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CONVERGENCE OF GENERALIZED MUSCL SCHEMES

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CONVERGENCE OF GENERALIZED MUSCL SCHEMES

STANLEY OSHER

Abstract. Semi-discrete generalizations of the second order extension of Godunov's scheme, known as the MUSCL scheme, are constructed, starting with any three point "E" scheme. They are used to approximate scalar conservation laws in one space dimension. For convex conservation laws, each member of a wide class is proven to be a convergent approximation to the correct physical solution. Comparison with another class of high resolution convergent schemes is made.

I. Introduction and Preliminaries.

Recently, there has been an enormous amount of activity related to the construction and analysis of "high resolution" schemes approximating hyperbolic systems of conservation laws. Some examples of the successful consequences of this activity can be found in the proceedings of the latest (sixth) AIAA Computational Fluid Dynamics Conference, [1], [10], [18]. Extensive bibliographies can also be found in these papers.

Our aim here is merely to construct and prove the convergence of a subclass of these schemes approximating scalar convex conservation laws. This subclass is based on an idea of van Leer [17]. He christened his algorithms MUSCL (Monotonic Upstream Centered Schemes for Conservation Laws) schemes, and with mixed emotions, we shall use his acronym here.

In future work with S. Chakravarthy, we shall extend this construction to systems in multi-dimensions, using triangle-based algorithms. This work in progress will stress the computational aspects of the algorithms, especially as they relate to the Euler equations of compressible gas dynamics. In earlier work with the same author, [12], we constructed, and proved convergence of, a class of high resolution schemes approximating scalar convex conservation laws. We also showed for certain high resolution approximations to systems, that limit solutions satisfy an entropy inequality.

We shall consider numerical aproximations to the initial value problem for a single conservation law in one space dimension

(1.1)(a) $$w_t + f(w)_x = 0, \quad t > 0, \quad -1 \leq x \leq 1,$$

with a periodic boundary condition:

(b) $$w(x + 1,t) = w(x,t)$$

and initial condition:

(c) $$w(x,0) = w_0(x).$$

It is well-known that solutions of (1.1) may develop discontinuities in finite time, even when the initial data are smooth. Because of this, we seek a weak solution of (1.1), i.e. a bounded measurable func-

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tion w, such that, for all $\phi \in C_0(R \times R^+)$,

(1.2)(a) $$\iint_{R\times R^+} (w\phi_t + f(w)\phi_x)\, dx\, dt = 0$$

(b) $$\lim_{t\to 0} \| w(x,t) - w_0(x) \|_{L^1} = 0.$$

Solutions of (1.2) are not necessarily unique. For physical reasons, the limit solution of the viscous equation, as viscosity tends to zero, is sought. In the scalar case, this solution must satisfy, for $\phi \in C_0^{\infty}(R \times R^+)$, $\phi \geq 0$, and all real constants c:

(1.3)(a) $$\iint (|w - c|\phi_t + \mathrm{sgn}(w - c)(f(w) - f(c))\phi_x)\, dx\, dt \leq 0$$

This is equivalent to the statement:

(1.3)(b) $$\frac{\partial}{\partial t} |w - c| + \frac{\partial}{\partial x} ((f(w) - f(c))\, \mathrm{sgn}(w - c)) \leq 0,$$

in the sense of distributions.

Such solutions are called entropy solutions. Kruzkov has shown in [9], that two entropy solutions satisfy

(1.4) $$\| w(x,t_1) - u(x,t_1) \|_{L^1} \leq \| w(x,t_0) - u(x,t_0) \|_{L^1}$$

for all $t_1 \geq t_0$. Hence, (1.3) guarantees the uniqueness of solutions to the scalar version of (1.2). Existence was also obtained there.

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Equations (1.3) may be viewed as the statement

$$\frac{\partial}{\partial t} V(w) + \frac{\partial}{\partial x} F(w) \leq 0 \tag{1.5}$$

for each of the convex "entropy" functions of solutions w to (1.1):

$$V(w) = |w - c|$$

and their associated flux functions

$$F(w) = (f(w) - f(c)) \operatorname{sgn}(w - c).$$

For scalar <u>convex</u> conservation laws whose solutions lie in the space BV, a <u>single</u> entropy inequality (1.5) for any strictly convex $V(w)$ and its associated entropy flux $F(w)$, satisfying

$$F'(w) = V'(w)f'(w) \tag{1.6}$$

is enough to imply existence and uniqueness of solutions to (1.1). This follows from the results of DiPerna [3].

Next we consider a semi-discrete, method of lines, approximation to (1.1). We break the interval (-1,1) into subintervals:

$$I_j = \{x \mid (j - \tfrac{1}{2}) \Delta \leq x < (j + \tfrac{1}{2}) \Delta\}$$

$j = 0, \pm 1, \ldots, \pm N$, with $(2N + 1) \Delta = 2$.

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Let $x_j = j\Delta$, be the center of each interval I_j, with end points $x_{j-1/2}, x_{j+1/2}$.

Define the step function for each $t > 0$, as

$$U_\Delta(x,t) = u_j(t),$$

for $x \in I_j$.

The initial data is discretized via the averaging operator T_Δ,

$$T_\Delta w_0(x) = \frac{1}{\Delta} \int_{I_j} w_0(s)\, ds = u_j(0) \quad \text{for} \quad x \in I_j.$$

For any step function, we define the difference operators

$$\Delta_\pm u_j = \pm(u_{j\pm1} - u_j)$$

$$D_\pm u_j = \frac{1}{\Delta}\Delta_\pm u_j$$

A method of lines, conservation form, discretization of (1.1), is a system of differential equations

$$\frac{\partial}{\partial t} u_j + D_+ h_{j-1/2} = 0, \quad j = 0, \pm1, \ldots, \pm N \tag{1.8}$$

$$U_\Delta(x,0) = T_\Delta w_0(x), \quad \text{for} \quad x \in I_j.$$

Here, the numerical flux defined by:

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(1.9) $$h_{j-1/2} = h(u_{j+k-1},\ldots,u_{j-k}).$$

for $k \geq 1$, is a Lipschitz continuous function of its arguments, satisfies the consistency condition:

$$h(w,w,\ldots,w) = f(w).$$

It is well known that bounded a.e. limits, as $\Delta \to 0$, of approximate solutions converge to weak solutions of (1.1), i.e. (1.2)(a) is satisfied. However, this does not also imply that the limit solutions will satisfy any of the entropy conditions (1.5), let alone the general condition (1.3). Some restrictions on h are required.

A simple class of flux functions h, for which (1.8) converges, for all f, to the unique entropy solution in $L^{\infty}(L^1(R);[0,T])$, as $\Delta \to 0$, for any $T > 0$, is the class of "E" schemes introduced in [11]. Such schemes satisfy the following:

A consistent scheme whose numerical flux satisfies

(1.10) $$\mathrm{sgn}(u_j - u_{j-1})[h_{j-1/2} - f(u)] \leq 0$$

for all u between u_{j-1} and u_j is said to be an E scheme. This is at present, the most general class of schemes known to converge in the nonconvex case.

It is clear that this class includes the widely known class of

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three point monotone schemes, i.e. those for which

$$h_{j-1/2} = h(u_j, u_{j-1}),$$

with $h_{j-1/2}$ nonincreasing in its first argument, nondecreasing in its second.

We denote partial derivatives of a numerical flux via

$$\frac{\partial}{\partial u_{j+y}} h(u_{j+k}, \ldots, u_{j-k+1}) = h_y.$$

Thus a three point scheme with a differentiable flux function is monotone iff

$$h_1 \leq 0 \leq h_0.$$

Any numerical flux can be written

$$h_{j+1/2} = 1/2(f(u_j) + f(u_{j+1})) - 1/2 Q_{j+1/2}(\Delta_+ u_j) \tag{1.11}$$

where $Q_{j+1/2}$ can be viewed as the viscosity of the scheme [15].

One particular three point scheme is due to Godunov, [5], and has a special significance in this theory. The flux for Godunov's scalar scheme can be defined by

$$h^G_{j-1/2} = h_G(u_j, u_{j-1}) = \min_{u_{j-1} \leq u \leq u_j} f(u), \quad \text{if} \quad u_{j-1} \leq u_j \tag{1.12}$$

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$$= \max_{u_{j-1} \geq u \geq u_j} f(u), \quad \text{if} \quad u_{j-1} > u_j.$$

One can thus, [11], characterize E schemes as precisely those for which:

(1.13)(a) $$h_{j-1/2} \leq h^G_{j-1/2}, \quad \text{if} \quad u_j < u_{j-1}$$

(b) $$h_{j-1/2} \geq h^G_{j-1/2}, \quad \text{if} \quad u_j > u_{j-1},$$

or, as Tadmor [15] pointed out, those which have at least as much viscosity as Godunov's scheme

(1.14) $$Q^G_{j-1/2} \leq Q_{j-1/2}.$$

It follows from [11], Lemma (2.1), that these approximations are, at most, first order accurate.

Together with an entropy inequality, a key estimate involved in many convergence proofs, is a bound on the variation. For any fixed $t \geq 0$, the x variation of $U_\Delta(x,t)$ is

$$B(U_\Delta) = \sum_j |\Delta_+ u_j(t)|.$$

If we can write

(1.15)(a) $$\Delta_+ h_{j-1/2} = -C_{j+1/2} \Delta_+ u_j + D_{j-1/2} \Delta_- u_j$$

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(b) $$C_{j+1/2} \geq 0$$

(c) $$D_{j-1/2} \geq 0$$

then it is easy to show, [12], using an argument of [19], that, for $t_1 \geq t_2 \geq 0$

(1.16) $$B(U_{\Delta}(\cdot,t_1)) \leq B(U_{\Delta}(\cdot,t_2)).$$

Harten in [7], pointed out for explicit methods, that this decomposition could be obtained for schemes which are higher order accurate. In our present method of lines context, it involves a five point, consistent, approximation:

(1.17)(a) $$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta}\Delta_+ h_{j-1/2} = C_{j+1/2}\Delta_+ u_j - D_{j-1/2}\Delta_- u_j$$

with

(b) $$C_{j+1/2} = C(u_{j+2},u_{j+1},u_j,u_{j-1}) \geq 0$$

(c) $$D_{j-1/2} = D(u_{j+1},u_j,u_{j-1},u_{j-2}) \geq 0,$$

both Lipschitz continuous functions of their arguments. (See also van Leer [16].)

In addition to (1.16) we have a maximum principle for (1.17), [12],:

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$$\min_k u_k(0) \leq u_j(t) \leq \max_k u_k(0) \tag{1.18}$$

for each j and all $t \geq 0$.

Morover, in [12] we also showed a limit on the possible accuracy of approximations of the type (1.17). Any scheme of the type (1.17) is at most first order accurate at nonsonic critical points of u. (A sonic point $\bar{u}$ is one such that $f'(\bar{u}) = 0$). Thus, although schemes of this type can be made to be as high as third order accurate, Lipschitz continuity implies a local degeneracy to first order accuracy at smooth maxima and minima. This local degeneracy, together with some results on initial boundary value problems in [6], indicate strongly that overall second order accuracy is the best possible.

Following Harten [7], we call algorithms of the type (1.17), total variation diminishing, or TVD schemes.

It is known, that although (1.17) generates a compact family of solutions, $U_\Delta(x,t)$, in $L^{\infty}(L^1(R),[0,T])$, (if $w_0 \in L^1 \cap L^{\infty} \cap BV$), limit solutions may not satisfy any of the entropy conditions (1.15), and hence may not even be unique (1.12) See e.g. [4], [11], and [15]. One way of avoiding this difficulty is to use an inequality obtained in [11].

Let $V(w)$ be any convex function. We showed in [11], Section III, that, for any solution of any scheme (1.8):

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$$(1.19)\qquad \Delta\left(\frac{d}{dt}V(u_j) + D_+F_A(u_j)\right) = \int_{u_j}^{u_{j+1}} dw\; V''(w)[h_{j+1/2} - f(w)],$$

where the approximate entropy flux is defined through

$$(1.20)\qquad F_A(u_j) = F(u_j) + V'(u_j)[h_{j-1/2} - f(u_j)].$$

Thus, a sufficient condition that any limit solution satisfy (1.5) for a fixed convex V, is that

$$(1.21)\qquad \int_{u_j}^{u_{j+1}} dw\; V''(w)[h_{j+1/2} - f(w)] \leq 0$$

In order that the above inequality be valid for all convex V, it is necessary and sufficient that $h_{j+1/2}$ correspond to an E scheme, which implies again that the approximation be at most first order accurate. Thus, for the schemes to be constructed in the following section we shall only obtain our entropy inequality (1.21) for a single $V(w)$, say $V(w) = \frac{1}{2}w^2$. The following Theorem summarizes the technical hypotheses needed for convergence.

THEOREM 1.1. <u>The sequence of approximate solutions converges a.e. as $\Delta \to 0$, to the unique solution of the scalar, convex conservation law (1.1) provided that the initial data is in BV and that inequalities (1.17) and (1.21) are valid for a single convex</u> $V(w)$.

The proof is analogous to that of Theorem (4.1) in [12].

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II. Construction of TVD MUSCL Schemes. The MUSCL scheme as discussed e.g., in [2] and [8] is a second order accurate extension of Godunov's method [5] that is based on ideas expressed by van Leer in [16].

We assume that $u_j(t)$ is known. The first step in MUSCL is the reconstruction of a piecewise linear description of the solution. In the interval I_j the result of this operation is

$$(2.1) \qquad z(x,t) = u_j(t) + (x - x_j)s_j(t), \quad \text{for} \quad (x - x_j) < \Delta/2.$$

Here $s_j(t)$ are slopes that satisfy

$$(2.2) \qquad s_j = w_x(x_j,t) + O(\Delta)$$

subject to some monotonicity constraints discussed below.

Godunov's scheme, in our semi-discrete context, is the following. The numerical flux $h^G(u_{j+1},u_j)$ is computed by solving the Riemann problem, i.e. the initial value problem (1.1) with initial data

$$w = u_j(t), \quad \text{for} \quad x < x_{j+1/2}$$

$$w = u_{j+1}(t), \quad \text{for} \quad x \geq x_{j+1/2}.$$

The resulting unique entropy condition satisfying solution is a function of the type

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$$w\left[\frac{x - x_{j+1/2}}{t - t_{j+1/2}}\right] = w(\zeta).$$

A closed form for w was recently obtained in [11].

Then Godunov's flux is defined through

$$h^G_{j+1/2} = f(w(0)) \tag{2.3}$$

which is the same as (1.12).

A fully discrete, explicit in time, Godunov scheme has the same numerical flux, after we impose a CFL restriction which prevents the interaction of solutions of adjacent Riemann problems The literal second order accurate extension to this fully discrete approach would be to compute the solution to the initial value problem (1.1), with initial data

$$w = u_j(t) + (x - x_j)s_j(t), \quad \text{for} \quad x < x_{j+1/2}$$

$$= u_{j+1}(t) + (x - x_j)s_{j+1}(t), \quad \text{for} \quad x \geq x_{j+1/2}.$$

Find $f(w(x_{j+1/2},s))$ for $t + \lambda\Delta \geq s \geq t$, with $\lambda > 0$ satisfying a CFL restriction, then compute

$$\frac{1}{\lambda\Delta}\int_t^{t+\lambda\Delta} f(w(x_{j+1/2},s))\,ds. \tag{2.4}$$

This is the MUSCL numerical flux.

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Unfortunately, obtaining the exact solution to this nonlinear initial value problem with piecewise linear initial data is a non-trivial business, even in the scalar case. However, at $s = t^+$, $x = x_{j+1/2}$, the solution is the same as for the Riemann problem with initial data:

$$w = u_j(t) + \frac{\Delta}{2} s_j(t), \quad \text{for} \quad x < x_{j+1/2}$$

$$w = u_{j+1} - \frac{\Delta}{2} s_{j+1}(t), \quad \text{for} \quad t \geq x_{j+1/2}$$

which is easily calculated, e.g., from the formula in [11]. Thus, the semi-discrete MUSCL extension of Godunov's algorithm comes through

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta} \Delta_- h^G(u_{j+1} - \frac{\Delta}{2} s_{j+1}, u_j + \frac{\Delta}{2} s_j). \tag{2.5}$$

We generalize this still further as follows. Let $h(u_{j+1}, u_j) = h_{j+1/2}$ be an arbitrary first order accurate numerical flux function. Then our generalized MUSCL algorithm is merely:

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta} \Delta_- h(u_{j+1} - \frac{\Delta}{2} s_{j+1}, u_j + \frac{\Delta}{2} s_j). \tag{2.6}$$

We have:

LEMMA 2.1. <u>At points</u> $w(x_j, t) = u_j(t)$ <u>in a neighborhood of which</u> $h(u_{j+1}, u_j)$ <u>is</u> C^2 <u>with Lipschitz continuous partial derivatives, and</u> $\frac{\Delta s_{j+1}}{\Delta_+ u_j} = 1 + O(\Delta) = \frac{\Delta s_j}{\Delta_+ u_j}$, <u>the algorithm (2.6) is at least second order</u>

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accurate for smooth functions w.

Proof. Let

$$u_{j+1/2} = \tfrac{1}{2}(u_{j+1} + u_j). \tag{2.7}$$

We shall show:

$$\Delta_- \left(h\left(u_{j+1} - \frac{\Delta}{2} s_{j+1}, u_j + \frac{\Delta}{2} s_j\right) - f(u_{j+1/2})\right) = o(\Delta^2) \tag{2.8}$$

at these points.

Using consistency of $h(u_{j+1}, u_j)$, the left side of (2.8) can be written as

$$\Delta_- \left[\int_0^1 h_1\left[u_{j+1/2} + \gamma \frac{\Delta_+ u_j}{2}\left[1 - \frac{\Delta s_{j+1}}{\Delta_+ u_j}\right], u_j + \frac{\Delta}{2} s_j\right] d\gamma \frac{\Delta_+ u_j}{2}\left[1 - \frac{\Delta s_{j+1}}{\Delta_+ u_j}\right]\right.$$

$$\left. + \int_0^1 h_0\left[u_j + \frac{\Delta}{2} s_j, u_{j+1/2} + \gamma \frac{\Delta_+ u_j}{2}\left[\frac{\Delta s_j}{\Delta_+ u_j} - 1\right]\right] d\gamma \left[\frac{\Delta_+ u_j}{2}\left[\frac{\Delta s_j}{\Delta_+ u_j}\right] - 1\right]\right].$$

The result is immediate.

Next we make an observation about the viscosity of this scheme, $Q_{j+1/2}$, as defined in (1.11).

LEMMA 2.2. If h is monotone, then the viscosity of algorithm (2.6) is a decreasing function of $\frac{s_{j+1}}{\Delta_+ u_j}$ and $\frac{s_j}{\Delta_+ u_j}$.

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The proof comes directly from (1.11).

Thus, as expected, if $\frac{s_j}{\Delta_+ u_j}$ and $\frac{s_{j+1}}{\Delta_+ u_j}$ are both restricted to be positive, the most viscous, hence the least accurate, case occurs when these values are both zero. The scheme then degenerates to first order accuracy. We can compress or smear out the solution locally by increasing or decreasing these ratios.

We wish this scheme to be TVD, which will give a natural restrictions on these above ratios.

LEMMA 2.3. <u>If h is a flux corresponding to an E scheme, then the scheme is TVD if</u>

$$0 \leq \frac{\Delta s_j}{\Delta_+ u_j}, \frac{\Delta s_{j+1}}{\Delta_+ u_j} \leq 1 \tag{2.9}$$

<u>for each</u> j.

<u>Proof</u>. We have

$$\frac{1}{\Delta}\Delta_+ h(u_j - \frac{\Delta}{2} s_j, u_{j-1} + \frac{\Delta}{2} s_{j-1}) \tag{2.10}$$

$$= \frac{1}{\Delta}[h(u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta}{2} s_j) - f(u_j + \frac{\Delta}{2} s_j)$$

$$+ f(u_j + \frac{\Delta}{2} s_j) - h(u_j - \frac{\Delta s_j}{2}, u_j + \frac{\Delta}{2} s_j)]$$

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$$+ \frac{1}{\Delta}[h(u_j - \frac{\Delta s}{2}, u_j + \frac{\Delta}{2} s_j) - f(u_j - \frac{\Delta s_j}{2})$$

$$+ f(u_j - \frac{\Delta s_j}{2}) - h(u_j - \frac{\Delta s_j}{2}, u_{j-1} + \frac{\Delta}{2} s_{j-1})]$$

Thus, in (1.15) we can write:

$$\text{(2.11)(a)} \quad C_{j+1/2} = - \frac{1}{\Delta (\Delta_+ u_j)} [(u_{j+1} - \frac{s_{j+1}}{2} \Delta u_j + \frac{\Delta}{2} s_j) - f(u_j + \frac{\Delta}{2} s_j)$$

$$+ f(u_j + \frac{\Delta}{2} s_j) - h(u_j - \frac{\Delta}{2} s_j, u_j + \frac{\Delta}{2} s_j)]$$

$$\text{(b)} \quad D_{j-1/2} = \frac{1}{\Delta (\Delta_- u_j)} [h(u_j - \frac{\Delta s_j}{2}, u_j + \frac{\Delta}{2} s_j) - f(u_j - \frac{\Delta}{2} s_j)$$

$$+ f(u_j - \frac{\Delta}{2} s_j) - h(u_j - \frac{\Delta}{2} s_j, u_{j-1} + \frac{\Delta}{2} s_{j-1})]$$

Both $C_{j+1/2}$, $D_{j-1/2}$ are nonnegative because of (1.10) and (2.9).

The restrictions (2.9) can be relaxed somewhat if h is monotone; we have:

LEMMA 2.4. <u>If h is a flux corresponding to a monotone scheme, then the scheme is TVD if</u>

$$\text{(2.12)(a)} \qquad 1 \geq \frac{(s_{j+1} - s_j)\Delta}{2 \Delta_+ u_j} \quad \text{near points where} \quad h_1 \neq 0$$

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(b) $$1 \geq \frac{(-s_{j+1} + s_j)\,\Delta}{2\,\Delta_+ u_j} \quad \text{near points where} \quad h_0 \neq 0$$

<u>for each</u> j.

<u>Proof</u>. We have

$$(2.13) \quad \frac{1}{\Delta}\Delta_+ h(u_j - \frac{\Delta}{2} s_j, u_{j-1} + \frac{\Delta}{2} s_{j-1})$$

$$= \frac{1}{\Delta}\,[h(u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta s_j}{2}) - h(u_j - \frac{\Delta s_j}{2}, u_j + \frac{\Delta s_j}{s})]$$

$$+ \frac{1}{\Delta}\,[h(u_j - \frac{\Delta s_j}{2}, u_j + \frac{\Delta s_j}{2}) - h(u_j - \frac{\Delta s_j}{2}, u_{j-1} + \frac{\Delta s_{j-1}}{2})].$$

Thus, in (1.15), we can write:

$$(2.14)(a) \quad C_{j+1/2} = -\frac{1}{\Delta\,(\Delta_+ u_j)}\, h_1(\tilde{u}_j, u_j + \frac{\Delta s_j}{2})[\Delta_+ u_j - \frac{\Delta}{2}(s_{j+1} - s_j)]$$

$$(b) \quad D_{j-1/2} = \frac{1}{\Delta\,(\Delta_- u_j)}\, h_0\,(u_j - \frac{\Delta s_j}{2}, \tilde{u}_j)[\Delta_- u_j + \frac{\Delta}{2}(s_j - s_{j-1})]$$

Both $C_{j-1/2}$ and $D_{j-1/2}$ are nonnegative because of monotonicity and (2.12).

<u>Remark 2.5</u>. These last two Lemmas can be made "local," i.e. in regions of monotonicity we may relax the restrictions from (2.9) to (2.12).

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III. Convergence of MUSCL Schemes. In order to prove convergence for convex f, we need only verify the discrete entropy inequality (1.21) for the schemes constructed in the previous section (Theorem (1.1)). We shall do this for the entropy

$$V(w) = \tfrac{1}{2} w^2$$

and for schemes satisfying the TVD hypothesis of Lemma (2.3), in addition to some other restrictions near sonic shock points.

Suppose $u_j < u_{j+1}$, i.e. we are at a rarefaction. We then have, letting $u_{j+1/2} = \tfrac{1}{2}(u_j + u_{j+1})$ as above

$$(3.1) \quad \int_{u_j}^{u_{j+1}} dw\left[h\left(u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta s_j}{2}\right) - f(w)\right]$$

$$= (\Delta_+ u_j)\left(h\left(u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta s_j}{2}\right) - f(u_{j+1/2})\right)$$

$$+ \int_{u_j}^{u_{j+1}} dw[f(u_{j+1/2}) - f(w)]$$

$$= [I] + [II].$$

Now, using the hypotheses of Lemma (2.3), we have:

$$(3.2) \qquad [I] \leq 0$$

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Next we have

$$(3.3)\quad [II] = \int_{u_j}^{u_{j+1}} dw \int_w^{u_{j+1/2}} \left\{\frac{\partial}{\partial s}(s - w)\right\} f'(s)\, ds$$

$$= \int_{u_j}^{u_{j+1}} dw\, (u_{j+1/2} - w)\, f'(u_{j+1/2})$$

$$- \int_{u_j}^{u_{j+1}} dw \int_w^{u_{j+1/2}} (s - w)\, f''(s)\, ds$$

$$= - \int_{u_j}^{u_{j+1}} dw \int_w^{u_{j+1/2}} (s - w)\, f''(s)\, ds$$

$$= - \tfrac{1}{2} \int_{u_j}^{u_{j+1}} ds\, f''(s)\, (\min|s - u_{j+1}|, |s - u_j|)^2 < 0.$$

Suppose $u_j > u_{j+1}$, i.e. we are at a shock. Let $\tilde{u}_{j+1/2}$ be chosen so that

$$\int_{u_j}^{u_{j+1}} f'(w)(w - \tilde{u}_{j+1/2})\, dw = 0$$

i.e.

$$(3.4)\qquad \tilde{u}_{j+1/2} = \frac{\int_{u_j}^{u_{j+1}} wf'(w)\, dw}{\int_{u_j}^{u_{j+1}} f'(w)} = \frac{\Delta_+ F(u_j)}{\Delta_+ f(u_j)}$$

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where $F(u)$ is the entropy flux corresponding to our entropy, $V(u) = \frac{1}{2}u^2$. Of course we need the denominator in (3.4) to be nonvanishing. In fact we restrict ourselves to the situation where

$$u_{j+1} \leq \bar{u}_{j+\frac{1}{2}} \leq u_j. \tag{3.5}$$

We note

$$\bar{u}_{j+\frac{1}{2}} - u_{j+\frac{1}{2}} = \frac{\int_{u_j}^{u_{j+1}} (w - u_{j+\frac{1}{2}})\, f'(w)\, dw}{\Delta_+ f(u_j)} \tag{3.6}$$

$$= \frac{\int_{u_j}^{u_{j+1}} (w - u_{j+\frac{1}{2}})\, (f'(w) - f'(u_{j+\frac{1}{2}}))\, dw}{\Delta_+ f(u_j)}$$

Thus, we have

$$|\bar{u}_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}| < \frac{1}{6} (\Delta_+ u_j)^2 \frac{\sup\limits_{u_{j+1} \leq u \leq u_j} f''(u)}{|s_{j+\frac{1}{2}}|}, \tag{3.7}$$

where $s_{j+\frac{1}{2}} = \dfrac{\Delta_+ f(u_j)}{\Delta_+ u_j}$.

We further restrict the s_j so that

$$0 \leq -\Delta s_j \leq 2 \max(\min((u_j - \bar{u}_{j+\frac{1}{2}}), (\bar{u}_{j-\frac{1}{2}} - u_j)), 0). \tag{3.8}$$

In view of (3.7), this restriction does not affect the second order

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accuracy of the scheme, except at sonic points.

We now write

$$(3.9)\int_{u_j}^{u_{j+1}} dw\left[h\left[u_{j+1} - \frac{\Delta s_{j+1}}{2},\ u_j + \frac{\Delta s_j}{2}\right] - f(w)\right]$$

$$= \Delta_+ u_j\left[h\left[u_{j+1} - \frac{\Delta s_{j+1}}{2},\ u_j + \frac{\Delta s_j}{2}\right] - f(\bar{u}_{j+1/2})\right]$$

$$+ \int_{u_{j+1}}^{u_j} dw\ [f(w) - f(\bar{u}_{j+1/2})]$$

$$= [III] + [IV].$$

Now, the assumptions (3.5), (3.8) imply

$$(3.10) \qquad [III] \leq 0.$$

An argument analagous to that of (3.3) suffices to show that $[IV] \leq 0$.

Finally, suppose $u_{j+1} < u_j$ and (3.5) is violated. The inequalities of Lemma (2.1), and (3.8) imply that s_j and s_{j+1} vanish. Then inequality (1.21) is immediate, since h is an E flux.

We may summarize all this in

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THEOREM 3.1. The sequence of approximate solutions satisfying (2.6) converges a.e. to the unique solution of the scalar convex conservation law (1.1) provided that the initial data is in B.V. and that for each j

$$0 \leq \frac{\Delta s_j}{\Delta_+ u_j} \leq 1$$

$$0 \leq \frac{\Delta s_j}{\Delta_- u_j} \leq 1$$

and, in addition if $u_j > u_{j+1}$,

$$-\Delta s_j \leq 2 \max (\min ((u_j - \bar{u}_{j+1/2}), \bar{u}_{j-1/2} - u_j)), 0).$$

IV. Steady Discrete Shocks. We now check for the existence of discrete, steady, shock solutions to the convergent MUSCL schemes constructed in the previous section, based on the Engquist-Osher flux, [4]. Since the scheme satisfies a maximum principle (Lemma (2.1) of [12]), any profile must be monotone.

Let $f''(u) > 0$, and for simplicity, we take $f'(0) = 0 = f(0)$. In this case, the monotone E-O scheme becomes

(4.1)(a) $\quad h^{EO}(u_{j+1}, u_j) = f_-(u_{j+1}) + f_+(u_j)$

where

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(b) $$f_-(u) = f(u) \quad \text{if} \quad u < 0$$

$$f_-(u) = 0 \quad \text{if} \quad u \geq 0$$

(c) $$f_+(u) = f(u) \quad \text{if} \quad u > 0$$

$$f_+(u) = 0 \quad \text{if} \quad u < 0$$

Let u^L, u^R be the left and right states for a physically correct, steady shock, i.e.

$$f(u^L) = f(u^R), \quad u^L > 0 > u^R.$$

A steady discrete shock $\{u_j\}$ will satisfy

(4.2)(a) $$\lim_{j \to -\infty} u_j = u^L$$

(b) $$\lim_{j \to +\infty} u_j = u^R$$

(c) $$h^{EO}\left[u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta s_j}{2}\right] = f(u^L)$$

$$= f_-\left[u_{j+1} - \frac{\Delta s_{j+1}}{2}\right] + f_+\left[u_j + \frac{\Delta s_j}{2}\right]$$

for all j.

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We shall seek steady, discrete, shocks of the same general form obtained in [4] for the E-O scheme, c.f. also [12], section V.

These are

(4.3)(a) $$u_j = u^L, \quad j \leq -1$$

(b) $$u_j = u^R, \quad j \geq 2$$

(c) $$u^L \geq u_0 \geq 0 > u_1 \geq u^R.$$

For the first order scheme, u_0 can be viewed as a smooth function of u_1 satisfying the above, and, in addition

$$f(u_0) + f(u_1) = f(u_L)$$

For the present scheme, we shall get a different one parameter family of intermediate states.

It follows for all $j \neq 0$, that (4.3) implies (4.2)(c).

For $j = 0$, we have the following equation:

(4.4) $$0 = f_-\left[u_1 - \frac{\Delta s_1}{2}\right] + f_+\left[u_0 + \frac{\Delta s_0}{2}\right] - f(u^L) = G(u_1, u_0)$$

One special solution is, again, $u_1 = u^R$, $u_0 = 0$.

For (u_0, u_1) close to $(0, u^R)$ and satisfying (4.3)(c), we have,

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from the hypotheses of Theorem (3.1):

$$(4.5) \qquad s_1 = a_{1/2} \frac{(u^R - u_1)}{\Delta}$$

$$s_0 = 0$$

Here $0 \leq a_{1/2}$ is assumed, in addition, to be a C^2 function of u_1 near $u_1 = u^R$, with $\frac{\partial a_{1/2}}{\partial u_1} = 0$.

A straightforward calculation gives us

$$(4.6)(a) \qquad \frac{\partial G}{\partial u_1}(u^R, 0) = f'(u_R)\left[1 + \frac{a_{1/2}}{2}\right] < 0$$

$$(b) \qquad \frac{\partial G}{\partial u_0}(u^R, 0) = 0$$

$$(c) \qquad \frac{\partial^2 G}{\partial u_1^2}(u^R, 0) = f''_-(u_R)\left[1 + \frac{a_1}{2}\right]^2 > 0$$

$$(d) \qquad \frac{\partial^2 G}{\partial u_0 \partial u_1}(u^R, 0) = 0$$

$$(e) \qquad \frac{\partial^2 G}{\partial u_0^2} = f''(0) > 0$$

A simple application of the implicit function theorem gives us:

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THEOREM 4.1. Given the hypotheses of Theorem (3.1) and the mild technical assumption concerning $a_{1/2}$, there exists a family of sharp, discrete, shock solutions to (2.6) of the type (4.3), with u_1 a smooth function of u_0, and $0 < u_0$, small enough. By symmetry, the same is true, with u_0 a smooth function of $u_1 < 0$, and $-u_1$ small enough.

V. Comparison With Other Convergent High Resolution Schemes. In [12] we constructed a class of convergent high resolution schemes, using flux limiters and second order accurate upwinding. Our first step was to use a first order E scheme flux, $h(u_{j+1},u_j)$, to construct a second order accurate TVD scheme. The resulting scheme is

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta}\Delta_- H(u_{j+2}, u_{j+1}, u_j, u_{j-1}), \tag{5.1}$$

with

$$H(u_{j+2}, u_{j+1}, u_j, u_{j-1}) = h(u_{j+1}, u_j) - \tfrac{1}{2}\,\psi\,(R^-_{j+1})(h(u_{j+1}, u_j) - f(u_j)) + \tfrac{1}{2}\,\psi(R^+_j)(f(u_{j+1}) - h(u_{j+1}, u_j)). \tag{5.2}$$

Here

$$R^+_j = \frac{f(u_j) - h(u_j, u_{j-1})}{f(u_{j+1}) - h(u_{j+1}, u_j)} \tag{5.3}$$

$$R^-_j = \frac{h(u_{j+1},u_j) - f(u_j)}{h(u_j, u_{j-1}) - f(u_{j-1})}$$

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The slope limiter $\psi(R)$ satisfies $\psi(1) = 1$ and the following inequalities

(5.4)(a) $$1 + \tfrac{1}{2}\left[\frac{\psi(R_j^-)}{R_j^-} - \psi(R_{j+1}^-)\right] \geq 0$$

(b) $$1 + \tfrac{1}{2}\left[\frac{\psi(R_j^+)}{R_j^+} - \psi(R_{j-1}^+)\right] \geq 0$$

Various slope limiters have been developed. See Sweby [13], for a numerical and theoretical analysis of their properties. Perhaps the simplest example is the so called min-mod, [14]:

(5.5) $$\psi(R) = \max(0, \min(R, 1)).$$

We have the following analogue of Lemma (2.2):

LEMMA 5.1. <u>If</u> h <u>is an E flux, then the viscosity of (5.1) is a decreasing function of</u> ψ.

Next we compare the schemes (2.6) and (5.1). If $f'(w) \neq 0$ and w is smooth at x_j, then it is easy to see that

$$H(u_{j+2}, u_{j+1}, u_j, u_{j-1}) = h\left[u_{j+1} - \frac{\Delta s_{j+1}}{2}, u_j + \frac{\Delta s_j}{2}\right] + O((\Delta_- u_j)^2 + (\Delta_+ u_j)^2)$$

if

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(5.6) $\Delta s_j = \psi(R_j^-)\,\Delta_- u_j + O((\Delta_- u_j)^2 + (\Delta_+ u_j)^2)$, if $h_1(w,w) \neq 0$

and/or

$$\Delta s_j = \psi(R_j^+)\,\Delta_+ u_j + O((\Delta_- u_j)^2 + (\Delta_+ u_j)^2), \quad \text{if} \quad h_0(w,w) \neq 0$$

Dropping the quadratic terms, inequality (2.9) gives

(5.7)(a) $$0 \leq \frac{\psi(R_j^-)}{R_j^-}, \quad \psi(R_j^-) \leq 1 \quad \text{if} \quad h_1(w,w) \neq 0$$

(b) $$0 \leq \frac{\psi(R_j^+)}{R_j^+}, \quad \psi(R_j^+) \leq 1 \quad \text{if} \quad h_0(w,w) \neq 0$$

These inequalities are a bit more restrictive than those needed in the monotone case. There we have

(5.8)(a) $$1 \geq \tfrac{1}{2}\left[-\frac{\psi(R_j^-)}{R_j^-} + \psi(R_{j+1}^-)\right] \quad \text{if} \quad h_1(w,w) \neq 0$$

(b) $$1 \geq \tfrac{1}{2}\left[-\frac{\psi(R_j^+)}{R_j^+} + \psi(R_{j-1}^+)\right] \quad \text{if} \quad h_0(w,w) \neq 0$$

which are the same as inequalities (5.4)(a),(b).

For the entropy condition to be proven valid, the flux was modified. We replaced H by H^{ac} (where a.c. stands for artificial

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compression), as follows.

First we modify the flux differences in (5.2):

(5.9)(a) $$(h(u_{j+1}, u_j) - f(u_j))^M$$

$$= (h(u_{j+1}, u_j) - f(u_j)) \left[1 + \frac{a^-_{j+1/2}(\Delta_+ h_1(u_j, u_j))(\Delta_+ u_j)}{h(u_{j+1}, u_j) - f(u_j)} \right]$$

(b) $$(f(u_{j+1}) - h(u_{j+1}, u_j))^M$$

$$= (f(u_{j+1}) - h(u_{j+1}, u_j)) \left[1 - \frac{a^+_{j+1/2}(\Delta_+ h_0(u_j, u_j))\Delta_+ u_j}{f(u_{j+1}) - h(u_{j+1}, u_j)} \right]$$

Here $a^{\pm}_{j+1/2}$ are both positive, chosen first so that the quantities in the brackets in (5.9) are between 0 and 2, (inequalities (5.13) below).

We next let:

$$R_j^{M+} = \frac{(f(u_j) - h(u_j, u_{j-1}))^M}{(f(u_{j+1}) - h(u_{j+1}, u_j))^M}$$

$$R_j^{M-} = \frac{(h(u_{j+1}, u_j) - f(u_j))^M}{(h(u_j, u_{j-1}) - f(u_{j-1}))^M}$$

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Then we define our scheme:

$$\frac{\partial u_j}{\partial t} = -\frac{1}{\Delta}\Delta_- H^{ac}(u_{j+2}, u_{j+1}, u_j, u_{j-1}) \tag{5.10}$$

with

$$H^{ac}(u_{j+2}, u_{j+1}, u_j, u_{j-1}) \tag{5.11}$$

$$= h(u_{j+1}, u_j) - \tfrac{1}{2}\,\psi(R^{M}_{j+1}^{-})\,(h(u_{j+1}, u_j) - f(u_j))^M$$

$$+ \tfrac{1}{2}\,\psi(R^{M}_{j}^{+})\,(f(u_{j+1}) - h(u_{j+1}, u_j))^M.$$

The resulting scheme is convergent if the following inequalities are valid:

$$\int_{u_j}^{u_{j+1}} f''(w)dw\,((\tfrac{1}{2}(\Delta_+ u_j))^2 - (w - \tfrac{1}{2}(u_{j+1} + u_j))^2) \tag{5.12}$$

$$- a^-_{j+1/2}\,(\Delta_+ u_j)^2\,[h_1(u_{j+1}, u_{j+1}) - h_1(u_j, u_j)]$$

$$- a^+_{j+1/2}\,(\Delta_+ u_j)^2\,[h_0(u_{j+1}, u_{j+1}) - h_0(u_j, u_j)] \le 0$$

$$|a^-_{j+1/2}(h_1(u_{j+1}, u_{j+1}) - h_1(u_j, u_j))| \le -\frac{(h(u_{j+1}, u_j) - f(u_j))}{\Delta_+ u_j} \tag{5.13(a)}$$

$$|a^+_{j+1/2}(h_0(u_{j+1}, u_{j+1}) - h_0(u_j, u_j))| \le \frac{f(u_{j+1}) - h(u_{j+1}, u_j)}{\Delta_+ u_j}, \tag{b}$$

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and if $\psi(R)$ is defined by (5.5).

Examples are given in [12].

It is interesting that this case allows compression to be added at shocks, i.e. the restrictions on the $a^{\pm}_{j+1/2}$ allow negative viscosity to be added to (5.2) if $u_j > u_{j+1}$, while, again for the proof of convergence, positive viscosity must be added for rarefactions, $u_j < u_{j+1}$. In the present MUSCL case the situation is (unnaturally) reversed. A comparison of the convergent MUSCL scheme with this one yields two main points:

(1) Away from sonic points, the fluxes differ only in $O((\Delta_{\pm} u_j)^3)$ terms

(2) At sonic points, the fluxes differ in $O((\Delta_{\pm} u_j)^2)$ terms.

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