>
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<tr>
<td>2</td>
<td>.981993E+00</td>
<td>-.220043E-02</td>
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<tr>
<td>3</td>
<td>.924128E+00</td>
<td>.172318E-02</td>
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<tr>
<td>4</td>
<td>.944983E+00</td>
<td>.305296E-01</td>
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<td>.971504E+00</td>
<td>-.134243E-01</td>
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<td>6</td>
<td>1.000000E+01</td>
<td>0</td>
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<td>7</td>
<td>.953668E+00</td>
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<tr>
<td>8</td>
<td>.975622E+00</td>
<td>-.367880E-02</td>
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</table>

Undamped Natural Frequency = 5.48780E+01
Damping = .959778E-02

<table>
<thead>
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<th>Coefficient</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>.709776E+00</td>
<td>.119457E+00</td>
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<td>2</td>
<td>.676447E+00</td>
<td>.170732E-01</td>
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<tr>
<td>3</td>
<td>.129852E+00</td>
<td>-.441760E-01</td>
</tr>
<tr>
<td>4</td>
<td>-.915645E+00</td>
<td>-.537032E-01</td>
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<td>-.857429E+00</td>
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<td>6</td>
<td>-.941759E+00</td>
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<tr>
<td>8</td>
<td>.955578E+00</td>
<td>.678429E-01</td>
</tr>
</tbody>
</table>

Undamped Natural Frequency = 5.54758E+01
Damping = .104780E-01

<table>
<thead>
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<th>Coefficient</th>
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<tbody>
<tr>
<td>1</td>
<td>.992678E+00</td>
<td>.668940E-01</td>
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<td>2</td>
<td>.100000E+01</td>
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<td>3</td>
<td>-.878481E+00</td>
<td>-.152797E+00</td>
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<td>4</td>
<td>-.856727E-01</td>
<td>.457745E-01</td>
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<tr>
<td>5</td>
<td>.638683E+00</td>
<td>.176693E+00</td>
</tr>
<tr>
<td>6</td>
<td>.692493E+00</td>
<td>.154775E+00</td>
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<td>7</td>
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</tr>
<tr>
<td>8</td>
<td>-.713608E+00</td>
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</table>

No. of Forces = 4       Noise Ratio = 0.2

Table 7
### Undamped Natural Frequency = $4.99656E+01$

Damping = $9.85967E-02$

<table>
<thead>
<tr>
<th>Coordinate</th>
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<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.42906E-01$</td>
<td>$8.64394E-03$</td>
</tr>
<tr>
<td>2</td>
<td>$1.00000E+01$</td>
<td>$0$</td>
</tr>
<tr>
<td>3</td>
<td>$9.98184E+00$</td>
<td>$-3.67753E-03$</td>
</tr>
<tr>
<td>4</td>
<td>$9.99433E+00$</td>
<td>$-1.15383E-03$</td>
</tr>
<tr>
<td>5</td>
<td>$9.97441E+00$</td>
<td>$-5.13542E-03$</td>
</tr>
<tr>
<td>6</td>
<td>$9.97768E+00$</td>
<td>$-4.46875E-03$</td>
</tr>
<tr>
<td>7</td>
<td>$9.96674E+00$</td>
<td>$-6.67799E-03$</td>
</tr>
<tr>
<td>8</td>
<td>$9.97072E+00$</td>
<td>$-5.87373E-03$</td>
</tr>
</tbody>
</table>

### Undamped Natural Frequency = $5.49997E+01$

Damping = $9.99536E-02$

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.00000E+01$</td>
<td>$0$</td>
</tr>
<tr>
<td>2</td>
<td>$9.99341E+00$</td>
<td>$5.35210E-03$</td>
</tr>
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<td>3</td>
<td>$-1.18561E-01$</td>
<td>$1.50597E-02$</td>
</tr>
<tr>
<td>4</td>
<td>$-9.99390E+00$</td>
<td>$-4.92155E-03$</td>
</tr>
<tr>
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<td>$-9.98124E+00$</td>
<td>$-1.51908E-02$</td>
</tr>
<tr>
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<td>$-9.98102E+00$</td>
<td>$-1.53676E-02$</td>
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<tr>
<td>7</td>
<td>$9.97187E+00$</td>
<td>$2.28076E-02$</td>
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<tr>
<td>8</td>
<td>$9.97264E+00$</td>
<td>$2.21846E-02$</td>
</tr>
</tbody>
</table>

### Undamped Natural Frequency = $5.54991E+01$

Damping = $9.99709E-02$

<table>
<thead>
<tr>
<th>Coordinate</th>
<th>Coefficient</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-9.97589E+00$</td>
<td>$-4.58045E-03$</td>
</tr>
<tr>
<td>2</td>
<td>$-9.96632E+00$</td>
<td>$-6.38380E-03$</td>
</tr>
<tr>
<td>3</td>
<td>$9.99379E+00$</td>
<td>$1.16695E-03$</td>
</tr>
<tr>
<td>4</td>
<td>$9.78273E-02$</td>
<td>$3.72708E-04$</td>
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<td>5</td>
<td>$-9.98702E+00$</td>
<td>$-2.49320E-03$</td>
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<td>6</td>
<td>$-9.98729E+00$</td>
<td>$-2.44781E-03$</td>
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<td>7</td>
<td>$1.00000E+01$</td>
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<td>8</td>
<td>$9.99868E+00$</td>
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</table>

**No. of Forces = 8**

**Noise Ratio = 0.0**

Table 8
Undamped Natural Frequency = .499798E+01
Damping = .966077E-02

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.251566E-01</td>
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<tr>
<td>2</td>
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<td>-.423328E-02</td>
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<tr>
<td>3</td>
<td>.100000E+01</td>
<td>0</td>
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<tr>
<td>4</td>
<td>.984109E+00</td>
<td>-.513716E-02</td>
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<td>5</td>
<td>.973499E+00</td>
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<tr>
<td>6</td>
<td>.971472E+00</td>
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<tr>
<td>7</td>
<td>.973073E+00</td>
<td>.201598E-01</td>
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<td>8</td>
<td>.969475E+00</td>
<td>.141926E-01</td>
</tr>
</tbody>
</table>

Undamped Natural Frequency = .550583E+01
Damping = .102792E-01

<table>
<thead>
<tr>
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<th>Coefficient 1</th>
<th>Coefficient 2</th>
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<tr>
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<td>.994625E+00</td>
<td>.324345E-01</td>
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<td>.276250E-01</td>
<td>-.977544E-02</td>
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<td>.100000E+01</td>
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<tr>
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<td>.946089E+00</td>
<td>.302677E-01</td>
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<td>6</td>
<td>.952970E+00</td>
<td>.644163E-01</td>
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<tr>
<td>7</td>
<td>.986397E+00</td>
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<tr>
<td>8</td>
<td>.959759E+00</td>
<td>-.215829E-01</td>
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Undamped Natural Frequency = .554904E+01
Damping = .999550E-02

<table>
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<th>Coefficient 2</th>
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<td>4</td>
<td>-.645294E-01</td>
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<td>.875027E+00</td>
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<td>6</td>
<td>.874207E+00</td>
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<td>8</td>
<td>-.890127E+00</td>
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No. of Forces = 8
Noise Ratio = 0.1

Table 9
Damping = .929434E-02

<table>
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<td>.995489E+00</td>
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<tr>
<td>3</td>
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<td>8</td>
<td>.100000E+01</td>
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Undamped Natural Frequency = .500069E+01
Damping = .786745E-02

<table>
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<th>Coefficient</th>
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<tbody>
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<td>1.00000E+01</td>
<td>0</td>
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<tr>
<td>2</td>
<td>.978124E+00</td>
<td>.865004E-01</td>
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<td>-.562677E-02</td>
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<td>4</td>
<td>-.949020E+00</td>
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<td>-.928190E+00</td>
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<td>6</td>
<td>-.959287E+00</td>
<td>-.153123E+00</td>
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<td>.969323E+00</td>
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<td>8</td>
<td>.982434E+00</td>
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Undamped Natural Frequency = .549181E+01
Damping = .786745E-02

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<td>3</td>
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<td>.105953E+00</td>
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<tr>
<td>4</td>
<td>-.618963E-01</td>
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<td>5</td>
<td>-.966994E+00</td>
<td>-.642726E-01</td>
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<tr>
<td>6</td>
<td>-.963982E+00</td>
<td>-.450693E-01</td>
</tr>
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<td>7</td>
<td>1.00000E+01</td>
<td>0</td>
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<tr>
<td>8</td>
<td>.969904E+00</td>
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No. of Forces = 8
Noise Ratio = 0.2

Table 10
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<th>Mode No.</th>
<th>Frequency (Hz)</th>
<th>Damping</th>
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<tr>
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<td>0.010</td>
</tr>
<tr>
<td>2</td>
<td>5.250</td>
<td>0.010</td>
</tr>
<tr>
<td>3</td>
<td>5.500</td>
<td>0.010</td>
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<tr>
<td>4</td>
<td>5.550</td>
<td>0.010</td>
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<tr>
<td>5</td>
<td>15.000</td>
<td>0.010</td>
</tr>
<tr>
<td>6</td>
<td>25.000</td>
<td>0.010</td>
</tr>
</tbody>
</table>

Table 11. System Frequencies, Damping Factors, and Mode Shapes (defined at 8 DOF only); Forces and Locations
Figure 1. Error Comparisons for Test Series 1
Noise/Signal = 0.2, Random Noise Excitation
Normalized Dependent to Independent Modal Vector Error

Figure 2. Error Comparisons for Test Series 2
Noise/Signal = 0.2, Random Noise
Chapter 4

4.1 Conclusions

In this thesis, the theory of a method used to determine an estimation of the modal parameters for a viscously-damped, linear structure has been developed. The MSMA method involves the application and measurement of several non-coherent forces. Verification of the method was demonstrated analytically through an extensive computer simulation study.

The computer simulation showed the MSMA method to be stable. With increasing amounts of noise the results did not diverge dramatically. Also, as the number of applied forces was increased, the results converged to the system values.

Several advantages of the MSMA method were demonstrated by the computer analysis. The first is the ability to locate a mode which has a node line in the vicinity of a shaker by applying several forces. The second is the small amount of user judgement and expertise involved. Basically the only user judgement needed is a knowledge of the approximate number of modes in the analysis range, and that is usually supplied by a previous finite element simulation or can be estimated by examination of a preliminary FRF. The third advantage is the relatively fast analysis time. With the small amount of user interaction the analysis time is basically dependent upon computation speed and total amount of data processed.
4.2 Recommendations for Further Work

There are still many features of the MSMA method that need to be examined.

a. A modal confidence factor (MCF) should be developed to systematically evaluate the validity of acquired modes. The MCF should be based on a comparison of the eigenvectors corresponding to the pseudo-DOF and the eigenvectors of their companion true-DOF.

b. A preliminary mode count procedure should be established. Instead of an initial count based on a finite element model, there should be a method of determining an estimate from the structure itself. One method may be to administer SPR tests at a couple of points to obtain a rough estimate, and then increase that by some margin.

c. To make this method more efficient and more capable of being put into small computers, numerical methods to reduce the size of the data block and required working space should be developed.

d. Since the $[\ddot{C}_i]$ and $[\ddot{K}_i]$ matrices are system constants, it might prove advantageous to use cross-spectrums of the forces and accelerations to reduce the statistical variance. One form of this technique would modify Eq. (2-9) to obtain

$$-[G(\ddot{X}) G(\ddot{X}) G(\ddot{X})] = \begin{bmatrix} G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \\ G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \\ G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \end{bmatrix} \begin{bmatrix} G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \\ G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \\ G(\ddot{X}) & G(\ddot{X}) & G(\ddot{X}) \end{bmatrix}$$

where $G(AB)$ is the cross-spectrum between $A$ and $B$. This equation can now be used to solve for the modal matrices in one of two
fashions. The first would be to average the cross-spectrum matrices for each frequency point into one matrix before carrying out the mathematical operations and subsequent solution for the modal matrices. The second would be to obtain an estimate for the modal matrices for each frequency point and then average those estimates together to obtain the overall estimate.

e. Though the equations from Chapter 2 are applicable to any M, C, and K matrices, the derivation of the equations was for a system with proportional damping. A formal derivation for non-proportional damping should be developed.

f. Alternative methods for the development of pseudo-DOF should be explored and developed. A method which uses data previously acquired and manipulates the data numerically, through a phasing scheme or some other approach, would be highly attractive in that it would eliminate additional testing of the structure to produce the pseudo-DOF.
APPENDIX

Computer Program Listing

PROGRAM RCMB2(TTY,INPUT,OUTPUT,TAPE2=TTY,TAPE5=INPUT,TAPE6=OUTPUT, TAPE1)*
* IN THE PROGRAM ALL INTEGERS ARE INPUT IN I5 FORMAT, AND ALL
* REAL NUMBERS IN F10 FORMAT.
* THE USER IS FIRST ASKED HOW MANY D.O.F. THE SYSTEM HAS (I.E.
* HOW MANY RESPONSE LOCATIONS).
* NEXT THE USER IS ASKED HOW MANY MODES ARE TO BE IN THE SYSTEM
* (THE NUMBER OF MODES DO NOT HAVE TO EQUAL THE NUMBER OF D.O.F.).
* THE FREQUENCIES AND DAMPING RATIOS ARE THEN INPUT.
* THE MODE SHAPES ARE INPUT AFTER THAT.
* THE NEXT INPUT IS THE INITIAL FREQUENCY, AND THE FINAL FREQUENCY
* GIVING THE FREQUENCY RANGE TO BE TESTED IN 256 INCREMENTS.
* THE PROGRAM WILL OUTPUT THE FREQUENCIES CORRESPONDING TO THE
* MAXIMUM AND MINIMUM AMPLITUDES IN THE IMAGINARY RESPONSE.
* THE USER HAS THE CHOICE OF CHOOSING A DIFFERENT RESPONSE POINT
* (THE REFERENCE POINT IS FIXED AT 1) OR JUST USING THE GIVEN
* VALUES FOR ESTIMATES.
* THE USER IS THEN ASKED HOW MANY MODES ARE TO BE EXAMINED (THIS
* CANNOT EXCEED THE NUMBER OF D.O.F.), AND IS THEN REQUESTED TO
* INPUT THE CORRESPONDING NUMBER OF RESPONSE LOCATIONS, AND THE
* EXCITATION LOCATION.
* THE PROGRAM RESPONDS WITH THE EIGENVALUES AND EIGEN VECTORS.
* THIS PROGRAM USES IMSLIBF ROUTINE 'EIGRF'
* AND 'GGUBFS'
* COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
COMMON/DATA/AI(256),AR(256),WW(256),F(10,512),UU(256)
COMMON/PA/S(30,10),SMBK(10,30),ASIN(10,10)
COMMON/AR/XX(30,30),SINT(10,10),SMIN(100),SMINVT(100)
DIMENSION Y(10,512),X(30,512),YX(30,30),XIN(30,30)
DIMENSION T(10,30),IRESRE(10)
DIMENSION IDL(10),YD(10,512)
DIMENSION ITYPE(20),IPHI(20),IRESLOC(20)
DIMENSION ZZZ(100),LB(29),MB(29),SY(400)
DIMENSION IJK(20),PDPI(20,10)
DIMENSION PMAG(20,20),THETA(20,20),WK(1000),XMEN(900)
COMPLEX AW(20),PHI(20,20),ZZ(20,20),Z(400)
DOUBLE PRECISION DSEED

* SETTING CONSTANTS
DSEED=1.00
TWOPI=8.*ATAN(1.)

* INPUTTING DATA
WRITE(2,80)
PRINT 80

80 FORMAT(1X,'ARE YOUR INPUT FREQUENCIES AND SHAPES ON A DATA',
1/1X,'FILE? [Y=1:N=0]')
PRINT 27
WRITE(2,27)
READ(2,400)I
PRINT 301,I
IF(I.NQ.1) GOTO 600
PRINT 1
READ(5,400)NDOF
PRINT 301,NDOF
PRINT 2
READ(5,400)NFR
PRINT 301,NFR
PRINT 3
DO 200 I=1,NFR
PRINT 4,I
READ(5,401),W(I),D(I)
PRINT 300,W(I),D(I)
W(I)=W(I)*TWOPI
200 CONTINUE
PRINT 5
INT=NDOF/7+1
IEXTRA=NDOF-(INT-1)*7
DO 201 I=1,NFR
PRINT 6,I
DO 201 J=1,INT
IEND=7
IF(JJ.EQ.INT) IEND=IEXTRA
IF(IEND.EQ.0) GOTO 201
READ(5,401)(SM((JJ-1)*7+J,I),J=1,IEND)
201 CONTINUE
DO 202 I=1,NFR
202 PRINT 300,(SM(J,I),J=1,NDOF)
GOTO 601
600 CONTINUE
WRITE(2,14)
PRINT 14

14 FORMAT(1X,'YOU MAY ENTER UP TO 20 MODES AND D.O.F. AND',
1/1X,'ANALIZE UP TO 10.1',//)
PRINT 1
WRITE(2,1)
PRINT 27
WRITE(2,27)
27 FORMAT(1X,'[ I5 FORMAT ]')
28 FORMAT(1X,'[ F10 FORMAT ]')
1 FORMAT(1X,'ENTER NUMBER OF DEGREES OF FREEDOM')
READ(2,400) NDOF
400 FORMAT(I5)
PRINT 301,NDOF
300 FORMAT(1X,19(1X,F6.3))
301 FORMAT(1X,4(2X,I5))
PRINT 2
WRITE(2,2)
PRINT 27
WRITE(2,27)
2 FORMAT(1X,'ENTER NUMBER OF MODES')
READ(2,400) NFR
PRINT 301,NFR
PRINT 3
WRITE(2,3)
PRINT 28
WRITE(2,28)
28 FORMAT(1X,'ENTER FREQUENCIES (IN HZ.3, DAMPING RATIOS')
DO 100 I=1,NFR
PRINT 4,1
WRITE(2,4)I
4 FORMAT(1X,'FREQUENCY NO. ,I3)
READ(2,401) W(I),D(I)
401 FORMAT(7(E10.4))
PRINT 300,W(I),D(I)
W(I)=TWOPI*W(I)
100 CONTINUE
PRINT 5
WRITE(2,5)
PRINT 28
WRITE(2,28)
5 FORMAT(1X,'ENTER MODE SHAPES')
INT=NDOF/7+1
IEXTRA=NDOF-(INT-1)*7
DO 101 I=1,NFR
PRINT 6,1
WRITE(2,6)I
6 FORMAT(1X,'ENTER MODE NO. ,I3,' SEVEN D.O.F. PER LINE')
DO 101 JJ=I,INT
IEND=7
IF(JJ.EQ.INT) IEND=IEXTRA
IF(IEND.EQ.0) GOTO 101
READ(2,401)(SM((JJ-1)*7+J,I),J=1,IEND)
101 CONTINUE
DO 102 I=1,NFR
102 PRINT 300,(SM(J,I),J=1,NDOF)
601 CONTINUE
DO 569 I=1,NFR
AM(I)=0.0
569 AM(I)=AM(I)+SM(J,I)*SM(J,I)
888 CONTINUE

*  SETTING PARAMETERS FOR RUN *

WRITE(2,7)
PRINT 7
WRITE(2,28)
WRITE(2,22)
7 FORMAT(1X,'ENTER INITIAL FREQUENCY RANGE IN HZ. ; INIT. FREQ. / ', 1'FINAL FREQ.')
READ(2,401) STFR,FIFR
PRINT 330,STFR,FIFR
330 FORMAT(1X,'*1,2(1X,F12.5))
STFR=TWOPI*STFR
FIFR=TWOPI*FIFR
FIST=FIFR-STFR
WRITE(2,29)
PRINT 29
WRITE(2,27)
PRINT 27
READ(2,400) ITWO
PRINT 301,ITWO
29 FORMAT(1X,'INPUT THE NUMBER OF FREQUENCY POINTS TO BE ANALIZED', 1/,1X,'[UP TO 00256]')
IDTWO=2*ITWO
ITT=ITWO-1
TWOM1=FLOAT(ITT)
WRITE(2,32)
PRINT 32
WRITE(2,28)
PRINT 28
READ(2,401) ANOISE
PRINT 300,ANOISE
32 FORMAT(1X,'INPUT THE PERCENT OF NOISE IN THE FUNCTIONS', 1/,1X,'[ 100% = 1. ]')

*  GENERATING A CLEAN TRANSFER FUNCTION REFERENCED TO DOF 1. *
*  PRINTS OUT ALL LOCAL MAXIMUMS AND MINIMUMS IN THE IMAGINARY *
*  PLANE. *

519 WRITE(2,16)
PRINT 16
PRINT 27
WRITE(2,27)

16 FORMAT(1X,'ENTER THE RESPONSE POINT FOR THE INITIAL RUN')
READ(2,400)JRESP
PRINT 301,JRESP

* * *

* INITIALIZE PARAMETERS TO KEEP FROM GETTING A WARNING *
* *

ANSOLD=0.
DIFOLD=0.
WWOLD=0.
DO 106 J=1,ITWO
   TT=FLOAT(J)
   WW(J)=STFR+(TT-1.)*FIST/TWOMI
   CALL TRAN(WW(J),1,ANS,NFR,JRESP)
   IF(J.EQ.1) GOTO 123
   DIF=ANS-ANSOLD
   IF(J.EQ.2) GOTO 124
   POS=DIF*DIFOLD
   IF(POS.GE.0) GOTO 124
   WRITE(2,12)WWOLD,ANSOLD
   PRINT 12,WWOLD,ANSOLD
124 DIFOLD=DIF
123 ANSOLD=ANS
   WWOLD=WW(J)/TWOPI
12 FORMAT(1X,'W =',E12.6,' AMP=',E12.6)
106 CONTINUE
WRITE(2,17)
PRINT 17
PRINT 27
WRITE(2,27)

17 FORMAT(1X,'IS ANOTHER RESPONSE POINT REQUIRED? [Y=1/N=0]')
READ(2,400)IANS
PRINT 301,IANS
IF(IANS.EQ.1) GOTO 519
WRITE(2,8)

8 FORMAT(1X,'ENTER NUMBER OF MODES TO BE EXAMINED')
PRINT 8
PRINT 27
WRITE(2,27)
READ(2,400)NME
PRINT 301,NME
WRITE(2,10)NME
PRINT 10,NME
PRINT 27
WRITE(2,27)

10 FORMAT(1X,'ENTER THE ',I3,
   1' DIFFERENT INDEPENDENT RESPONSE LOCATIONS;')
   1/X,'ONE PER LINE')
DO 103 I=1,NME
READ(2,400) IRESLOC(I)
PRINT 301,IRESLOC(I)
103 CONTINUE
WRITE(2,26)
PRINT 26
WRITE(2,27)
PRINT 27

26 FORMAT(1X,'ENTER THE NUMBER OF FORCED INDEPENDENT RESPONSE','
1/X,'LOCATIONS')
READ(2,400)NF
PRINT 301,NF
DO 509 I=1,NF
WRITE(2,40)
PRINT 40
READ(2,400)IFOR(I)
PRINT 301,IFOR(I)
509 CONTINUE

40 FORMAT(1X,'INPUT FORCE LOCATION')
WRITE(2,21)
PRINT 21
WRITE(2,27)

21 FORMAT(1X,'ENTER THE NUMBER OF DEPENDENT RESPONSE LOCATIONS','
1/X,'ONE PER LINE')
DO 522 I=1,NDL
READ(2,400)IDL(I)
522 PRINT 301,IDL(I)
505 CONTINUE

603 CONTINUE
WRITE(2,602)
PRINT 602
WRITE(2,27)
PRINT 27
READ(2,400) IQUEST
PRINT 301,IQUEST
602 FORMAT(1X,'DO YOU WANT SWEPT SINE [1]',
1/X,'OR RANDOM [2] FORCING?')
IF(IQUEST.NE.1.AND.IQUEST.NE.2) GOTO 603
IF(IQUEST.EQ.2) GOTO 605
WRITE(2,606),NF
PRINT 606,NF
DO 604 I=1,NF
WRITE(2,28)
READ(2,401)LB(I)
PRINT 300,LB(I)
604 CONTINUE
606 FORMAT(1X,'ENTER THE',I2,' PHASE SHIFTS')
WRITE(2,608)
PRINT 608
WRITE(2,27)
PRINT 27
READ(2,400)IAVE
PRINT 301,IAVE
608 FORMAT(1X,'HOW MANY AVERAGES DO YOU WANT?')
DO 576 KK=1,IAVE
DO 576 I=1,NDOF
DO 577 J=1,NF
K=IFOR(J)
577 IF(K.EQ.I) GOTO 578
DO 579 J=1,IDTWO
579 F(I,J)=0.0
GOTO 580
578 CONTINUE
PHII=LB(J)
CALL ASINE(FIST,IDTWO,STFR,FIFR,PHII,TWOM1,XMEN)
DO 582 II=1,IDTWO
582 Z(II)=CMPLX(XMEN(II),0.0)
CALL FFT(IDTWO,Z,TWOPI)
DO 581 J=1,ITWO
JJJ=J+(KK-1)*ITWO
JJ=JJJ+IAVE*ITWO
F(K,JJJ)=REAL(Z(J))
F(K,JJ)=AIMAG(Z(J))
581 CONTINUE
580 CONTINUE
576 CONTINUE
GOTO 607
605 CONTINUE
WRITE(2,608)
PRINT 608
WRITE(2,27)
PRINT 27
READ(2,400)IAVE
PRINT 301,IAVE
DO 568 KK=1,IAVE
DO 568 I=1,NDOF
DO 570 J=1,NF
K=IFOR(J)
570 IF(K.EQ.I) GOTO 571
DO 572 J=1,IDTWO
572 F(I,J)=0.0
GOTO 573
571 CONTINUE
DO 574 J=1,ITWO
      JJJ=J+(KK-1)*ITWO
      JJ=JJJ+IAVE*ITWO
      ARAN=GGUBFS(DSEED)
      ARAN=ARAN*TWOPI
      F(K,JJJ)=COS(ARAN)
      F(K,JJ)=SIN(ARAN)
574 CONTINUE
573 CONTINUE
568 CONTINUE
607 CONTINUE

Y is an (NME)*(512) matrix containing the acceleration responses
the real parts are in the first 256 columns and the imaginary are
in the second 256 columns.

X is an (2*NME+1)*(512) matrix containing the velocity responses
in the first NME rows (in the same fashion as the acceleration
responses), the displacement responses in the second NME rows.
The final row contains 1. in the first 256 columns and 0. in
the final 256.

DO 500 K=1,IAVE
DO 500 I=1,NME
      IN=IRESLOC(I)
      CALL FRF(ITWO,TFM1,STFR,FF,IN,ANoise,DSEED,K,IAVE)
DO 500 J=1,ITWO
      JJJ=J+(K-1)*ITWO
      JJ=JJJ+IAVE*ITWO
      II=I+NME
      III=II+NME
      X(II,J)=AI(J)
      X(II,JJJ)=AR(J)
      X(I,J)=WW(J)*X(II,JJ)
      Y(I,J)=WW(J)*X(II,JJJ)
      Y(I,JJJ)=WW(J)*X(II,JJ)
500 CONTINUE

IDTAVE=IDTWO*IAVE
DO 575 I=1,NF
      II=I+NME+NME
      K=IFOR(I)
DO 575 J=1,IDTAVE
      X(I,J)=F(K,J)
575 CONTINUE

IF(NDL.EQ.0) GOTO 530
DO 523 K=1,IAVE
DO 523 I=1,NDL
IN=IDL(I)
CALL FRF(ITWO,TWOMI,STFR,FIFR,NFR,IN,ANOISE,DSEED,K,IAVE)
DO 523 J=1,ITWO
JJJ=J+(K-1)*ITWO
JJ=JJJ+IAVE*ITWO
YD(I,JJJ)=-WW(J)*WW(J)*AR(J)
YD(I,JJ)=-WW(J)*WW(J)*AI(J)
523 CONTINUE
530 CONTINUE
N3=2*NME+NF
N2=2*NME
N0=NME-NF
N03=3*N0
NF3=3*NF

* XIN=(X)*(TRANSPOSE(X))
XIN IS PUT INTO AN ONE DIMENSIONAL ARRAY XMEN SO THAT IT CAN BE
INVERTED EASILY, THE INVERTED XMEN IS THEN PUT BACK INTO XIN

DO 501 I=1,N3
DO 501 J=1,N3
XIN(I,J)=0.
DO 501 K=1,IDTAVE
XIN(I,J)=XIN(I,J)+X(I,K)*X(J,K)
501 CONTINUE
13 FORMAT(1X,17(E8.2))
ID=0
DO 510 J=1,N3
DO 510 I=1,N3
ID=ID+1
XMEN(ID)=XIN(I,J)
510 CONTINUE
CALL MINV(XMEN,N3,DETER,LB,MB)
ID=0
DO 511 J=1,N3
DO 511 I=1,N3
ID=ID+1
XIN(I,J)=XMEN(ID)
511 CONTINUE

* YX=(Y)*(TRANSPOSE(X))
T=(YX)*(XIN); XIN BEING THE INVERTED FORM

DO 502 I=1,NME
DO 502 J=1,N3
YX(I,J) = 0.
DO 502 K=1,N3
  YX(I,J) = YX(I,J) + Y(I,K)*X(J,K)
DO 503 J=1,N3
  T(I,J) = 0.
DO 503 K=1,N3
  T(I,J) = T(I,J) - YX(I,K)*XIN(K,J)

* THE T MATRIX IS COMPOSED OF THREE SUBMATRICES;
* THE FIRST IS B(NME,NME), THE NEXT IS K(NME,NME), THE
* LAST IS GAMMA(NME,1).
* SY IS A MATRIX COMPOSED OF B,K, AN IDENTITY MATRIX, AND A
* NULL MATRIX, AS FOLLOWS:
* 
* \[ SY = \begin{bmatrix} B & K \\ -I & 0 \end{bmatrix} \]
* 
* THE NEGATIVE OF SY IS PUT INTO AN EIGEN SOLVER AND THE VALUES
* RETURNED ARE OUTPUT.
* 
* THE EIGENVALUES AND VECTORS ARE ALSO GENERATED USING ONLY THE
* K MATRIX.
* 
* IT=-NME
DO 504 J=1,2*NME
  IT=IT+NME
DO 504 I=1,NME
  IT=IT+1
504 SY(IT) = T(I,J)
IT=0
N2=2*NME
DO 506 I=1,N2
  IT=IT+NME
DO 506 J=1,NME
  IT=IT+1
SY(IT) = 0.
IF(I.EQ.J) SY(IT) = -1.
506 CONTINUE
514 CONTINUE
DO 518 I=1,N2
  SY(I) = -SY(I)
518 CONTINUE
CALL EIGRF(SY,N2,N2,2,AW,Z,N2,WK,IER)
* SCALING EIGENVECTORS
1R=0
TO FIND THE MODE SHAPE COEFFICIENTS FOR THE DEPENDENT D.O.F.
USING THE EQUATION:

\[
\text{PHI(\text{DEP})} \cdot \text{INV[PHI(\text{IND})]} = [Y \cdot \text{TRANS}(X)] \cdot \text{INV}[X \cdot \text{TRANS}(X)]
\]

WHERE [\text{Y}] = THE ACCELERATION RESPONSES FOR THE DEPENDENT D.O.F.
[X] = THE ACCELERATION RESPONSES FOR THE INDEPENDENT D.O.F.

PDPI = YX \cdot YY \quad \text{WHERE YY HAS BEEN INVERTED}

IF(NDL.EQ.0) GOTO 529
DO 524 I=1,NME
DO 524 J=1,NME
XIN(I,J)=0.0
DO 524 K=1,IDTAVE
524 XIN(I,J)=XIN(I,J)+Y(I,K)*Y(J,K)
DO 525 I=1,NDL
DO 525 J=1,NME
YX(I,J)=0.0
DO 525 K=1,IDTAVE
525  YX(I,J)=YX(I,J)+YD(I,K)*Y(J,K)
    IT=0
    DO 526 J=1,NME
    DO 526 I=1,NME
    IT=IT+1
526  XMEN(IT)=XIN(I,J)
    CALL MINV(XMEN,NME,DETER,LB,MB)
    IT=0
    DO 527 J=1,NME
    DO 527 I=1,NME
    IT=IT+1
527  XIN(I,J)=XMEN(IT)
    DO 528 I=1,NDL
    DO 528 J=1,NME
    PDPI(I,J)=0.0
528  PDPI(I,J)=PDPI(I,J)+YX(I,K)*XIN(K,J)
    ND2=2*NDL
    DO 532 I=1,NDL
    DO 532 J=1,N2
    PHI(I,J)=0.0
532  PHI(I,J)=PHI(I,J)+PDPI(I,K)*ZZ(KK,J)
529  CONTINUE
*
*  PRINT OUT THE CONVERGENCE CRITERIA FOR THE EIGEN SOLUTION
*  ( LESS THAN ONE INDICATES GOOD CONVERGENCE)
*  WRITE(2,11)WK(1)
*  PRINT 11,WK(1)
*  SORTING THE DEPENDENT AND INDEPENDENT D.O.F. INTO
*  CONSECUTIVE ORDER
*
* ITOT=NDL+NME
    DO 533 I=1,ITOT
    DO 534 J=1,NME
    IF(IRESLOC(J).NE.I) GOTO 534
    IPHI(I)=J
    ITYPE(I)=1
    GOTO 533
534  CONTINUE
    IF(NDL.EQ.0) GOTO 533
    DO 541 J=1,NDL
    IF(IDL(J).NE.I) GOTO 541
    IPHI(I)=J
    ITYPE(I)=2
541  CONTINUE
CONTINUE

PRINTING OUT THE EIGEN SOLUTIONS

DO 547 K=1,N2
AW(K)=AW(K)/TWOPI
SIG=REAL(AW(K))
DEL=AIMAG(AW(K))
SQ=SQRT(SIG*SIG+DEL*DEL)
DMP=SIG/SQ
PRINT 24,K,SQ,DMP
24 FORMAT(///,1X,'MODE NO.=',I3,2X,'NATURAL UNDAMPED FREQUENCY=',E12.6,/,33X,'DAMPING =',E12.6,
1//,1X,'COORDINATE',9X,'COEFFICIENT',/)
DO 535 I=1,ITOT
J=IPH(I)
JJ=J+NME
IF(ITYPE(I).EQ.2) GOTO 536
PRINT 23,I,ZZ(JJ,K)
GOTO 535
536 PRINT 23,I,PHI(J,K)
535 CONTINUE
IF(NMEID.EQ.0) GOTO 43
w DO 44 I=1,NMEID
II=I+NME+NME
44 PRINT 23,IRESRE(I),ZZ(II,K)
43 CONTINUE
547 CONTINUE
23 FORMAT(4X,I2,12X,E12.6,2X,E12.6)

SETTING THE EIGEN PROBLEM FOR THE STIFFNESS MATRIX
XMEN(NME,NME) = T(NME,[NME TO 2*NME])
IT=0

DO 515 J=1,NME
DO 515 I=1,NME
JJ=J+NME
IT=IT+1
XMEN(IT)=T(I,JJ)
515 CONTINUE
PRINT 31
31 FORMAT(///,1X,'K MATRIX')
DO 516 I=1,NME
PRINT 18,(XMEN(I+(J-1)*NME),J=1,NME)
18 FORMAT(3X,10E12.6)
516 CONTINUE
CALL EIGRF(XMEN,NME,NME,2,AW,Z,NME,WK,IER)
IR=0
DO 537 J=1,NME
DO 537 I=1,NME
IR=IR+1
ZZ(I,J)=Z(IR)
537 CONTINUE
DO 555 J=1,NME
ZMAGO=0.0
DO 555 II=I,NME
PMAG(II,J)=REAL(ZZ(II,J))
IF(II.GT.NME) GOTO 45
IF(ABS(PMAG(II,J)).GT.ZMAGO) IJK(J)=II
IF(ABS(PMAG(II,J)).GT.ZMAGO) ZMAGO=ABS(PMAG(II,J))
45 CONTINUE
DO 556 J=1,NME
DO 556 II=I,NME
K=IJK(J)
ZZ(II,J)=ZZ(II,J)/ABS(PMAG(K,J))
556 CONTINUE

* FINDING THE MODE SHAPE COEFFICIENTS FOR THE DEP. D.O.F. *
* *
* IF(NDL.EQ.0) GOTO 544
* DO 538 I=1,NDL
* DO 538 J=1,NME
* PHI(I,J)=0.0
* DO 538 K=1,NME
* PHI(I,J)=PHI(I,J)+PDPI(I,K)*ZZ(K,J)
* 538 CONTINUE
* 544 CONTINUE
* *
* COMPUTING THE DAMPING FACTORS
* DIAG[INV[PHI(IND)]*B(MATRIX)*PHI(IND)] = [2*DAMP(I)*AW(I)]
* *
* IT=0
* DO 508 J=1,NME
* DO 508 I=1,NME
* IT=IT+1
* ZZZ(IT)=Z(IT)
* 508 PDPI(I,J)=Z(IT)
* DO 542 I=1,NME
* DO 542 J=1,NME
* XIN(I,J)=0.0
* DO 542 K=1,NME
542 XIN(I,J)=XIN(I,J)+T(I,K)*PDPI(K,J)
CALL MINV(ZZZ,NME,DETER,LB,MB)
IT=0
DO 543 J=1,NME
DO 543 I=1,NME
IT=IT+1
543 PDPI(I,J)=ZZZ(IT)
DO 520 I=1,NME
DO 520 J=1,NME
THETA(I,J)=0.0
DO 520 K=1,NME
520 THETA(I,J)=THETA(I,J)+PDPI(I,K)*XIN(K,J)
*
*
CHANGING EIGENVALUES TO HERTZ AND OUTPUTTING SOLUTIONS
*
*
DO 548 K=1,NME
AW(K)=AW(K)/(TWOPI*TWOPI)
RAW=AW(K)
ICOUNT=0
IF(RAW.LT.0) ICOUNT=1
RAW=ABS(RAW)
RAW=SQR(RAW)
DAMP=THETA(K,K)/(2.*TWOPI*RAW)
PRINT 20,K,RAW,DAMP
IF(ICOUNT.EQ.1) PRINT 25
25 FORMAT(1X,'OMEGA SQUARED WAS NEGATIVE',//)
20 FORMAT(1X,'MODE NO. ', I2, ' NATURAL FREQ.(HZ)= ', E12.6,
12X,'DAMPING= ',E12.6,/)
DO 539 I=1,ITOT
J=IPHI(I)
IF(ITYPE(I).EQ.2) GOTO 540
PRINT 23,I,ZZ(J,K)
GOTO 539
540 PRINT 23,I,PHI(J,K)
539 CONTINUE
IF(NMEID.EQ.0) GOTO 548
DO 47 I=1,NMEID
II=I+NME
47 PRINT 23, IRESRE(I),ZZ(II,K)
548 CONTINUE
WRITE(2,19)
PRINT 19
WRITE(2,27)
19 FORMAT(1X,'DO YOU WANT TO REPEAT THIS WITH THE SAME FUNCTIONS?',
1' Y=1 N=0')
READ(2,400)IT
PRINT 301,IT
IF(IT.EQ.1) GOTO 888
11 FORMAT(2X,E12.6,2X,E12.6)
END

SUBROUTINE TRAN(F,IX,Y,NFR,J)
COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
Y=0.
DO 10 I=1,NFR
Y=Y+(SM(J,I)*SM(IX,I)*2.*D(I)*(F/W(I))/
1((1-(F/W(I))**2)**2+(2.*D(I)*F/W(I))**2))
10 CONTINUE
RETURN
END

SUBROUTINE FRF(ITWO,TWOMI,STFR,FIFR,NFR,I,ANOISE,DSEED,KAY,IAVE)
COMMON/FREQ/AM(20),W(20),D(20),SM(20,20),IFOR(10),NF
COMMON/DATA/AI(256),AR(256),WW(256),F(10,512),UU(256)
DOUBLE PRECISION DSEED

* DEVELOPES THE DISPLACEMENT FRFS
* RANGE=FIFR-STFR
DELT=RANGE/TWOMI
NTIM=ITWO
X=0.0
DO 101 K=1,ITWO
KFOUR=K+(KAY-1)*ITWO
KKK=KFOUR+IAVE*ITWO
TK=FLOAT(K)
WW(K)=STFR+(TK-1.)*DELT
AI(K)=0.
AR(K)=0.
DO 101 KK=1,NF
ISH=IFOR(KK)
101 CONTINUE

DELT=RANGE/TWOMI
NTIM=ITWO
X=0.0
DO 101 K=1,ITWO
KFOUR=K+(KAY-1)*ITWO
KKK=KFOUR+IAVE*ITWO
TK=FLOAT(K)
WW(K)=STFR+(TK-1.)*DELT
AI(K)=0.
AR(K)=0.
DO 101 KK=1,NF
ISH=IFOR(KK)
101 CONTINUE

101 CONTINUE
X = SQRT(X)
C = ANOISE * X / ITWO
DO 102 K = 1, ITWO
DSE = GGUBFS(DSEED)
ANOI = 2. * (2. * DSE - 1.) * C
AMAG = AI(K) * AI(K) + AR(K) * AR(K)
AMAG = SQRT(AMAG)
AI(K) = AI(K) + ANOI * AI(K) / AMAG
AR(K) = AR(K) + ANOI * AR(K) / AMAG
102 CONTINUE
RETURN
DO 110 J = 1, NTIM
T = FLOAT(J)
T = 8. * ATAN(1.) * T / RANGE
UU(J) = 0.0
DO 110 K = 1, ITWO
AN = FLOAT(K) - 1.
IF(K, EQ, 1) THEN
AR(K) = AR(K) / 2.
ENDIF
UU(J) = UU(J) + AR(K) * COS(T * DELT * AN) - 2. * AI(K) * SIN(T * DELT * AN)
IF(K, EQ, 1) THEN
AR(K) = 2. * AR(K)
ENDIF
110 CONTINUE
WRITE(1, 111) (UU(J), J = 1, NTIM)
111 FORMAT (4E14.6)
RETURN
END

SUBROUTINE MINV(AMAT, N, D, L, M)
DIMENSION AMAT(900), L(29), M(29)
*
MATRIX INVERSION SUBROUTINE
*
42 CONTINUE
D = 1.
NK = N
DO 80 K = 1, N
NK = NK + N
L(K) = K
M(K) = K
KK = NK + K
BIGA = AMAT(KK)
DO 20 J = K, N
IZ = N * (J - 1)
DO 20 I = K, N
IH = IZ + I
10 IF(ABS(BIGA) - ABS(AMAT(IJ))) 15, 20, 20
15 BIGA = AMAT(IJ)
L(K) = I
M(K) = J
20 CONTINUE
J=L(K)
IF(J-K) 35,35,25
25  KI=K-N
DO 30  I=1,N
KI=KI+N
HOLD=-AMAT(KI)
JI=KJ+J-K
AMAT(KI)=AMAT(JI)
30  AMAT(JI)=HOLD
35  I=M(K)
IF(I-K) 45,45,38
38  JP=N*(I-1)
DO 40  J=1,N
JK=NK+J
JI=JP+J
HOLD=-AMAT(JK)
AMAT(JK)=AMAT(JI)
40  AMAT(JI)=HOLD
45  IF(BIGA) 48,46,48
46  D=0.0
RETURN
48  DO 55  I=1,N
IF(I-K) 50,55,50
50  IK=NK+I
AMAT(IK)=AMAT(IK)/(-BIGA)
55  CONTINUE
DO 65  I=1,N
IK=NK+I
HOLD=AMAT(IK)
IJ=I-N
DO 65  J=1,N
IJ=IJ+N
IF(I-K) 60,65,60
60  IF(J-K) 62,65,62
62  KJ=IJ-I+K
AMAT(IJ)=HOLD*AMAT(KJ)+AMAT(IJ)
65  CONTINUE
KJ=K-N
DO 75  J=1,N
KJ=KJ+N
IF(J-K) 70,75,70
70  AMAT(KJ)=AMAT(KJ)/BIGA
75  CONTINUE
D=D*BIGA
AMAT(KK)=1./BIGA
80  CONTINUE
K=N
100  K=(K-1)
IF(K) 150,150,105
105  I=L(K)
IF(I-K) 120,120,108
108 JQ=N*(K-1)
JR=N*(I-1)
DO 110 J=1,N
JK=JQ+J
HOLD=AMAT(JK)
JI=JR+J
AMAT(JK)=-AMAT(JI)
110 AMAT(JI)=HOLD
120 J=M(K)
IF(J-K) 100,100,125
125 KI=K-N
DO 130 I=1,N
KI=KI+N
HOLD=AMAT(KI)
JI=KI-K+J
AMAT(KI)=-AMAT(JI)
130 AMAT(JI)=HOLD
GOTO 100
150 RETURN
END

SUBROUTINE ASINE(FIST,IDTWO,STFR,FIFR,PHI,TWOM1,XMEN)
DIMENSION XMEN(512)

*** DEVELOPES TIME HISTORY OF SWEPT-SINE EXCITATION ***

DEL=FIST/TWOM1
TO=1./((TWOM1+1.)*DEL)
DO 10 I=1,IDTWO
TI=FLOAT(I)
F=STFR+DEL*(TI-1.)
T=TO*(TI-1.)/TWOM1
10 XMEN(I)=(3.-T/TO)*SIN(F*T+PHI)
RETURN
END

SUBROUTINE FFT(IDTWO,Z,TWOP1)
COMPLEX Z(512),U,T,W

*** DEVELOPES FRF OF THE SWEPT-SINE EXCITATION ***

II=IDTWO
M=0
PI=TWOP1/2.
1 M=M+1
II=II/2
IF(I.I.GE.2) GOTO 1
MV2=IDTWO/2
NM1=IDTWO-1
J=1
DO 7 I=1,NM1
IF(I.GE.J) GOTO 5
T=Z(J)
Z(J)=Z(I)
Z(I)=T
5 K=MV2
6 IF(K.GE.J) GOTO 7
J=J-K
K=K/2
GOTO 6
7 J=J+K
DO 20 L=1,M
LE=2**L
LE1=LE/2
U=(1.,0.)
W=CMPLX(COS(PI/LE1),-SIN(PI/LE1))
DO 20 J=1,LE1
DO 10 I=J,IDTWO,LE
IP=I+LE1
T=Z(IP)*U
Z(IP)=Z(I)-T
10 Z(I)=Z(I)+T
20 U=U*W
RETURN
END
Bibliography


