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PERFORMANCE ANALYSIS OF A
CONCATENATED CODING SCHEME FOR ERROR CONTROL

Technical Report

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PERFORMANCE ANALYSIS OF A
CONCATENATED CODING SCHEME FOR ERROR CONTROL

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ABSTRACT

In this paper, a concatenated coding scheme for error control in data communications is analyzed. In this scheme, the inner code is used for both error correction and detection, however the outer code is used only for error detection. A retransmission is requested if the outer code detects the presence of errors after the inner code decoding. Probability of undetected error is derived and bounded. A particular example, proposed for NASA Planetary Program, is analyzed.

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1. Introduction

Consider a concatenated coding scheme for error control for a binary symmetric channel with bit-error-rate $\varepsilon < 1/2$ as shown in Figure 1. Two linear block codes, $C_f$ and $C_b$, are used. The inner code $C_f$, called frame code, is an $(n,k)$ code with minimum distance $d_f$. The frame code is designed to correct $t$ or fewer errors and simultaneously detect $\lambda(\lambda \geq t)$ or fewer errors where $t+\lambda+1 \leq d_f$. The outer code $C_b$ is an $(n_b,k_b)$ code with

$$n_b = mk.$$  

The outer code is designed for error detection only.

The encoding is done in two stages. A message of $k_b$ bits is first encoded into a codeword of $n_b$ bits in the outer code $C_b$. Then the $n$-bit word is divided into $m$ $k$-bit segments. Each $k$-bit segment is encoded into an $n$-bit word in the frame code $C_f$. This $n$-bit word is called a frame. Thus, corresponding to each $k_b$-bit message at the input of the outer code encoder, the output of the frame code encoder is a sequence of $m$ frames. This sequence of $m$ frames is called a block. A two dimensional block format is depicted in Figure 2.

The decoding consists of error correction in frames and error detection in $m$ decoded $k$-bit segments. When a frame in a block is received, it is decoded based on the frame code $C_f$. The $n-k$ parity bits are then removed from the decoded frame, the $k$-bit decoded segment is stored in a buffer. If there are $t$ or fewer transmission errors in a received frame, the errors will be corrected and the decoded segment is error free. If there are more than $t$ errors in a received frame, the decoded segment contains undetected errors. After $m$ frames of a block have been decoded, the buffer contains $m$ $k$-bit decoded segments. Then error detection is performed on these $m$ decoded segments based on the outer code $C_b$. If no error is detected, the $m$ decoded segments are assumed to be error free and are accepted (with the $n_b-k_b$ parity bits removed) by the receiver. If
the presence of errors is detected, the \( m \) decoded segments are discarded and the receiver requests a retransmission of the rejected block. Retransmission and decoding process continues until a transmitted block is successfully received. Note that a successfully received block may be either error free or contains undetectable errors.

The error control scheme described above is actually a combination of forward-error-correction (FEC) and automatic-repeat-request (ARQ), called a hybrid ARQ scheme \([1]\). The retransmission strategy determines the system throughput, it may be one of the three basic modes namely, stop-and-wait, go-back-\( N \) or selective-repeat. In this report, we are only concerned with the reliability of the proposed error control scheme. The reliability is measured in terms of the probability of undetected error after decoding.

An example scheme, proposed for NASA Planetary Program, is analyzed.

2. Probability of Undetected Error for the Frame Code

For a codeword \( \bar{v} \) in the frame code \( C_f \), let \( w(\bar{v}), w^{(1)}(\bar{v}) \) and \( w^{(2)}(\bar{v}) \) denote the weight of \( \bar{v} \), the weight of information-part of \( \bar{v} \) and the weight of parity-part of \( \bar{v} \) respectively. Clearly \( w(\bar{v})=w^{(1)}(\bar{v})+w^{(2)}(\bar{v}) \). If a decoded frame contains an undetectable error pattern, this error pattern must be a nonzero codeword in \( C_f \). Let \( \bar{e}_0 \) be a nonzero error pattern after decoding. Since \( \bar{e}_0 \) is in \( C_f \), we have

\[
w^{(1)}(\bar{e}_0) + w^{(2)}(\bar{e}_0) \geq d_f,
\]

and

\[
w^{(1)}(\bar{e}_0) \geq 1.
\]

The probability \( P_f(\bar{e}_0, \epsilon) \) that a decoded frame contains a nonzero error vector \( \bar{e}_0 \) after decoding is given by \([2]\),

\[
P_f(\bar{e}_0, \epsilon) = \sum_{i=0}^{t} \sum_{j=0}^{\min(t-i,n-w)} \binom{t-i}{i} \binom{n-w}{j} \epsilon^{w-i+j}(1-\epsilon)^{n-w+i-j}, \tag{3}
\]
Let \( p_{ud}(f)(\varepsilon) \) denote the probability of undetected error for the frame code. Let \( \{A_w(f), 0 \leq w \leq n\} \) be the weight distribution of \( C_f \) where \( A_w(f) \) denotes the number of codewords in \( C_f \) with weight \( w \). Then, it follows from (3) that

\[
p_{ud}(f)(\varepsilon) = \sum_{w=0}^{n} A_w(f) \sum_{i=0}^{\min(t-i,n-w)} \binom{n-w}{i} \binom{n-w-1}{j} \varepsilon^{w-i+j}(1-\varepsilon)^{n-w+i-j}.
\]

Hence, if we know the weight distribution of the frame code, we can compute \( p_{ud}(f)(\varepsilon) \) from (4).

Let \( Q_t(w,\varepsilon) \) denote the right-hand side of (3). For \( w < n-1-j \),

\[
\frac{\frac{w+1}{i} \binom{n-w-1}{j} \varepsilon^{w+1-i+j}(1-\varepsilon)^{n-w-1+i-j}}{\binom{n-w}{i} \binom{n-w-1}{j} \varepsilon^{w-i+j}(1-\varepsilon)^{n-w+i-j}} = \frac{(w+1)(n-w-j)\varepsilon}{(w+1-i)(n-w)(1-\varepsilon)} \leq \frac{(w+1)\varepsilon}{(w+1-t)(1-\varepsilon)}.
\]

Since \( w \geq 2t+1 \), we have that

\[
\frac{w+1}{w+1-t} \leq \frac{2t+2}{t+2}.
\]

It follows from (5) and (6) that, for \( \varepsilon \leq \frac{t+2}{3t+4} \),

\[
Q_t(w+1,\varepsilon) \leq Q_t(w,\varepsilon).
\]

For \( 1 \leq i \leq k \), let

\[
W(i) = \min\{w(\overline{v}) : \overline{v} \in C_f \text{ and } w^{(1)}(\overline{v}) = i\}.
\]

Then we see readily that

\[
W(i) \geq \max(d_f,i).
\]

If \( C_f \) is an even-weight code and \( i \) is odd, then

\[
W(i) \geq \max(d,i+1).
\]

It follows from (3), (7) and (9) that, for \( 0 \leq \varepsilon \leq (t+2)/(3t+4) \),

\[
P_{f}(\overline{e}_0,\varepsilon) \leq Q_t(W^{(1)}(\overline{e}_0),\varepsilon) \leq Q_t(\max(d_f,W^{(1)}(\overline{e}_0)),\varepsilon).
\]
For $\epsilon<<1/n$, we can see from (3) and (11) that

$$P_f(e_0, \epsilon) \leq \begin{cases} (d_f)^{e_t-d+1} (1-e)^{n-d_f+t} & \text{for } w(1)(e_0)<d, \\ t^{w(1)(e_0)} t^{w(1)(e_0)-t(1-e)n-w(1)(e_0)+t} & \text{otherwise.} \end{cases}$$  \quad (12)

(13)

3. Probability of Undetected Error for the Outer Code

Recall that a codeword in the outer code $C_b$ consists of $m$ k-bit segments. At the receiver, error detection is performed on every $m$ decoded segments based on $C_b$. Let $P_b(e, \epsilon)$ denote the probability that the decoded word contains an undetectable error pattern $\bar{e}$ (a nonzero codeword in $C_b$). For a nonzero codeword $\bar{v}$ in $C_b$, we define the weight configuration of $\bar{v}$ as the sequence of nonzero weights of component segments of $\bar{v}$, arranged in ascending order. For an undetectable error pattern $\bar{e}$ with weight configuration $(i_1, i_2, \ldots, i_h)$, it follows from (11) that

$$P_b(e, \epsilon) \leq \prod_{k=1}^{h} Q_t(W(i_k), \epsilon)[1 - P_{ud}(e)]^{m-h}. \quad (14)$$

It follows from (12), (13) and (14) that the power of $\epsilon$ in the right-hand side of (14) is

$$\sum_{k=1}^{h} W(i_k) - th. \quad (15)$$

Suppose that the frame code $C_f$ is an even-weight code. For any positive integer $i$, define $W'(i)$ as follows:

$$\begin{cases} W'(i) = d, & \text{if } i \leq d_f; \\ W'(i) = i, & \text{if } i > d_f \text{ and } i \text{ is even}; \\ W'(i) = i+1, & \text{otherwise}. \end{cases} \quad (16)$$

For an error pattern $\bar{e}$ of weight configuration $(i_1, i_2, \ldots, i_h)$, define the order $O_e(\bar{e})$ of $\bar{e}$ as follows:

$$O_e(\bar{e}) = \sum_{k=1}^{h} W'(i_k) - th. \quad (17)$$
If $0_{e}(c) > w(c)$, we say that $e$ has a good weight configuration and $e$ is a good error pattern. On the other hand, if $0_{e}(c) < w(c)$, $e$ is said to be a bad error pattern.

Let $P_{ud}(c)$ be the probability of undetected error for the outer code $C_b$. In order to obtain a good evaluation of $P_{ud}(c)$, we need to estimate the proportion of bad error patterns and that of good error patterns to the error patterns of the same weight for several low weights. Unless the former is negligibly smaller than the latter, the error correction of the frame code does not make sense.

For $1 < j_1 < j_2 < ... < j_h < m$, consider the set of codewords in $C_b$ where nonzero bits are confined in $j_1$-th segment, $j_2$-th segment,..., and the $j_h$-th segment. This set of codewords forms a subcode of $C_b$, called a $(j_1, j_2, ..., j_h)$-subcode of $C_b$. If $C_b$ is a cyclic or shortened cyclic code, all $(j_1, j_2, ..., j_h)$-subcodes of $C_b$ for a given $h$ with the same $j_2 - j_1, j_3 - j_2, ..., j_h - j_{h-1}$ are equivalent codes and are called $h$-segment $(j_2 - j_1, j_3 - j_2, ..., j_h - j_{h-1})$ subcodes of $C_b$.

Consider a $(j_1, j_2, ..., j_h)$-subcode of $C_b$. Let $i_1, i_2, ..., i_h$ be a set of integers for which $0 < i_k < n$ with $1 < k < h$. Let $A_{i_1, i_2, ..., i_h}$ denote the number of codewords in the $(j_1, j_2, ..., j_h)$-subcode whose weight in the $j_k$-th segment is $i_k$ for $1 < k < h$. If we know segment-weight distributions for all $h$-segment subcodes of $C_b$, we can obtain an upper bound on $P_{ud}(c)$ from (14). If $e < 1/n_b$, a small $h$ is sufficient to attain a good evaluation of $P_{ud}(c)$. This will be illustrated by an example in the next section.

4. Examples

Consider the concatenated coding scheme proposed for the NASA planetary command system in which both the inner (frame) code and outer code are shortened Hamming codes. The frame code $C_f$ is a distance-4 Hamming code with generator polynomial,
\( \overline{g}(X) = (X+1)(X^6+X+1) = X^7+X^6+X^2+1 \),

where \( X^6+X+1 \) is a primitive polynomial of degree 6. The maximum length of this code is 63. This code is used for single error correction. The code is capable of detecting all the error patterns of double and odd number errors. The outer code is also a distance-4 shortened Hamming code with generator polynomial,

\[
\overline{g}(X) = (X+1)(X^{15}+X^{14}+X^{13}+X^{12}+X^4+X^3+X^2+X+1)
= X^{16}+X^{12}+X^5+1,
\]

where \( X^{15}+X^{14}+X^{13}+X^{12}+X^4+X^3+X^2+X+1 \) is a primitive polynomial of degree 15. This code is the X.25 standard for packet-switched data networks. The natural length of this code is \( 2^{15}-1 = 32,767 \). But maximum length \( n_b \) being considered is 2,048 bits. The 16 parity bits of this code is used for error detection only.

To evaluate \( P(b)(E) \), we need to know the weight configurations and orders of error patterns for \( C_b \). The weight configurations and orders of errors patterns of weights 4, 6 and 8 are listed in Table 1. The order of an error pattern \( e \), \( O(e) \), is at least

\[
O(e) = \left\lfloor \frac{w(e)}{4} \right\rfloor,
\]
which occurs for the weight configuration

\[(4,4,\ldots,4,w(\overline{e})-4\left\lfloor \frac{w(\overline{e})}{4} \right\rfloor+4),\]

where \( \lfloor x \rfloor \) denotes the integer no greater than \( x \).

Suppose that \( n>7 \) and

\[
e \leq \frac{1}{2n}.
\]

Then \( (1-e)^{n-1/2} \) and \( (1-e)/e \geq 13 \). Note that

\[
Q_1(w,e)^{1/w} = \frac{e}{1-e} \left[ \frac{w(1-e)^{n+1}}{e} \right]^{1/w} \left[ 1 + \frac{e}{w(1-e)} + \frac{n-w}{w} \left( \frac{e}{1-e} \right)^2 \right]^{1/w},
\]

which decreases monotonically as \( w \) increases for \( 4 \leq w \leq n \). Hence
for 4 < w < w'. It is easy to check that

$$Q_1(4, \varepsilon) \leq Q_1(4, \varepsilon)^{1/2} \leq Q_1(4, \varepsilon)^{1/3} \leq Q_1(6, \varepsilon)^{1/6}.$$  

It follows from (14), (21) and (22) that

$$P_b(\varepsilon, c) = \begin{cases} Q_1(4, \varepsilon)^{|w(e)|/4}, & \text{if } w(e) \text{ is a multiple of } 4, \\
Q_1(4, \varepsilon)^{|w(e)|/4} - 1 Q_1(6, \varepsilon), & \text{otherwise}. 
\end{cases}$$  

Let

$$\varepsilon \leq Q_1(4, \varepsilon)^{1/4}.$$  

Then, it follows from (20) that

$$\varepsilon \leq 1.5 \varepsilon^{3/4}.$$  

Let \( \{A_i(b)\} \) be the weight distribution of the outer code \( C_b \). Suppose that \( A_4^1 \) and \( A_6^1 \) of the one-segment subcode of \( C_b \) are known besides \( \{A_i^b\} \). Then it follows from (14), (21) to (24) and Table 1 that

$$P_{ud}(\varepsilon) \leq m A_4^1 \varepsilon^4 + (A_4^1)^{-m} A_4^{1-8} + m A_6^{1 Q_1(6, \varepsilon)}$$

$$+ (A_6^1)^{-m} A_6^{1-8} + \sum_{i=2}^{[\frac{n_b}{4}]} A_i^b \varepsilon^{4i}$$

$$+ \sum_{i=2}^{(n_b-2)/4} A_i^b \varepsilon^{4i} - 4 Q_1(6, \varepsilon).$$  

where \( m \) is the number of segments per block (see Figure 2) and \( 2 \leq m \leq 52 \).

Let \( A_i^{1, j} \) be the number of codewords of weight \( i \) in two-segment (1, j) subcode of \( C_b \) for \( 1 \leq j \leq m \). If \( A_4^1, A_6^1, A_8^1, A_6^{1-8} \) with \( 1 \leq j \leq m \) and \( 4 \leq i \leq 8 \) are known, then \( P_{ud}(\varepsilon) \) can be bounded as follows:
\[ p_{ud}(\varepsilon) \leq \sum_{j=2}^{m} (m-j)(A_{4}^{1,j} - 2A_{4}^{1})\varepsilon^{8} \]

\[ + [A_{4}^{(b)} - mA_{4}^{1}] - \sum_{j=2}^{m} (m-j)(A_{4}^{1,j} - 2A_{4}^{1})\varepsilon^{12} \]

\[ + m A_{6}^{1} Q_{1}(6,\varepsilon) + \sum_{j=2}^{m} (m-j)(A_{6}^{1,j} - 2A_{6}^{1})\varepsilon^{8} \]

\[ + [A_{6}^{(b)} - mA_{6}^{1}] - \sum_{j=2}^{m} (m-j)(A_{6}^{1,j} - 2A_{6}^{1})\varepsilon^{12} \]

\[ + m A_{8}^{1} Q_{1}(8,\varepsilon) + \sum_{j=2}^{m} (m-j)(A_{8}^{1,j} - 2A_{8}^{1})\varepsilon^{8} \]

\[ + [A_{8}^{(b)} - mA_{8}^{1}] - \sum_{j=2}^{m} (m-j)(A_{8}^{1,j} - 2A_{8}^{1})\varepsilon^{12} \]

\[ \frac{[n_{b} - 2]/4}{\varepsilon^{41 - 4}} Q_{1}(6,\varepsilon) + \sum_{i=3}^{[n_{b}/4]} A_{4i+2}^{(b)} \varepsilon^{4i-4} Q_{1}(6,\varepsilon) + \sum_{i=3}^{[n_{b}/4]} A_{4i}^{(b)} \varepsilon^{4i} . \]

(28)

If \( A_{ij}^{1,j} \) for \( 2 \leq j \leq m \) and \( 4 \leq i_{1} + i_{2} \leq 8 \) are also known, a better bound on \( p_{ud} \) can be obtained,

\[ p_{ud}(\varepsilon) \leq a_{4}^{-4} + (a_{1,3} + a_{3,1} + a_{2,2})\varepsilon^{8} + \sum_{i=2}^{m} A_{4i}^{(b)}\varepsilon^{12} \]

\[ + a_{6}^{1} Q_{1}(6,\varepsilon) + (a_{2,4} + a_{4,2} + a_{3,3})\varepsilon^{8} \]

\[ + (a_{1,5} + a_{5,1})\varepsilon^{4} Q_{1}(6,\varepsilon) + A_{6}^{(b)}\varepsilon^{12} \]

\[ + a_{8}^{1} Q_{1}(8,\varepsilon) + (a_{1,7} + a_{7,1})^{4} Q_{1}(8,\varepsilon) \]

\[ + \sum_{i=2}^{m} A_{4i+2}^{(b)} \varepsilon^{4i-4} Q_{1}(6,\varepsilon) + \sum_{i=3}^{[n_{b}/4]} A_{4i}^{(b)} \varepsilon^{4i} , \]

(29)

where

\[ a_{i} = m A_{i}^{1} , \]

\[ a_{i_{1},i_{2}} = \sum_{j=2}^{m} (m-j) A_{i_{1},i_{2}}^{1,j} , \]

\[ \bar{A}_{i}^{(b)} = A_{i}^{(b)} - a_{i} - \sum_{i_{1}+i_{2}=i} a_{i_{1},i_{2}} . \]

(30)
The weights $A^1_i$, $A^1_{ij}$ and $A^1_{i1i2}$ can be computed from those of the dual code of $C_b$ by the MacWilliams' identities [3].

If finer weight structure of $C_f$ and $C_b$ is known, further improvement on bounding $P_{ud}(\varepsilon)$ can be made. Let $A_{i,j}$ denote the number of codewords $\overline{v}$ in $C_f$ such that $w^{(1)}(\overline{v}) = i$, $w^{(2)}(\overline{v}) = j$ and the vector consisting of the first $k$ bits of $\overline{v}$ and $n_b-k$ zeros is a codeword in one-segment subcode of $C_b$. Better bounds on $P_{ud}(\varepsilon)$ can be obtained by replacing $m A^1_4 \varepsilon^4$, $m A^1_6 Q_1(6,\varepsilon)$ and $m A^1_8 Q_1(8,\varepsilon)$ by $n-k \sum_{j=0}^{n-k} A^{(f,b)}_{4+j} Q_1(4+j,\varepsilon)$, $n-k \sum_{j=0}^{n-k} A^{(f,b)}_{6+j} Q_1(6+j,\varepsilon)$ and $n-k \sum_{j=0}^{n-k} A^{(f,b)}_{8+j} Q_1(8+j,\varepsilon)$ respectively in (21), (28) and (29).

For moderate code length $n$, it may be feasible to compute $A^{(f,b)}_{i,j}$. Evaluation based on (27) for $\varepsilon = 10^{-5}$ is given in Table 2. Evaluation of $P_{ud}(\varepsilon)$ based on (28) or (29) will be made in the next report.

REFERENCES


### Table 1

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Figure 1 A concatenated coding system
Figure 2 Block format