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SHOCK CONDITIONS AND SHOCK WAVE STRUCTURES IN A STEADY FLOW IN A DISSIPATIVE FLUID

Paul Germain and Jean-Pierre Guiraud

Translation of "Conditions de Choc et Structure des ondes De Choc Dans un Ecoulement Stationnaire de Fluide Dissipatif", Office National d'etudes et de Recherches Aeronautiques, Chatillon, France, Publ. No. 105, 1962, 45 pages.

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MINS: / CAMBER/ CORRECTION/ CURVATURE/ DISSIPATION/ FLOW/ FLUIDS/ FORMULAS/ LOSSES/ NUMBERS/ PRESSURE/ RADII/ RANKINE-HUGONIOT RELATION/ RECIPROCATION / REYNOLDS NUMBER/ SHOCK/ STRUCTURES/ THICKNESS/ TOTAL ENERGY SYSTEMS/ WAVES

ABA: Author

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16. Abstract

This work differs from conventional studies on the same subject in that the thickness and camber terms, usually neglected, are here taken into account. More precisely, calling ε the reciprocal of the Reynolds number based on the shock wave curvature radius, the ε terms of the first order are systematically taken into account. The most important result is a system of formulas giving a correction of order ε for the various RANKINE-HUGONIOT conditions. The suggested formulas may for instance have to be used instead of the conventional ones to evaluate the loss of the total pressure across the detached shock wave which is found at the nose of a very small probe in supersonic flow.

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Paul Germain* and Jean-Pierre Guiraud**

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SUMMARY

This work differs from conventional studies on the same subject in that the thickness and camber terms, usually neglected, are here taken into account. More precisely, calling ε the reciprocal of the Reynolds number based on the shock wave curvature radius, the ε terms of the first order are systematically taken into account. The most important result is a system of formulas giving a correction of order ε for the various RANKINE-HUGONIOT conditions. The suggested formulas may for instance have to be used instead of the conventional ones to evaluate the loss of the total pressure across the detached shock wave which is found at the nose of a very small probe in supersonic flow.

INTRODUCTION

The present paper will make a contribution to the classical shock wave theory. Strictly speaking, a shock wave in a perfect fluid is a surface along which the various parameters which characterize the flow (especially pressure) experience this continuity. The shock conditions are relationships which relate the various discontinuities. Such a discontinuity surface in fact is only a perfect fluid model of a zone with a very small thickness, which exists in a real fluid which is slightly dissipative.

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within it these quantities experience substantial variations but these are continuous. The flow is always continuous, no matter how small the effects of the coefficients are which characterize the various dissipation mechanisms. In an exact sense, how can we legitimately speak of a shock wave in such a fluid? Is it possible to write down shock conditions which complement the classical conditions taking into account the dissipative effect? Such is the purpose of this paper. /4

This has led to various studies which have now become classical [2, 6, 7, 9] and provide a valid model when the coefficients of the dissipation mechanisms are evanescent. Recently, the question has been discussed by Sedov-Mikhailova and Chernii [8] and also by Probstein and Kemp [5], in order to improve the description when the dissipation mechanisms are weak and not evanescent. This was treated in a memoire which discusses a number of interesting questions. For reasons to be discussed below, the responses given by these various authors are not satisfactory.

Our discussion will be divided into four parts. In the first part which continues in a sense our Introduction, we will analyze the work of Probstein and Kemp. In addition to its intrinsic interest, this will allow us to discuss the relationship between our problem and modern aeronautics. The formulation of the problem is then described exactly and intuitive analysis is performed in order to show the kind of results obtained and the terms which have to be implemented in order to obtain these results. The second part has the purpose of setting the simple properties of a class of functions, called N.S., which allows one to formulate in a mathematically explicit manner the term which is loosely called shock wave and shock conditions in a fluid which has dissipation mechanisms, for example, a fluid which satisfies the Navier-Stokes conditions. In the third part we will use a global method to formulate the shock conditions in the most general way. We will use a symbolic notation in the case of a steady flow. The calculations are made up to first order in $\frac{1}{Re}$. The result shows that some of the additional terms which are introduced in the conditions (compared with the classical Rankine-Hugoniot conditions) require a precise knowledge of the development of the flow inside the zone in which the rapid

but continuous variations of the various characteristic quantities take place.

The study of this "internal structure" of the shock wave is discussed in the last part. We indicate how one should set up differential equations which govern this structure and we proceed with the solution of these equations. Independent of the global method one can then obtain the shock conditions and one can prove the coherence of the general theory which is presented here. In this article we will limit ourselves to the consideration of first order terms, but verification of coherence was made for all orders. /5

1. PRELIMINARY GENERAL REMARKS

With the construction of artificial satellites and rockets having slow re-entry, there is increasing interest in the prediction of the aerodynamic characteristics of flying machines which fly at a very high speed and a very high altitude. Even though the stresses encountered and the heat flux received are small, because of the low atmospheric density, it may be important to take them into account for long flights.

Two aerodynamic regimes can be considered to be well known at the present time, at least from the engineer-designer point of view. First of all we have the molecular regime and the continuous regime. The molecular regime is the regime for which the average mean free path is sufficiently large with respect to the characteristic dimensions of the obstacle, and for this the gas molecules can be considered as independent of one another. From the point of view of calculations of these forces the aerodynamic aspect of this regime does not introduce any difficulties. The only question which is not entirely clear is the mechanism of the reflection of molecules at the surface of the object. At the other extreme, the continuous regime is so well known that it is not necessary

to discuss it.

Therefore, for the aerodynamic designer, the aerodynamic regime which is most important for extended flight at high altitudes may be the intermediate regime. This is a situation which exists when the flight is not high enough so that the rarefaction of the air is sufficient so that one can utilize the very simple molecular regime properties, but is not low enough so that one could use these numerous results for the continuous regime.

The intermediate regime which we will now discuss is characterized by the requirement for describing the microscopic structure of the gas medium and equilibrium. This is a structure which varies greatly from the Maxwell-Boltzmann structure, which is adapted to extreme regimes. The Boltzmann equation states the compromise which is realized between the tendency to move away from the Maxwell-Boltzmann distribution due to the motion of the molecules by inertia, and the trend to return to this cononical distribution due to the regularization effect of the collisions. In principle it should be able to predict the variation of the microscopic structure of the gas medium and consequently to determine the aerodynamic characteristics of the obstacles in the intermediate regime.

In any case, the extreme complexity of the Boltzmann equation (integral-differential equation which is highly nonlinear) has discouraged all attempts to obtain explicit solutions which are different from the Maxwell-Boltzmann solution. Experimentation at high speeds and in a rarefied atmosphere is difficult. There are at least two decades of aerodynamics of the intermediate regime which are entirely unknown.

But the development in this area is rapid, and we now have both theoretical and experimental reasons which lead us to assume

that the variation of the aerodynamic characteristics is perfectly known from one of these extreme regimes to the other. If this is confirmed, together with a good knowledge of the /6 continuous regime and the molecular regime, then approximation formulas will make it possible to predict the initial part of the variation from either one of the other two regimes. For the engineer, this will be sufficient information for interpolation with enough certainty of the variation of the aerodynamic characteristics in the intermediate regime. The various recently developed models may be sufficiently powerful in order to make a direct study of the intermediate regime and also to support interpolation methods.

1.1. CRITICAL EXAMINATION OF THE MEMOIRE OF PROBSTEIN AND KEMP (P and K)

Recently the study of heat transfer of the stagnation point was discussed in a memoire* of Probstein and Kemp [5] in order to extend slightly the results of the continuous regime already known into the intermediate regime. In reality, the work of these authors goes even further. Figures 6 and 7 of their memoire show this. It is possible to show that an interpolation within the intermediate regime between the continuous regime and the molecular regime is possible with minimum difficulty. Nevertheless, in spite of this technical success, and one might say even because of this, the work of P and K is not without criticism, both in the conceptual and fundamental sense. The purpose of the present article is to examine one of the points of the P and K memoire which can be criticized. We will attempt to shed light on the controversial question [4, 5, 8] of shock conditions within a dissipation medium. But before starting the present study, we have to somewhat analyze the work of P and K. After this is done, the reader will undoubtedly feel that the question formulated is a fundamental one. We are considering the problem of the flow over a sphere or a circular cylinder near the stagnation point. We will follow the investigation

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* In the following we will call this memoire by its initials P and K.

of P and K about the influence of rarefaction. One is then led to distinguishing less than eight flow regimes mentioned above. Regimes 1 and 8 are naturally the two extreme regimes mentioned before. The mean free path and the viscosity are directly related ($\mu \cong \rho a \lambda$ with μ = viscosity and ρ = specific mass, a = speed of sound, λ = mean free path). The continuous regime is a fluid regime which is almost perfect, that is, it has only slight dissipative mechanisms. It follows from this that the viscous fluid effects and the heat conduction are only felt in the boundary layer near the obstacle. Naturally they are felt through the shock waves because the concept of a shock wave, discontinuity surface are conditions of a perfect fluid, which have to be replaced by the path layer with a very high gradient if one takes into account the dissipation mechanisms. Also, the shock wave effects are one order higher than the boundary layer effects so that in regime (1), which one can call the limit boundary layer, the shock waves can be modeled by a discontinuity surface. The remaining six regimes establish a division which may be artificial of the intermediate regime. Using approximation formulas, one can distinguish in reality three sub-regimes within the intermediate regime. There is the almost continuous regime, or sliding regime for which one takes into account terms on the order of $M_\infty/\sqrt{R_{e_\infty}}$ (M_∞ : Mach number of the incident flow, $R_{e_\infty} = \rho_\infty U_\infty R/\mu_\infty$: Reynolds number of the flow upstream based on the radius of the obstacle). There is the almost molecular regime for which one takes into account terms on the order of R_{e_∞}/M_∞ . There is the transition regime proper, for which one considers all of the characteristics of the intermediate regime. The regime (7) of P and K is the "First-order collision regime", but does not differ from the almost molecular regime. /7
One also has to identify regime (2) or the "Vorticity interaction regime" and the almost continuous regime mentioned above. The remaining four regimes: "viscous layer regime (3), "incipient merged layer regime" (4), "fully merged layer regime" (5), "transitional regime" (6) with the only real transition regime. From the point of view of kinetic theory of gases, this division into six sub-regimes is not necessary.

In fact, the point of view of P and K is different because it is not based on the concepts of kinetic theory of gases, but on the theory of continuous media and the Navier-Stokes equations. The best proof of this is that in regime (2), in which we have found the sliding regime of the kinetic theory of gases, the sliding conditions and temperature jump conditions at the wall are not taken into account by P and K, and also not in regimes (3) and (4).

Since we have now discussed this, the rather artificial division of P and K is based on the substitution of the initial problem by four different problems which are assumed to be directly related to the evolution of the first problem as a functon of the degree of rarefaction of the air. Let us discuss this in more detail and also the four problems. The problem of regime (1) is so well known that it is not necessary to discuss. The problem of regime (2) is the same as for regime (1) but there is only a changed condition at the boundary layer edge, so as to consider the effect produced on it by an external vortex, a vortex produced by the shock wave. The problem of regime (3) is the problem of a flow of a viscous fluid between an obstacle and a shock wave, when one assumes that the Rankine-Hugoniot conditions are valid over the shock wave. The problem of regime (4) is the same as that for regime (3), but for different conditions over the shock, which makes it possible to take into account the thickness.

Naturally the question can be posed as to how the four problems discussed are related to the evolution of the solution of the initial problem as a function of the degree of rarefaction of the air. This is a very difficult question and it does not seem possible to give a satisfactory answer, at least from the theoretical point of view. The only remark which we can make, and which was stated by P and K, is that recent studies have shown that the conclusions derived from the Navier-Stokes equations agree well with experimental results in the rarefaction domains, where from the beginning one would expect a good agreement. Based on this remark, we can replace the initial problem by a less difficult

problem and which is probably well posed: the variation of the flow of a viscous fluid over an obstacle having a rounded nose when the dissipation mechanism coefficients (viscosity and conductivity) increase from a value of zero. The question has special importance if one wishes to enlarge the area of debate and if one wants more than a simple analysis of an expedient means of predicting the aerodynamic characteristics at high speeds and high altitudes. In effect, this is one of the aspects of a fundamental problem which arises and which is being discussed more and more by applied mathematicians: how to study problems with nonlinear limits, in order to obtain explicit solutions and using asymptotic expansion. This is certainly the problem raised by the memoire of P and K, even though it is not strictly stated in these terms. It is necessary to expand the heat flux for example, according to powers of $1/\sqrt{R_{e_\infty}}$, and the expansion starts with a term in $1/\sqrt{R_{e_\infty}}$. The only valid criterion for distinguishing among regimes is the following: determination of each coefficient of expansion is related to solving a special problem, and one is certainly free to assign a flow regime to each one. Certainly, the preceding criterion is the only one that an applied mathematician can accept.

1.2 EXACT DEFINITION OF THE PROBLEM BEING STUDIED /8

Of course it is necessary to examine the memoire of P and K from this point of view. This is a problem of singular perturbations, because a unique formal development can only be valid close to an obstacle and also close to a shock wave. These are two regions where there are large gradients. Our purpose here is more limited because we will propose to only study the vicinity of the shock wave. But we will operate with a maximum of generality by posing the problem as a question in the following form. When the fluid has dissipative mechanisms (essential viscosity and heat conduction), which are small but not zero characterized by a small parameter ε, and if we have a perfect fluid, then for this fluid, how can one develop using asymptotic expansions in ε, a representation of the real fluid close to (Σ), where (Σ) is the shock wave?

The answer to this question has to give conditions over the shock wave (Σ) to be applied for regime (4). Conditions of this type were obtained by P and K but they are different from the ones established here and which are certainly exact, because they are obtained using an exact systematic procedure. The differences essentially come from the fact that P and K did not have a special technique for solving this problem. They wrote down the Navier-Stokes equations in integral form using intuitive ideas about the orders of magnitude which led to approximations. It is difficult to control the validity of these approximations and especially the significance of them with respect to the question being discussed. Again we have shown that obtaining terms of a higher order in an asymptotic expansion is most of all a technical procedure in contrast to determining the first term. The first term often is obtained using intuitive methods. Only a posteriori will the result be justified by the technique.

Let us now directly consider the question being discussed, by trying to develop as intuitively as possible the solution method. In Figure 1 a) we show (Σ) which is the discontinuity surface model of the perfect fluid. In a real fluid, it is part of a band B which lies between (Σ_1) and (Σ_2). Inside of this, the gradients are much larger than outside of it. In 1 b) we schematically show the profile of the densities when one moves perpendicular to (Σ). Outside of B, we can make a formal expansion according to the powers of the small parameter ε which characterizes the magnitude of the dissipation mechanisms.

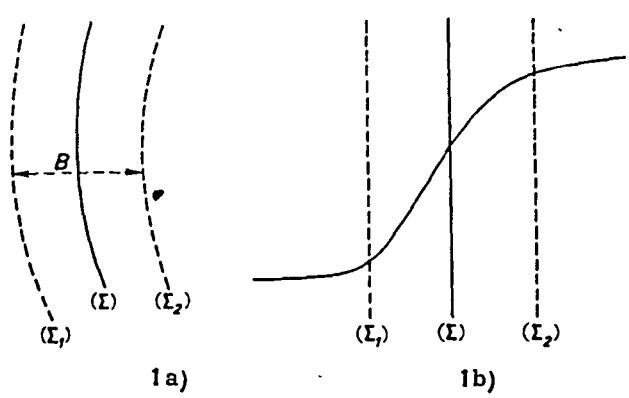

Figure 1.

Let

$$\begin{cases} p = p^{(o)} + \epsilon p^{(1)} + \ldots\ldots & + \epsilon^n p^{(n)} + \ldots\ldots, \\ \rho = \rho^{(o)} + \epsilon \rho^{(1)} + \ldots\ldots & + \epsilon^n \rho^{(n)} + \ldots\ldots, \\ h = h^{(o)} + \epsilon h^{(1)} + \ldots\ldots & + \epsilon^n h^{(n)} + \ldots\ldots, \\ S = S^{(o)} + \epsilon S^{(1)} + \ldots\ldots & + \epsilon^n S^{(n)} + \ldots\ldots, \\ \mathbf{V} = \mathbf{V}^{(o)} + \epsilon \mathbf{V}^{(1)} + \ldots\ldots & + \epsilon^n \mathbf{V}^{(n)} + \ldots\ldots, \end{cases} \qquad (1)$$

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be such expansions for the pressure, the specific mass, the specific enthalpy, the specific entropy and the velocity vector. When one limits oneself to variables with index 0, which is exact for a perfect fluid ($\varepsilon=0$) and which constitutes the zero order approximation for the real fluid, the variables in question will satisfy the perfect fluid equation

$$\begin{cases} \dfrac{\partial \rho^{(o)}}{\partial t} + \operatorname{div}(\rho^{(o)} \mathbf{V}^{(o)}) = 0, \\ \rho^{(o)} \dfrac{D\mathbf{V}^{(o)}}{Dt} + \operatorname{grad} p^{(o)} = 0, \\ \dfrac{DS^{(o)}}{Dt} = 0, \end{cases} \qquad (2)$$

and the Rankine-Hugoniot conditions over (Σ):

$$\begin{cases} [\rho^{(o)} w^{(o)}] = 0, \\ [\mathbf{U}^{(o)}] = 0, \\ [p^{(o)} + \rho^{(o)} w^{(o)2}] = 0, \\ [h^{(o)} + \dfrac{w^{(o)2}}{2}] = 0. \end{cases} \qquad (3)$$

In the equation (3) the [ ] indicates a difference between the values of the expression inside the bracket, directly adjacent to either side of (Σ); w is the vector velocity component perpendicular to (Σ) and $\mathbf{U}$ designates its projection on the plane tangent to (Σ).

We will designate the n order of approximation as being the result obtained by taking into account the first n terms in equation (1). The approximation of rank n is formed by variables having an index n. The determination of the approximation of rank n includes the solution of a linear system of partial differential equations with suitable boundary conditions, especially on (Σ).

The expansions (1) are certainly not valid close to (Σ), in the sense that they do not represent the real flow in this vicinity, that is, within the band B. Therefore, one has two expansion systems similar to (1): one to the right of $(\Sigma 2)$, and the other to the left of (Σ_1), which respectively are their areas of representation. Even though the expansions discussed are not representative within the band, nevertheless they can be extended respectively to the left of (Σ_2) and to the right of (Σ_1), and in particular, right up to (Σ), where they do not coincide. If we do not now discuss the question of which expansion is representative (1) close to (Σ), we can say that we have only a single system of expansion, valid to either side of (Σ), which experiences discontinuities through this surface. The question posed therefore is the following: How to express the discontinuities of the variables having index n through (Σ)? We will indicate such discontinuities in the following form

$$[p^{(n)}],\ [\rho^{(n)}],\ [h^{(n)}],\ [S^{(n)}],\ [V^{(n)}]\cdot \tag{4}$$

One would expect that knowledge of the shock conditions over (Σ), that is knowledge of all of the discontinuities (4), allows one to determine each of the terms in the expansion one after the other (1), using suitable boundary conditions at the various boundaries of the flow and the partial differential equation system which we mentioned. This then leads to a representation of the real fluid flow, which is valid except close to (Σ). The exceptional band becomes

wider as ε becomes smaller. Inside this band it is necessary to carry out a special study. We state that one has to determine the internal structure of the shock wave.

It is important to note that we are not especially interested in the vicinity of (Σ); it is quite sufficient to have only the information provided by the expansions (1). Then we have the situation represented in Figure 2 where (0) is the approximation curve of order 0, that is, the perfect fluid model, (1) is the approximation curve of order 1 corresponding to retaining two terms in the expansions (1). (∞) is the curve obtained by retaining the complete series (1), assuming that they converge. Therefore, it can be completely legitimate to disregard completely the internal structure of the shock wave, but with the reservation that one must not lose sight of the fact that the fictitious fluid flow obtained is not representative for the real flow except for a more or less close vicinity of (Σ).

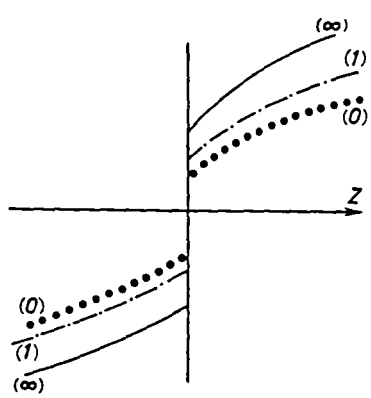

Figure 2

However, it is necessary to determine the expansions (1). For this purpose it is of fundamental importance to know the shock conditions for the approximations of various rank, that is, the expressions (4) for each value of n. For this purpose, we are led to pose the following question: is it possible to determine the discontinuities experienced by the characteristics of the fluid, for the rank and approximation, without taking into account the internal structure of the shock wave? We will see that the answer is negative and this is fundamental because as a consequence, it is definitely not possible to disregard the internal structure, even if at the end of the calculation we are not interested in it, in contrast to what may have been stated above. In reality, for the perfect fluid model,

that is for the zero approximation, it is not necessary to take it into account, but this is a very exceptional circumstance. Starting with approximation of rank 1, the determination of the shock conditions involves very greatly the internal structure of the shock wave. Sedov, Mikhailova, Chernii [8] nevertheless assumed the opposite and gave shock conditions with a first order of approximation which were wrong. This error was corrected by P and K which were very well aware of the fact that it is necessary to take into /11 account the internal structure of the shock wave using a fictitious expression for its thickness. But because there was no technique for the problem, these authors were not able to do this in a completely correct manner.

1.3 INTERNAL STRUCTURE AND SHOCK CONDITIONS

Let us consider the one dimensional model shown in Figure 3 which represents the profile of densities close to (Σ) $(z = 0)$. The solid curve shows the behavior of a real fluid. The dash curve which is interrupted is the result of using expansions (1). A study of the internal structure, therefore, amounts to studying how the solid curve can lead to a continuous agreement between the two parts of the interrupted curve in dashed lines.

In order to study this internal structure, we are led to substituting a variable ζ for the variable z which is related to z with the relationship

$$\varepsilon \zeta = z, \tag{5}$$

if we note that the width of the band B in Figure 3 is effectively on the order of ε. Let us now perform this variable change in the two functions shown in this figure: One corresponds to expansion (1) and is shown by dashed lines. The other one corresponds to the effective value taken on in the flow. Let us now assume that one could expand the result obtained according to powers of ε and let us now consider in the two expansions terms of rank 0, rank 1... of rank n. The results obtained are shown in Figures 4 and 5 which we will now briefly discuss. Let us first discuss the nature of the curves obtained in Figure 4. When ζ is finite, z is on the

order of ε so that the approximation of order 0 based on (1) is independent of ζ to either side of (Σ) which results in Figure 4a.

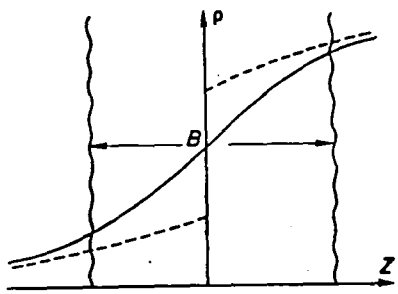

Figure 3

The approximation of rank 1 obviously is a linear function of ζ to both sides of (Σ), and Figure 4b results. The approximation of rank n is a polynomial of degree n in ζ which results in Figure 4c. In this way, the curves derived from Figure 4 are perfectly determined for any value of ζ by the local behavior of the quantity under study and which is represented schematically in Figure 3 (ρ, for example) according to the definition given by (1) to either side of the value $z = 0$. Obviously, curves in Figure 5 are not very simple, but their relationship with the other curve corresponding to Figure 4 is easy to discern. When ζ is on the order of a few tens of units, the curves of Figures 4 and 5 must coincide for the corresponding cases. This is because outside of B, the deviation between the solid curve and the interrupted curve is exponentially small in ε. Thus,

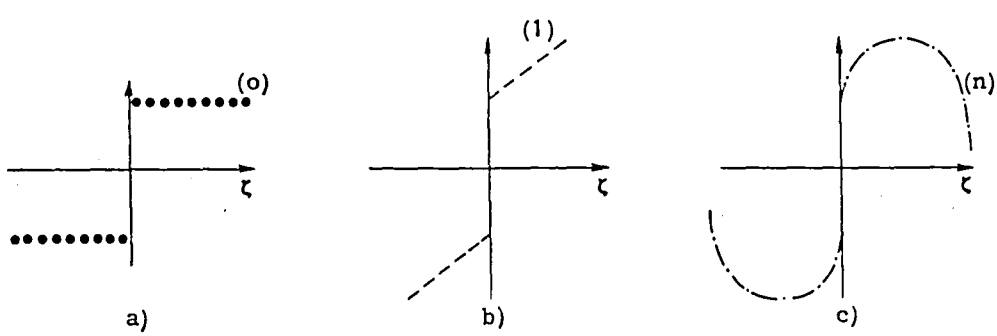

Figure 4

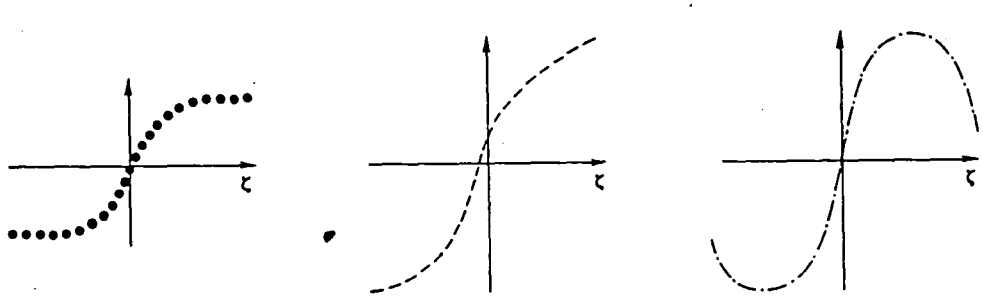

Figure 5

for each order of approximation, the behavior close to $z = \pm 0$ of expansion (1) and the behavior close to $\zeta = \pm \infty$ of the internal structure are closely related. The approximation of rank n of the internal structure is equivalent to a polynomial of order n (up to an exponentially small term), both for $\zeta \cong +\infty$ and $\zeta \cong -\infty$. The constant terms of these two polynomials are exactly the values taken on in (Σ) on either side by the approximation of rank n defined from (1). We, therefore, have a rule which will be discussed again, proven and used in the fourth part and which will provide a first method for obtaining the shock conditions. These will be a direct consequence of the internal shock structure.

1.4 THE FIRST DEVELOPMENT OF A GLOBAL METHOD

Disregarding any technical considerations, we would now like to discuss the principles of the global method which will be developed in the third part. Without disregarding the intuitive considerations, at least for the simple case of the mass conservation equation, we will give almost the final formulation of the corresponding shock condition. As shown in Figure 6, we will consider a small cylinder perpendicular to (Σ) which cuts a surface element dA out of the surface which is limited by the closed curve (Γ). By $\mathbf{n}$ we designate the unit vector perpendicular to (Γ) drawn in the plane tangent to (Σ) and which is directed toward the outside. The terminal bases of the cylinder with area dA are located at a distance of (Σ) which is sufficient so that the deviation between the approximation of order n based on expansions (1) and the behavior of the reel fluid are exponentially small in ε. By w, we will denote the component of $\mathbf{V}$ along the cylinder generators. $\mathbf{u} = \mathbf{V} - w\mathbf{Z}$ is the projection on the plane perpendicular to the generators. The distance to (Σ) along the generators (algebraic distance) is naturally called z. $\mathbf{Z}$ designates the unit vector of one of these generators. The mass flux penetrating into the cylinder through the two bases is $-\{\rho w dA\}$, where $\{\ \}$ designates the difference to be taken between these two bases. It is, therefore,

clear that the mass balance for the cylinder is written as follows where H is a geometric factor such that $ds=Hds_o$ which is an arc element of the curve parallel to (Γ) to the distance z of (Σ):

$$\{\rho w \, dA\} + \int_\Gamma ds_o \int_P^Q \rho \, \mathbf{U} \cdot \mathbf{n} \, H dz = 0. \tag{6}$$

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Now f designates an arbitrary characteristic of the flow. We call its approximation of order ∞, which is a formal approximation using expansions (1). It is clear that according to the position of the terminal bases inside the flow, in (6) one can replace $\{\rho w \, dA\}$ by $\{\rho \check{w} \, dA\}$. After this, let us introduce the notation $(\ )_\pm$ in order to indicate the difference between the terminal base and the section perpendicular on (Σ), from the side $z > 0$ or of the side $z < 0$. We then have

$$\{\rho \check{w} \, dA\} = [\rho \check{w}] \, dA_o + (\rho \check{w} \, dA)_+ - (\rho \check{w} \, dA)_- . \tag{7}$$

After this, it is clear that even after the definition of the approximation of order infinity based on expansion (1) the following is true

$$\begin{cases} (\rho \check{w} \, dA)_- + \int_\Gamma ds_o \int_M^P \rho \check{\mathbf{U}} \cdot \mathbf{n} \, H dz = 0, \\ (\rho \check{w} \, dA)_+ + \int_\Gamma ds_o \int_M^Q \rho \check{\mathbf{U}} \cdot \mathbf{n} \, H dz = 0. \end{cases} \tag{8}$$

From (6), (7) and (8) and the remark made above that there is a time at which the shock condition on ρ w is the following, for order infinity

$$[\rho \check{w}] = - \lim_{dA_o \to 0} \frac{1}{dA_o} \int_\Gamma ds_o \int_P^Q (\rho \mathbf{U} - \rho \check{\mathbf{U}}) \cdot \mathbf{n} \, H dz = 0. \tag{9}$$

The shock condition corresponding to the mass conservation was thus obtained up to any desired approximation. Obtaining the other conditions does not involve any new ideas but the calculation is slightly more complicated. Here we will not discuss this in this introduction, but we will give equation (9) a slightly different form.

Let us note that $\rho U - \check{\rho U}$ is exponentially small in ε when (z) satisfies both >> ε and << 1. The points P and Q which intersect (Σ) at M in Figure 6 naturally satisfy this condition. But according to the definition of the approximation of order ∞ for the internal structure, we can modify the integral $\int_P^Q$ by an exponentially small amount by substituting for ρU its internal approximation of order ∞, which is $\hat{\rho U}$. But then $\zeta = z/\varepsilon$ is in fact the variable upon which the expression under the $\int_P^Q$ sign depends without these conditions, and except for an exponentially small error, the integral $\int_P^Q$ of (10) is equal to the following expression

$$\epsilon(\rho U)^* (M) = \epsilon \int_{-\infty}^{+\infty} (\hat{\rho U} - \check{\rho U}) \cdot n \, \hat{H} d\zeta \cdot \qquad (10)$$

Finally, the shock condition (9) can be written in the form /14

$$\boxed{[\rho w] = -\epsilon \lim_{dA_0 \to 0} \frac{1}{dA_0} \int_\Gamma (\rho U)^* \cdot n \, ds.} \qquad (11)$$

It should be noted that for the zero order of approximation, the second term is 0 and one finds as required the Rankine-Hugoniot conditions. It is clear that the second term will involve the internal structure in an essential manner. When developing the internal structure as in (1), we could develop $(\rho U)^*$ according to the powers of ε and from this we could obtain the development of $[\rho w]$. It is then obvious that obtaining the shock conditions up to the approximation of rank n will involve internal structure up to order n-1, but not of order n. For the problem of P and K where one is restricted to the first order, it is sufficient to have the internal structure up to the approximation of order 0, which has been studied many times [2], [6], [7], [9].

It should be noted that the second term in (11) provides a local measure which is mathematically well defined of the shock wave thickness. There is a thickness for each conservation equation, that is, for each shock condition. In fact, (11) gives a displacement thickness because (10) is formally almost identical with the definition

thickness in the boundary layer theory. In this theory, the displacement thickness expresses the mass flux above the mass flux of the external flow which traverses a section perpendicular to the boundary layer. In the present case, the second term of (10) expresses the mass flux which traverses the lateral wall of the cylinder inside the shock wave, above the same mass flux relative to the outside flow.

It remains to give an explicit expression for the second term of (12) by transforming the curvilinear integral into a surface integral. This is an easy calculation in vectorial analysis which will be done later on.

The fundamental concepts and ideas which will be used during parts 3 and 4 have now been introduced in paragraphs 1.3 and 1.4. We will now give a precise definition of the functions which are involved in this work and will discuss a few properties which are particularly important for study.

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2. THE CLASS OF NS FUNCTIONS

2.1. DEFINITIONS

Now we have to mathematically give an exact definition of the functions which represent the various characteristics of a viscous fluid flow near a surface which is a shock wave for the flow for the perfect fluid model. These functions depend on four variables x_1, x_2, z, ε which have the following meaning: x_1 and x_2 are the curvilinear coordinates along the shock waves, z is the third coordinate which is 0 on the shock, ε is a small parameter which characterizes the dissipation mechanisms. In this part, x_1 and x_2 will play the role of auxiliary parameters. In most cases, we will not use them for writing down these functions.

The function $f(z,\varepsilon)$ depends on the parameters x_1 and x_2 for a fixed z and small ε. It behaves as a regular function of z and ε.

The two functions $f^+(z,\varepsilon)$ and $f^-(z,\varepsilon)$ obtained in this way for $z > 0$ and $z < 0$ obviously are not an extension of each other. See the discussion (I-1). Also, in the shock zone which is studied by introducing the variable ζ already defined in (I-5) we have

$$\varepsilon\ \zeta = z. \tag{1}$$

Agreement between the upstream and downstream flow has to be implemented with this definition. According to already known results about the shock wave structure, this suggests defining the quantity as follows.

DEFINITION 1 (class NS)

The function $f(z,\varepsilon)$ may depend on parameters x_1 and x_2 and appears in the class NS ($f \varepsilon NS$) if f is defined in the range

$K(a,\epsilon_0)$: $-a \leqslant z \leqslant a$, $0 \leqslant \epsilon \leqslant \epsilon_0$ $(a > 0,\ \epsilon_0 > 0)$, and if in this range we can write f in the form

$$f(z,\epsilon) = \frac{1}{2}\left\{f^+(z,\epsilon) + f^-(z,\epsilon)\right\} + \frac{1}{2}\left\{f^+(z,\epsilon) - f^-(z,\epsilon)\right\} \operatorname{tgh} \zeta + \tilde{f}(\zeta,\epsilon)\ , \tag{2}$$

The functions f^+ and f^- appear in the class A, $\tilde{f}$ in the class B which are defined above. We say that the functions $(f^+, f^-, \tilde{f})$ define f on K.

DEFINITION 2 (class A)

A function $k(z,\varepsilon)$ belongs to class A if it is defined and limited for any value of z when $\epsilon \leqslant \epsilon_0$, and which can be developed in series of a power of ε

$$k(z,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n k_n(z),\ \epsilon \leqslant \epsilon_0\ , \tag{3}$$

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The quantities $k_n(z)$ are expandable in series of powers of z in the interval $|z| < a$, and are zero outside a finite interval. These expansions are assumed to be differentiable an infinite number of times with respect to z, x_1 and x_2.

Therefore, for the functions f^+ and f^- introduced in the definition 1 we have

$$f^{\pm}(z,\epsilon) = \sum_{n=0}^{\infty} \epsilon^n f^{\pm}(z) , \tag{4}$$

$$f_n^{\pm}(z) = \sum_{p=0}^{\infty} (\pm z)^p \; f_{n,p}^{\pm} , \quad |z| < a . \tag{5}$$

DEFINITION 3 (class B)

A function $\tilde{f}(\zeta,\epsilon)$ is defined for any value of ζ when $0 < \epsilon < \epsilon_0$. For small ε $\tilde{f}$ allows an asymptotic expansion

$$\tilde{f} \sim \sum_{n=0}^{\infty} \epsilon^n \tilde{f}_n (\zeta) . \tag{6}$$

For sufficiently large $|\zeta|$, $\tilde{f}$ and $\tilde{f}_n$ are exponentially decreasing (uniformly in ε, as far as f is concerned), that is, there are two positive constants A and α independent of ε such that for example

$$\tilde{f}(\zeta,\epsilon) < Ae^{-\alpha|\zeta|} . \tag{7}$$

Finally, properties (6) and (7) are also valid for the derivatives of $\tilde{f}$ with respect to x_1, x_2 ζ.

Two functions $f \varepsilon NS$ which coincide in a domain $K(a, \varepsilon_o)$ are considered to be identical. If $(f^+, f^-, \tilde{f})$ defines a function, $(g^+, g^-, \tilde{g})$ defines the same function f if $g^+ - f^+, g^- - f^-, \tilde{f} - \tilde{g}$ are zero over a domain $K(a, \varepsilon_o)$. It should be noted that (2) defines the extension of f for any $(0 < \epsilon < \epsilon_0)$

Now that we have made these definitions, we will give the exact definition of useful properties in the following. Here we will not again discuss the proofs in detail.

2.2. PROPERTIES OF THE FUNCTIONS OF CLASS NS

We can immediately verify the following:

LEMMA 1. If $k(z,\epsilon) \in A$, $\tilde{g}(\zeta,\epsilon) \in B$, then

$$k(\epsilon\zeta,\epsilon)\, \tilde{g}(\zeta,\epsilon) , \; \mathrm{tgh}\, \zeta\, k(\epsilon\zeta,\epsilon)\, \tilde{g}(\zeta,\epsilon)$$

are functions of class B.

From this we derive in particular that if f^+ and $f^- \epsilon A$, $\tilde{f}_1(\zeta,\epsilon)$ and $\tilde{f}_2(\zeta,\epsilon) \epsilon B$, then

$$\frac{1}{2}(f^+ + f^-) + \frac{1}{2}(f^+ - f^-)\left(\operatorname{tgh}\zeta + \tilde{f}_1(\zeta)\right) + \tilde{f}_2(\zeta), \qquad (8)$$

which defines a function of class NS.

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LEMMA 2. If $f_1(z,\epsilon)$ and $f_2(z,\epsilon) \epsilon NS$, then there are also exists $f_1 + f_2$, $f_1 f_2$, and derivatives with respect to x_1 and x_2 of these functions.

The only part of this statement which is not obvious is this state for the product of the functions. If we write

$$f_1 = P_1 + Q_1 \operatorname{tgh}\zeta + \tilde{f}_1,$$

$$f_2 = P_2 + Q_2 \operatorname{tgh}\zeta + \tilde{f}_2,$$

$$f = f_1 f_2 = P + Q \operatorname{tgh}\zeta + \tilde{f},$$

then we have $P = P_1 P_2 + Q_1 Q_2$,

$$Q = P_1 Q_2 + P_2 Q_1,$$

$$\tilde{f} = \tilde{f}_1 \tilde{f}_2 + \tilde{f}_1(P_2 + Q_2 \operatorname{tgh}\zeta) + \tilde{f}_2(P_1 + Q_1 \operatorname{tgh}\zeta) + Q_1 Q_2 (\operatorname{tgh}^2\zeta - 1),$$

and the result appears as a direct consequence of lemma 1.

LEMMA 3. (Outside development)

For any $z \neq 0$ so that $|z| < a$, we have the following asymptotic expansions

$$f(z,\epsilon) \sim \begin{cases} \sum_{n=0}^{\infty} \epsilon^n f_n^+(z), & \text{if } z > 0, \\ \sum_{n=0}^{\infty} \epsilon^n f_n^-(z), & \text{if } z < 0, \end{cases} \qquad (9)$$

By definition, (9) defines the external expansion of $f(z,\varepsilon)$. We then have the hypotheses that the two series written in (9) are convergent within $|z| < a$ and within them exactly define $f^+(z,\varepsilon)$ and $f^-(z,\varepsilon)$.

Then within $K(a,\epsilon_0)$, we have the uniqueness of the functions $f^+, f^-, \tilde{f}$ which satisfy (2) for a given function f given of class NS. In effect, if f = 0, we find of necessity according to the preceding lemma that $f^+ = f^- = 0$ and consequently $\tilde{f} = 0$.

LEMMA 4. (interior expansion)

For any fixed ζ, we have the following asymptotic expansion

$$f(z,\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n \bar{f}_n(\zeta) \tag{10}$$

and the functions $\bar{f}_n$ are defined by

$$\bar{f}_n(\zeta) = \frac{1}{2}\left(P_n^+(\zeta) + P_n^-(\zeta)\right) + \frac{1}{2}\left(P_n^+(\zeta) - P_n^-(\zeta)\right) \operatorname{tgh}\zeta + \tilde{f}_n(\zeta), \tag{11}$$

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P_n^+ and P_n^- are the polynomials in ζ of degree n

$$P_n^{\pm}(\zeta) = \sum_{p=0}^{n} (\pm\zeta)^p f^{\pm}_{n-p,p}. \tag{12}$$

The proof is immediate.

LEMMA 5.

For any function $f(z,\varepsilon) \in NS$, we can write down the identity

$$f(\delta,\epsilon) - f(-\delta,\epsilon) = \sum_{p=0}^{\infty} \delta^p [f]_p(\epsilon) + \tilde{g}\left(\frac{\delta}{\epsilon}, \epsilon\right), \tag{13}$$

if $0 < \delta < a$. In this identity $\tilde{g}(\zeta,\epsilon)$ is exponentially decreasing in $\zeta (\zeta > 0)$ and

$$[f]_p(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n (f^+_{n,p} - f^-_{n,p}). \tag{14}$$

Proof is simple. In effect, we have

$$f(\delta,\epsilon) - f(-\delta,\epsilon) = f^+(\delta,\epsilon) - f^-(-\delta,\epsilon) + \tilde{g}\left(\frac{\delta}{\epsilon},\epsilon\right),$$

if

$$\tilde{g}(\zeta,\epsilon) = \frac{1}{2}\left\{f^+(\zeta,\epsilon) - f^-(\zeta,\epsilon)\right\}(\operatorname{tgh}\zeta - 1) +$$

$$+\frac{1}{2}\left\{f^+(-\zeta,\epsilon) - f^-(-\zeta,\epsilon)\right\}(\operatorname{tgh}\zeta - 1) + \tilde{f}(\zeta,\epsilon) - \tilde{f}(-\zeta,\epsilon).$$

The result is a a consequence of (4) and (5) and also of the reasoning given under lemma 1.

LEMMA 6.

If δ is a positive number less than a, any function $f \epsilon NS$ can be associated with a function $f^*(\varepsilon)$ such that

$$\int_{-\delta}^{+\delta} f(z,\epsilon)\,dz = \epsilon f^*(\epsilon) + \sum_{p=1}^{\infty} \frac{\delta^p}{p} [F]^+_{p-1}(\epsilon) + \tilde{h}\left(\frac{\delta}{\epsilon},\epsilon\right), \tag{15}$$

where $\tilde{h}(\zeta,\epsilon)$ decreases exponentially in $\zeta\,(\zeta > 0)$ and

$$[F]^+_{p-1}(\epsilon) = \sum_{n=0}^{\infty} \epsilon^n (f^+_{n,p-1} + f^-_{n,p-1}). \tag{16}$$

In order to prove this lemma, we can introduce a function $\hat{f}(z,\epsilon)\,\epsilon NS$ defined as the derivative with respect to z of

$$\frac{1}{2}\left\{F^+(z,\epsilon) + F^-(z,\epsilon)\right\} + \frac{1}{2}\left\{F^+ - F^-\right\} \operatorname{tgh} \zeta, \tag{17}$$

This is a formula in which /19

$$F^+(z,\epsilon) = \int_0^z f^+(t,\epsilon)\,dt,\quad F^-(z,\epsilon) = \int_0^z f^-(t,\epsilon)\,dt. \tag{18}$$

Consequently,

$$\int_{-\delta}^{+\delta} f(z,\epsilon)\,dz = \int_{-\delta}^{+\delta}\left(f(z,\epsilon) - \hat{f}(z,\epsilon)\right)dz + \int_{-\delta}^{+\delta} \hat{f}(z,\epsilon)\,dz.$$

But we can write

$$\int_{-\delta}^{+\delta} \hat{f}(z,\epsilon)\,dz = F^+(\delta,\epsilon) - F^-(-\delta,\epsilon) + \tilde{h}_1\left(\frac{\delta}{\epsilon},\epsilon\right),$$

$$\int_{-\delta}^{+\delta}\left\{f(z,\epsilon) - \hat{f}(z,\epsilon)\right\}dz = \int_{-\infty}^{+\infty}(f - \hat{f})\,dz + \tilde{h}_2\left(\frac{\delta}{\epsilon},\epsilon\right),$$

$\tilde{h}_1(\zeta,\epsilon)$ and $h_2(\zeta,\epsilon)$ are two functions of $\zeta\,(\zeta > 0)$ which decrease exponentially.

The results (15) and (16) are derived directly

with

$$\epsilon f^*(\epsilon) = \int_{-\infty}^{+\infty} \left\{ f(z,\epsilon) - \hat{f}(z,\epsilon) \right\} dz = \epsilon \int_{-\infty}^{+\infty} \left\{ f(\epsilon\zeta,\epsilon) - \hat{f}(\epsilon\zeta,\epsilon) \right\} d\zeta \cdot \tag{19}$$

We can complete this result by stating that

LEMMA 7.

The function $f^*(\varepsilon)$ which corresponds linearly with $f(z,\varepsilon)$ has the following asymptotic expansion for small ε

$$f^*(\epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n f_n^* , \tag{20}$$

with

$$f_n^* = \int_{-\infty}^{+\infty} \tilde{f}_n(\zeta)\, d\zeta - \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\zeta}{\mathrm{Ch}^2\zeta} \int_0^{\zeta} \left\{ P_n^+(t) - P_n^-(t) \right\} dt \cdot \tag{21}$$

This result is obtained by writing formula (19) explicitly and taking into account the properties of functions of class A for F^+ and F^- and of B for $\tilde{f}$. We find that f_n^* can be calculated from the n + 1 first terms of the internal expansion of f.

In all the preceding definitions, the variables x_1 and x_2 are introduced as parameters. In particular, $[f]_p$ (lemma 5), $[F]_p$, f^*, (lemma 6) are functions of ε and the parameters x_1 and x_2.

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2.3. SHOCK CONDITIONS

We can now indicate how the preceding concepts can be used in order to obtain the shock conditions which relate the functions of class NS which satisfy a relationship of the following form

---

* (1) Let us recall that f can be considered defined for any value of z when $0 < \epsilon < \epsilon_0$ by (2). We can also verify that $f^*(\epsilon)$ only depends on values of f within $K(a, \epsilon_0)$

$$(Af)_{z1}^{z2} + \int_{z1}^{z2} bg\,dz = 0 \cdot \tag{22}$$

Here f and $g \varepsilon NS$, A and B are linear independent operators, independent of z which leave the class NS invariant. We find that the Navier-Stokes equations easily lead to relationships of type (22).

The results of lemmas 5 and 6 then lead one to write $(0 < \delta < a)$

$$[Af]_o(\epsilon) + \epsilon(bg)^* + \sum_{p=1}^{\infty} \delta^p \left\{ [Af]_p(\epsilon) + \frac{1}{p}[bg]_{p-1}^{+}(\epsilon) \right\} = \tilde{h}\left(\frac{\delta}{\epsilon}, \epsilon\right) \cdot \tag{23}$$

The second term has an asymptotic expansion in ε which is identically at zero for the asymptotic sequence of the ε^n. This is the same for the first one and consequently if we set $(bg)_{-1}^* = 0$:

$$[Af_n]_o + (bg)_{n-1}^* + \sum_{p=1}^{\infty} \delta^p \left\{ [Af_n]_p + \frac{1}{p}[bg_n]_{p-1}^{+} \right\} = 0 \,. \tag{24}$$

The first term of (24) represents an expansion into a whole series of δ which are identically zero. Therefore, all of the coefficients are zero.

From this we derive that

$$[Af]_p(\epsilon) + \frac{1}{p}[bg]_{p-1}^{+}(\epsilon) = 0$$

and that

$$\sum_{n=0}^{\infty} \epsilon^n (bg)_{n-1}^* = -\sum_{n=0}^{\infty} \epsilon^n [Af_n]_o = -[Af]_o(\epsilon) = \epsilon(bg)^*(\epsilon)$$

We therefore can formulate the following theorem:

THEOREM: If the functions f and g for class NS satisfy (22) for any pair of values z_1 and z_2, then

$$[Af]_o(\epsilon) + \epsilon(bg)^*(\epsilon) = 0 \tag{25}$$

and

$$[Af]_p(\epsilon) + \frac{1}{p}[bg]_{p-1}^{+}(\epsilon) = 0 \tag{26}$$

By definition (25) represents the shock conditions at z = 0 which relate the quantities f and g which satisfy (22).

We find that in this case, the asymptotic expansion of type (20) of the quantity $(bg)^*$ is a series expansion which is whole and convergent. /21

The shock conditions of rank 0, 1...n...are the conditions obtained by making the coefficients of power 0, 1...n in ε in equation (25).

$$[Af_o]_o = 0 \qquad \text{rank } 0$$

$$[Af_1]_o + (bg)^*_o = 0 \qquad \text{rank } 1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[Af_n]_o + (bg)^*_{n-1} = 0 \qquad \text{rank } n$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

3. DETERMINATION OF THE SHOCK CONDITIONS USING THE GLOBAL METHOD

3.1 NOTATION

The surface of the shock wave, a discontinuity surface of the perfect fluid model, is called (Σ). On the surface we will consider a system of curvilinear coordinates x_1, x_2 for which the coordinate lines are the lines of curvature. By adding the distance z to (Σ)measured along the normal perpendicular to it, we obtain a system of orthogonal curvilinear coordinates (x_1, x_2, z). We will assume that the lengths are measured with a certain unit L so that x_1, x_2, z have no dimensions. In the same way, the pressure p, the specific mass ρ, the velocity V, the specific enthalpy h and the viscosity coefficient λ, μ and the conductivity coefficient k are assumed to be measured using the units $p_o = \rho_o V_o^2$, ρ_o, V_o, h_o, λ_o, μ_o, k_o: We assume that p, ρ, h, γ, λ, μ, k are dimensionless. The small parameter ε which we have discussed will be

$$\epsilon = \frac{1}{R_e} = \frac{\mu_o}{\rho_o V_o L} . \qquad (1)$$

The unit vectors tangent to the coordinate lines will be called $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the elementary displacement will be called

$$d\mathbf{M} = h_1 \mathbf{e}_1 dx_1 + h_2 \mathbf{e}_2 dx_2 + \mathbf{e}_3 dz , \tag{2}$$

the square of the linear element is

$$ds^2 = (h_1 dx_1)^2 + (h_2 dx_2)^2 + dz^2 . \tag{3}$$

The partial derivatives with respect to x_1 and x_2 will be called

$$\frac{\partial f}{\partial x_1} = f_{,1} , \quad \frac{\partial f}{\partial x_2} = f_{,2} . \tag{4}$$

If we use $\mathbf{N} = \mathbf{e}_3$ for the unit vector normal to (Σ) directed towards positive z, we have

$$\mathbf{N}_{,1} = K_1 \mathbf{e}_1 , \quad \mathbf{N}_{,2} = K_2 \mathbf{e}_2 , \tag{5}$$

where $K_1/H_1, K_2/H_2$ are the principal curvatures of (Σ) so that if an arbitrary point in space is written in the form

$$\mathbf{M} = \mathbf{m} + z\mathbf{N} , \tag{6}$$

the formulas (2), (5) and (6) give

$$h_1 = H_1 + K_1 z , \quad h_2 = H_2 + K_2 z , \tag{7}$$

H_1 and H_2, which do not depend on x_1 and x_2, are the coefficients /23 of the first quadratic form of (Σ):

$$dm^2 = (H_1 dx_1)^2 + (H_2 dx_2)^2 . \tag{8}$$

The velocity vector $\mathbf{V}$ is written in the form

$$\mathbf{V} = u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + w\mathbf{e}_3 . \tag{9}$$

3.2 NAVIER-STOKES EQUATIONS

The unit tensor is designated by $\mathbf{1}$ and the viscous stress tensor is called $\epsilon\mathbf{t}$. The unit heat flux density is $\epsilon\mathbf{q}$; (S) is an arbitrary surface with an external normal $\mathbf{N}$. Then the Navier-Stokes equations can be written in the form

$$
\left\{
\begin{aligned}
&\iint_{(S)} \rho \mathbf{V} \cdot \mathbf{N} \, dS = 0 , \\
&\iint_{(S)} (p \mathbf{1} + \rho \mathbf{V} \mathbf{V} + \epsilon \mathbf{t}) \cdot \mathbf{N} \, dS = 0 , \\
&\iint_{(S)} \left\{ \left(h + \frac{V^2}{2} \right) \rho \mathbf{V} + \epsilon (\mathbf{t} \cdot \mathbf{V} + \mathbf{q}) \right\} \cdot \mathbf{N} \, dS = 0 .
\end{aligned}
\right. \tag{10}
$$

In the case of stationary flow, this notation directly translates the laws of conservation of mass, momentum and energy.

If $\mathbf{D}$ is the deformation tensor with a trace $\theta = \nabla \cdot \mathbf{V}$ we have

$$
\left\{
\begin{aligned}
&\mathbf{t} = -2\mu \, \mathbf{D} - \lambda \theta \mathbf{1} \\
&\mathbf{q} = -k \, \mathrm{grad}\, T,
\end{aligned}
\right. \tag{11}
$$

and one often sets

$$
\lambda = \mu_v - \frac{2}{3} \mu , \tag{12}
$$

where μ_v is the volume viscosity coefficient which is zero if one uses the Stokes hypothesis.

Later on we will explain the calculations in the system of selected coordinates. But we will determine the shock conditions by using the dyad form shown above.

3.3. SELECTION OF A CONTROL SURFACE. SYMBOLIC FORM OF THE SHOCK CONDITIONS

The control surface (S) of equations (10) is selected in the form of a cylindrical surface as shown in Figure 7. The terminal bases are (A_δ) and $(A_{-\delta})$. The lateral surface is formed by the generators of length 2δ perpendicular to (Σ). They are supported at the centers by a closed curve (Γ) which delimits an area (A_o) on (Σ).

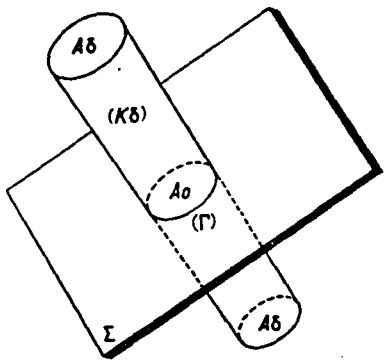

Figure 7

Using the preceding notation, and when $d\sigma$ designates the arc element of (Γ), dS, the area element of $(A_{\pm\delta})$ or of (A_o) as the case may be, $\mathbf{n}$ is the normal to (Γ), any of the formulas can be written in the form

$$\iint_{(A_{+\delta})} \mathbf{A}\cdot\mathbf{N}\, dS - \iint_{(A_{-\delta})} \mathbf{A}\cdot\mathbf{N}\, dS + \int_{(\Gamma)} d\sigma\, \mathbf{n}\cdot\int_{-\delta}^{+\delta} \mathbf{b}\, dz = 0\,, \qquad (12)$$

where $\mathbf{A}$ and $\mathbf{b}$ are either tensors or vectors.

Equation (12) is of the form (2-22), so that the shock conditions are as follows in the symbolic notation and in integral form:

$$\iint_{(A_o)} [\mathbf{A}]\cdot\mathbf{N}\, dS + \epsilon\int_{(\Gamma)} \mathbf{b}^*\cdot\mathbf{n}\, d\sigma = 0\,. \qquad (13)$$

In order to obtain the local shock conditions from (13), it is sufficient to transform the curvilinear integral into a surface integral, so that the first term in (13) can be written as a single integral taken over (A_o). Application of a classical analysis theorem allows one then to conclude, since (A_o) is arbitrary, that the quantity to be integrated is identically 0 and it is continuous in x_1, x_2 according to hypotheses. In order to perform this transformation, it is useful to distinguish the case where $\mathbf{b}$ is a vector or where $\mathbf{b}$ is a tensor.

First case: $\mathbf{b}$ is a vector:

We can set

$$\begin{cases} \mathbf{b}^* = b_1^* \mathbf{e}_1 + b_2^* \mathbf{e}_2 + b_3^* \mathbf{N}, \\ \mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{N}, \end{cases} \tag{14}$$

and also

$$\mathbf{n}\, d\sigma = H_2 \mathbf{e}_1\, dx_2 - H_1 \mathbf{e}_2\, dx_1 ,$$

therefore /25

$$\int_{(\Gamma)} \mathbf{b}^* \cdot \mathbf{n}\, d\sigma = \int_{(\Gamma)} b_1^* H_2\, dx_2 - b_2^* H_1\, dx_1$$

$$= \iint_{(A_0)} \left\{ (b_1^* H_2)_{,1} + (b_2^* H_1)_{,2} \right\} \frac{ds}{H_1 H_2} \tag{15}$$

For these conditions, (13) leads to the shock equation

$$[A_3] + \frac{\epsilon}{H_1 H_2} \left\{ (b_1^* H_2)_{,1} + (b_2^* H_1)_{,2} \right\} = 0 . \tag{16}$$

Second case: $\mathbf{b}$ is a tensor:

Using the summation convention for silent indices, we can here set

$$\mathbf{b}^* = b_{ij}^* \mathbf{e}_i \otimes \mathbf{e}_j \quad , \quad \mathbf{A} = A_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \; ; \tag{17}$$

Formula (15) must therefore be replaced by the following:

$$\int_{(\Gamma)} \mathbf{b}^* \cdot \mathbf{n}\, d\sigma = \iint_{(A_0)} \left\{ (b_{i1}^* H_2 \mathbf{e}_i)_{,1} + (b_{i2}^* H_1 \mathbf{e}_i)_{,2} \right\} \frac{dS}{H_1 H_2} \tag{18}$$

Let us recall that with the present notation, the expressions for the first derivatives of the vectors are as follows:

$$\mathbf{e}_{1,1} = -\frac{h_{1,2}}{h_2} \mathbf{e}_2 - K_1 \mathbf{e}_3 \;,\; \mathbf{e}_{2,1} = \frac{h_{1,2}}{h_2} \mathbf{e}_1 \;,\; \mathbf{e}_{3,1} = K_1 \mathbf{e}_1 \;,$$

$$\mathbf{e}_{1,2} = \frac{h_{2,1}}{h_1} \mathbf{e}_2 \;,\; \mathbf{e}_{2,2} = -\frac{h_{2,1}}{h_1} \mathbf{e}_1 - K_2 \mathbf{e}_3 \;,\; \mathbf{e}_{3,2} = K_2 \mathbf{e}_2 \;,$$

$$\mathbf{e}_{1,3} = 0 \;,\; \mathbf{e}_{2,3} = 0 \;,\; \mathbf{e}_{3,3} = 0 .$$

Under these conditions, (13) leads to the scalar shock equations

$$\begin{cases} [A_{13}] + \dfrac{\epsilon}{H_1H_2}\left\{(b^*_{11}H_2)_{,1} + (b^*_{12}H_1)_{,2} + b^*_{21}H_{1,2} + b^*_{31}H_2K_1 - b^*_{22}H_{2,1}\right\} = 0, \\ [A_{23}] + \dfrac{\epsilon}{H_1H_2}\left\{(b^*_{21}H_2)_{,1} + (b^*_{22}H_1)_{,2} - b^*_{11}H_{1,2} + b^*_{32}H_1K_2 + b^*_{12}H_{2,1}\right\} = 0, \\ [A_{33}] + \dfrac{\epsilon}{H_1H_2}\left\{(b^*_{31}H_2)_{,1} + (b^*_{32}H_1)_{,2} - b^*_{11}H_2K_1 - b^*_{22}H_1K_2\right\} = 0. \end{cases} \tag{19}$$

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3.4. SHOCK CONDITIONS

It is only necessary to use the preceding results (16) and (19) in the Navier-Stokes equations written down as (10). We will now examine the case of the mass equation, the momentum equation and finally the energy equation.

3.4.1. Mass equation

This case is particularly simple because $\mathbf{A}$ and $\mathbf{b}$ are vectors

$$\mathbf{A} = \rho \mathbf{V} \quad , \quad \mathbf{b} = \rho u_1 \frac{h_2}{H_2}\mathbf{e}_1 + \rho u_2 \frac{h_1}{H_1}\mathbf{e}_2 ,$$

and (16) shows that, considering (9), that

$$[\rho w] = -\frac{\epsilon}{H_1H_2}\left\{\left((\rho u_1 \frac{h_2}{H_2})^* H_2\right)_{,1} + \left((\rho u_2 \frac{h_1}{H_1})^* H_1\right)_{,2}\right\}. \tag{20}$$

This is a general equation. If we set $\varepsilon = 0$, this gives the zero order condition $[\rho w] = 0$ which relates the values of ρ w in the external approximation of order zero. This is one of the usual shock conditions of Rankine-Hugoniot in a perfect fluid. If as we have done here, one retains the terms of order ε, the first term has to be evaluated with the two first terms of the external expansion of ρ w. The expression in { } then only involves the first term of the expansion of each of the starred quantities, that is, the term which is independent of ε in the

expansion mentioned. According to lemma 7 of part 2, this term can be determined from the first term of the internal expansion of the corresponding quantity. The classical theory [2,6,7,9] of the shock wave structure has exactly the intent of calculating the first term of the internal expansion. In particular, we can see that $(\bar{u}_1)_o$ and $(\bar{u}_2)_o$ are independent of ζ and respectively equal to U_1 and U_2, which are tangential velocity components of the flow evaluated on (Σ) in a perfect fluid. Consequently, the shock condition translates the mass conservation as follows for the degree of approximation considered:

$$[\rho w] = -\frac{\epsilon}{H_1 H_2} \left\{ (\overset{*}{\rho} U_1 H_2)_{,1} + (\overset{*}{\rho} U_2 H_1)_{,2} \right\} . \tag{21}$$

In order to remove any possible ambiguity, let us discuss this equation in detail. The comments given for this simple case are also valid for other shock conditions which will be obtained in the following two paragraphs. The external expansions of ρ and w are

$$\rho = \begin{cases} \rho^{(o)+} + \epsilon \rho^{(1)+} + \dots , & z > 0 \\ \rho^{(o)-} + \epsilon \rho^{(1)-} + \dots , & z < 0 \end{cases} \tag{22}$$

$$w = \begin{cases} w^{(o)+} + \epsilon w^{(1)+} + \dots , & z > 0 \\ w^{(o)-} + \epsilon w^{(1)-} + \dots , & z < 0 \end{cases}$$

from which we find the external expansion of ρ w: /27

$$\rho w = \begin{cases} \rho^{(o)+} w^{(o)+} + \epsilon (\rho^{(o)+} w^{(1)+} + \rho^{(1)+} w^{(o)+}) + \dots , & z > 0 \\ \rho^{(o)-} w^{(o)-} + \epsilon (\rho^{(o)-} w^{(1)-} + \rho^{(1)-} w^{(o)-}) + \dots , & z < 0 \end{cases}$$

and we have

$$[\rho w] = \rho^{(o)+} w^{(o)+} - \rho^{(o)-} w^{(o)-} + \\ + \epsilon \left\{ \rho^{(o)+} w^{(1)+} - \rho^{(o)-} w^{(1)-} + \rho^{(1)+} w^{(o)+} - \rho^{(1)-} w^{(o)-} \right\} + \dots ,$$

The various expressions which appear in this relationship are functions of x_1 and x_2. In this way, we have explicitly written down the first term of (21). We will now explicitly determine

the second term, that is essentially $\rho^*(x_1,x_2)$. According to the second part, we have

$$\overset{*}{\rho}(x_1,x_2) = \overset{*}{\rho}^{(o)}(x_1,x_2) + 0(\epsilon)$$

$$= \int_{-\infty}^{+\infty} \left\{ \bar{\rho}^{(o)}(\zeta) - \frac{1}{2}(\rho^{(o)+} + \rho^{(o)-}) - \frac{1}{2}(\rho^{(o)+} - \rho^{(o)-})\left(\mathrm{tgh}\zeta + \frac{\zeta}{\mathrm{Ch}^2\zeta}\right) \right\} d\zeta + 0(\epsilon), \tag{23}$$

where $\bar{\rho}^{\circ}(\zeta)$ is the first term of the internal expansion of ρ defined by

$$\rho(x_1,x_2,\epsilon\zeta,\epsilon) = \bar{\rho}(x_1,x_2,\zeta,\epsilon) = \bar{\rho}^{(o)}(x_1,x_2,\zeta) + 0(\epsilon) \cdot$$

The function $\bar{\rho}^{(o)}(\zeta)$ is precisely the one given by the classical studies on the internal structure [2,6,7,9]. Since it satisfies the condition

$$\bar{\rho}^{(o)}(\pm\infty) = \rho^{(o)\pm} \quad ,$$

the integral of (23) certainly is convergent.

With this precise discussion, formula (21) is written explicitly in the following form

$$\left\{ \begin{array}{l} \rho^{(o)+} w^{(o)+} - \rho^{(o)-} w^{(o)-} = 0 \\ H_1H_2 \left\{ \rho^{(o)+} w^{(1)+} - \rho^{(o)-} w^{(1)-} + \rho^{(1)+} w^{(o)+} - \rho^{(1)-} w^{(o)-} \right\} = \\ \qquad\qquad = -\dfrac{\partial}{\partial x_1}(\rho^{*(o)} U_1 H_2) - \dfrac{\partial}{\partial x_2}(\rho^{*(o)} U_2 H_1) \ , \end{array} \right. \tag{24}$$

and the various expressions in this formula are obviously functions of x_1 and x_2 but not of z and of ε.

3.4.2. Momentum equation

It is now convenient to apply (19) with

$$\mathbf{A} = p\mathbf{1} + \rho\mathbf{V}\mathbf{V} + \epsilon\mathbf{t} \ , \ \mathbf{b} = \mathbf{A}\cdot\left(\frac{h_2}{H_2}\ \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{h_1}{H_1}\ \mathbf{e}_2 \otimes \mathbf{e}_2\right) \ ; \tag{25}$$

but before giving the explicit form of these equations, one must determine how $\mathbf{t}$ depends on the derivatives of the velocity components. In this system of coordinates which we are using, we have according to (11) /28

$$\begin{cases} \mathbf{t} = t_{ij}\, \mathbf{e}_i \otimes \mathbf{e}_j \,, \\ t_{ij} = -2\mu D_{ij} - \lambda \theta \delta_{ij} \,, \end{cases} \tag{26}$$

with

$$\begin{cases} D_{11} = \dfrac{u_{1,1}}{h_1} + \dfrac{u_2 h_{1,2}}{h_1 h_2} + \dfrac{K_1 w}{h_1} \,, \\ D_{22} = \dfrac{u_{2,2}}{h_2} + \dfrac{u_1 h_{2,1}}{h_1 h_2} + \dfrac{K_2 w}{h_2} \,, \\ D_{33} = \dfrac{\partial w}{\partial z} \,, \\ 2D_{21} = 2D_{12} = \dfrac{u_{2,1}}{h_1} + \dfrac{u_{1,2}}{h_2} - \dfrac{u_1 h_{1,2}}{h_1 h_2} - \dfrac{u_2 h_{2,1}}{h_1 h_2} \,, \\ 2D_{23} = 2D_{32} = \dfrac{\partial u_2}{\partial z} + \dfrac{w_{,2}}{h_2} - \dfrac{K_2 u_2}{h_2} \,, \\ 2D_{31} = 2D_{13} = \dfrac{\partial u_1}{\partial z} + \dfrac{w_{,1}}{h_1} - \dfrac{K_1 u_1}{h_1} \,, \\ \theta = D_{11} + D_{22} + D_{33} \,. \end{cases} \tag{27}$$

If we only consider terms of order ε in the shock conditions as we will do here, then in order to calculate the term in ε in $[\epsilon \mathbf{t}]$, it is appropriate to only consider in (26) and (27) the terms which are independent of ε, that is to calculate (26) and (27) with the values of u_1, u_2, w (and possibly T if we assume that λ and μ depend on temperature) in a perfect fluid flow. With this rule we can explicitly write down the $[A_{i3}]$ in (19). It is still necessary to find $(b_{ij})^*$ explicitly and it is sufficient to do this with the approximation of order zero. There are three categories of terms: $p^*\delta_{ij}, (\rho u_i u_j)^*, (\epsilon t_{ij})^*$. The terms of the first category can be immediately written down. Among the terms of the

second category, one can distinguish three types: $(\rho u_1^2)^* = \overset{*}{\rho} U_1^2$, $(\rho u_1 u_2)^* = \overset{*}{\rho} U_1 U_2$. Finally, $(\rho w u_1)^* = U_1 (\rho w)^* = 0$ (these evaluations are based on the fact that u_1, u_2, ρ w are constant in the internal structure of order zero). Among the terms of the third category, it is only necessary to retain those of order zero. For this purpose, one writes down the internal expression $\overline{\epsilon t_{ij}}$ of ϵt_{ij} and one only retains terms which are independent of ε. We then find:

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$$\begin{cases} \overline{\epsilon t_{11}} = -\overline{\lambda}\dfrac{\partial \overline{w}}{\partial \zeta} + 0(\epsilon) = \overline{\epsilon t_{22}}, \\ \overline{\epsilon t_{33}} = -(\overline{\lambda}+2\overline{\mu})\dfrac{\overline{\partial w}}{\partial \zeta} + 0(\epsilon), \\ \overline{\epsilon t_{23}} = -\overline{\mu}\dfrac{\partial \overline{u_2}}{\partial \zeta} + 0(\epsilon) = 0(\epsilon) = \overline{\epsilon t_{32}}, \\ \overline{\epsilon t_{13}} = -\overline{\mu}\dfrac{\partial \overline{u_1}}{\partial \zeta} + 0(\epsilon) = 0(\epsilon) = \overline{\epsilon t_{31}}, \\ \overline{\epsilon t_{12}} = 0(\epsilon). \end{cases} \tag{28}$$

Therefore, one only has to retain

$$\begin{cases} (\epsilon t_{11})^{*(o)} = (\epsilon t_{22})^{*(o)} = -\left(\lambda \dfrac{\partial w}{\partial \zeta}\right)^* \\ (\epsilon t_{33})^{*(o)} = -\left((\lambda+2\mu)\dfrac{\partial w}{\partial \zeta}\right)^* \end{cases} \tag{29}$$

In these expressions $\frac{\partial w}{\partial \zeta}$ designates the derivative of w expressed as a function of ζ, that is, the derivative of the function which gives the velocity profile in the classical internal structure of the shock wave [2,6,7,9].

Considering this, the shock conditions are easily written as follows*:

*There is a sign error in formula (7) of reference [1]. This error is corrected in the present formula (30).

$$
\left\{
\begin{aligned}
[\rho w u_1] &= \epsilon \left\{ \left[\mu \left(\frac{\partial u_1}{\partial z} + \frac{w_{,1}}{H_1} - \frac{K_1 U_1}{H_1} \right) \right] + \frac{1}{H_1} \left(\lambda \frac{\partial w}{\partial \zeta} - p \right)^*_{,1} - \right. \\
&\left. - \frac{1}{H_1 H_2} \left[(\overset{*}{\rho} U_1^2 H_2)_{,1} + (\overset{*}{\rho} U_1 U_2 H_1)_{,2} + \overset{*}{\rho} U_1 U_2 H_{1,2} - \overset{*}{\rho} U_2^2 H_{2,1} \right] \right\}, \quad \text{a)} \\
[\rho w u_2] &= \epsilon \left\{ \left[\mu \left(\frac{\partial u_2}{\partial z} + \frac{w_{,2}}{H_2} - \frac{K_2 U_2}{H_2} \right) \right] + \frac{1}{H_2} \left(\lambda \frac{\partial w}{\partial \zeta} - p \right)^*_{,2} - \right. \\
&\left. - \frac{1}{H_1 H_2} \left[(\overset{*}{\rho} U_1 U_2 H_2)_{,1} + (\overset{*}{\rho} U_2^2 H_1)_{,2} + \overset{*}{\rho} U_1 U_2 H_{2,1} - \overset{*}{\rho} U_1^2 H_{1,2} \right] \right\}, \quad \text{b)} \\
[p + \rho w^2] &= \epsilon \left\{ \left[(\lambda + 2\mu) \frac{\partial w}{\partial z} + \lambda \left[w \left(\frac{K_1}{H_1} + \frac{K_2}{H_2} \right) + \frac{(H_2 U_1)_{,1} + (H_1 U_2)_{,2}}{H_1 H_2} \right] \right] + \right. \\
&\left. + \overset{*}{\rho} \left(\frac{K_1 U_1^2}{H_1} + \frac{K_2 U_2^2}{H_2} \right) + \left(p - \lambda \frac{\partial w}{\partial \zeta} \right)^* \left(\frac{K_1}{H_1} + \frac{K_2}{H_2} \right) \right\}. \quad \text{c)}
\end{aligned}
\right. \tag{30}
$$

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Let us repeat that the first terms have to be calculated with the external expansions for the approximation of order 1. The second terms have to be calculated with the internal expansions at the approximation of order zero if these are terms with asterisks. This has to be done with external expansions of the approximation of the order zero also for the terms [ ]. In order to remove any ambiguity, let us state that

$$
\left(\lambda \frac{\partial w}{\partial \zeta} \right)^* = \int_{-\infty}^{+\infty} \bar{\lambda}^{(o)}(\zeta) \, \frac{\partial \bar{w}^{(o)}(\zeta)}{\partial \zeta} \, d\zeta \, , \tag{31}
$$

which corresponds to the general model (23) considering the fact that

$$
\left(\frac{\partial \bar{w}^{(o)}}{\partial \zeta} \right)_{\zeta = \pm\infty} = 0 \, .
$$

3.4.3. Energy equation

This time one has to apply (16) with

$$\mathbf{A} = \left(h + \frac{V^2}{2}\right) \rho \mathbf{V} + \epsilon (\mathbf{t} \cdot \mathbf{V} + \mathbf{q}) \ , \ \mathbf{b} = \mathbf{A} \cdot \left(\frac{h_2}{H_2} \ \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{h_1}{H_1} \ \mathbf{e}_2 \otimes \mathbf{e}_2\right) \ , \quad (32)$$

but since the mechanism is now well known, we can proceed very rapidly. Let us only say that the term $\epsilon \mathbf{q}$ gives a contribution of order ε to the discontinuity terms (term between [ ]). This contribution is calculated using the perfect fluid values. Conversely, there is no contribution to be retained in $(\epsilon \mathbf{q})^*$ because $\epsilon \bar{\mathbf{q}}_1 = 0(\epsilon)$ and $\epsilon \bar{\mathbf{q}}_2 = 0(\epsilon)$ according to (11) and because $(b_3)^*$ does not occur in (16). Without any difficulty, one obtains the following result:

$$\begin{aligned}
\left[\rho w \left(h + \frac{V^2}{2}\right)\right] = \epsilon \Bigg\{ \Bigg[& k \frac{\partial T}{\partial z} + (\lambda + 2\mu) \, w \frac{\partial w}{\partial z} + \\
& + \lambda w \left(w \left(\frac{K_1}{H_1} + \frac{K_2}{H_2} \right) + \frac{(H_2 U_1)_{,1} + (H_1 U_2)_{,2}}{H_1 H_2} \right) + \\
& + \mu \left(\frac{U_1 w_{,1}}{H_1} + \frac{U_2 w_{,2}}{H_2} + U_1 \frac{\partial u_1}{\partial z} + U_2 \frac{\partial u_2}{\partial z} - \frac{K_1 U_1^2}{H_1} - \frac{K_2 U_2^2}{H_2} \right) \Bigg] \\
& - \frac{1}{H_1 H_2} \Bigg[\left\{ \left(\rho \left(h + \frac{V^2}{2}\right) - \lambda \frac{\partial w}{\partial \zeta} \right)^* U_1 H_2 \right\}_{,1} + \\
& + \left\{ \left(\rho \left(h + \frac{V^2}{2}\right) - \lambda \frac{\partial w}{\partial \zeta} \right)^* U_2 H_1 \right\}_{,2} \Bigg] \Bigg\} .
\end{aligned} \quad (33)$$

3.5. CONCLUSION /31

The formulas (21), (30) and (33) translate the shock conditions when one takes into the account the terms of order ε and only those, and the higher order terms are ignored. Let us recall their interpretation. If ε is assumed to be zero, one finds the usual Rankine-Hugoniot conditions. If ε is not zero, these formulas, therefore, give a correction to the R-H conditions, a correction which is of order $\epsilon = \frac{1}{R_e}$. The Reynolds number is calculated with a length related to the average radius of curvature of the shock wave. The first terms contain the discontinuities of certain expressions which have to be evaluated with the terms of the external expansions up to order ε inclusive. In the second terms, we have, with a factor of ε, on the one hand the discontinuities (terms in [ ]) which have to be evaluated up to the approximation of order zero for the external flow (liquid fluid), and on the other hand, the terms with asterisks which are calculated from the results of the study of the classical internal structure [ ] (first term of order zero of the internal expansions). More precisely, there are four such terms with asterisks:

$$\rho^*, \left(\lambda \frac{\partial w}{\partial \zeta}\right)^*, \; p^*, \left(\rho\left(h + \frac{V^2}{2}\right)\right)^*, \tag{34}$$

which have to be evaluated as a function of x_1 and x_2, as integrals in ζ over the interval $-\infty$, $+\infty$, of functions of x_1, x_2, ζ. The general calculation formula is

$$\begin{aligned} f^*(x_1, x_2) = \int_{-\infty}^{+\infty} \Big\{ & f^{(o)}(x_1, x_2, \zeta) - \frac{1}{2}\left(f^{(o)+}(x_1, x_2) + f^{(o)-}(x_1, x_2)\right) - \\ & - \frac{1}{2}\left(f^{(o)+}(x_1, x_2) - f^{(o)-}(x_1, x_2)\right)\left(\operatorname{tgh}\zeta + \frac{\zeta}{\operatorname{ch}^2\zeta}\right) \Big\} \, d\zeta, \end{aligned} \tag{35}$$

and we find that for $\left(\lambda \frac{\partial w}{\partial \zeta}\right)^*$ the terms $f^{(o)\pm}$ become zero.

The formulation (21), (30) and (33) of the shock conditions is essentially related to the selected system of coordinates. The symbolic formulation (13) is intrinsic on the other hand. If it is advantageous for other reasons, there is no disadvantage for performing calculations in another system of coordinates on (Σ). For this purpose, one will write (13) by taking an elementary dS for (A_o), for example, limited by an infinitesimally small parallelogram which is constructed using coordinate lines on the surface. The term $[\mathbf{A}].\mathbf{N}\,dS$ is immediately interpreted as the variation of the flux of a mechanical quantity (mass flux, momentum flux, energy flux) through the area element dS. This is the variation which is experienced when one passes through the shock waves, more precisely the discontinuity surface (Σ). The term (A_o) obviously translates what escapes along the edges of the area (A_o) within the thickness of the shock wave, beyond what escapes if one were to use the values of the perfect fluid model to either side of it. This excess is precisely translated by the operation*. In the calculation of this term we have all of the terms involved in the Navier-Stokes equation and the curvilinear integral expresses an average value with respect to z of the various contributions of the flux through a cylinder with a base (Γ) and with a height ε perpendicular to (Σ). From this we obtain the following intrinsic rule:

$$\epsilon \int_{(\Gamma)} \mathbf{b}^* . \mathbf{n}$$

The variation of the flux density of a mechanical quantity (mass, momentum, energy) when passing through the shock wave (Σ) is equal to the sum of the two contributions if one ignores the terms of order equal to or greater than 2 in ε. The first contribution is the one from viscous stresses and the heat flux which is adjacent to the shock wave on either side, and which is evaluated in the immediate vicinity to either side of the shock wave with the data supplied by the perfect fluid model relative to surface elements along the two faces of (Σ). This first contribution automatically is of order ε. The second contribution can be evaluated formally by proceeding just like when one establishes the Navier- /32

Stokes equations from the integral form using the method of elementary parallelepiped volume elements limited by coordinate surfaces. This contribution corresponds to the lateral part of the elementary volume surface. This lateral portion is a cylinder perpendicular to (Σ) and has the height ε. However, the following precautions have to be observed.

1. In the stress terms and the heat flux terms, only those have to be retained which are associated with a gradient perpendicular to (Σ) by dividing them by ε;

2. replace the integration along the generator of the cylinder by an integration of an average value obtained by first performing the operation *. In other words, one uses the product of this average value and ε, and the contribution obtained in this way is of order ε. In this calculation, it should not be forgotten that the operation * has no effect on the tangential velocity components. This rule is the natural extension of the corresponding rule which is applied when one establishes the Rankine-Hugoniot conditions using the method which can be "physical" and which is used in all elementary courses. This is a rule which was used by Sedov, Mikhailova and Chernii [8], but only the first contribution is shown there which gives an erroneous result. Probstein and Kemp [5] were not successful in giving a satisfactory form to the second contribution.

4. STUDY OF THE INTERNAL STRUCTURE /33

The results given before show that for determining the shock conditions of order 1, it is necessary to know the internal structure of order zero and also the shock conditions of order n assume that the structure is known up to order n - 1, even though explicit calculations have not been made. In this part, we will show how one can form differential systems which allow one to study the various approximations. In order to simplify the analysis, we will only give the explicit calculation in the case of the order 1 structure. We will discuss the main properties of the differential system formed in this way and we will show that the internal flow "agrees" well with the external flow. One of the consequences of this study is to provide a new method, independent of the one which was discussed in the third part, and which gives the shock conditions of order 1.

4.1. EQUATIONS OF THE INTERNAL STRUCTURE OF ORDER 0 AND 1

The Navier-Stokes equations can be written in the form

$$\begin{cases} \operatorname{div}\ (\rho \vec{V}) = 0, \\ \operatorname{grad}\ p + \operatorname{div}\ (\rho V V + \epsilon t) = 0, \\ \operatorname{div}\ \left(\left\{ h + \frac{v^2}{2} \right\} \rho V + \epsilon \left\{ tV + q \right\} \right) = 0 \cdot \end{cases} \tag{1}$$

This involves the derivation operator (nabla) which is given by the following using internal variables:

$$\frac{1}{\epsilon} e_3 \frac{\partial}{\partial \zeta} + \frac{e_1}{h_1} \frac{\partial}{\partial x_1} + \frac{e_2}{h_2} \frac{\partial}{\partial x_2}, \tag{2}$$

so that symbolically, each equation (1) can be written in the form:

$$e_3 \cdot \frac{\partial \bar{A}}{\partial \zeta} = -\epsilon \left(\frac{e_1}{\bar{h}_1} \cdot \frac{\partial \bar{A}}{\partial x_1} + \frac{e_2}{\bar{h}_2} \cdot \frac{\partial \bar{A}}{\partial x_2} \right) \cdot \tag{3}$$

$\mathbf{A}$ designates a vector or a tensor of order 2, $\bar{\mathbf{A}}$ is the same vector or tensor with internal variables.

If in $\bar{\mathbf{A}}$ one separates the terms which contain ε as a factor from those which do not contain, by writing

$$\bar{\mathbf{A}} = \bar{\mathbf{B}} + \epsilon \bar{\mathbf{L}} , \tag{4}$$

and if we note that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are independent of ζ we can integrate this equation and we find that

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$$\mathbf{e}_3 \cdot \bar{\mathbf{B}}(\zeta) = - \epsilon \mathcal{L} (\mathbf{A}), \tag{5}$$

with

$$\mathcal{L}(\mathbf{A}) = \mathbf{e}_3 \cdot \bar{\mathbf{L}} + \mathbf{e}_1 \cdot \int^{\zeta} \frac{1}{h_1} \frac{\partial \bar{\mathbf{A}}}{\partial x_1} \partial\zeta + \\ + \mathbf{e}_2 \cdot \int^{\zeta} \frac{1}{h_2} \frac{\partial \bar{\mathbf{A}}}{\partial x_2} \partial\zeta . \tag{6}$$

Equation (5) is the symbolic equation of the internal structure. From this equation, we formally derive the equations of the internal structure of the various ranks by setting

$$\bar{\mathbf{A}} = \sum_{n=0}^{\infty} \epsilon^n \bar{\mathbf{A}}^{(n)} , \quad \bar{\mathbf{B}} = \sum_{n=0}^{\infty} \epsilon^n \bar{\mathbf{B}}^{(n)} ,$$
$$\bar{\mathbf{L}} = \sum_{n=0}^{\infty} \epsilon^n \bar{\mathbf{L}}^{(n)} , \quad \bar{\mathcal{L}} = \sum_{n=0}^{\infty} \epsilon^n \bar{\mathcal{L}}^{(n)} ,$$

and by equating the coefficients of the various powers of ε we find

$$\mathbf{e}_3 \cdot \bar{\mathbf{B}}^{(n)} = - \mathcal{L}^{(n-1)} \quad ; \quad \left(\mathcal{L}^{(-1)} = 0 \right) . \tag{7}$$

We must now give the explicit form of the calculations. We will only do this for the terms of rank 0 and 1.

4.1.1 <u>This equation</u>

In this case $\mathbf{A}$ and the vector $\rho\mathbf{V}$ and $\mathbf{L}$ is zero. Consequently, (5) is written as

$$\overline{\rho w} + \epsilon \bar{C}_o = 0 , \tag{8}$$

$\bar{C}_o$ designates a primitive with respect to ζ

$$\bar{c}_o = \left(\frac{\mathbf{e}_1}{\bar{h}_1}\frac{\partial}{\partial x_1} + \frac{\mathbf{e}_2}{\bar{h}_2}\frac{\partial}{\partial x_2}\right)\cdot\left(\overline{\rho u_1}\,\mathbf{e}_1 + \overline{\rho u_2}\,\mathbf{e}_2 + \overline{\rho w}\,\mathbf{e}_3\right)$$

$$= \frac{1}{\overline{h_1 h_2}}\left\{(\overline{h_2 u_1 \rho})_{,1} + (\overline{h_1 u_2 \rho})_{,2}\right\} + \overline{\rho w}\left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2}\right); \tag{9}$$

as one can see by carrying out the calculation by taking into account the expressions for the derivatives of the unit vectors $\mathbf{e}_i$ given in 3.3., or by applying the formula which gives the divergence of a vector field in curvilinear coordinates. The equations which correspond to (7) with ranks of 0 and 1 respectively are given in (17a) and (18a).

4.1.2. Momentum equation /35

Here A is a tensor of second order

$$\mathsf{A} = p\mathsf{1} + \rho\mathsf{V}\mathsf{V} + \epsilon\mathsf{t} ,$$

and in order to separate B and L, it is first useful to explicitly write down the components of the dissipative terms. The formulas (3-26) show that using the internal variables we have

$$
\left\{
\begin{aligned}
-\epsilon\bar{t}_{11} &= \bar{\lambda}\frac{\partial\bar{w}}{\partial\zeta} + \epsilon\bar{\lambda}\left\{\frac{(\overline{h_2u_1})_{,1} + (\overline{h_1u_2})_{,2}}{\overline{h_1h_2}} + \bar{w}\left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2}\right)\right\} \\
&\quad + 2\epsilon\bar{\mu}\left(\frac{\bar{u}_{1,1}}{\bar{h}_1} + \frac{\bar{u}_2\bar{h}_{1,2}}{\bar{h}_2} + \frac{K_1\bar{w}}{\bar{h}_1}\right), \\
-\epsilon\bar{t}_{22} &= \bar{\lambda}\frac{\partial\bar{w}}{\partial\zeta} + \epsilon\bar{\lambda}\left\{\frac{(\overline{h_2u_1})_{,1} + (\overline{h_1u_2})_{,2}}{\overline{h_1h_2}} + \bar{w}\left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2}\right)\right\} \\
&\quad + 2\epsilon\bar{\mu}\left(\frac{\bar{u}_{2,2}}{\bar{h}_2} + \frac{\bar{u}_1\bar{h}_{2,1}}{\bar{h}_1} + \frac{K_2\bar{w}}{\bar{h}_2}\right), \\
-\epsilon\bar{t}_{33} &= (\bar{\lambda}+2\bar{\mu})\frac{\partial\bar{w}}{\partial\zeta} + \epsilon\bar{\lambda}\left\{\frac{(\overline{h_2u_1})_{,1} + (\overline{h_1u_2})_{,2}}{\overline{h_1h_2}} + \bar{w}\left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2}\right)\right\} \\
-\epsilon\bar{t}_{31} &= -\epsilon\bar{t}_{13} = \bar{\mu}\frac{\partial\bar{u}_1}{\partial\zeta} + \epsilon\bar{\mu}\left(\frac{\bar{w}_{,1}}{\bar{h}_1} - K_1\frac{\bar{u}_1}{\bar{h}_1}\right), \\
-\epsilon\bar{t}_{32} &= -\epsilon\bar{t}_{23} = \bar{\mu}\frac{\partial\bar{u}_2}{\partial\zeta} + \epsilon\bar{\mu}\left(\frac{\bar{w}_{,2}}{\bar{h}_2} - K_2\frac{\bar{u}_2}{\bar{h}_2}\right), \\
-\epsilon\bar{t}_{12} &= -\epsilon\bar{t}_{21} = \epsilon\bar{\mu}\left\{\frac{\bar{h}_2}{\bar{h}_1}\left(\frac{\bar{u}_2}{\bar{h}_2}\right)_{,1} + \frac{\bar{h}_1}{\bar{h}_2}\left(\frac{\bar{u}_1}{\bar{h}_1}\right)_{,2}\right\}.
\end{aligned}
\right. \tag{10}
$$

With this, we then have /36

$$
\left\{
\begin{aligned}
\mathbf{e}_3 \cdot \mathbf{B} &= \left(\overline{\rho w u_1} - \bar{\mu}\,\frac{\partial \bar{u}_1}{\partial \zeta}\right)\mathbf{e}_1 + \left(\overline{\rho w u_2} - \bar{\mu}\,\frac{\partial \bar{u}_2}{\partial \zeta}\right)\mathbf{e}_2 + \\
&\quad + \left(\bar{p} + \overline{\rho w^2} - (\bar{\lambda} + 2\bar{\mu})\,\frac{\partial \bar{w}}{\partial \zeta}\right)\mathbf{e}_3 \,, \\
\mathbf{e}_3 \cdot \mathbf{L} &= D_1\,\mathbf{e}_1 + D_2\,\mathbf{e}_2 + D_3\,\mathbf{e}_3 \\
&= -\bar{\mu}\left(\frac{\bar{w}_{,1}}{\bar{h}_1} - K_1\,\frac{\bar{u}_1}{\bar{h}_1}\right)\mathbf{e}_1 - \bar{\mu}\left(\frac{\bar{w}_{,2}}{\bar{h}_2} - K_2\,\frac{\bar{u}_2}{\bar{h}_2}\right)\mathbf{e}_2 - \\
&\quad - \bar{\lambda}\left\{\frac{(\overline{h_2 u_1})_{,1} + (\overline{h_1 u_2})_{,2}}{\overline{h_1 h_2}} + \left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2}\right)\bar{w}\right\}\mathbf{e}_3 \,, \\
\mathcal{L} - \mathbf{e}_3 \cdot \mathbf{L} &= (\bar{C}_1 - \bar{V}_1)\,\mathbf{e}_1 + (\bar{C}_2 - \bar{V}_2)\,\mathbf{e}_2 + (\bar{C}_3 - \bar{V}_3)\,\mathbf{e}_3 \,.
\end{aligned}
\right. \tag{11}
$$

The C_i and V_i are primitives of the expressions, which are easy to write down explicitly in general, but here we will only give them for order zero, using a property which will be established in paragraph 4.2, that is that $\bar{u}_1^{(o)}$ and $\bar{u}_2^{(o)}$ are independent of ζ and equal to U_1 and U_2, the values of u_1 and u_2 on the shock wave for a perfect fluid.

Then we find:

$$
\left\{
\begin{aligned}
\bar{c}_1^{(o)} &= \frac{(H_2 U_1^2 \bar{\rho}^{(o)})_{,1} + (H_1 U_1 U_2 \bar{\rho}^{(o)})_{,2}}{H_1 H_2} + \left(2\frac{K_1}{H_1} + \frac{K_2}{H_2}\right) \overline{\rho w}^{(o)} U_1 + \\
&\quad + \bar{\rho}^{(o)} U_2 \left(\frac{U_1 H_{1,2} - U_2 H_{2,1}}{H_1 H_2}\right) + \frac{\bar{p}^{(o)}_{,1}}{H_1}, \\
\bar{c}_2^{(o)} &= \frac{(H_2 U_1 U_2 \bar{\rho}^{(o)})_{,1} + (H_1 U_2^2 \bar{\rho}^{(o)})_{,2}}{H_1 H_2} + \left(\frac{K_1}{H_1} + 2\frac{K_2}{H_2}\right) \overline{\rho w}^{(o)} U_2 + \\
&\quad + \bar{\rho}^{(o)} U_1 \frac{U_2 H_{2,1} - U_1 H_{1,2}}{H_1 H_2} + \frac{\bar{p}^{(o)}_{,2}}{H_2}, \\
\bar{c}_3^{(o)} &= \frac{(H_2 U_1 \overline{\rho w}^{(o)})_{,1} + (H_1 U_2 \overline{\rho w}^{(o)})_{,2}}{H_1 H_2} + \left(\frac{K_1}{H_1} + \frac{K_2}{H_2}\right) \overline{\rho w^2}^{(o)} - \\
&\quad - \bar{\rho}^{(o)} \left(\frac{K_1 U_1^2}{H_1} + \frac{K_2 U_2^2}{H_2}\right),
\end{aligned}
\right. \tag{12}
$$

/37

$$
\left\{
\begin{aligned}
\bar{v}_1^{(o)} &= \frac{1}{H_1} \left(\lambda \overline{\frac{\partial w}{\partial \zeta}}^{(o)}\right)_{,1} \\
\bar{v}_2^{(o)} &= \frac{1}{H_2} \left(\lambda \overline{\frac{\partial w}{\partial \zeta}}^{(o)}\right)_{,2} \\
\bar{v}_3^{(o)} &= 2\mu \overline{\frac{\partial w}{\partial \zeta}}^{(o)} \left(\frac{K_1}{H_1} + \frac{K_2}{H_2}\right).
\end{aligned}
\right. \tag{13}
$$

We find that in (12), considering the properties of the internal structure of order zero which will be discussed in paragraph 4.2, $\overline{\rho w}^{(o)}$ is independent of ζ and equal to the value taken on by ρ w on the shock wave in a perfect fluid.

The equations corresponding to (7) with respective ranks of 0 and 1 are given by (17b, c,d) and (18b,c,d).

4.1.3. Energy equation

In this case, $\mathbf{A}$ is a vector

$$\mathbf{A} = \left(h + \frac{V^2}{2}\right) \rho \mathbf{V} + \epsilon \left(\mathbf{t} \cdot \mathbf{V} + \mathbf{q} \right), \tag{14}$$

$\mathbf{q}$ is given by an expression in terms of internal variables as follows

$$-\epsilon \mathbf{q} = \bar{k} \frac{\partial \bar{T}}{\partial \zeta} \mathbf{e}_3 + \epsilon \bar{k} \left(\frac{\mathbf{e}_1}{\bar{h}_1} \bar{T}_{,1} + \frac{\mathbf{e}_2}{\bar{h}_2} T_{,2} \right) .$$

We now have:

$$\left\{ \begin{aligned} \mathbf{e}_3 \cdot \mathbf{B} &= \left(\bar{h} + \frac{\bar{V}^2}{2}\right) \rho \bar{w} - (\bar{\lambda} + 2\bar{\mu}) \bar{w} \frac{\partial \bar{w}}{\partial \zeta} - \\ &\quad - \bar{\mu} \left(\bar{u}_1 \frac{\partial \bar{u}_1}{\partial \zeta} + \bar{u}_2 \frac{\partial \bar{u}_2}{\partial \zeta} \right) - \bar{k} \frac{\partial \bar{T}}{\partial \zeta} , \\ \mathbf{e}_3 \cdot \mathbf{L} = D_4 &= - \bar{\lambda} \bar{w} \frac{(\overline{h_2 u_1})_{,1} + (\overline{h_1 u_2})_{,2}}{\overline{h_1 h_2}} - \overline{\lambda w^2} \left(\frac{K_1}{\bar{h}_1} + \frac{K_2}{\bar{h}_2} \right) - \\ &\quad - \bar{\mu} \bar{u}_1 \left(\frac{\bar{w}_{,1}}{\bar{h}_1} - \frac{K_1 \bar{u}_1}{\bar{h}_1} \right) - \bar{\mu} \bar{u}_2 \left(\frac{\bar{w}_{,2}}{\bar{h}_2} - \frac{K_2 \bar{u}_2}{\bar{h}_2} \right) , \\ \mathcal{L} - \mathbf{e}_3 \cdot \mathbf{L} &= \bar{C}_4 - \bar{V}_4 , \end{aligned} \right. \tag{15}$$

and $\overline{C_4}$, $\overline{V_4}$ are the primitives with respect to ζ of the expression /38
which is easily written down explicitly in general. However, we will only give it here up to order zero, always taking into account that $\bar{u}_1^{(o)} = U_1$ and $\bar{u}_2^{(o)} = U_2$, U_1 and U_2 are independent of ζ:

$$\left\{\begin{aligned} \bar{c}_4^{(o)} &= \frac{\left(U_1 H_2 \overline{\rho\,(h+\frac{V^2}{2})}^{(o)}\right)_{,1} + \left(U_2 H_1 \overline{\rho\,(h+\frac{V^2}{2})}^{(o)}\right)_{,2}}{H_1\,H_2} + \\ &\quad + \left(\frac{K_1}{H_1} + \frac{K_2}{H_2}\right) \overline{\rho w\,(h+\frac{V^2}{2})}^{(o)} , \\ \bar{v}_4^{(o)} &= \frac{\left(U_1 H_2 \overline{\lambda \frac{\partial w}{\partial \zeta}}^{(o)}\right)_{,1} + \left(U_2 H_1 \overline{\lambda \frac{\partial w}{\partial \zeta}}^{(o)}\right)_{,2}}{H_1\,H_2} + \\ &\quad + \left(\frac{K_1}{H_1} + \frac{K_2}{H_2}\right)\left(k \overline{\frac{\partial T}{\partial \zeta}}^{(o)} + \overline{(\lambda+2\mu)\, w \frac{\partial w}{\partial \zeta}}^{(o)}\right). \end{aligned}\right. \tag{16}$$

The equations corresponding to (7) with respective ranks 0 and 1 are given by (17e) and (18e).

4.2. STUDY OF THE INTERNAL STRUCTURES OF ORDER 0 AND 1

According to the preceding results, the internal structure of order 0 is defined by the nonlinear system.

$$\left\{\begin{aligned} &\overline{\rho w}^{(o)} = M_o, && \text{a)} \\ &M_o \bar{u}_1^{(o)} - \bar{\mu}^{(o)} \overline{\frac{\partial u_1}{\partial \zeta}}^{(o)} = P_1^{(o)} , && \text{b)} \\ &M_o \bar{u}_2^{(o)} - \bar{\mu}^{(o)} \overline{\frac{\partial u_2}{\partial \zeta}}^{(o)} = P_2^{(o)} , && \text{c)} \\ &\overline{p + \rho w^2}^{(o)} - \overline{(\lambda + 2\mu)}^{(o)} \frac{\partial w^{(o)}}{\partial \zeta} = P_3^{(o)} , && \text{d)} \\ &M_o \overline{(h+\frac{V^2}{2})}^{(o)} - \bar{\mu} \bar{u}_1^{(o)} \overline{\frac{\partial u_1}{\partial \zeta}}^{(o)} - \bar{\mu} \bar{u}_2^{(o)} \overline{\frac{\partial u_2}{\partial \zeta}}^{(o)} - \overline{(\lambda + 2\mu)\, w}^{(o)} \overline{\frac{\partial w}{\partial \zeta}}^{(o)} - k \overline{\frac{\partial T}{\partial \zeta}}^{(o)} = E_o , && \text{e)} \end{aligned}\right. \tag{17}$$

in which $M_0, P_1^\circ, P_2^\circ, P_3^\circ, E_0$ designates the constants. According to the lemma of the second part, the unknown functions $\bar{\rho}^{(o)}, \bar{w}^{(o)}, \bar{u}_1^{(o)} \ldots$ have to strive towards the finite limits when ζ increases indefinitely in absolute value. This condition, considering (17b) and (17c), implies that $\bar{u}_1^{(o)}$ and $\bar{u}_2^{(o)}$ remain constant: the tangential components remain invariant and equal to U_1 and U_2 in the internal structure of order 0. The problem is then reduced to solving a system of two equations, eseentially (17b) and (17c), with two unknowns, for example, $\bar{p}^o$, $\bar{T}^o$. The other thermodynamic quantities $\bar{\rho}^\circ, \bar{\mu}^\circ, \bar{\lambda}^\circ, \ldots$ are derived from them using the state equation and the law of variation of dissipative /39
coefficients as a function of temperature. $\bar{w}^o$ is derived from (17a). This problem is a classical one [2]. One finds that for a system of values of the constants (M, P, E) which insure the possibility of a shock wave, there is one solution and only one solution (naturally defined up to a translation in ζ) and that the various values of the dynamic and thermodynamic quantities for $\zeta = +\infty$ and $\zeta = -\infty$ will satisfy the zero order shock conditions. This is as it has to be according to the theorem of the second part, section 2.3. One also can show that the functions $\rho^{-(o)}(\zeta)$, $\bar{T}^{(o)}(\zeta)$ obtained in this way have the properties of the first term of the internal expansion of a function of class NS (lemma 4--formula 2-11).

We will now propose to study the internal structure of rank 1. The equations which govern this are written in the following form according to the preceding results:

$$
\left\{
\begin{array}{ll}
\overline{\rho w}^{(1)} + \overline{C}_o^{(o)} = 0 \,, & \text{a)} \\[2ex]
\overline{\rho w u_1}^{(1)} - \overline{\mu \dfrac{\partial u_1}{\partial \zeta}}^{(1)} + \overline{C}_1^{(o)} + \overline{D}_1^{(o)} - \overline{V}_1^{(o)} = 0 \,, & \text{b)} \\[2ex]
\overline{\rho w u_2}^{(1)} - \overline{\mu \dfrac{\partial u_2}{\partial \zeta}}^{(1)} + \overline{C}_2^{(o)} + \overline{D}_2^{(o)} - \overline{V}_2^{(o)} = 0 \,, & \text{c)} \\[2ex]
\overline{p + \rho w^2}^{(1)} - \overline{(\lambda + 2\mu) \dfrac{\partial w}{\partial \zeta}}^{(1)} + \overline{C}_3^{(o)} + \overline{D}_3^{(o)} - \overline{V}_3^{(o)} = 0, & \text{d)} \\[2ex]
\overline{\rho w \left(h + \dfrac{V^2}{2} \right)}^{(1)} - \overline{(\lambda + 2\mu)\, w \dfrac{\partial w}{\partial \zeta}}^{(1)} - \overline{k \dfrac{\partial T}{\partial \zeta}}^{(1)} - & \text{e)} \\[2ex]
\qquad - \overline{\mu}^{(o)} \left(U_1 \overline{\dfrac{\partial u_1}{\partial \zeta}}^{(1)} + U_2 \overline{\dfrac{\partial u_2}{\partial \zeta}}^{(1)} \right) + \overline{C}_4^{(o)} + \overline{D}_4^{(o)} - \overline{V}_4^{(o)} = 0, &
\end{array}
\right.
\qquad (18)
$$

These are formulas in which the $\overline{C}_i^{(o)}$ and the $\overline{V}_j^{(o)}$ are primitives of the functions $\overline{c}_i^{(o)}$ and $\overline{v}_j^{(o)}$, whose explicit expressions were written down in (9), (12), (13), (16). Because of the properties which result from solving (17) and which were discussed above, these are functions of ζ. For large values of ζ, they have the behavior of polynomials with a degree which is at least equal to one. In contrast to (17), this system (18) is linear, because a quantity like $\overline{a\,b\,c}^{(1)}$ is linear with respect to $\overline{a}^{(1)}$, $\overline{b}^{(1)}$, $\overline{c}^{(1)}$:

$$\overline{abc}^{(1)} = \overline{ab}^{(o)}\, \overline{c}^{(1)} + \overline{ca}^{(o)}\, \overline{b}^{(1)} + \overline{bc}^{(o)}\, \overline{a}^{(1)} \,.$$

In order to simplify the notation, we will write down the system in the form

$$\mathfrak{A}\ (\mathfrak{X}) + (\mathfrak{F}) = 0 \,, \qquad (19)$$

where $\mathfrak{A}$ designates a linear operator which operates on the flow $\mathfrak{X}$ defined by $w^{(1)}, \rho^{(1)}, p^{(1)} \ldots$ and $(\mathfrak{F})$ which is a vector with the components

$$\mathfrak{F}_i = \overline{C}_i^{(o)} + \overline{D}_i^{(o)} - \overline{V}_i^{(o)} \,, \qquad (\overline{D}_o^{(o)} = 0 \,,\ \overline{V}_o^{(o)} = 0)$$

4.2.1. The associated homogeneous linear system /40

Now we must study the homogeneous linear system associated with (18). According to lemma (4) of the second part, $\overline{w}^{(1)}, \overline{p}^{(1)}, \overline{p}^{(1)}\ldots$ is limited by polynomials of the first degree at the most in ζ for large values of ζ which are positive or negative. Consequently, only the solutions of the homogeneous system which satisfy this property have to be retained.

Let $\hat{w}, \hat{\rho}, \hat{u}$ ... be a solution of $\alpha(\xi) = 0$.
Of necessity

$$\left\{\begin{array}{l} \overline{w}^{(o)}\hat{\rho} + \hat{w}\,\overline{p}^{o} = 0 \quad , \qquad \hat{u}_1 = 0 \quad , \quad \hat{u}_2 = 0 \, , \\ \hat{p} + M_o\,\hat{w} - \overline{(\lambda + 2\mu)}^{(o)} \dfrac{\partial \hat{w}}{\partial \zeta} - \widehat{(\lambda + 2\mu)} \dfrac{\partial \overline{w}^{(o)}}{\partial \zeta} = 0 \, , \\ M_o(\hat{h} + \overline{w}^{(o)}\,\hat{w}) - \overline{(\lambda + 2\mu)\,w}^{(o)} \dfrac{\partial \hat{w}}{\partial \zeta} - \overline{(\lambda + 2\mu)} \dfrac{\partial \overline{w}^{(o)}}{\partial \zeta}\,\hat{w} - \\ \qquad - \widehat{(\lambda + 2\mu)}\,\overline{w \dfrac{\partial \overline{w}^{(o)}}{\partial \zeta}} - \hat{k} \dfrac{\partial \overline{T}^{(o)}}{\partial \zeta} - \overline{k}^{(o)} \dfrac{\partial \hat{T}}{\partial \zeta} = 0 \, , \end{array}\right. \tag{20}$$

and consequently the system to be studied is of the form

$$\frac{\partial \hat{w}}{\partial \zeta} = \alpha\,\hat{w} + \beta\,\hat{T} \, , \quad \frac{\partial \hat{T}}{\partial \zeta} = \gamma\,\hat{w} + \delta\hat{T} \, , \tag{21}$$

where α, β, γ, δ are functions of ζ defined and limited for all values of ζ.

Any solution of (21) is defined for any value of ζ. We assume here, which can be verified rigorously, that its behavior for values of ζ with a large absolute value is obtained by considering the behavior of the solutions of the linear homogeneous systems with constant coefficients formed from (21), by replacing α, β, γ, δ by their values taken on respectively for $\zeta = +\infty$ and $\zeta = -\infty$. If one considers (20), it is clear that the systems with constant coefficients are precisely those which one would obtain when one would study the behavior of the integrals of (17) for $\zeta = +\infty$ and

$\zeta = -\infty$. In the classical theory of the internal structure of order 0, one establishes that for $\zeta = -\infty$ all of the solutions are exponentially decreasing. For $\zeta = +\infty$, the solutions in general are exponentially increasing with the exception of the single family $\hat{\mathcal{F}}$ of solutions which are exponentially decreasing, up to a multiplication factor. We therefore can state the following:

Lemma 1. The system (20) has a solution $(\hat{\mathcal{F}})$, defined up to a multiplication constant which decreases exponentially at infinity. Any other solution is exponentially increasing.

4.2.2. Solution of the nonhomogeneous system (19)

Considering the previous result, it is sufficient here to study a particular solution. Each of the components $\mathcal{F}_i$ of the vector $(\mathcal{F})$ can be written in the form

$$\mathcal{F}_i = \frac{1}{2}\left(P^+(\mathcal{F}_i) + P^-(\mathcal{F}_i)\right) + \frac{1}{2}\left(P^+(\mathcal{F}_i) - P^-(\mathcal{F}_i)\right) \operatorname{tgh} \zeta + \tilde{\mathcal{F}}_i \,. \quad (22)$$

$P^+(\mathcal{F}_i)$ and $P^-(\mathcal{F}_i)$ are polynomials of degree 1 and $\tilde{\mathcal{F}}$ is an exponentially decreasing function of ζ. Let $\mathcal{A}^+$ and $\mathcal{A}^-$ be the differential operators with constant coefficients obtained by setting $\zeta = +\infty$ and $\zeta = -\infty$ in the coefficients of $\mathcal{A}$. One then insures that the systems /41

$$\mathcal{A}^+(\mathcal{F}) = P^+(\mathcal{F}), \qquad \mathcal{A}^-(\mathcal{F}) = P^-(\mathcal{F}), \quad (23)$$

have solutions whose components w, ρ, p...are polynomials of degree 1 in ζ. We will call these solutions $P^+(\mathcal{F})$ and $P^-(\mathcal{F})$ which are obtained by substitution and identification*. If we now set

$$\mathcal{F}_i = \frac{1}{2}\left(P^+(\mathcal{F}_i) + P^-(\mathcal{F}_i)\right) + \frac{1}{2}\left(P^+(\mathcal{F}_i) - P^-(\mathcal{F}_i)\right) \operatorname{tgh} \zeta + \tilde{\mathcal{F}}_i \quad (24)$$

the system of the $\tilde{\mathcal{F}}_i$ satisfies the symbolic equation

$$\mathcal{A}(\tilde{\mathcal{F}}) + \tilde{\mathcal{G}} = 0 \quad (25)$$

and it is easy to see that $\tilde{\mathcal{G}}$ is a function of ζ which is exponentially decreasing at infinity. Here we will assume without a

*One is led to linear systems whose determinants are not zero when the velocity is not sonic, a condition which is obviously excluded.

proof the following lemma:

LEMMA 2. The system (25), in which $\tilde{\mathfrak{F}}$ is a given function which decreases exponentially at infinity, itself has at least one exponentially decreasing solution at infinity.

We can now collect the preceding results and state the following:

THEOREM. The differential system which defines the internal structure of rank 1 has a family of solutions which depend on an arbitrary constant. These are solutions which behave like polynomials of degree 1 in ζ for large values of positive and negative ζ.

We should note that the preceding statement assumes that $(\mathfrak{F})$ is well determined. In fact, we have seen that each component $\mathfrak{F}_i$ of $(\mathfrak{F})$ is only defined up to an additive constant. Therefore, the most general solution of the internal structure of rank 1 corresponding to a given zero order structure depends on 6 constants.

4.3. RELATIONSHIP WITH THE EXTERNAL FLOW--SHOCK CONDITIONS

In order to provide coherence of the results, it is necessary that the solution found in this way has the properties of the terms of rank 1 of the internal structure of a system of NS functions. For this purpose, it is necessary to verify that the coefficients of ζ in each of the components $\overline{w}^{(1)}, \overline{\rho}^{(1)}, \overline{p}^{(1)}$, de $P^{+}(\mathfrak{z})$ and $P^{-}(\mathfrak{z})$ are identical to the derivatives $\frac{\partial w^{(o)}}{\partial z}, \frac{\partial \rho^{(o)}}{\partial z}, \frac{\partial p^{(o)}}{\partial z}, \ldots$ taken for $z = +0$ and $z = -0$ of the corresponding quantities $w^{(o)}, \rho^{(o)}, p^{(o)}, \ldots$ in perfect fluid flow. This proof results immediately because the equations of perfect fluids are the following in symbolic form

$$\mathbf{e}_3 \cdot \frac{\partial \mathbf{A}^{(o)}}{\partial z} = -\left(\frac{\mathbf{e}_1}{h_1} \cdot \frac{\partial \mathbf{A}^{(o)}}{\partial x_1} + \frac{\mathbf{e}_2}{h_2} \cdot \frac{\partial \mathbf{A}^{(o)}}{\partial x_2} \right), \qquad (26)$$

In succession $A^{(o)}$ is equal to $(\rho V)^{(o)}$, $(p)^{(o)} I + (\rho V V)^{(o)}$ and /42
$\left((h + \frac{V^2}{2})\rho V\right)^{(o)}$; . But the values of the second terms of (26) for $z = +0$ are precisely the values of $-\bar{c}_i^{(o)}$ for $\zeta = +\infty$. One can easily see that the first terms of (26) are identical with the product of the coefficients $\bar{w}^{(1)}, \bar{\rho}^{(1)}, \bar{p}^{(1)}, \ldots$ in $\alpha^{\pm}(\zeta)$ and the coefficients of ζ in the polynomial part of the same $\bar{w}^{(1)}$, $\bar{\rho}^{(1)}, \bar{p}^{(1)}$ for $z = \pm 0$. This result follows from the fact that the coefficients of the components of $P^{\pm}(\zeta)$ in (26) are precisely the $\bar{c}_i^{(o)}$ for $\zeta \pm \infty$. Thus, for the mass conservation equation (26) leads to the following

$$\left(\rho^{(o)} \frac{\partial w^{(o)}}{\partial z} + w^{(o)} \frac{\partial \rho^{(o)}}{\partial z}\right)_{z=\pm o} + (\bar{c}_o{}^{(o)})_{\zeta=\pm\infty} = 0, \qquad (26')$$

and (23) leads to

$$\left(\rho^{(o)} \frac{\partial \bar{w}^{(1)}}{\partial \zeta} + \bar{w}^{(o)} \frac{\partial \bar{\rho}^{(1)}}{\partial \zeta}\right)_{\zeta=\pm\infty} + (\bar{c}_o{}^{(o)})_{\zeta=\pm\infty} = 0, \qquad (23')$$

naturally we find that $(\rho^{(o)})_{z=\pm o} = (\bar{\rho}^{(o)})_{\zeta=\pm\infty}$ as well as the relationship similar to $\bar{w}^{(o)}$.

If we are now interested in the constant terms in each of the components $w^{(1)}, \ldots$ of $P^{+}(\zeta)$ and $P^{-}(\zeta)$, we are led to finding the shock conditions using a different method than the one used in the third part, that is, one which agrees with the hypothesis used in this third part. The various functions which characterize the characteristics of the flow are NS functions and this is not contradictory, and this is true up to at least order of 1. We have verified the fact that the same thing holds true up to order n for arbitrary n. Here we will be content with giving this proof for the fourth equation (18) only for the terms of rank 1. The procedure is simple: the first term of the equation is uniformly limited in ζ if one isolates a term of the form ζ (a + b thζ) and the form of a and b is exactly the one which we have discussed. Once we have subtracted this, let us then subtract the values of the two first terms for $\zeta = +\infty$ and $\zeta = -\infty$ for each term. It is clear that $\overline{p + \rho w^2}^{(1)}$ in this way gives a result which is none other than $[\overline{p+\rho w^2}^{(1)}]$. As for the term $\overline{(\lambda + 2\mu)\frac{\partial w}{\partial \zeta}}^{(1)}$, the conditions are not as clear but if we write

$$\overline{(\lambda+2\mu)\frac{\partial w}{\partial \zeta}}^{(1)} = \overline{(\lambda+2\mu)}^{(o)}\ \overline{\frac{\partial w}{\partial \zeta}}^{(1)} + \overline{(\lambda+2\mu)}^{(1)}\ \overline{\frac{\partial w}{\partial \zeta}}^{(o)} \quad ,$$

and note that $\overline{\frac{\partial w}{\partial \zeta}}^{(o)}$ is zero at infinity and also what we have said about the terms in ζ in $P\pm(\zeta)$, we find that $\left(\overline{\frac{\partial w}{\partial \zeta}}^{(1)}\right)_{\zeta=\pm\infty} = \left(\frac{\partial w^{(o)}}{\partial z}\right)_{z=\pm o}$ We see that the term $\overline{(\lambda+2\mu)\frac{\partial w}{\partial \zeta}}^{(1)}$ gives a result equal to $\left[(\lambda+2\mu)\frac{\partial w}{\partial \zeta}\right]_o$, using the rule defined at the beginning of this paragraph. The term $D_3^{(o)}$ obviously gives $[D_3]_o$, and we only have to examine what the result from $C_3^{(o)}$ and $V_3^{(o)}$ is. The quantities $C_i^{(o)}$ and $V_i^{(o)}$ are primitives of $c_i^{(o)}$ and $v_i^{(o)}$. It is clear that the result has to be c^* and v^*. If we refer to (12) and (13) taking into account that $\overline{\rho\ w}^{(o)}$ is independent of ζ and, therefore, gives zero contribution due to the operation * we find that

$$-c_3^* + v_3^* = \left(\frac{K_1}{H_1}+\frac{K_2}{H_2}\right)\left\{\left(2\mu\frac{\partial w}{\partial \zeta}\right)^* - (\rho w^2)^*\right\} + \rho^*\left(\frac{K_1U_1^2}{H_1}+\frac{K_2U_2^2}{H_2}\right)$$

$$= \left(p - \lambda\frac{\partial w}{\partial \zeta}\right)^*\left(\frac{K_1}{H_1}+\frac{K_2}{H_2}\right) + \rho^*\left(\frac{K_1U_1^2}{H_1}+\frac{K_2U_2^2}{H_2}\right) \qquad /43$$

when we use (17d).

If we collect these various results, we find the third condition (3.30).

In summary, the solutions discussed when explaining theorem 4.2 satisfy all of the desired matching and coherence conditions with results obtained up to the order considered.

4.4. CONCLUSION AND FINAL REMARKS

By specifying a perfect fluid flow, which allows the presence of shock wave (Σ), one can determine the internal structure of order zero*. From this result, it is possible to then without difficulty write down the shock conditions* of rank 1. If one knows the external flow of rank 1, the quantities which define

this flow on (Σ) will verify the rank 1 shock conditions. It is then possible to find a solution of the internal structure of rank 1 which is compatible with the given external flow. But this solution is not unique. It depends on an arbitrary constant. The difference between the two solutions decreases exponentially in ζ. It is then clear that the shock conditions of rank 2 depend on this constant, even though the explicit calculations were not given here.

This allows one to assume that this indetermination of the internal structure of rank 1 is undoubtedly limited to an overdetermination of the external flow of rank 2. But this is a suggestion which seems difficult to prove. The verification will then involve the properties of a global problem and goes beyond the scope of this article in which we only consider the local problem.

This does not affect the essential point of the results obtained nor their coherence. The shock conditions and the equations for the shock structure for any rank n, at least in principle, have been developed. The complete results were given explicitly for rank 1**.

NOTES ADDED UPON CORRECTING THE PROOFS

* The internal structure of order 0 is in fact determined up to a translation in ζ (see system 4.17). The effect of this translation obviously affects the evaluation of quantities such as ρ^*, p^*,.... Therefore, there is an indeterminancy (depending on the parameter) in writing down the order 1 shock conditions. This indeterminancy comes from the fact that we have specified the position of the shock at $z = 0$. In the global problem, the position of the shock is not arbitrary. When writing down the shock conditions, it is usual to find an indeterminancy which corresponds to the overabundance of conditions relative to the boundary problems which govern the external development.

** Since writing this manuscript in December 1960, the results of this work have been expanded to the case of non-steady flow and shock conditions were given explicitly using intrinsic notations (C.R.A. Sci. t.252, p. 1101-1103, February 20, 1961). Since /44
this time, we have become aware of the memoir by R. R. Chow and L. Ting: Higher order theory of curved shock, JAS 1961, p. 428, which discussed the same problem using methods which are very close in their technical details.

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