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STATISTICAL MODELING OF SPACE SHUTTLE ENVIRONMENTAL DATA

Final Report

by

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STATISTICAL MODELING OF SPACE SHUTTLE ENVIRONMENTAL DATA

FINAL REPORT
(For Period July 24, 1981 - October 23, 1983)

Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

Under Contract NAS8-34502

Prepared by

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Acknowledgements</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Preface</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reports</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A Note on the Ratio of Positively Correlated Gamma Variates</td>
<td>1</td>
</tr>
<tr>
<td>J. D. Tubbs and O. E. Smith</td>
<td></td>
</tr>
<tr>
<td>2. A Method for Determining if Unequal Shape Parameters are Necessary</td>
<td>13</td>
</tr>
<tr>
<td>J. D. Tubbs</td>
<td></td>
</tr>
<tr>
<td>3. A Differential Equations Approach to the Modal Location for a Family of Bivariate Gamma Distribution</td>
<td>35</td>
</tr>
<tr>
<td>D. W. Brewer, J. D. Tubbs, and O. E. Smith</td>
<td></td>
</tr>
<tr>
<td>4. Analysis of Wind Gust Data</td>
<td>62</td>
</tr>
<tr>
<td>J. D. Tubbs</td>
<td></td>
</tr>
</tbody>
</table>
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J. D. Tubbs and D. W. Brewer
Principal Investigators
PREFACE

The report contains the results of several papers related to modeling using a class of the bivariate gamma distribution. The separate papers contain loosely related subjects pertaining to this problem. Since the separate papers were prepared at different times during the contract period and have been submitted for publication in the open literature and each paper is intended to be self-contained, there is some redundancy in tables and illustrations.

Each of the papers in this report were extensions and/or generalizations of the results given in NASA TM-82483, entitled "A Bivariate Gamma Probability Distribution with Application to Gust Modeling," by O. E. Smith, S. I. Adelfang, and J. D. Tubbs. A modification of this paper is currently under review by Communications in Statistics.

The first paper in this report, entitled "A Note on the Ratio of Positively Correlated Gamma Variates," has been accepted for publication in Communications in Statistics and it presents some new analytical results using a class of the bivariate gamma distribution. Comparable results were available in the open literature using a different class of the bivariate gamma.

The second paper is entitled "A Method for Determining if Unequal Shape Parameters are Necessary in a Bivariate Gamma Distribution" and is an application of the results given in the first paper and addresses questions concerning
hypothesis tests for equality of shape parameters from correlated gamma distributed variates. This paper is currently under review by Technometrics.

The third paper, entitled "A Differential Equations Approach to the Modal Location for a Family of Bivariate Gamma Distribution," contains extensive analytical results for the location of the mode as a function of the free parameters. To the authors' knowledge this is the only such representation for a non-gaussian bivariate distribution. This paper has been submitted to SIAM J. on Scientific and Statistical Computing.

The fourth paper is a report summarizing the analysis of some wind gust data using the analytical results developed in relationship to the modeling application.
A NOTE ON THE RATIO OF POSITIVELY CORRELATED GAMMA VARIATES

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ABSTRACT

Mielke and Flueck (1976) derived the density function and corresponding moments for the ratio of correlated gamma distributed variates. They considered a class of bivariate gamma distributions suggested by Cherian (1941) and David and Fix (1961). Recently, Lee, Holland, and Flueck (1979) derived some additional distributional results using this class of functions. This paper derives similar results using a different class of bivariate gamma distributions.

1. INTRODUCTION

Mielke and Flueck (1976) derived the distributional results for the ratio, R, of correlated gamma distributed variables. There are several classes of the bivariate gamma distribution [three are summarized in Mardia (1970) and an additional two in Johnson and
Kotz (1972). Mielke and Flueck (1976) derived the distributional results for the ratio, \( R \), of correlated gamma distributed variables using the Cherian-David-Fix class of bivariate gamma random variables [Cherian (1941) and David and Fix (1961)]. That is, let \( X, Y, \) and \( P \) denote independent gamma random variables with common scale parameter \( \lambda \) and respective shape parameters \( \alpha = \xi, \beta = \xi, \) and \( \xi, \) for \( 0 < \xi < \min(\alpha, \beta) \).

Then it can be shown that the bivariate probability density function for \( U = X + P \) and \( V = Y + P \) is given by

\[
f_{U,V}(u,v) = \frac{\exp(-uv)}{K} \int_0^{\min(u,v)} p^{\xi-1}(u-p)^{\alpha-1}(v-p)^{\beta-1}e^p dp \quad (1.1)
\]

for \( K = \Gamma(\alpha-\xi)\Gamma(\beta-\xi)\Gamma(\xi) \)

when the scale parameter \( \lambda \) is assumed to be unity. Mielke and Flueck (1976) showed that (1.1) can be written as

\[
f_{U,V}(u,v) = \begin{cases} 
\frac{\alpha-1}{\Gamma(\alpha)} \frac{\beta-1}{\Gamma(\beta)} \frac{e^{-(u+v)}}{\Gamma(\alpha)\Gamma(\beta-\xi)} F_1^*(\xi,1+\xi-\alpha;\beta,u/v,-u) & \text{if } 0 < u < v \\
\frac{\alpha-1}{\Gamma(\alpha)} \frac{\beta-1}{\Gamma(\beta)} \frac{e^{-(u+v)}}{\Gamma(\alpha-\xi)\Gamma(\beta)} F_1^*(\xi,1+\xi-\beta;\alpha,v/u,-v) & \text{if } 0 < v < u 
\end{cases} \quad (1.2)
\]

where \( F_1^*(a,b,c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(b)_n}{(c)_{m+n}} x^m y^n \) is a "degenerate" two variable hypergeometric function [Gradshteyn and Ryzhik (1967), p. 1067] and \( (a)_n = \Gamma(a+n)/\Gamma(a) \). Thus, \( U \) and \( V \) are gamma random variables with shape parameters \( \alpha \) and \( \beta \) and positive dependence parameter \( \xi \). In particular, \( E(U) = Var(U) = \alpha, E(V) = Var(V) = \beta, \) and \( Cov(U,V) = \xi \).

Mielke and Flueck (1976) derived the density function for \( R = U/V \) using a change of variables. That is,
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\begin{align*}
  f_R(r) = \begin{cases} 
  \frac{\Gamma(x+\alpha-\xi, 1+\xi-\alpha, r/1+r, r)}{B(\alpha-\xi, \beta)} & \text{if } 0 < r < 1 \\
  \frac{\Gamma(x+\alpha-\xi, 1+\xi-\alpha, 1/r, 1/1+r)}{B(\alpha-\xi, \beta)} & \text{if } 1 < r
\end{cases}
\end{align*}

(1.3)

where \( F_1(a,b,c,d;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m} (c)_{n}}{(d)_{m+n} m! n!} x^m y^n, |x| < 1, |y| < 1 \) is a two variable hypergeometric function [Gradshteyn and Ryzhik, (1967), p. 1053], and \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \). In addition, they show that the integral moments of \( R \) are given by

\begin{align*}
  E(R^s) = \sum_{j=0}^{s} \frac{(a-\xi)(\xi)}{(\beta-j)_{s-j}} \frac{1}{(\beta-1)^{s-1}} \quad \text{for } s \geq 0.
\end{align*}

(1.4)

In particular,

\begin{align*}
  E(R) = \frac{a\beta-\xi}{\beta(\beta-1)} , \quad \beta > 1
\end{align*}

(1.5)

\begin{align*}
  E(R^2) = \frac{\xi(\xi+1)}{\beta(\beta+1)} + \frac{2(a-\xi)\xi}{\beta(\beta-1)} + \frac{(a-\xi+1)(a-\xi)}{(\beta-1)(\beta-2)}, \quad \beta > 2
\end{align*}

Recently, Lee, Holland, and Flueck (1979) were able to obtain comparable results for density of \( R \) using the Cherian-David-Fix class of densities by expressing \( f_R \) as a weighted difference of hypergeometric functions. The purpose of this paper is to derive comparable results for \( R \) using a different class of the bivariate gamma distribution. This class is a special case of the one suggested by Jensen (1970) as modified by Gunst and Webster (1973). The next section contains a brief discussion of this class of distributions. In section 3 the derivation of \( f_R \) is given using this class of functions. Section 4 outlines a possible application for the probability function in the area of hypothesis testing for the equality of shape parameters in the presence of correlation.
2. GUNST AND WEBER CLASS OF BIVARIATE GAMMAS

Gunst and Weber (1973) proposed a computationally feasible method for deriving the joint density function for the bivariate chi-square distribution. Since the chi-square is a special case of the gamma, this method was used for the bivariate gamma case. That is, a bivariate gamma density function for $U$ and $V$ with common scale parameter $\lambda = 1$ and shape parameters $\alpha, \beta$, $(\alpha < \beta)$ is given by

$$f(u,v) = \frac{u^{\alpha-1}v^{\beta-1}e^{-(u+v)/(1-\eta)}}{(1-\eta)^\alpha \Gamma(\alpha) \Gamma(\beta)\eta^j k_l}$$

(2.1)

where $\eta = \rho \sqrt{\beta/\alpha}$, $\rho$ is the correlation coefficient between the variables $U$ and $V$. Gunst and Webster (1973) suggested this class of densities in that they are computationally tractable and do not involve mathematical functions, such as Laguerre polynomials or convoluted sums [Jensen (1970) and Kibble (1941)]. Smith, Adelfang, and Tubbs (1982) discuss this class of densities in greater detail.

In the next section the distributional properties for the ratio, $R$, are derived using the Gunst-Webster class of bivariate gammas.

3. RATIO OF CORRELATED GAMMA VARIATES

By letting $R = U/V$ and $S = U+V$, the joint pdf for $R$ and $S$ can easily be shown to be

$$f_{R,S}(r,s) = c_1 \sum \frac{s^{a+j-1} e^{-s/(1-\eta)}}{(1+r)^{a+j-k+1} e^{-s/(1-\eta)}}$$

(3.1)

where $c_1 = [(1-\eta)^\alpha \Gamma(\alpha) \Gamma(\beta)]^{-1}$, $c_2 = \frac{n^j k_l \Gamma(\beta-a+k)}{(1-\eta)^{2j+k} \Gamma(\beta+j+k) j! k!}$. Hence, by integrating over $S$ the pdf for $R$ becomes
\[ f_R(r) = (1-\eta)^\beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k B(a+j, \beta+j+k) r^{a+j-1}(1+r)^{\alpha+2j+k} \] (3.2)

where \( z_j = (a) j! / j! \), \( c_k = (\beta-a) k! / k! \), \( (a)_n = \Gamma(a+n)/\Gamma(a) \), and

\[ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \]

Whenever the shape parameters are equal then the density function for \( R \) is given by

\[ f_R(r) = (1-\eta)^\alpha \sum_{j=0}^{\infty} c_j B(a+j, \alpha+j) r^{a+j-1}(1+r)^{2a+2j} \] (3.3)

From (3.2) it can be shown that the \( \alpha \)th raw moment for \( R \) is given by

\[ E(R^m) = (1-\eta)^\beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k B(a+j+m, \beta+j+k-m)/B(a+j, \beta+j+k) \] (3.4)

if \( m < \beta \). In which case, it follows that

\[ E(R) = (1-\eta)^\beta \sum_{j=0}^{\infty} c_j c_k (a+j)/(\beta+j+k-1) \] (3.5)

and

\[ E(R^2) = (1-\eta)^\beta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k (a+j)(a+j+1)/(\beta+j+k-1)(\beta+j+k-2) \] (3.6)

Whenever \( \eta = 0 \), then

\[ E(R) = \alpha/\beta), \quad E(R^2) = \alpha(\alpha+1)/(\beta-1)(\beta-2) \] (3.7)

which agrees with the values given by Mielke and Flueck (1976) whenever \( \xi = 0 \) and with Lee, Holland, and Flueck (1979) whenever \( a = 0 \).

Lee, Holland, and Flueck (1979) discuss some of the mathematical properties for the density of \( R \) for various values of
They demonstrated that the density can be 0 at r=1 whenever either of the shape parameters is less than one. However, in the Gunst-Webster construction by assuming that \( a > 1 \) and \( a < \beta \) the density function given in equation (3.2) is stable. Figures 1-4 illustrate the various shapes that \( f_R(r) \) has as a function of the three parameters.

A definite computational advantage of equation (3.2) versus equation (1.3) stems from the ability to compute the tail probabilities for \( R \). By letting \( a = a+j \) and \( b = \beta + j + k \), we have

\[
F_R(r_0) = (1-\eta)^\beta \sum_{j=1}^{c} c_j \sum_{k=1}^{c_k} P[F_{2a+b} \leq br_0/a] 
\]

(3.8)

where \( F_{r,s} \) denotes a random variable from an F-distribution with \( r \) and \( s \) degrees of freedom. Note if \( \eta = 0 \), then (3.7) becomes

\[
F_R(r_0) = P[F_{2a+2\beta} \leq br_0/a] 
\]

(3.9)

which agrees with the well known results concerning the ratio of independent chi-squares. Furthermore, if \( \eta \neq 0 \) and \( a = \beta \) then (3.7) becomes

\[
F_R(r_0) = (1-\eta)^\beta \sum_{j=1}^{c} c_j P[F_{2(a+j)} + 2(a+j) \leq r_0] 
\]

(3.10)

which is similar to an expression given by Johnson and Kotz (1972), Chapter 40, Section 3.

4. APPLICATION

In this section an application is given for computing the cdf of \( R \), given by equation (3.7). Diagram 1 defines the area given in equation (4.1).
FIGURE 1. DENSITY FUNCTION FOR $\tau = u/w \quad \eta = 1.5, \beta = 1.5$

FIGURE 2. DENSITY FUNCTION FOR $\tau = u/w \quad \eta = 1.0, \beta = 3.0$
Figure 3. Density function for $R = U/V, \alpha = 3.0, \beta = 3.0$

Figure 4. Density function for $R = U/V, \alpha = 3.0, \beta = 3.0$
By letting \( \cot \theta_0 = u/v = r_0 \) and \( G(\theta) = 1-F_R(r_0) \), one has

\[
G(\theta) = (1-\eta) \beta \sum_j \sum_k c_j c_k F[F_{2b,2a} \leq (a/b)\tan \theta]
\] (4.1)

Figure 5 contains the graph of the function \( G(\theta) \) versus \( \theta \) for \( \alpha = 1 \) and \( \beta = 1, 2, \) or 3 and \( \eta = 0, .25, .50, \) and .75. From this figure and other cases which are not included one observes that whenever \( \alpha = \beta \) then \( G(45^\circ) = .5 \) and \( G(45^\circ) < .5 \) whenever \( \alpha < \beta \). This observation and additional properties were used in developing a test for the hypothesis

\[ H_0: \alpha = \beta \quad \text{vs.} \quad H_A: \alpha < \beta \] (4.2)

The procedure is presented in Tubbs (1983) and uses the Cramer-Von Mises criteria for testing (4.2). That is, define

\[
W_n = n \int (F_R(r) - F_n(r))^2 dF_R(r)
\] (4.3)

where \( F_R(r) \) is the cdf for the null distribution given in (3.10). \( F_n(r) \) is the empirical distribution for \( r_i = u_i/v_i \) and the \( r_i \)'s are arranged in increasing order. Whenever \( H_0 \) is true, then \( W_n \) is distribution free and has a convenient computational form given by

\[
W_n = \frac{1}{12n} + \sum_{i=1}^{n} \left( Z_i - \frac{(2i-1)}{2n} \right)^2
\] (4.4)

where \( Z_i = F_R(r_i) \). \( H_0 \) is rejected if \( W_n \) exceeds a specified critical point. Tubbs (1983) considers the properties of this test procedure in greater detail.

5. CONCLUSIONS AND SUMMARY

This paper derives both the density and the distribution functions for the ratio of positively correlated gamma variates using a modification of Jensen's bivariate gamma distribution. The expression for the moments differ from those given by either Mielke and Flueck (1976) or Lee, Holland, and Flueck (1979).
However, all the expressions are identical whenever the variates are uncorrelated. A principal advantage found in this representation stems from the ability to compute the CDF of the ratio. The value of the CDF for the ratio was shown to have potential application to the problem of testing for equality of shape parameters in a particular family of the bivariate gamma distribution.

Diagram 1
FIGURE 5. $G(\theta)$ vs. $\theta$
6. **BIBLIOGRAPHY**


A METHOD FOR DETERMINING IF UNEQUAL SHAPE PARAMETERS ARE NECESSARY IN A BIVARIATE GAMMA DISTRIBUTION

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ABSTRACT

A procedure for aiding an experimentalist in deciding between four and five parameters in a Jensen's type bivariate gamma distribution is presented. The procedure is based upon the properties of the CDF for the ratio of correlated gamma distributed variates. The criteria of interest is posed in a test of hypothesis setting and results are presented using the Cramér-Von Mises test of fit.

1. INTRODUCTION

Smith and Adelfang (1981) discuss the applicability of a bivariate gamma distribution as a parametric model for wind gust amplitude and length. In modeling this bivariate data with a gamma distribution, it was necessary to find a distribution that would allow for correlation between the random variables X and Y when the marginal distributions are univariate gammas with possibly unequal shape and scale parameters. That is, \( X \sim G(y_x, \beta_x) \) and
\[ Y \sim G(\gamma_y, \beta_y) \text{ where the probability density function for} \]
\[ Z \sim G(\gamma, \beta) \text{ is given by} \]
\[ f_Z(z) = \beta^\gamma z^{\gamma-1} e^{-\beta z} / \Gamma(\gamma). \quad (1.1) \]

A brief survey of the open literature reveals that there are several classes of the bivariate gamma distribution. One need only consult Mardia (1970) and Johnson and Kotz (1972) to find five classes of the bivariate gamma distribution [Kibble (1941), Cherion (1941), McKay (1934), Jensen (1970), and Moran (1969)]. Of these classes only Jensen (1970) and Moran (1969) allow for unequal shape parameters and both of these have computational limitations which affect their utility to the experimentalist. Recently, McAllister, Lee, and Holland (1981) and McAllister (1983) have addressed the limitations with Jensen's model and provided results which overcome many of the computational difficulties. However, at the time of Smith et al. (1983) development these results were not available. Hence, they modified a bivariate chi-square model given by Gunst and Webster (1973). This allows for possibly unequal shape parameters and is computationally tractable. The model is not as general as that given by Jensen (1970), however, one can derive the bivariate model given by Kibble (1941) as a special case whenever the shape parameters are equal. In this paper, the unequal shape parameter model will be referred to as the five-parameter model and the equal shape
case as the four-parameter model. Smith, Adelfang, and Tubbs (1983) discuss the properties of these distributions and it is apparent that the four-parameter has numerous computational advantages over the five-parameter model. So if one assumes that the data is correctly modeled by this class of the bivariate gamma distribution, a question of practical interest becomes, How does one decide if the five-parameters are really necessary? The purpose of this paper is to present a procedure which would aid the experimentalist in answering the above question. The problem is posed in a hypothesis testing setting. That is, test the hypothesis

\[ H_0: \gamma_x = \gamma_y \quad (1.2) \]

versus

\[ H_1: \gamma_x < \gamma_y \quad (1.3) \]

It should be noted that the proposed method is not an omnibus test of fit for the bivariate gamma against all other possible models. Instead the procedure is intended for deciding between the four or five parameter models as given in Smith, Adelfang, and Tubbs (1983).

The next section contains the distributional results needed for the test of hypothesis (1.2). The test procedure is given in section 3 and evaluated in section 4. Section 5 contains a summary and remarks concerning some of the limitations of the procedure.
2. DISTRIBUTIONAL RESULTS

Smith, Adelfang, and Tubbs (1983) modified a bivariate-Chi square distribution given by Gunst and Webster (1973) and obtained the density function given by

\[ f(x, y) = \frac{K_1}{K_2} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{jk} x^j (ny)^{j+k} \]

(2.1)

where

\[ K_1 = x^{\gamma_x - 1} y^{\gamma_y - 1} \exp\{-\frac{(x+y)}{(1-\rho)}\}, \]

\[ K_2 = (1-\rho)^{\gamma_x} \Gamma(\gamma_y) \Gamma(\gamma_y - \gamma_x), \]

\[ c_{jk} = n^{j+k} \Gamma(\gamma_y - \gamma_x + k) / [(1-\rho)^2]^{j+k} \Gamma(\gamma_y + j + k) j! k! \]

and \( x = x_{\beta_x}, y = y_{\beta_y}, \) \( \beta_x, \beta_y \) are known scale parameters, \( \eta = \rho \sqrt{\gamma_y / \gamma_x}, \) \( \rho \) is the correlation coefficient between the variables \( X \) and \( Y. \) The joint probability distribution function is given by

\[ F(x_o, y_o) = \Pr[X \leq x_o, Y \leq y_o] = J \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{jk} H(\gamma_x + j, x_o / (1-\rho)) \]

\[ \cdot H(\gamma_y + j + k, y_o / (1-\rho)) \]

(2.2)

where

\[ J = (1-\rho)^{\gamma_y} \Gamma(\gamma_x) \Gamma(\gamma_y - \gamma_x), \]

\[ d_{jk} = n^{j+k} \Gamma(\gamma_y - \gamma_x + k) / \Gamma(\gamma_y + j + k) j! k! \]

\[ H(a, x) = \int_0^x t^{a-1} e^{-t} dt. \]
Equations (2.1) and (2.2) are for the unequal shape parameters and will be referred to as the five-parameter model. It should be re-emphasized that this model is not completely general in that one assumes that $\gamma_y > \gamma_x$ and the correlation between variables $X$ and $Y$ are restricted to the interval $[0, \eta \sqrt{\gamma_x/\gamma_y}]$ for $\eta \in [0,1]$.

If $\gamma_x = \gamma_y = \gamma$ then it can be shown that (2.1) and (2.2) reduce to the well known functions given by Kibble (1941). That is, the density function is given by

$$f(x,y) = (xy)^{-1} \exp\left\{-\frac{(x+y)/(1-\eta)}{\gamma}\right\} \cdot \sum_{j=0}^{\infty} \frac{(\eta xy)/(1-\eta)^j}{\Gamma(\gamma+j)} j!$$

(2.3)

and the distribution function becomes

$$F(x,y) = \frac{(1-\eta)^{\gamma}}{\Gamma(\gamma)} \sum_{j=0}^{\infty} \frac{\eta^j}{\Gamma(\gamma+j)} j! \cdot H(\gamma+j,x/(1-\eta)) H(\gamma+j,y/(1-\eta)).$$

(2.4)

Equations (2.3) and (2.4) will be referred to as the four parameter model. A comparison of the distribution function given in (2.2) and (2.4) reveals that there are distinct differences in terms of the computational complexity. Thus for computational reasons the experimentalist would like to know how much greater does $\gamma_y$ have to exceed $\gamma_x$ before equation (2.2) is really necessary. Ideally he would like to answer this question before using both (2.2) and (2.4) then selecting the results which are more gratifying. In order to address this issue, this
paper considers the problem of testing hypothesis (1.2) versus (1.3) using an univariate random variate given by the ratio of X to Y, R = X/Y. Tubbs and Smith (1983) derive the density and distribution functions for R whenever the bivariate density is either (2.1) or (2.3). That is, if equation (2.1) holds then the density function for R is given by

\[ f_R(r) = (1-\eta) \sum_{j=0}^{\infty} c_j c_k B(a, b)^{-1} r^{a-1} (1+r)^{a+b} \]  

(2.5)

where \( B(a, b) = r(a)r(b)/r(a+b) \), \( c_j = (a)_n j^n/j! \), \( c_k = (b-a)_n k^n/k! \), \( a = \gamma_x + j \), \( b = \gamma_y + j + k \), and \( (a)_n = r(a+n)/r(a) \). The distribution function for R is given by

\[ F_R(r_0) = P[X/Y \leq r_0] = (1-\eta) \sum_{j=0}^{\infty} c_j c_k Pr[F_{2a, 2b} \leq br_0/a] \]  

(2.6)

where \( F_{r,s} \) denotes a random variable from an F-distribution with \( r \) and \( s \) degrees of freedom. The corresponding functions whenever \( \gamma_x = \gamma_y = \gamma \) are given by

\[ f_R(r) = (1-\eta) \sum_{j=0}^{\infty} c_j B(a, a)^{-1} r^{a-1} (1+r)^{2a} \]  

(2.7)

and

\[ F_R(r_0) = (1-\eta) \sum_{j=0}^{\infty} c_j Pr[F_{2a, 2a} \leq r_0] \]  

(2.8)

where \( a = \gamma + j \).
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3. HYPOTHESIS TESTING

Since \( R = \frac{X}{Y} \) is a univariate random variable it is informative to graph \( F_R(r) \) versus \( r \). However, since \( r > 0 \) a more meaningful graph can be produced by letting \( \theta = \cot^{-1} r \) and \( G(\theta) = 1 - F_R(r_0) \) where \( \theta_0 = \cot^{-1} r_0 \). The area corresponding to \( F_R(r_0) \) is shown in diagram 1. Furthermore, it follows that

\[
G(\theta_0) = (1-n) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_j c_k \Pr[F_{2a,2b} \leq (a/b) \tan \theta_0]
\]

(3.1)

in the five-parameter model and

\[
G(\theta_0) = (1-n) \sum_{j=0}^{\infty} c_j \Pr[F_{2\alpha,2\beta} \leq \tan \theta_0]
\]

(3.2)

in the four-parameter case.

Since \( \theta \) is restricted to the finite interval \((0, \pi/2)\), it is somewhat instructive to plot \( G(\theta) \) versus \( \theta \) as functions of the free parameters, \( Y_x, Y_y \) and \( n \). As in Tubbs and Smith (1983) the scale parameters are assumed to be known and hence equal to one. This restriction will be addressed later in the paper. Figures 1-3 contain some of the illustrative cases.

From these plots one observes that \( G(45^\circ) = .5 \) whenever the four-parameter model holds and \( G(45^\circ) < .5 \) in the five-parameter models. Rather than just using this observation a function was selected to measure the distance between these distribution functions. The Cramér-Von Mises
DIAGRAM 1

\[ u = v r_0 \]

\[ c(\theta_0) = 1 - F_R(r_0) \]
FIGURE 1. $F_R(0)$ vs. $\theta$
FIGURE 2. $F_{R}(\theta)$ vs. $\theta$

$\alpha = 2.0$
$\beta = 2.0$

$\alpha = 2.0$
$\beta = 3.0$

$\alpha = 2.0$
$\beta = 4.0$

MATERIAL IS OF POOR QUALITY
FIGURE 3. $F_R(0)$ vs. $\theta$
type goodness-of-fit procedure was selected since the test is distribution free whenever the parameters are specified. Furthermore the test statistic is easy to compute.

Let

$$W_n = n \int_0^{\pi/2} (G(\theta) - G_n(\theta))^2 dG(\theta)$$  \hspace{1cm} (3.3)$$

where $G(\theta)$ is given in (3.2), $G_n(\theta)$ is the empirical distribution function of $\theta_i = \tan^{-1}(r_i)$, $r_i = X_i/Y_i$ are arranged in increasing order. Whenever hypothesis (1.2) is true, then $W_n$ has the convenient computational form given by

$$W_n = 1/12n + \sum_{i=1}^{n} \left( z_i - \frac{(2i-1)}{2n} \right)^2$$  \hspace{1cm} (3.4)$$

where $z_i = G(\theta_i)$. Furthermore, from Anderson and Darling (1951) one can reject (1.2) whenever $W_n$ exceeds a specified critical point. These critical points are given from Anderson and Darling's asymptotic distribution. Stephen (1976) defines a procedure for modifying the critical points for small samples, however, the underlying problem of modeling bivariate data will probably dictate large sample sizes.

4. EVALUATION OF THE TEST PROCEDURE

In this section the procedure defined in the previous section is evaluated. The evaluation is performed in two parts. The intent of the first part was to determine whether or not the procedure even works. That is, are the
apparent visual differences between the function \(G(\theta)\) as seen in Figures 1-3 significant in the "Cramér-Von Mises" metric. The second part of the evaluation concerns the robustness of the procedure to the nuisance parameters.

In the first part, let

\[
D_n(\delta) = n \int_0^{\pi/2} \{G(\theta) - A(\theta)\}^2 dG(\theta)
\]

where \(G(\theta)\) is given in (3.2) and \(A(\theta)\) is given by (3.1) when \(\gamma_y = \gamma_x + \delta\), for \(\delta > 0\). For positive integers \(n\), compute.

\[
a_n(\delta) = P_x[W_n > D_n(\delta)].
\]

If the alternative hypothesis given by

\[
H_1: \gamma_x < \gamma_y = \gamma_x + \delta
\]

holds, then the expected value of \(W_n\) in (3.3) is given by \(D_n(\delta)\). Hence, \(a_n(\delta)\) is the expected type I error of testing hypothesis (1.2) as a function of \(\delta\). Table 1 contains the value of \(a_n(\delta)\) for various values of the parameters. The \(a_n(\delta)\)'s were computed using Tiku's approximation to the asymptotic distribution of \(W_n\) [Tiku (1965)].

For example, from Table 1 one would expect the test to reject integer \((\delta=1)\) differences between the shapes for \(X\) and \(Y\) at the 95% significant level whenever \(n > 50\).

The procedure used to generate the values in Table 1 is somewhat unconventional, however, they do indicate that
the test procedure would be sensitive to differences in the shape parameters that exceed unity. A Monte Carlo simulation was also performed. The results are not reported in the interest of space and since the simulation was quite limited. A detail simulation is very expensive due to the computational costs in computing the null distribution G(θ) needed in evaluating type I errors. It is especially costly to simulate any type II errors. In spite of these restrictions upon the simulation's merit, the results were supportive of the expected results given in Table 1.

The second part of the evaluation is concerned with the question of robustness of the test to the unspecified parameters, namely, ρ and βx, βy. In order to determine the sensitivity of the test to the misspecified correlation coefficient ρ, the following distance was evaluated for different values of γx = γy.

\[ D_n(ρ) = n \int_0^{π/2} \left( G(θ) - B(θ) \right)^2 dG(θ) \]  

(4.4)

where G(θ) is given in equation (3.2) when ρ = 0, and B(θ) is given by equation (3.2) whenever ρ > 0, for ρ = .25(.25).75. Table 4 contains the type 1 errors \( \alpha_n(ρ) \) given by

\[ Pr[W_n > D_n(ρ)] = \alpha_n(ρ) \]  

(4.5)

for different values of n and γx = γy = γ. \( \alpha_n(ρ) \) in
Table 1. Tail Probabilities for $\alpha_n(\delta)$*

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*if $\alpha_n(\delta) < .01$, then the entry is left blank.
(4.5) is the expected type I error as a function of the nuisance parameter $\rho$. It should be mentioned that the distribution for $W_n$ is not the same as that given by Anderson and Darling asymptotic approximation since the nuisance parameter $\rho$ is unspecified [cp. Stephen (1976)], however, it does not appear feasible to follow Darling's procedure for computing the exact distribution whenever $\rho$ and $\beta_x$, $\beta_y$ are replaced by their consistent estimators. In spite of this shortcoming, equation (4.5) is used. However, Stephens (1976) showed that the asymptotic approximation given by Anderson and Darling is conservative as compared to his fitted distribution in the family of normal distributions [Stephens (1976) Table 4, p. 367] and the extreme value distributions [Stephens (1977) Table 1, p. 687]. Thus, it seems reasonable that equation (4.5) is also conservative, that is, if $a_n(\rho)$ is the true value for the l.h.s. of equation (4.5), then $a(\rho) < a_n(\rho)$.

Table 2. Type I Errors for Unsatisfied

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From Table 2, it follows that the procedure is only sensitive to \( \rho \) whenever \( \rho = .75 \) and \( n > 50 \). This observation was also supported in the simulation study.

In order to determine the sensitivity of the test to the scale parameters, the distance given by

\[
D_n(s) = n \int_0^n (G(\theta) - C(\theta))^2 dG(\theta)
\]

(4.6)

where \( G(\theta) \) is given in (3.2) and \( C(\theta) \) is given by (3.7) whenever \( \tan \theta = sr \), \( s = \frac{\beta_x}{\beta_y} = .90(0.02)1.10 \). Errors in either of the scale parameters can be considered by varying \( s \) in (3.2). Table 3 contains the expected type I errors given by

\[
Pr[W_n > D_n(s)] = \alpha_n(s)
\]

(4.7)

for different values of \( n, \gamma, \) and \( \rho \).

From Table 3 one observes that the procedure appears to be resilient to errors in the scale parameter and that one might have a type I error when \( \gamma = 3, \rho = .75, \) and \( n = 100 \) at the 95% significance level. In addition it also appears that the results are symmetric about \( s = 1 \).

5. CONCLUSIONS AND SUMMARY

A procedure is outlined for determining whether a four or five parameter bivariate gamma model is appropriate. The procedure was evaluated and three
### Table 3. Type 1 Errors for Misspecified Scales

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The values of s = (.96,1.04) were omitted since the Type 1 error was 1.0 for all parameters. Likewise whenever n=20.
different functions were evaluated in order to determine the procedure's feasibility and sensibility to the nuisance parameters. Admittedly, the evaluation is very limited and there are several limitations which would prohibit this type of procedure as an omnibus test of fit. However, the results appear to be promising to the experimentalist interested in obtaining insight into the stated problem.

There are several nonparametric procedures for testing (1.2) versus (1.3) and perhaps these are not as sensitive to the nuisance parameters. However, the proposed procedure is based upon "measuring" significant departures of the parametric distributions function which are vital to the modelers' primary objective.
6. REFERENCES


A DIFFERENTIAL EQUATIONS APPROACH TO THE MODAL LOCATION FOR A FAMILY OF BIVARIATE GAMMA DISTRIBUTIONS

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ABSTRACT

Analytical and numerical computational methods are given for determining the location of the mode as a function of the parameters of a class of the bivariate gamma distribution.

I. INTRODUCTION

Smith, Adelfang, and Tubbs (1983) derived some computational results for a family of bivariate distributions. In their paper they consider the location of the mode as a function of the shape parameters, \( \gamma_1 \) and \( \gamma_2 \), and the dependence coefficient \( \eta \). The purpose of this paper is to consider this problem in greater detail. That is, the paper will consider analytical and numerical computational
methods for locating the modal values for the class of density functions given in Smith, Adelfang, and Tubbs (1983). The general density function is given by:

\[ f(t_1, t_2; \gamma_1, \gamma_2, n) = \frac{t_1^{-1} t_2^{-1} e^{-(t_1+t_2)} e^{-d \gamma_1^t \gamma_2^{\gamma_1} e^{(\gamma_2-\gamma_1)^{\gamma_1}}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}}{(1-n)^{\gamma_1^t \gamma_2^{\gamma_1} e^{(\gamma_2-\gamma_1)^{\gamma_1}}} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}} \]  

(1.1)

where \( a_{jk} = \frac{n^{j+k} e^{(\gamma_2-\gamma_1+k)(t_1 t_2)^j t_2^k}}{(1-n)^{2j+k} e^{(\gamma_2+j+k)^{j+k}}} \).

\( t_1 = \beta_1 x, t_2 = \beta_2 y, \beta_1, \beta_2 \) are scale parameters, \( \gamma_2 > \gamma_1 > 1 \) are shape parameters, and \( 0 < n < 1 \) is associated with the correlation coefficient \( \rho \) by the equation \( n = \rho \sqrt{\gamma_2 / \gamma_1} \). We will assume without loss of generality that \( \beta_1 = \beta_2 = 1 \).

We will concentrate on the special case \( \gamma_1 = \gamma_2 = \gamma \) of (1.1) for which the distribution function reduces to

\[ f(t_1, t_2; \gamma, n) = \frac{t_1^{-1} t_2^{-1} e^{-(t_1+t_2)/(1-n)}}{(1-n)^{\gamma} e^{(\gamma_2-\gamma_1)^{\gamma}}} \sum_{j=0}^{\infty} \frac{n^j (t_1 t_2)^j}{(1-n)^{2j} e^{(\gamma+j)^{j}}} \]  

(1.2)

This is the form given by Kibble (1941).

Smith and Adelfang (1981) used the above class of density functions in modeling wind gust data for the ascent flight of the Space Shuttle. A parametric model was selected in that the parameters are used to establish engineering constraints for the shuttle payload system. Thus, the modal location and value
were of interest to this particular application. The authors are not aware of any other results, either analytical or numerical, for the modal location for non-Gaussian multivariate distributions. The closest related work is in the area of density and mode estimation [e.g. Sager (1978, 1979), de Beaville (1978), and Eddy (1980)].

In Section 2 we will derive some qualitative results concerning the behavior of the modal location of (1.2) as a function of $(\gamma, n)$. Section 3 presents analogous results for another borderline case $\gamma_1 = 1, \gamma_2 > 2$ of (1.1). In Section 4 we present some numerical procedures based on the theoretical investigations of the previous sections. The general case $\gamma_2 > \gamma_1 > 1$ is considered in Section 5. We present some numerical tabulations for the modal location of (1.1) as a function of $(\gamma_1, \gamma_2, n)$ and consider some numerical interpolations from the borderline cases considered in Sections 2 and 3.

2. EQUAL SHAPE PARAMETERS - ANALYTICAL METHODS

Lemma 1. The function $f(t_1, t_2; \gamma, n)$ defined by (1.2) attains its maximum in the region $R_+^2 = \{(t_1, t_2); t_1 \geq 0, t_2 \geq 0\}$ on the line $t_1 = t_2$.

Proof: Since $f$ is integrable and continuous over $R_+^2$, it is clear that $f$ attains its maximum on $R_+^2$. Choose any constant $c > 0$. Let $h(t) = f(t, c-t; \gamma, n)$, $0 < t < c$. Then from (1.2) we have
h(t) = \sum_{j=0}^{\infty} K_j(\gamma, \eta) t^{\gamma+j-1} (c-t)^{\gamma+j-1},

where \text{K}_j(\gamma, \eta) > 0 is independent of t. Therefore,

h'(t) = \sum_{j=0}^{\infty} K_j(\gamma, \eta) (\gamma+j-1) t^{\gamma+j-2} (c-t)^{\gamma+j-2} (c-2t).

Since h'(t) > 0 for 0 < t < c/2 and h'(t) < 0 for c/2 < t < c, h attains its maximum at t = c/2. Therefore f(t_1, t_2) attains its maximum along any line t_1 + t_2 = c at the point (c/2, c/2). This completes the proof.

Define g(t; \gamma, \eta) = f(t, t; \gamma, \eta). Then by Lemma 1 it is sufficient to find the point on \tau \geq 0 at which g attains its maximum value. Using (1.2) one can show that

\[ g(t; \gamma, \eta) = c(\gamma, \eta) e^{-2t/(1-\eta)} h(t) \] (2.1)

where \[ c(\gamma, \eta) = [(1-\eta)(\sqrt{\eta})^{\gamma-1} I_\gamma(\eta)]^{-1}, \] and \[ h(t) = t^{\gamma-1} I_{\gamma-1}(p(\eta)t), \]

where \( I_\mu(z) \) denotes the modified Bessel function with index \( \mu \), and \( p(\eta) = 2\sqrt{\eta}/(1-\eta) \).

Using [Abramowitz and Stegun (1964), Eqn. 9-6-28] it is not difficult to show that \( h'(t) = p(\eta) t^{\gamma-1} I_{\gamma-2}(p(\eta)t) \), therefore \( f(\tau, \tau; \gamma, \eta) \) is the mode at the bivariate gamma distribution given by (1.2) if and only if \( g'(\tau) = 2g(\tau)/(1-\eta) \) or

\[ \sqrt{\eta} I_{\gamma-2}(p(\eta)\tau) = I_{\gamma-1}(p(\eta)\tau), \] (2.2)

where \( p(\eta) = 2\sqrt{\eta}/(1-\eta) \).

With the aid of (2.1), we may prove the following theorem.

Theorem 1. For fixed \( \gamma > 1 \), let \( \tau(\eta) \) denote the value at which \( f(\tau(\eta), \tau(\eta); \gamma, \eta) \) is a maximum. Then \( \tau \) is continuously differentiable for \( 0 \leq \eta < 1 \) and satisfies the initial value problem

\[ 38 \]
\[
\tau'(n) = \left(\tau/2n\right)(2\tau - 2\gamma + 3)^{-1} - (1+n)(1-n)^{-1}
\]
(2.3)

\[
\tau(0) = \gamma - 1.
\]

Proof. It is easy to show directly from (2.1) that \( g \) attains its maximum at \( t = \gamma - 1 \) when \( n = 0 \), so that \( \tau(0) = \gamma - 1 \).

Furthermore \( \partial g / \partial t \) is continuously differentiable for \( 0 < n < 1 \) and computation shows that \( \partial^2 g / \partial t^2 \neq 0 \) at \( t = \gamma - 1 \) and \( n = 0 \). Therefore, \( \tau(n) \) is continuously differentiable in a neighborhood of \( n = 0 \) by the implicit function theorem. The proof will be completed by differentiating both sides of (2.1) with respect to \( n \).

After some simplification and solving for \( \tau'(n) \) this yields

\[
\tau'(n) = (1-n)I_{Y-2}(p(n)\tau)/(4nq(n)) - (1+n)\tau/2n(1-n)
\]
(2.4)

where \( q(n) = I_{Y-1}(p(n)\tau) - \sqrt{n} \) \( I_{Y-2}(p(n)\tau) \).

By [Abramowitz and Stegun (1964), Eqn. 9-2-26],

\[
I_{Y-1}'(p(n)\tau) = I_{Y-2}(p(n)\tau) - (Y-1)I_{Y-1}(p(n)\tau)/p(n)\tau
\]

\[
I_{Y-2}(p(n)\tau) = I_{Y-1}(p(n)\tau) + (Y-2)I_{Y-2}(p(n)\tau)/p(n)\tau
\]

Substituting these expressions into \( q(n) \) and using (2.2) yields after some simplification

\[
q(n) = (1-n)(2\tau - 2\gamma + 3)I_{Y-2}(p(n)\tau)/2\tau.
\]

Substituting this expression into (2.4) completes the proof of Theorem 1.

The nonlinear differential equation (2.3) cannot be solved in general in closed form. Some numerical solutions are given.
in Section 4. However (2.3) does give information regarding the qualitative and limiting behavior of \( \tau(n) \) for \( \gamma > 1 \). In the special case \( \gamma = 3/2 \), (2.3) reduces to a linear differential equation which can be solved directly by standard methods.

**Corollary 1.** If \( \gamma = 3/2 \), then

\[
\tau(n) = \frac{1 - n}{4\sqrt{n}} \ln \left( \frac{1 + \sqrt{n}}{1 - \sqrt{n}} \right).
\]

This result can also be obtained directly from (2.2) using the fact that \( I_\mu(z) \) can be expressed in terms of hyperbolic functions when \( \mu = \pm 1/2 \).

Since the differential equation (2.3) is singular at \( n = 0 \), its numerical solution requires some additional knowledge of the behavior of \( \tau(n) \) near \( n = 0 \). This is provided by the following corollary.

**Corollary 2.** The function \( \tau(n) \) is continuously differentiable at \( n = 0 \) and satisfies

\[
\tau'(0) = -(\gamma - 1)/\gamma, \quad \gamma > 1.
\]

**Proof.** The continuous differentiability of \( \tau \) at \( n = 0 \) was considered in the proof of Theorem 1. Choose \( n > 0 \), then by the mean value theorem there is a number \( \xi \in (0, n) \) such that

\[
\tau(n) = \tau(0) + n\tau'(\xi) = \gamma - 1 + n\tau'(\xi).
\]

Substituting this expression into (2.3) and simplifying yields

\[
\tau'(n) = \frac{(\gamma - 1 + n\tau'(\xi))(1 + (1 + \eta)\tau'(\xi))}{(1 - \eta)(1 + 2n\tau'(\xi))}.
\]
Letting \( n \to 0 \) and using the continuity of \( \tau'(n) \) we have

\[
\tau'(0) = -(y-1)(1+\tau'(0)).
\]

Solving this equation for \( \tau'(0) \) yields (2.5)

We will write \( \tau(n,y) \) when we wish to emphasize the dependence of the modal location on \( y \). Theorem 1 and Corollary 2 may be used to obtain several of the qualitative and asymptotic properties of the function \( \tau(n,y) \) in the region \( 0 < n < 1, \ y > 1 \). These are summarized in the following theorem.

**Theorem 2.** The modal location function \( \tau(n,y) \) has the following properties:

(i) \( \tau(n,y) \) is a decreasing function of \( n \) for fixed \( y > 1 \);

(ii) \( \lim_{n \to 0} \tau(n,y) = \max\{y - \frac{3}{2}, 0\} \) for \( y > 1 \);

(iii) \( \tau(n,y) - (y - \frac{3}{2}) \) is a decreasing function of \( y \) for fixed \( n \in (0, 1) \) and \( y > 1 \);

(iv) \( \lim_{y \to \infty} \tau(n,y) - (y - \frac{3}{2}) = \frac{1-n}{2(n+1)} \), for \( 0 \leq n \leq 1 \).

**Proof:** We will show that \( \tau'(n) < 0 \) for \( 0 \leq n < 1 \). Suppose not, then since \( \tau'(0) < 0 \) by Corollary 2, there is a point \( \xi > 0 \) such that \( \tau' (\xi) = 0 \) and \( \tau'(n) < 0 \) for \( 0 < n < \xi \). Let \( w(n) = \tau(n) - (y - \frac{3}{2}) \) and \( z(n) = \frac{1-n}{2(n+1)} \), then from (2.3) it is easy to see that

\[
\tau'(n) = \frac{\tau(n)}{4n} \left[ \frac{1}{w(n)} - \frac{1}{z(n)} \right],
\]

so \( \tau'(\xi) = 0 \) if and only if \( w(\xi) = z(\xi) \).
Let $h = w - z$. Note that $z'(\eta) = -(1+\eta)^{-2}$ so that $h'(0) = w'(0) - z'(0) = \tau'(0) + 1 > 0$ and $h(0) = w(0) - z(0) = 0$.

Therefore since $h(\xi) = 0$ and $h(\eta) > 0$ for $0 < \eta < \xi$, we must have $h'(\xi) < 0$. However, $h'(\xi) = w'(\xi) - z'(\xi) = \tau'(\xi) - z'(\xi) = -(1+\xi)^{-2} > 0$. This contradiction proves (i).

Furthermore, we have that $w(\eta) > z(\eta)$ for $0 < \eta < 1$.

We will now consider the proof of (iii). Fix $\gamma_1 > \gamma_2 > 1$ and let $f(\eta) = w(\eta, \gamma_1) - w(\eta, \gamma_2)$ where as before $w(\eta, \gamma) = \tau(\eta, \gamma) - (\gamma^{-\frac{3}{2}})$. We wish to show that $f(\eta) < 0$ for $0 < \eta < 1$.

Clearly $f(0) = 0$ and by (2.5) $f'(0) = \frac{1}{\gamma_1} - \frac{1}{\gamma_2} < 0$. Assume to obtain a contradiction that there is a point $\xi \in (0, 1)$ such that $f(\xi) = 0$. If, in addition, we assume $\xi$ is the first such point, then $f(\eta) < 0$ for $0 < \eta < \xi$ so $f'(\xi) > 0$. However, using (2.6) at both $\gamma_1$ and $\gamma_2$ and the fact that $w(\xi, \gamma_1) = w(\xi, \gamma_2)$ it is not difficult to show that

$$f'(\xi) = \left(\frac{\gamma_1 - \gamma_2}{\eta}\right) \left[\frac{1}{w(\xi)} - \frac{1}{z(\xi)}\right].$$

Since $\gamma_1 > \gamma_2$ and $w(\xi) > z(\xi)$, it follows that $f'(\xi) < 0$. This contradiction completes the proof of (iii).

Now we turn to the proof of (ii). First consider the case $1 < \gamma < 3/2$. Since $\tau$ is decreasing in $\eta$ and positive for $0 \leq \eta < 1$ we know that $\tau^* = \lim_{\tau+1} \tau(\eta)$ exists, where the limits at 1 are always from the left. Assume to obtain a contradiction that $\tau^* > 0$. Then it is not difficult to show using (2.3) that
\[ \tau'(n) \leq \frac{1}{4n} - \frac{(1+n)}{2n(1-n)} \]

Therefore, for \( \frac{1}{2} \leq n < 1 \) we have
\[ \tau'(n) \leq \frac{1}{2} - \frac{\tau^*}{2(1-n)}. \]

Integrating both sides of this inequality from \( \frac{1}{2} \) to \( n \) yields
\[ \tau(n) \leq \tau\left(\frac{1}{2}\right) + \frac{1}{2} + \frac{\tau^*}{2} \ln(1-n) \]
for \( \frac{1}{2} \leq n < 1 \). However, this implies that \( \tau(n) \to -\infty \) as \( n \to 1 \), a contradiction.

The case \( \gamma > \frac{3}{2} \) follows easily from (iii) and the proof of (i) because for \( \gamma \geq \frac{3}{2} \)
\[ z(n) \leq \tau(n) - (\gamma - \frac{3}{2}) \leq \tau(n, \frac{3}{2}) \]
and both \( z(n) \) and \( \tau(n, \frac{3}{2}) \) approach zero as \( n \to 1 \).

Finally, we consider the proof of (iv). Let \( u(n, \gamma) = w(n, \gamma) - z(n) \) for \( 0 \leq n \leq 1 \) and \( \gamma \geq \frac{3}{2} \). From the proof of (i) we know that \( u(n, \gamma) \geq 0 \). From (2.9) we obtain
\[ w'(n, \gamma) \leq \frac{1}{4n} \left[ 1 - \frac{w(n, \gamma)}{z(n)} \right] + \frac{(\gamma - \frac{3}{2})}{4n} \left[ \frac{z(n) - w(n, \gamma)}{w(n, \gamma)z(n)} \right] \]
so that
\[ w'(n, \gamma) \leq -(\gamma - \frac{3}{2})u(n, \gamma). \]

Therefore,
\[ u'(n, \gamma) = w'(n, \gamma) - z'(n) \leq -(\gamma - \frac{3}{2})u(n, \gamma) + 1. \]
From this inequality we obtain
\[
\frac{d}{dn} \left[ u(n, \gamma)e^{(\gamma-\frac{3}{2})n} \right] \leq e^{(\gamma-\frac{3}{2})n}
\]
\[
u(n, \gamma)e^{(\gamma-\frac{3}{2})n} - u(0, \gamma) \leq \frac{1}{(\gamma-\frac{3}{2})} \left[ e^{(\gamma-\frac{3}{2})n} - 1 \right].
\]
Therefore
\[
0 \leq u(n, \gamma) \leq \frac{1}{(\gamma-\frac{3}{2})}.
\]
This implies that \( u(n, \gamma) \rightarrow 0 \) as \( \gamma \rightarrow \infty \) and completes the proof of Theorem 2.

3. UNEQUAL SHAPE PARAMETERS--THE CASE \( \gamma_1 = 1 \)

In this section we consider another "borderline" case of the general bivariate gamma distribution, the case \( \gamma_1 = 1 \). For technical reasons we will limit our discussion to the range \( \gamma_2 \geq 2 \) and for brevity let \( \gamma_2 = \gamma \). Then the function given by (1.1) reduces to

\[
f(t_1, t_2; 1, \gamma, n) = \frac{t_2^{\gamma-1}e^{-s_2}}{(1-n)\Gamma(\gamma-1)} \sum_{j=0}^{\infty} e^{-s_1} \frac{c_j}{j!}
\]

where

\[
c_j = \sum_{k=0}^{\infty} s_3^{j+k} \frac{\Gamma(\gamma+k-1)}{k!\Gamma(\gamma+j+k)}, \quad j = 0, 1, 2, \ldots
\]

and where \( s_1 = \frac{t_1}{1-n}, \quad s_2 = \frac{t_2}{1-n}, \) and \( s_3 = ns_2 \).

The following lemma allows us to restrict our attention to the line \( t_1 = 0 \).
Lemma 2. The function \( f(t_1, t_2; 1, \gamma, n) \) given by (3.1) for \( \gamma \geq 2 \), takes on its maximum value in the region \( t_1 \geq 0 \), \( t_2 \geq 0 \) on the line \( t_1 = 0 \).

**Proof:** Since \( f \) is continuous and integrable in the first quadrant, we know it takes on its maximum value at some point \((t_1^*, t_2^*)\). We will prove that \( t_1^* = 0 \) by showing that for any fixed \( t_2 > 0 \), \( f(t_1, t_2) \) is a decreasing function of \( t_1 \). This is equivalent to showing that the function

\[
g(s) = \sum_{j=0}^{\infty} e^{-s} c_j \frac{s^j}{j!}
\]

is a decreasing function on \( s \geq 0 \) where \( c_j \) is given by (3.2).

Note that

\[
g'(s) = -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^{j-1}}{j!} = -e^{-s} \sum_{j=0}^{\infty} c_j \frac{s^j}{j!} + e^{-s} \sum_{j=0}^{\infty} c_{j+1} \frac{s^j}{j!}
\]

\[
= -e^{-s} \sum_{j=0}^{\infty} \frac{s^j}{j!} (c_j - c_{j+1}).
\]

Therefore \( g'(s) < 0 \) for \( s \geq 0 \) if \( c_{j+1} < c_j \) for \( j = 0, 1, 2, \ldots \).

To this end note that

\[
c_j + 1 = \sum_{k=0}^{\infty} s^{j+1+k} \frac{\Gamma(\gamma+k+1)}{k! \Gamma(\gamma+j+1+k)}
\]

\[
= \sum_{k=1}^{\infty} s^{j+k} \frac{\Gamma(\gamma+k)}{(k-1)! \Gamma(\gamma+j+k)}
\]

\[
= \sum_{k=1}^{\infty} s^{j+k} \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+j+k)} \cdot \frac{k}{\gamma+k-2} < c_j
\]
since \( \gamma \geq 2 \) implies that \( \frac{k}{\gamma+k-2} \leq 1 \) for \( k = 1, 2, 3, \ldots \). This completes the proof of Lemma 2.

According to the preceding lemma, the mode of the bivariate gamma distribution in this case is the point \((0, \mu)\) where \( \mu \) is the point on \( t > 0 \) where the following function is a maximum:

\[
g(t) = t^{\gamma-1} e^{-t/(1-n)} h(t)
\]

where

\[
h(t) = \sum_{k=0}^{\infty} \left( \frac{n t}{1-n} \right)^k \frac{\Gamma(\gamma+k-1)}{k! \Gamma(\gamma+k)} = \sum_{k=0}^{\infty} \left( \frac{n t}{1-n} \right)^k \frac{1}{k! \Gamma(\gamma+k-1)}
\]

Note that

\[
\frac{d}{dt} \left( \left( \frac{n t}{1-n} \right)^{\gamma-1} h(t) \right) = \frac{d}{dt} \left( \sum_{k=0}^{\infty} \left( \frac{n t}{1-n} \right)^{\gamma+k-1} \frac{1}{k! \Gamma(\gamma+k-1)} \right)
\]

\[
= \frac{n}{1-n} \sum_{k=0}^{\infty} \left( \frac{n t}{1-n} \right)^{\gamma+k-2} \frac{1}{k!}
\]

\[
= \frac{n}{1-n} \left( \frac{n t}{1-n} \right)^{\gamma-2} e^{nt/(1-n)}.
\]

Therefore,

\[
h(t) = \left( \frac{1-n}{nt} \right)^{\gamma-1} \int_0^{nt} \left( \frac{n s}{1-n} \right)^{\gamma-2} e^{ns/(1-n)} \, ds
\]

\[
= \frac{1}{t^{\gamma-1}} \int_0^t s^{\gamma-2} e^{ns/(1-n)} \, ds
\]

so that the function we wish to maximize is

\[
g(t) = e^{-t/(1-n)} \int_0^t s^{\gamma-2} e^{ns/(1-n)} \, ds. \quad (3.3)
\]
Lemma 4. Let \( \mu(\eta) \), or when necessary \( \mu(\eta, \gamma) \), denote the value for which \( f(0, \mu(\eta, \gamma), 1, \gamma, \eta) \) is a maximum where \( f \) is defined by (3.1) and (3.2). Then \( \mu(0) = \gamma - 1 \), \( \mu(\eta) \geq \gamma - 2 \) for \( 0 \leq \eta < 1 \), and satisfies the equation

\[
g(\mu) = (1-\eta)\mu^{\gamma-2}e^{-\mu}, \quad 0 \leq \eta < 1,
\]

where \( g \) is defined by (3.3).

Proof: It is easy to see from (3.3) that \( g \) attains its maximum on \([0, \infty)\) at a point \( t^* > 0 \) for which \( g'(t^*) = 0 \) and \( g''(t^*) \leq 0 \). Differentiating (3.3) we obtain

\[
g'(t) = -\frac{1}{1-\eta} g(t) + t^{\gamma-2}e^{-t}
\]

and

\[
g''(t) = -\frac{1}{1-\eta} g'(t) + t^{\gamma-3}e^{-t}(\gamma-2-t).
\]

Therefore \( g'(\mu) = 0 \) implies (3.4) and \( g''(\mu) < 0 \) implies that \( \mu \geq \gamma - 2 \). Since when \( \eta = 0 \), \( g(t) = e^{-t} \frac{t^{\gamma-1}}{\gamma-1} \) it is easy to see that \( \mu(0) = \gamma - 1 \). This completes the proof of the lemma.

With the aid of these preliminaries we may prove the following theorem in the spirit of Theorem 1.

Theorem 3. For fixed \( \gamma \geq 2 \), let \( \mu(\eta) \) denote the value of which \( f(0, \mu(\eta), 1, \gamma, \eta) \) is a maximum. Then \( \mu \) is continuously differentiable on \( 0 \leq \eta < 1 \), \( \mu'(0) = -1 + \frac{1}{\gamma} \), and on \( 0 < \eta < 1 \) \( \mu \) satisfies the initial value problem

\[
\]
\[ \mu'(\eta) = -\frac{\mu(\mu - (\gamma - 1) + \eta)}{\eta(1-\eta)(\mu - (\gamma - 2))} ; \tag{3.5} \]

\[ \mu(0) = \gamma - 1. \]

**Proof:** As in the proof of Theorem 1 the continuous differentiability of \( \mu \) in a neighborhood of \( \eta = 0 \) may be proved by applying the implicit function theorem to (3.4). This differentiability will be extended to all of \([0,1)\) by proving that (3.5) holds. Let \( g(t, \eta) \) denote the function defined by (3.3) and let \( g_t \) and \( g_\eta \) denote its partial derivatives with respect to \( t \) and \( \eta \), respectively. Then differentiating both sides of (3.4) with respect to \( \eta \) we obtain

\[ g_t(\mu, \eta)\mu' + g_\eta(\mu, \eta) = (1-\eta)e^{-\mu}\gamma^3(\gamma - 2 - \mu)\mu' - e^{-\mu}\gamma^2. \tag{3.6} \]

By definition \( g_t(\mu, \eta) = 0 \) and direct differentiation of (3.4) and integration by parts yields for \( 0 < \eta < 1 \) that

\[ g_\eta(t, \eta) = \frac{t}{(1-\eta)^2} g(t, \eta) \]

\[ + \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} \int_0^t s^{\gamma - 1} e^{\eta s/(1-\eta)} ds \]

\[ = \frac{t}{(1-\eta)^2} g(t, \eta) \]

\[ + \frac{1}{(1-\eta)^2} e^{-t/(1-\eta)} \left[ \frac{1-\eta}{\eta} t^{\gamma - 1} e^{\eta t/(1-\eta)} \right. \]

\[ - \frac{1-\eta}{\eta} (\gamma - 1) \int_0^t s^{\gamma - 2} e^{\eta s/(1-\eta)} ds \]

48
= \frac{t}{(1-n)^2} g(t, n) + \frac{1}{n(1-n)} t^{\gamma-1} e^{-t} - \frac{\gamma-1}{n(1-n)} g(t, n).

Therefore, using (3.4) we obtain

\[ g_n(t, n) = -\frac{\mu}{1-n} \mu^{\gamma-2} e^{-\mu} + \frac{1}{n(1-n)} \mu^{\gamma-1} e^{-\mu} - \frac{\gamma-1}{n} \mu^{\gamma-2} e^{-\mu} \]

\[ = \mu^{\gamma-2} e^{-\mu} \left(-\frac{\mu}{1-n} + \frac{\mu}{n(1-n)} - \frac{\gamma-1}{n}\right) \]

\[ = \mu^{\gamma-2} e^{-\mu} \frac{\mu - (\gamma-1)}{n}. \]

Substituting this expression into (3.6) and simplifying yields (3.5). For \( n = 0 \), an easy calculation shows that

\[ g_n(t, 0) = -\frac{1}{\gamma(\gamma-1)} e^{-t} \gamma^t \]

from which substitution into (3.6) with \( n = 0 \) and \( \mu = \gamma - 1 \) shows that \( \mu'(0) = -1 + \frac{1}{\gamma} \). This completes the proof of Theorem 3.

The following corollary exploits the fact that (3.5) reduces to a linear differential equation when \( \gamma = 2 \).

**Corollary 3.** If \( \gamma = 2 \), then

\[ \mu(n) = \frac{1-n}{n} \ln(\frac{1}{1-n}), \quad 0 < n < 1. \]

**Proof.** For \( \gamma = 2 \) equation (3.5) reduces to

\[ \mu' + \frac{\mu}{n(1-n)} = \frac{1}{n} \]
which is easily solved in closed form by standard methods to show the desired result. This result is also easily derived directly from (3.4).

It is interesting to note that the translated modal location function \( v(n) = \mu(n) - (y - 2) \) satisfies the differential equation

\[
v'(n) = \frac{v(n)}{n} + \frac{y - 2}{n} \left( \frac{1}{v(n)} - \frac{1}{1-n} \right)
\]

\( v(0) = 1, \ v'(0) = -1 + \frac{1}{y} \),

whereas the translated modal location function \( w(n) = \tau(n) - (y - \frac{3}{2}) \) of Section 2 satisfies the analogous differential equation

\[
w'(n) = \frac{w(n)}{4n} + \frac{y - \frac{3}{2}}{2} \left( \frac{1}{w(n)} - \frac{2(1+n)}{1-n} \right)
\]

\( w(0) = \frac{1}{2}, \ w'(0) = -1 + \frac{1}{y} \).

For this reason \( \mu \) behaves in a manner similar to \( \tau \). Its properties are stated in the following theorem. Since the proof of this theorem is entirely analogous to the proof of Theorem 2, it is omitted.

**Theorem 4.** The modal location function \( \mu(n, y) \) has the following properties:

(i) \( \mu(n, y) \) is a decreasing function of \( n \) for fixed \( y \geq 2 \);

(ii) \( \lim_{n \to 1} \mu(n, y) = y - 2 \) for \( y \geq 2 \);
(iii) \( \mu(n, \gamma) - (\gamma-2) \) is a decreasing function of \( \gamma \) for 
\( \gamma \geq 2 \) and fixed \( n \in (0,1) \);
(iv) \( \lim_{\gamma \to \infty} (\mu(n, \gamma) - (\gamma-2)) = 1-n \) for \( 0 \leq n \leq 1 \).

4. NUMERICAL RESULTS

In this section we present some quantitative results based on the results of the previous sections. Table 1 shows the value of the modal location function for equal shape parameters \( \tau(n, \gamma) \) for various values of \( n \) and \( \gamma \). Table 2 shows values of the translated modal location function \( w(n, \gamma) = \tau(n, \gamma) - (\gamma-\frac{3}{2}) \). This table illustrates the qualitative behavior of this function derived in Theorem 2. The limiting values of \( n = 1 \) and \( \gamma = \infty \) are taken from Theorem 2.

The values in Tables 1 and 2 were computed using Theorem 1. Specifically, a fourth-order Runge-Kutta algorithm was used to compute an approximate solution of the differential equation (2.3) on the interval \( 0 \leq n < 1 \) for each specified value of \( \gamma \). Since equation (2.3) is singular at \( n = 0 \), Corollary 2 was used to replace the initial condition \( \tau(0) = \gamma-1 \) by the approximate initial condition

\[
\tau(h) = \gamma-1 - (\frac{\gamma-1}{\gamma}) h
\]

where \( h \) is the step size of the numerical method. Figure 1 shows the data of Table 2 in graphical form and illustrates the behavior of the function \( w(n, \gamma) \) derived in Theorem 2.
Tables 3 and 4 show the corresponding results for the modal location function \( \mu(\eta, \gamma) \) for the case \( \gamma_1 = 1, \gamma_2 = \gamma \) and its associated translate \( v(\eta, \gamma) = \mu(\eta, \gamma) - (\gamma - 2) \). These tables were computed by the same methods as Tables 1 and 2 except using the results of Section 3. Figure 2 illustrates the qualitative behavior of the function \( v(\eta, \gamma) \) as indicated by Theorem 4.

Note that the differential equations (2.3) and (3.5) allow the modal location to be computed recursively in \( \eta \) for a fixed value of \( \gamma \) as a dynamic process in a time scale measured by the modified correlation coefficient \( \eta \). Error in the computation is introduced through the discretization of this continuous evolutionary process. A more conventional computation of the modal location would require an independent calculation for each value of \( \eta \) with error introduced through the truncation of the series representation (1.2) of the distribution function. This error becomes particularly troublesome as \( \eta + 1 \).
Table 1. Selected values of the modal location function $\tau(\eta, \gamma)$ for equal shape parameters.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>1.000</td>
<td>0.0908</td>
<td>0.0720</td>
<td>0.0525</td>
<td>0.0322</td>
<td>0.0110</td>
<td>0.000</td>
</tr>
<tr>
<td>1.3</td>
<td>3.000</td>
<td>0.2765</td>
<td>0.2268</td>
<td>0.1724</td>
<td>0.1117</td>
<td>0.0412</td>
<td>0.000</td>
</tr>
<tr>
<td>1.5</td>
<td>5.000</td>
<td>0.6650</td>
<td>0.3931</td>
<td>0.3116</td>
<td>0.2169</td>
<td>0.0958</td>
<td>0.000</td>
</tr>
<tr>
<td>2.0</td>
<td>1.000</td>
<td>0.9491</td>
<td>0.8412</td>
<td>0.7277</td>
<td>0.6127</td>
<td>0.5279</td>
<td>0.500</td>
</tr>
<tr>
<td>3.0</td>
<td>2.000</td>
<td>1.9334</td>
<td>1.8034</td>
<td>1.6862</td>
<td>1.5398</td>
<td>1.5268</td>
<td>1.500</td>
</tr>
<tr>
<td>10.0</td>
<td>9.000</td>
<td>8.9152</td>
<td>8.7754</td>
<td>8.6697</td>
<td>8.5891</td>
<td>8.5624</td>
<td>8.500</td>
</tr>
</tbody>
</table>

Table 2. Selected values of the translated modal location function $w(\eta, \gamma)$ for equal shape parameters.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>0.500</td>
<td>0.4908</td>
<td>0.4720</td>
<td>0.4525</td>
<td>0.4322</td>
<td>0.4110</td>
<td>0.400</td>
</tr>
<tr>
<td>1.3</td>
<td>0.500</td>
<td>0.4765</td>
<td>0.4268</td>
<td>0.3724</td>
<td>0.3117</td>
<td>0.2412</td>
<td>0.200</td>
</tr>
<tr>
<td>1.5</td>
<td>0.500</td>
<td>0.4660</td>
<td>0.3931</td>
<td>0.3116</td>
<td>0.2169</td>
<td>0.0958</td>
<td>0.000</td>
</tr>
<tr>
<td>2.0</td>
<td>0.500</td>
<td>0.4491</td>
<td>0.3412</td>
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<td>0.1127</td>
<td>0.0279</td>
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</tr>
<tr>
<td>3.0</td>
<td>0.500</td>
<td>0.4334</td>
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<td>0.1862</td>
<td>0.0939</td>
<td>0.0268</td>
<td>0.000</td>
</tr>
<tr>
<td>10.0</td>
<td>0.500</td>
<td>0.4152</td>
<td>0.2754</td>
<td>0.1697</td>
<td>0.0891</td>
<td>0.0264</td>
<td>0.000</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.500</td>
<td>0.4091</td>
<td>0.2692</td>
<td>0.1667</td>
<td>0.0882</td>
<td>0.0263</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Table 3. Selected values of the modal location function $\mu(n, \gamma)$ for $\gamma_1 = 1$, $\gamma_2 = \gamma \geq 2$.

<table>
<thead>
<tr>
<th>$\gamma \backslash n$</th>
<th>0</th>
<th>.1</th>
<th>.3</th>
<th>.5</th>
<th>.7</th>
<th>.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.0000</td>
<td>.9482</td>
<td>.8322</td>
<td>.6931</td>
<td>.5160</td>
<td>.2558</td>
<td>.0000</td>
</tr>
<tr>
<td>3.0</td>
<td>2.0000</td>
<td>1.9310</td>
<td>1.7767</td>
<td>1.5936</td>
<td>1.3702</td>
<td>1.1111</td>
<td>1.0000</td>
</tr>
<tr>
<td>10.0</td>
<td>9.0000</td>
<td>8.9085</td>
<td>8.7170</td>
<td>8.5155</td>
<td>8.3081</td>
<td>8.1011</td>
<td>8.0000</td>
</tr>
</tbody>
</table>

Table 4. Selected values of the translated modal location function $v(n, \gamma)$ for $\gamma_1 = 1$, $\gamma_2 = \gamma \geq 2$.

<table>
<thead>
<tr>
<th>$\gamma \backslash n$</th>
<th>0</th>
<th>.1</th>
<th>.3</th>
<th>.5</th>
<th>.7</th>
<th>.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.0000</td>
<td>.9482</td>
<td>.8322</td>
<td>.6931</td>
<td>.5160</td>
<td>.2558</td>
<td>.0000</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0000</td>
<td>.9310</td>
<td>.7767</td>
<td>.5936</td>
<td>.3702</td>
<td>.1111</td>
<td>.0000</td>
</tr>
<tr>
<td>10.0</td>
<td>1.0000</td>
<td>.9085</td>
<td>.7170</td>
<td>.5155</td>
<td>.3081</td>
<td>.1011</td>
<td>.0000</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.0000</td>
<td>.9000</td>
<td>.7000</td>
<td>.5000</td>
<td>.3000</td>
<td>.1000</td>
<td>.0000</td>
</tr>
</tbody>
</table>
Figure 1. Qualitative behavior of the translated modal location function $w(n, \gamma)$ for equal shape parameters.
Figure 2. Qualitative behavior of the translated modal location function \( V(\gamma, \xi) \) for \( \gamma_1 = 1, \gamma_2 = \gamma_2 \geq 2 \).
5. UNEQUAL SHAPE PARAMETERS

In this section we briefly consider the mode of the general bivariate gamma distribution given by (1.1). By setting the partial derivatives of \( f(t_1, t_2; \gamma_1, \gamma_2, n) \) with respect to \( t_1 \) and \( t_2 \) equal to zero, one finds that \( f \) attains its maximum at the point \((t_1, t_2)\) whose coordinates satisfy

\[
t_1 = \frac{1-n}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk} (\gamma_1^{j+k-1}) \tag{5.1}
\]

and

\[
t_2 = \frac{1-n}{S} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk} (\gamma_2^{j+k-1}) \tag{5.2}
\]

where \( S = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_{jk} \)

and \( a_{jk} \) given as in (1.1) depends on \( t_1 \) and \( t_2 \).

Table 5 shows selected values of the modal location for the case \( \gamma_2 = 3 \). They were computed by truncating each of the series in (5.1) and (5.2) to about fifty terms and simultaneously iterating on these equations until an approximate solution is obtained. These computations become unreliable as \( n \to 1 \) and the truncation error becomes unacceptable.

Figure 3 gives a graphical representation of the change in modal location with \( n \) and \( \gamma_1 \) for fixed \( \gamma_2 = 3 \). It is interesting to note for a fixed \( n \) the extent to which the modal location may be approximated by linear interpolation between the borderline cases discussed in Sections 2 and 3.
More specifically, we have the empirical approximations

\[ t_1 = \frac{\gamma_1^{-1}}{\gamma_2^{-1}} \tau(n, \gamma_2) \]  

(5.3)

and

\[ t_2 = \mu(n, \gamma_2) + \frac{\gamma_1^{-1}}{\gamma_2^{-1}} (\tau(n, \gamma_2) - \mu(n, \gamma_2)) \]  

(5.4)

where \( \tau \) and \( \mu \) are as defined in Sections 2 and 3, respectively. This empirical relationship is a subject for further investigation.
Table 5. Location of the mode using (5.1) and (5.2) with $\gamma_2 = 3$. Approximate values computed using (5.3) and (5.4) are denoted by *. When these values are equal to two decimal places, only one is given.

<table>
<thead>
<tr>
<th>$\eta \backslash \gamma_1$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0, 2.00)</td>
<td>(.50, 2.00)</td>
<td>(1.00, 2.00)</td>
<td>(1.50, 2.00)</td>
<td>(2.00, 2.00)</td>
</tr>
<tr>
<td>.25</td>
<td>(0, 1.82)</td>
<td>(.46, 1.82)</td>
<td>(.92, 1.83)</td>
<td>(1.38, 1.83)</td>
<td>(1.84, 1.84)</td>
</tr>
<tr>
<td>.50</td>
<td>(0, 1.59)</td>
<td>(.42, 1.62)</td>
<td>(.84, 1.64)</td>
<td>(1.26, 1.66)</td>
<td>(1.69, 1.69)</td>
</tr>
<tr>
<td>.75</td>
<td>(0, 1.31)</td>
<td>(.40, 1.38)</td>
<td>(.70, 1.44)*</td>
<td>(1.18, 1.51)*</td>
<td>(1.58, 1.58)</td>
</tr>
<tr>
<td>.85</td>
<td>(0, 1.18)</td>
<td>(.41, 1.26)</td>
<td>(.77, 1.36)*</td>
<td>(1.16, 1.45)*</td>
<td>(1.54, 1.54)</td>
</tr>
</tbody>
</table>
Figure 3. Graphical representation of the values in Table 5.
REFERENCES


ANALYSIS OF WIND GUST DATA

J. D. Tubbs
Department of Mathematical Sciences
University of Arkansas
Fayetteville, Arkansas

ABSTRACT

This paper summarizes the analysis of wind gust data using statistical and mathematical procedures which were developed for the bivariate gamma distribution.

1. INTRODUCTION

Adelfang and Smith (1981) discuss the use of the gamma distribution in modeling gust data at Cape Canaveral, Florida. Smith and Adelfang (1981) treated gust amplitude and length scale as the variables of the bivariate gamma distribution. Smith, Adelfang, and Tubbs (1983) presented some useful analytical and computational results for a class of the bivariate gamma and applied some of these results to the wind gust data. The purpose of this paper is to analyze the wind gust data using some additional analytical results obtained for the bivariate gamma distribution.

2. DATA

The data used in this paper consists of absolute gust
magnitude and gust length for both the zonal and meridional components. The 150 wind profiles were filtered using the band pass filter for wavelengths within 420-2470 meter band. Data were available for the reference altitudes: 4Km, 6Km, 8Km, 10Km, 12Km, and 14Km. The data was paired into bivariate components for both the zonal and meridional components, denoted by the pairs \((A_u, L_u)\) and \((A_v, L_v)\), respectively.

3. ANALYTICAL PROCEDURES

The data were partitioned according to reference altitudes, then the 150 observations were analyzed using both univariate and multivariate techniques. Simple descriptive univariate techniques were generated using PROC UNIVARIATE in SAS. These procedures were used to help in the assessment of the marginal distribution. The multivariate descriptive procedures consisted of bivariate scatter plots and contour plots.

Goodness of fit tests consisted of a univariate test for marginal normality generated by SAS, two tests for bivariate normality as discussed in Meredith and Tubbs (1981), and a bivariate test for the gamma distribution. The latter procedure is a bivariate Chi-square type test which uses the computational results for the distribution function as presented in Smith, Adelfang, and Tubbs (1983).

Parameter estimates for the bivariate gamma distribution were evaluated. These estimates were then used in
generating the three-dimensional bivariate gamma density function plots and the modal locations were estimated using the results given by Brewer, Tubbs, and Smith (1983).

4. RESULTS

The results are summarized in Tables 1-7. Additional results are given in Appendices A and B.

Tables 1-6 summarize the results for both the test of fit and the parameter estimates for the bivariate gamma distribution. There are two main tests for bivariate normality and both of these are discussed in Meredith and Tubbs (1981). The first is a procedure proposed by Rincon-Gallardo et al. (1979). Since this procedure transforms the data to a univariate test for uniformity three different tests for uniformity are used. The second test for bivariate normality is based upon a procedure proposed by Cox and Small (1978).

The bivariate test for the gamma distribution is a Chi-square type test of fit. Thus, this procedure has the usual difficulties of selecting the number of cells and cell location that are associated with this type of test. In the interest of time and space a fixed procedure was applied for all the data sets. Namely, the marginal distributions were partitioned according to the .05, .25, .50, .75, and .90 quantiles based upon the gamma parameter estimates. This particular choice affected the results for some of the data sets, however, it seemed a reasonable global choice.
The univariate gamma parameters were estimated using a maximum likelihood procedure presented by Greenwood and Durand (1960) and discussed in Tubbs and Brewer (1981).

Appendix A contains the results for the univariate descriptive statistics. Appendix B contains plots for each data set. The density functions were generated using the gamma parameter estimates. The contour plots are level slices of the density function and are not equal probability contours. The location of the mode is denoted by the symbol + and this value is computed using the analytical results given in Brewer et al. (1983). Table 7 summarizes the results for the modal location.
Table 1. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 4 Km.

<table>
<thead>
<tr>
<th>Multivariate Test</th>
<th>(Au,Lu)</th>
<th>(Av,Lv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cramer-Von Mises</td>
<td>.2062</td>
<td>.2144</td>
</tr>
<tr>
<td>Watson's U^2</td>
<td>.2023*</td>
<td>.2058*</td>
</tr>
<tr>
<td>K - S</td>
<td>.0618</td>
<td>.0551</td>
</tr>
<tr>
<td>Cox</td>
<td>11.45**</td>
<td>26.07***</td>
</tr>
<tr>
<td>Gamma</td>
<td>Chi-Square</td>
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</tr>
</tbody>
</table>

Univariate Test

<table>
<thead>
<tr>
<th>Normality</th>
<th>(Au)</th>
<th>(Lu)</th>
<th>(Av)</th>
<th>(Lv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normality</td>
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<td>.090**</td>
<td>.077*</td>
</tr>
</tbody>
</table>

Parameter Estimates

<table>
<thead>
<tr>
<th></th>
<th>( \hat{\gamma} )</th>
<th>( \hat{\beta} )</th>
<th>( \hat{\rho} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au</td>
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<td>2.430</td>
<td>2.280</td>
</tr>
<tr>
<td>Lu</td>
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<td>1.007</td>
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</tr>
<tr>
<td>Av</td>
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<td>1.891</td>
<td>.3415</td>
</tr>
<tr>
<td>Lv</td>
<td>4.429</td>
<td>.006</td>
<td>.3415</td>
</tr>
</tbody>
</table>

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.
Table 2. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 6 Km.

<table>
<thead>
<tr>
<th>Multivariate Test</th>
<th>(AuLu)</th>
<th>(Av,Lv)</th>
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</thead>
<tbody>
<tr>
<td>Cramer-Von Mises</td>
<td>.3806</td>
<td>.2942</td>
</tr>
<tr>
<td>Watson's U²</td>
<td>.2411*</td>
<td>.2208*</td>
</tr>
<tr>
<td>K - S</td>
<td>.0897</td>
<td>.0623</td>
</tr>
<tr>
<td>Cox</td>
<td>24.4***</td>
<td>31.8***</td>
</tr>
<tr>
<td>Gamma</td>
<td>72.03***</td>
<td>57.44***</td>
</tr>
<tr>
<td>Chi-Square</td>
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<table>
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</tr>
</thead>
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<td>Au</td>
<td>.047</td>
<td></td>
</tr>
<tr>
<td>Lu</td>
<td>.081*</td>
<td></td>
</tr>
<tr>
<td>Av</td>
<td>.054</td>
<td></td>
</tr>
<tr>
<td>Lv</td>
<td>.072</td>
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<table>
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<th>̂γ</th>
<th>̂β</th>
<th>̂ρ</th>
</tr>
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<tbody>
<tr>
<td>Au</td>
<td>2.577</td>
<td>1.916</td>
<td>.4217</td>
</tr>
<tr>
<td>Lu</td>
<td>4.102</td>
<td>.005</td>
<td></td>
</tr>
<tr>
<td>Av</td>
<td>3.168</td>
<td>2.030</td>
<td>.2506</td>
</tr>
<tr>
<td>Lv</td>
<td>4.895</td>
<td>.005</td>
<td></td>
</tr>
</tbody>
</table>

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.
Table 3. Summary for Wind Gust Statistic Using Band Filter 420-2470 Altitude = 8 Km.

<table>
<thead>
<tr>
<th>Multivariate Test</th>
<th>(Au, Lu)</th>
<th>(Av, Lv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cramer-Von Mises</td>
<td>1.090***</td>
<td>.490*</td>
</tr>
<tr>
<td>Normality</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Watson's U²</td>
<td>.721***</td>
<td>.409***</td>
</tr>
<tr>
<td>K - S</td>
<td>.102*</td>
<td>.083</td>
</tr>
<tr>
<td>Cox</td>
<td>10.67**</td>
<td>8.67*</td>
</tr>
<tr>
<td>Gamma</td>
<td>34.74</td>
<td>52.57**</td>
</tr>
<tr>
<td>Chi-Square</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Univariate Test</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Normality</td>
<td></td>
</tr>
<tr>
<td>Au</td>
<td>1.14**</td>
</tr>
<tr>
<td>Lu</td>
<td>1.08**</td>
</tr>
<tr>
<td>Av</td>
<td>1.07**</td>
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<tr>
<td>Lv</td>
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<table>
<thead>
<tr>
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<th>( \gamma )</th>
<th>( \beta )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au</td>
<td>3.023</td>
<td>2.113</td>
<td>.3396</td>
</tr>
<tr>
<td>Lu</td>
<td>4.149</td>
<td>.005</td>
<td></td>
</tr>
<tr>
<td>Av</td>
<td>2.922</td>
<td>1.811</td>
<td>.4253</td>
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<tr>
<td>Lv</td>
<td>4.614</td>
<td>.005</td>
<td></td>
</tr>
</tbody>
</table>

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.
Table 4. Summary for Wind Gust Statistic Using Band Filter 420-2470 Altitude = 10 Km.

<table>
<thead>
<tr>
<th>Multivariate Test</th>
<th>(Au, Lu)</th>
<th>(Av, Lv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normality</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cramer-Von Mises</td>
<td>0.129</td>
<td>0.753**</td>
</tr>
<tr>
<td>Watson's U²</td>
<td>0.126</td>
<td>0.469**</td>
</tr>
<tr>
<td>K - S</td>
<td>0.052</td>
<td>0.104*</td>
</tr>
<tr>
<td>Cox</td>
<td>8.23*</td>
<td>8.58*</td>
</tr>
<tr>
<td>Gamma</td>
<td>Chi-Square</td>
<td>45.04**</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Univariate Test</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Normality</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Au</td>
<td>0.073*</td>
<td></td>
</tr>
<tr>
<td>Lu</td>
<td>0.063</td>
<td></td>
</tr>
<tr>
<td>Av</td>
<td>0.073*</td>
<td></td>
</tr>
<tr>
<td>Lv</td>
<td>0.109**</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>γ</th>
<th>β</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au</td>
<td>3.041</td>
<td>1.909</td>
<td>0.4345</td>
</tr>
<tr>
<td>Lu</td>
<td>5.203</td>
<td>0.006</td>
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</tr>
<tr>
<td>Av</td>
<td>2.522</td>
<td>1.300</td>
<td>0.3890</td>
</tr>
<tr>
<td>Lv</td>
<td>4.543</td>
<td>0.005</td>
<td></td>
</tr>
</tbody>
</table>

* denotes that test is significant at .05 level.  
** denotes that test is significant at .01 level.  
*** denotes that test is significant at .001 level.
Table 5. Summary for Wind Gust Statistic
Using Band Filter 420-2470 Altitude = 12 Km.

<table>
<thead>
<tr>
<th>Multivariate Test</th>
<th>(Au,Lu)</th>
<th>(Av,Lv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cramer-Von Mises</td>
<td>.465*</td>
<td>.398</td>
</tr>
<tr>
<td>Watson's U²</td>
<td>.391**</td>
<td>.329**</td>
</tr>
<tr>
<td>K - S</td>
<td>.077</td>
<td>.076</td>
</tr>
<tr>
<td>Cox</td>
<td>25.04***</td>
<td>23.70***</td>
</tr>
<tr>
<td>Gamma Chi-Square</td>
<td>52.54**</td>
<td>41.24*</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Univariate Test</th>
<th>(Au,Lu)</th>
</tr>
</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>Au</td>
<td>.116**</td>
</tr>
<tr>
<td>Lu</td>
<td>.112**</td>
</tr>
<tr>
<td>Av</td>
<td>.042**</td>
</tr>
<tr>
<td>Lv</td>
<td>.085**</td>
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</tbody>
</table>

Parameter Estimates

<table>
<thead>
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<th>̂γ</th>
<th>̂β</th>
<th>̂ρ</th>
</tr>
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<tbody>
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<td>Au</td>
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<td>.969</td>
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<td>Lu</td>
<td>4.113</td>
<td>.005</td>
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<tr>
<td>Av</td>
<td>2.612</td>
<td>1.051</td>
<td>.3066</td>
</tr>
<tr>
<td>Lv</td>
<td>4.023</td>
<td>.005</td>
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</tbody>
</table>

* denotes that test is significant at .05 level.
** denotes that test is significant at .01 level.
*** denotes that test is significant at .001 level.
Table 6. Summary for Wind Gust Statistic  
Using Band Filter 420-2470  Altitude = 14 Km

<table>
<thead>
<tr>
<th>Test Type</th>
<th>Statistic</th>
<th>Au</th>
<th>Lu</th>
<th>Av</th>
<th>Lv</th>
</tr>
</thead>
<tbody>
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<td><strong>Multivariate Test</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cramer-Von Mises</td>
<td>1.359***</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Watson's U²</td>
<td>6.647***</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K - S</td>
<td>1.134**</td>
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<td></td>
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</tr>
<tr>
<td>Cox</td>
<td>14.6***</td>
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<tr>
<td>Gamma</td>
<td>Chi-Square 48</td>
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<td></td>
<td>48.29**</td>
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**Univariate Test**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Au</th>
<th>Lu</th>
<th>Av</th>
<th>Lv</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normality</td>
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</tr>
<tr>
<td></td>
<td>.090**</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.096**</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.085**</td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>.061</td>
<td></td>
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</tbody>
</table>

**Parameter Estimates**

<table>
<thead>
<tr>
<th>Statistic</th>
<th>(\hat{\gamma})</th>
<th>(\hat{\beta})</th>
<th>(\hat{\rho})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Au</td>
<td>3.803</td>
<td>1.549</td>
<td>.2420</td>
</tr>
<tr>
<td>Lu</td>
<td>4.725</td>
<td>.005</td>
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<tr>
<td>Av</td>
<td>3.324</td>
<td>1.161</td>
<td>.3171</td>
</tr>
<tr>
<td>Lv</td>
<td>5.057</td>
<td>.006</td>
<td></td>
</tr>
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</table>

* denotes that test is significant at .05 level.  
** denotes that test is significant at .01 level.  
*** denotes that test is significant at .001 level.
Table 7. Modal Location

<table>
<thead>
<tr>
<th>Altitude</th>
<th>Variables</th>
<th>Method I</th>
<th>Method II</th>
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</thead>
<tbody>
<tr>
<td>4000</td>
<td>(Au, Lu)</td>
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<td>0.939</td>
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<tr>
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<td>(Av, Lv)</td>
<td>0.874</td>
<td>0.873</td>
</tr>
<tr>
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<td>(Au, Lu)</td>
<td>0.747</td>
<td>0.745</td>
</tr>
<tr>
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<td>(Av, Lv)</td>
<td>1.008</td>
<td>1.007</td>
</tr>
<tr>
<td>8000</td>
<td>(Au, Lu)</td>
<td>0.875</td>
<td>0.874</td>
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<tr>
<td></td>
<td>(Av, Lv)</td>
<td>0.965</td>
<td>0.960</td>
</tr>
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<td>10000</td>
<td>(Au, Lu)</td>
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<tr>
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<td>(Av, Lv)</td>
<td>1.064</td>
<td>1.058</td>
</tr>
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<td>12000</td>
<td>(Au, Lu)</td>
<td>1.083</td>
<td>1.080</td>
</tr>
<tr>
<td></td>
<td>(Av, Lv)</td>
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<td>1.400</td>
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<td>(Au, Lu)</td>
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<td>1.713</td>
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<tr>
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<td>(Av, Lv)</td>
<td>1.868</td>
<td>1.865</td>
</tr>
</tbody>
</table>

* Method I  Truncation of a double series.
  Method II  Interpolation.
5. SUMMARY

The data sets are discussed according to reference altitude.

4 Km. The normal or the gamma are not rejected for the zonal (u) components. As discussed in Meredith and Tubbs (1981) the Cox and Small procedure is sensitive to symmetry and is not recommended for this data. The gamma distribution was rejected for the v-component and normality was accepted. However, marginal normality was rejected at the .01 level for the absolute gust magnitude (Av).

6 Km. The bivariate gamma was rejected in both wind components and normality was not rejected.

8 Km. Normality was rejected for both wind components. The bivariate gamma was accepted in the u-component but not for the v-component.

10 Km. The u-component appears to be normal whereas the gamma is accepted in the v-component.

12 Km. Both distributions appear to be suspect for the u-component and the gamma is accepted for v. Normality is also rejected for v by considering the marginal distributions.

14 Km. Neither distribution is acceptable for u and the gamma is perhaps better for v.
6. CONCLUSIONS

The wind gust data was analyzed using some new procedures for the bivariate gamma distribution. The analysis was meant to be informative, in that it represents examples for some of the analytical procedure. The analysis is not meant to be completely thorough. Hence, there are still some unresolved questions concerning the applicability of the bivariate gamma for modeling wind gust data. One suspects that neither the normal nor the gamma are completely appropriate, however, perhaps both could provide acceptable results for defining engineering constraints.

As mentioned in the paper the test for gamma is a Chi-square type procedure which has inherent problems which does not lend itself to easy data independent analysis. Instead it requires judicious selection of parameters. This analysis did not take advantage of this option, hence, the rejection of the gamma could be attributable to poor cell location choices.

Every data set was analyzed using a test for equality of shape parameters as proposed by Tubbs (1983). This hypothesis of equal shape parameters was rejected in every case.
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7. REFERENCES


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76
APPENDIX A

Univariate summary statistics generated using PROC UNIVARIATE is the Statistical Analysis System.
The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

\[ \text{ALTITUDE} = 4000 \]
GENERAL INT. EL
OF POOR QUALITY.

VIA ALL NO

<table>
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<th>QUANTILES (DEF=4)</th>
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<tbody>
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<td>150 100% MAX 3.67 50% 3.463</td>
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<td>22 1.29004 SUM 209.576 75% 33 1.7225 95% 2.863</td>
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<td>22 0.70616 WIND 0.49073 50% 44 1.295 50% 2.339</td>
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</tr>
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<td>32 0.61626 WIND 0.25246 25% 31 0.6775 10% 0.59</td>
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<tr>
<td>32 360.247 CPS 74.555 0.1% MIN 0.05 5% 0.3665</td>
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<td>32 59.126 CPS 0.057076 0.1% MAX 0.06925</td>
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<tr>
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<td>+1</td>
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<tr>
<td>+2</td>
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</tr>
</tbody>
</table>

MULTIPLE STR LEAF BY 10%-0.1
The next four pages contain the summary statistics for each of the univariate variables $A_u$, $L_u$, $A_v$, and $L_v$, respectively. The reference altitude is

\[ \text{ALTITUDE} = 6000 \]
MULTIPLY STAT. LSL BY 10**01
### Moments

<p>| | | | | | | |</p>
<table>
<thead>
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<td>711.3</td>
<td>90%</td>
<td>1262.96</td>
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<td>5%</td>
<td>225.55</td>
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### Quantiles (df=4)

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### Entries

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### Normal Probability Plot

- **P**:<br>
- **50%**:<br>
- **+**:<br>
- ***+***:<br>
- ****:<br>
- ++++:<br>
- -2 -1 +0 +1 +2

*Multiply stem:leaf by 10**0.02
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<td>STD DEV</td>
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<tr>
<td>LIN: c=0</td>
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<td>D:COL:AL</td>
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<tr>
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**ROYAL PROBABILITY PLOT**

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X: LEAF
22 2
21
20
19
16
17 0
16 8
15 169
14 116799
13 127
12 003395
11 111223455567689
10 0012445689
9 02334566
8 00112223445666577969
7 11223566568999
6 0112333344445555666
5 0224447669999
4 00111444555567688
3 13366
2 16
1 2579
```

```
MULTIPLY SITW. LEAP BY 10**402
```
The next four pages contain the summary statistics for each of the univariate variables $A_u$, $L_u$, $A_v$, and $L_v$, respectively. The reference altitude is

$$\text{ALTITUDE} = 8000$$
**VARIABILITY IN VOLATILITY OF POOR QUALITY**

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<th>95%</th>
<th>1972.5</th>
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<td>75% Q3</td>
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<td>703.95</td>
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<td>0.15506</td>
<td>KB</td>
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<td>25%</td>
<td>Q1</td>
<td>503.075</td>
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<td>CSS</td>
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<td>PROB</td>
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<td>95%</td>
<td>550.7</td>
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<tr>
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**NORMAL PROBABILITY PLOT**

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**MATERIAL**

MULTIPLY SIZE VIEW BY 10**02
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<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
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<tbody>
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<td>4.75</td>
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**Notes:**
- Q1, Q2, Q3, and Q4 represent the first, second, third, and fourth quartiles, respectively.
- The values listed are in arbitrary units for demonstration purposes.

**Graph:**
- A box plot is shown with the following characteristics:
  - The box represents the interquartile range (IQR) from Q1 to Q3.
  - The median is indicated by a line within the box.
  - Whiskers extend to the adjacent values (Q1 - 1.5 * IQR and Q3 + 1.5 * IQR).
  - Outliers are marked with asterisks.

**Analysis:**
- The data distribution is symmetrical with the median close to the center.
- No significant outliers are present within the given range.
The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

\[ \text{ALTITUDE} = 10,000 \]
**ORIGINAL PAGE IS OF POOR QUALITY**

### VARIABLES

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**MULTIPLY SHT. LEAF BY 10**
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VARIANCE

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QUANTILES

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VARIANCE

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QUANTILES

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VAT

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OBSERVED PROBABILITY DISTRIBUTION

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OBSERVED EXP.

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NORMAL PROBABILITY PLOT

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MULITPLY STELL LEAF BY 10**+02
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<th>Value</th>
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<th>90%</th>
<th>50%</th>
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NORMAL PROBABILITY PLOT

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<th>95%</th>
<th>90%</th>
<th>50%</th>
<th>25%</th>
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<td>8.25+</td>
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<td>10.5+</td>
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-2 -1 0 1 2

STEM AND LEAF

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<td>0.15</td>
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<td>4 7</td>
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<tr>
<td>3 8</td>
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### TOTAL PROBABILITY PLOT

- **MULTIPLY STEM LEAF BY 10** + **02**

- **98**
The next four pages contain the summary statistics for each of the univariate variables $A_u$, $L_u$, $A_v$, and $L_v$, respectively. The reference altitude is

\[ \text{ALTITUDE} = 12,000 \]
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**Actual Possibility Grid**

- **Grid Locations**
  - Upper Left: 0.25
  - Upper Right: 0.4675
  - Lower Left: 1.8575
  - Lower Right: 2.07

- **Grid Values**
  - 0.25: **Grid Location**
  - 0.4675: **Grid Location**
  - 1.8575: **Grid Location**
  - 2.07: **Grid Location**

- **Additional Grid Values**
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**Notes:**
- Grid values represent different probabilities or outcomes.
- The grid is used to visualize the distribution or frequency of certain events or data points.
The next four pages contain the summary statistics for each of the univariate variables Au, Lu, Av, and Lv, respectively. The reference altitude is

ALTITUDE - 14,000
### Variables of Poor Quality

**Variables (N=150)***

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<th>Description</th>
<th>Mean</th>
<th>Median</th>
<th>Std Dev</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Variance (95% CI)</th>
<th>Median (95% CI)</th>
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**Stem and Leaf Plot**

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  1  0011111222333444  5555555566666666669999
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```

**Q-Q Plot**

![Q-Q Plot](image)
### Variables: 100% quality

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### Normal Probability Plot

The plot shows the normal probability distribution with a straight line indicating the expected normal distribution. Points on the plot are marked with various symbols indicating the observed data compared to the expected distribution. The y-axis represents the normal probability, and the x-axis represents the observed data values. The plot confirms the normal distribution of the data.
### Original PCE Is of Poor Quality

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#### NORMAL PROBABILITY PLOT

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### Percentage of Area

- 0.5%: Z < -1.645
- 1%: Z < -1.282
- 5%: Z < -1.645
- 10%: Z < -1.282

---

**Note**: Values are multiplied by 10^{-4} or 10^{-5}.
APPENDIX B

Plots for the bivariate gamma density function, scatter plots, and contours are given for each data set.
ALTITUDE = 6000

BIVARIATE GAMMA DENSITY
SCATTER PLOT OF RAW DATA
REFERENCE ALTITUDE = 6000

OF POOR QUALITY
ORIGIN PLACED 19
CONTOUR PLOTS

VARIABLES (AV-LV)
ALTITUDE = 6000

OF POUR QUATITY
ORIGINAL PAGE 19
SCATTER PLOT OF RAW DATA

REFERENCE ALTITUDE = 8000

ORIGINAL PAGE IS OF POOR QUALITY

123
BIVARIATE GAMMA DENSITY

VARIABLES (AV, LV)
ALTITUDE = 8000
BIVARIATE GAMMA DENSITY

VARIABLES (AU, U)
ALTITUDE = 10000
BIVARIATE GAMMA DENSTY
CONTOUR PLOTS

VARIABLES (AVLV)
ALTITUDE = 10000

ORIGINAL PAGE 133
BIVARIATE GAMMA DENSITY

VARIABLES (A(U,U))
ALTITUDE = 12000

OF POOR QUALITY

134
SCATTER PLOT OF RAW DATA
REFERENCE ALTITUDE = 12000
End of Document