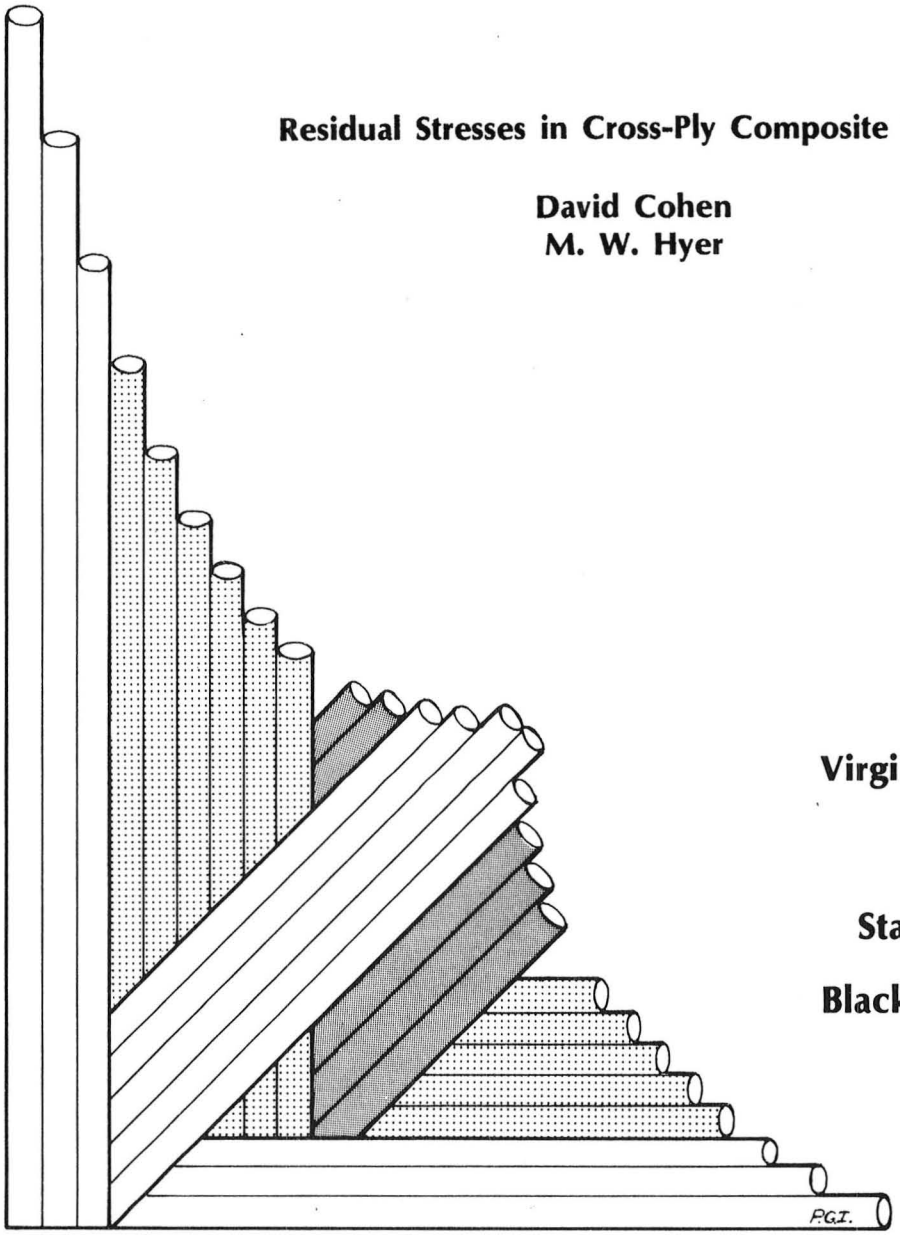
NASA-CR-173,540

CCMS-84-04
VPI-E-84-16

VIRGINIA TECH

CENTER FOR COMPOSITE MATERIALS AND STRUCTURES

NASA-CR-173540 ~~IR-42~~
1984 00177700



Residual Stresses in Cross-Ply Composite Tubes

David Cohen
M. W. Hyer

Virginia Polytechnic Institute
and
State University

Blacksburg, Virginia
24061

College of Engineering

Virginia Polytechnic Institute and State University

Blacksburg, VA 24061

VPI-E-84-16
CCMS-84-04

June, 1984

Residual Stresses in Cross-Ply Composite Tubes

David Cohen[1]
M. W. Hyer[2]

Department of Engineering Science & Mechanics

Interim Report 42
The NASA-Virginia Tech Composites Program

NASA Cooperative Agreement NAG-1-343

Prepared for: S. S. Tompkins
Applied Materials Branch
National Aeronautics & Space Administration
Langley Research Center
Hampton, VA 23665

[1]Graduate Research Associate, Department of Engineering Science and Mechanics

[2]Professor, Department of Engineering Science and Mechanics

N84-25768#

ABSTRACT

The residual thermal stresses in 4-layer cross-ply tubes are studied. The tubes considered had a small radius to wall-thickness ratios and so elasticity solutions were used. The residual thermal stress problem was considered to be axisymmetric and three elasticity solutions were derived and the results compared with the results using classical lamination theory. The comparison illustrates the limitations of classical lamination theory. The three elasticity solutions derived were: plane stress, plane strain, and generalized plane strain, the latter being the most realistic. The study shows that residual stresses in both the hoop and axial direction can be significant. Stacking arrangement effects the residual stress to some extent, as do the material properties of the individual lamina. The benefits of hybrid construction are briefly discussed.

TABLE OF CONTENTS

	Page
ABSTRACT	ii
LIST OF FIGURES	v
Chapter	
1 INTRODUCTION	1
2 LITERATURE REVIEW	3
Elasticity Methods	3
Shell Approximation Methods	5
Comparison of the Elasticity and Shell Approximation Methods	6
3 GOVERNING EQUATIONS	9
Geometry	9
Constitutive Relations	12
Equilibrium	24
Strain-Displacement Relations	25
Compatibility Equations	26
Boundary Conditions	27
Multiple Layers	28
Assumptions and Simplifications for Computing Residual Thermal Stresses	29
Plane-Strain Solution	37
Multiple Layer Plane-Strain Solution	41
Plane-Stress Solution	42
Multiple Layer Plane-Stress Solution	46
Generalized Plane-Strain	47

Multiple Layer Generalized Plane-Strain Solution..... 51

Details of Multiple Layer Solution..... 52

Laminated Shell Theory (LST)..... 54

4 RESULTS..... 60

5 CONCLUSIONS..... 94

REFERENCES..... 95

APPENDIX A..... 97

LIST OF FIGURES

Figure		Page
1	Tube geometry and coordinate systems used	10
2	Detail of laminate and material coordinate systems	11
3	Residual Axial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube	62
4	Residual Hoop Stress in a $(0/90/0/90)_t$ Laminated Composite Tube	63
5	Residual Radial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube	65
6	Residual Axial Stress in a $(90/0/90/0)_t$ Laminated Composite Tube	68
7	Residual Hoop Stress in a $(90/0/90/0)_t$ Laminated Composite Tube	69
8	Residual Radial Stress in a $(90/0/90/0)_t$ Laminated Composite Tube	70
9	Residual Axial Stress in a $(0_2/90_2)_t$ Laminated Composite Tube	71
10	Residual Hoop Stress in a $(0_2/90_2)_t$ Laminated Composite Tube	72
11	Residual Radial Stress in a $(0_2/90_2)_t$ Laminated Composite Tube	73
12	Residual Axial Stress in a $(0/90_2/0)_t$ Laminated Composite Tube	74
13	Residual Hoop Stress in a $(0/90_2/0)_t$ Laminated Composite Tube	75
14	Residual Radial Stress in a $(0/90_2/0)_t$ Laminated Composite Tube	76
15	Residual Axial Stress in a $(90/0_3)_t$ Laminated Composite Tube	77
16	Residual Hoop Stress in a $(90/0_3)_t$ Laminated Composite Tube	78

17 Residual Radial Stress in a $(90/0_3)_t$ Laminated Composite Tube..... 79

18 Residual Axial Stress in a $(0/90/0_2)_t$ Laminated Composite Tube..... 80

19 Residual Hoop Stress in a $(0/90/0_2)_t$ Laminated Composite Tube..... 81

20 Residual Radial Stress in a $(0/90/0_2)_t$ Laminated Composite Tube..... 82

21 Residual Axial Stress in a $(0/90_3)_t$ Laminated Composite Tube..... 83

22 Residual Hoop Stress in a $(0/90_3)_t$ Laminated Composite Tube..... 84

23 Residual Radial Stress in a $(0/90_3)_t$ Laminated Composite Tube..... 85

24 Residual Axial Stress in a $(90_3/0)_t$ Laminated Composite Tube..... 86

25 Residual Hoop Stress in a $(90_3/0)_t$ Laminated Composite Tube..... 87

26 Residual Radial Stress in a $(90_3/0)_t$ Laminated Composite Tube..... 88

27 Residual Hoop Stress in a $(90_4)_t$ Laminated Composite Tube..... 89

28 Residual Radial Stress in a $(90_4)_t$ Laminated Composite Tube..... 90

29 Normalized Residual Hoop Stress in a $(0/90/0/90)_t$ Laminated Composite Tube vs. R/t Ratios..... 91

30 Residual Radial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube vs. R/t Ratios..... 92

31 Residual Hoop Stress in a $(0/90/0/90)_t$ Hybrid Laminated Composite Tube..... 93

INTRODUCTION

The use of composite materials in engineering structures has gained wide acceptance in the last decade. The success of the NASA space-shuttle program has provided the impetus for the development of large structures for space applications. In order to increase payload efficiency, these structures will have to be fabricated from lightweight, highly efficient components. For the following reasons these qualities are uniquely offered by composite tubes: (1) Tubes are components of high structural efficiency, as measured by torsional and bending stiffness per unit weight. (2) Composite materials are characterized by high stiffness and strength per weight ratios. (3) Composite tubes can be fabricated with relative ease compared to other structural members (such as I or T section beams) using filament winding, braiding, or other manufacturing techniques. (4) Composite tubes eliminate the free-edge problem, a problem which exists in other structural components.

Since space structures operate under zero gravity conditions, they are subjected to relatively mild mechanical loads However environmental considerations, such as extremely low (-250°F) temperatures, are of great concern. The space environment will subject the material to a temperature difference as great as -500°F degrees (from the curing temperature). This is a substantial temperature differential and will undoubtedly result in high residual stresses. Therefore, in order to achieve optimal structure design, it is of great importance to be able to calculate the thermal stress magnitudes in composite tubes.

This paper reports the first step in the investigation of residual stresses in composite tubes. Specific interest is in the residual stresses in a tube operating at -250°F. With the tube having a cure temperature of, say, 250°F, this represents a severe temperature effect. In reality the effects of mechanical loads must be superposed on these stresses. These loads, however, will be light and the primary effect in the tube may well be the thermal effect. Thus the problem studied here will be the determination of the thermally induced stresses in cross-ply tubes. The problem can be considered axisymmetric. Single-material tubes as well as tubes with layers of various materials are studied. In the following work, cross-ply tubes are studied using an elasticity approach. These solutions are compared with results from a laminated shell theory based on Donnell's shell approximation.

This report begins by presenting a review of the literature relevant to the present study. Elasticity theory and shell theory solutions are reviewed. Next, the governing equations used in the elasticity solutions are introduced and the appropriate simplifications and assumptions use to derive the solution for one-layer transversely isotropic tube are outlined. The procedure for the solution of a multiple layered cross-ply tube is then presented. Following this the shell theory solution to the problem is discussed. Finally, numerical results and conclusions are presented.

LITERATURE REVIEW

The residual stresses in orthotropic hollow cylinders can be studied using an exact elasticity solution, an approximate approach (such as shell theory), or by numerical methods. Since the elasticity approach provides the most accurate solution to the problem, it is a very useful tool in determining the accuracy of the approximate and numerical methods. Unfortunately, elasticity solutions are more complex and thus difficult to obtain.

In the following discussion, literature related to the thermo-mechanical response of anisotropic hollow cylinders will be reviewed. Specific attention will be directed to the analysis of residual stresses in composite tubes. In addition, studies relating to mechanical loading of tubes will be mentioned. The method used in investigating mechanical loads are similar to those employed in thermal problems. Most of the work found in the literature relates to hollow composite tube subjected to combined mechanical loads.

The review begins by discussing the elasticity method. Following this is a review of the shell theory approximate method. Finally, the two methods will be compared.

Elasticity Methods

Sherrer [1] used an elasticity solution to determine the state of stress in a multi-layer filament-wound cylinder subjected to a combined axial, torsion, and pressure loads. In his model the fiber and matrix are treated as separate entities. Consequently, his final expressions

were more complex than if the fiber and matrix had been treated as homogeneous orthotropic material, as is commonly done in the analysis of composite materials. Pagano et al. [2], used a plane-strain elasticity solution to investigate the uniformity of the state of stress in an anisotropic helical-wound composite tube subjected to an extensional force. In their study the laminae were treated as homogeneous orthotropic layers. This resulted in considerably simplified expressions. Pagano and Whitney [3] applied a modified plane-strain elasticity solution, in combination with shell theory, to study the effect of the degree of material anisotropy and shell geometry on the state of stress across the cylinder thickness under simulated laboratory loadings and end constraints. Their approach consisted of combining elasticity solution to an infinitely long cylinder and a shell analysis to a tube of finite length. At a certain distance from the constrained ends, the shell's stress resultants were assumed to be statically equivalent to the stresses calculated by the elasticity solution. Since the problem is easily solved by shell theory, this type of analysis simplified the boundary value problem. Weng [4] used a plane-strain elasticity method to solve the thermal problem in hollow anisotropic cylinders. Weng examined the thermal stress in grades ATJ and ZTA graphite cylinders subjected to thermal gradients in a radial direction. The bulk graphite was treated as transversely isotropic material with the hoop-radial plane taken as the plane of material isotropy. This condition constitutes a special problem in fiber reinforced composite tubes with axial fiber orientation. His solution was similar to the one generally

employed in the analysis of isotropic cylinders subjected to radial temperature gradients (see Timoshenko and Goodier, page 448 [5]). In Weng's solution the constitutive relations were generalized to transversely isotropic materials.

The above studies demonstrated that elasticity solutions can be obtained for certain anisotropic hollow cylinder problems. However, these solutions become more complex for angle-ply cylinders. In addition, elasticity solutions to thermal problems involve the solution to nonhomogeneous ordinary (or partial) differential equations and thus adds further complexities to the problem. In some cases, depending on cylinder geometry and loading conditions, one can utilize an approximate method such as first or higher-order shell theory in order to simplify the solution to anisotropic hollow cylinder problems.

Shell Approximation Methods

A number of investigators have utilized shell analysis to quantify the state of stresses in hollow anisotropic cylinders under combined loads. One advantage of a shell theory approach, as opposed to the elasticity method, is the relative ease with which boundary value problems can be solved. Whitney and Halpin [6] used Donnell's shell approximation analysis to characterize the response of laminated composites tubes subjected to combined thermal and mechanical loads. Pao et al. [7, 8] demonstrated that by applying Flugge's higher order shell theory to orthotropic laminated tubes, the thermal loading problem could be solved. Pao also compared the results obtained with a higher order

solution. He found that the Donnell method was fairly accurate if the shell radius-to-thickness ratio, R/t, was large. However, error of up to 10% occurred in the hoop force and moment resultants when this ratio dropped below 10. Stavsky and Smolash [9] used a shell approximation to derive the thermoelastic equations for an orthotropic semi-infinite hollow cylinder shell. In addition, they also developed the heat transfer equations which govern the temperature gradient through a shell.

It appears that a shell approximate method can be useful in the solution of circular hollow composite shell problems. However, one should be aware of the various factors which can contribute to error in the analysis. In the following section we shall discuss some of the factors and compare the elasticity and shell approximation solution methods.

Comparison of the Elasticity and Shell Approximate Methods

Shell geometry is one factor which determines the accuracy of the method employed. Whitney et al. [10] showed that shell approximation yielded accurate results, as compared to an elasticity solution, for $R/t > 10$ in a laminated tube subjected to combined mechanical loads. It has also been found in previous studies [2, 3, 10] that for large values of R/t, the state of stresses predicted in each layer of a laminate tube is similar to those predicted for a laminated flat coupon of infinite width subjected to the equivalent membrane forces (i.e., using classical lamination theory). Rizzo and Vicario [11] used a three-dimensional

finite element analysis to determine the effect of R/t and length-to-radius ratios, L/R, on the state of stress in composite tube specimens. They found that for a boron-epoxy tube the stress was nearly uniform away from the grip (for R/t > 25) and hence could be accurately characterized by a classical lamination theory formulation.

The effect of lamina fiber angle, relative to the tube axis, and stacking sequence on the state of stress in a thermally loaded composite shell was investigated by Stavsky and Smolash [9]. A similar problem was addressed by Pagano and Whitney [3] in composite tubes subjected to combined mechanical loads. Both studies found that fiber angle and lamination sequence strongly effected the state of stress in composite shells under combined mechanical and thermal loads. Whitney [12] and Zukas [13] have pointed out that in laminated composite tubes transverse thermal expansion as compared to in-plane fiber direction thermal expansion is relatively large. Consequently, the assumption of zero strain in the radial direction employed by Donnell shell theory will result in considerable inaccuracy in the stress calculation. Whitney presented a modified shell theory which incorporated the effect of transverse normal thermal strain. In comparing his modified method with an exact elasticity solution, he found a good level of agreement between the two for an angle ply tube of R/t = 10.5. However, an analysis of the same tube using Donnell shell theory resulted in a considerable amount of error.

From the literature review, it is clear that for values of R/t less than 10, an elasticity solution or a shell theory of higher order than

Donnell's shell theory must be used. Whitney has presented a higher order shell theory. The elasticity theory Whitney referenced was a private communication with Pagano and no written reference was available. Thus the elasticity solution in the following section is presented for the purpose of documenting the elasticity approach to the problem.

GOVERNING EQUATIONS

In the following sections the solutions used to compute residual stresses in composite tubes are outlined. The discussion begins with a description of the laminated composite tube geometry. This is followed by a presentation of the equations which govern the behaviour of a composite tube. Finally, the analytical solutions to the thermomechanical problem of composite tubes using various elasticity solutions and a classical shell theory approach are presented and compared.

Geometry

In the following analyses, three sets of orthogonal axis systems will be referred to; polar cylindrical (r, θ, and x), laminate global (x', y', and z'), and lamina principal (1, 2, and 3) coordinates.

Figures 1 and 2 show the tube geometry and the coordinate systems. The tube's inner radius is denoted as r_i, the outer radius is r_o, and the tube's mean radius, R. Note that

$$R = \frac{r_i + r_o}{2}$$

and R locates the distance of the tubes wall's geometric midsurface relative to the tube centerline.

The cylindrical r-θ-x system measures the location of a material point within the tube relative to a reference point on the tube's centerline. The directions of the r-θ-x coordinates are colinear with the three base vectors $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_x$ shown in fig. 1. This local ortho-

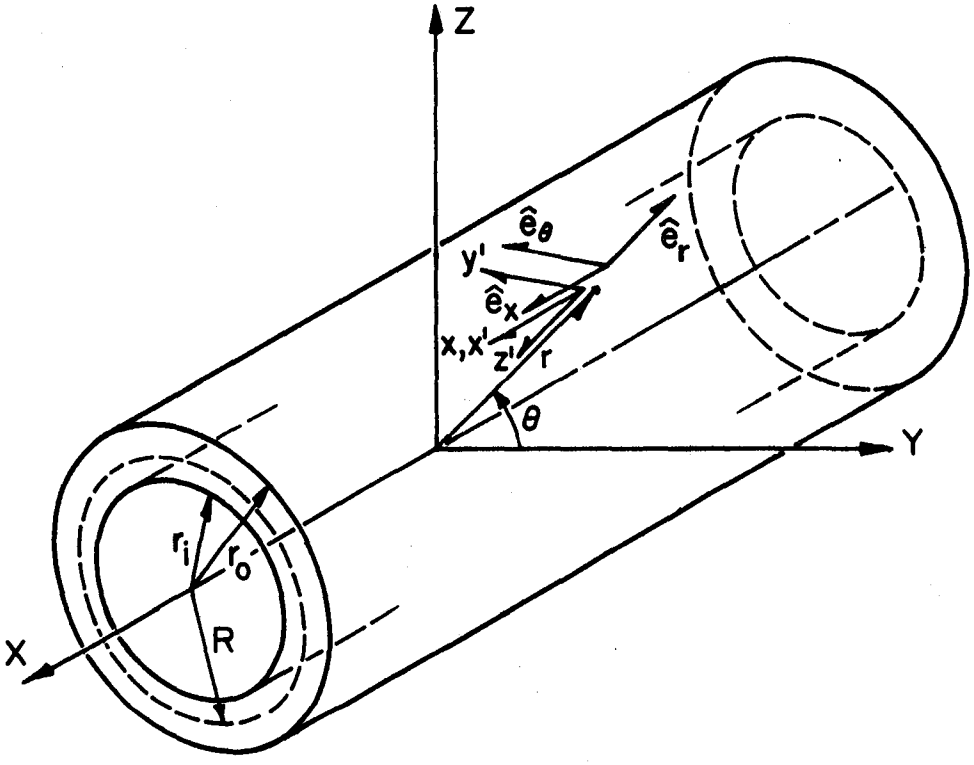


Fig. 1. Tube geometry and coordinate systems used.

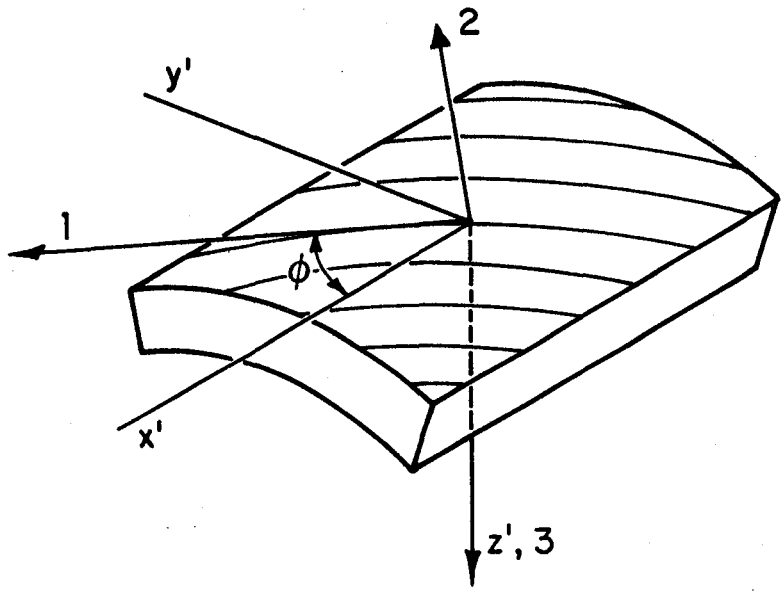


Fig. 2. Detail of laminate and material coordinate systems.

gonal base vector system is located in space by the cylindrical coordinates r, θ, and x. The quantity r is the radial distance of the point from the tube's centerline, θ is the circumferential position of the point, and x measures the position of the point along the tube.

The laminate global coordinates x', y' and z' are, except for the prime notation, those commonly used in the study of composite material analysis [14]. They are an off-axis coordinate system. The laminate x',y',z' coordinates are fixed at the laminate's midsurface, i.e., at a distance R from the cylinder's centerline. It should be noted that

$$r = R - z'.$$

The lamina's principal coordinates 1, 2 and 3 coincide with the material principal directions. The 1 and 2 axes represent the fiber and matrix directions, respectively. The 3 axis is oriented normal to the lamina plane and is colinear with the r and z' axis. The orientation of the lamina's fiber (axis 1) relative to the global x' axis is measured by the angle ϕ as shown by Fig. 2. The above conventions were chosen in order to unify the notations employed by elasticity and shell theories while maintaining the common convention used in the area of laminated composite analysis.

Constitutive Relations

The thermoelastic constitutive relations for an orthotropic lamina can be written relative to the lamina coordinate system as

$$\begin{Bmatrix} \varepsilon_1-\alpha_1\Delta T \\ \varepsilon_2-\alpha_2\Delta T \\ \varepsilon_3-\alpha_3\Delta T \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ & & \frac{1}{E_3} & 0 & 0 & 0 \\ & & & \frac{1}{G_{23}} & 0 & 0 \\ & \text{Symmetric} & & & \frac{1}{G_{13}} & \\ & & & & & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} \quad (1)$$

where α_1, α_2 and α_3 are the lamina's linear coefficients of thermal expansion in the 1, 2 and 3 directions, respectively. The quantities ε_1, ε_2, ε_3, γ_{23}, γ_{13}, and γ_{12} are the total strains. The quantities σ_1, ..., τ_{12} are the stresses. The lamina's moduli in the 1, 2 and 3 directions are designated by E_i (i = 1,2,3), the shear moduli for the i-j planes by G_{ij} (i,j = 1,2,3), and the material's Poissons' ratios are given by ν_{ij} (i,j = 1,2,3). The following three reciprocal relations hold:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \qquad i,j = 1,2,3. \qquad (2)$$

The temperature change with respect to some reference temperature is designated by ΔT, where $\Delta T > 0$ corresponds to a temperature rise. The square material matrix is designated by [S] and is referred to as the compliance matrix, i.e.,

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \quad (3)$$

Due to eq. 2, the above matrix is symmetric. The lamina stress-strain constitutive relations can be written in an inverted form as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1-\alpha_1\Delta T \\ \varepsilon_2-\alpha_2\Delta T \\ \varepsilon_3-\alpha_3\Delta T \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{4}$$

where $[C] = [S]^{-1}$ and $[C]$ is termed the lamina stiffness matrix. The inverse of $[S]$ gives the components of $[C]$ as follows:

$$C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{S} \qquad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S}$$

$$C_{22} = \frac{S_{33}S_{11} - S_{13}^2}{S} \qquad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}$$

$$C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{S} \qquad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S} \tag{5}$$

$$C_{44} = \frac{1}{S_{44}}, \quad C_{55} = \frac{1}{S_{55}}, \quad C_{66} = \frac{1}{S_{66}},$$

where

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2S_{23}S_{13}.$$

In terms of the engineering constants the C_{ij} components may be expressed as follows:

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{E_2E_3C}, \quad C_{12} = \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3C}, \quad C_{13} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2C}$$

$$C_{22} = \frac{1-\nu_{13}\nu_{31}}{E_1E_3C}, \quad C_{23} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2C}, \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{E_1E_2C} \tag{6}$$

$$C_{44} = G_{23}, \quad C_{55} = G_{13}, \quad C_{66} = G_{12}$$

and

$$C = \frac{1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

As a consequence of the fact that $S_{ij} = S_{ji}$, it can be shown that $C_{ij} = C_{ji}$.

Since the equilibrium and kinematic equations of elasticity are generally written in cylindrical coordinates, it is desirable to express the constitutive relations in terms of the stresses and strains referred

to cylindrical coordinate. This is achieved by writing the stresses in the 1-2-3 system in terms of the stresses in the x'-y'-z' system, and writing the strains in the 1-2-3 system in terms of strains in the x'-y'-z' system. From that point, transformation of stresses and strains from the x'-y'-z' system to the r,θ,x system is made.

The relation between the stresses in the 1-2-3 system and stresses in the x'-y'-z' system is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = [T_1] \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \sigma_{z'} \\ \tau_{y'z'} \\ \tau_{x'z'} \\ \tau_{x'y'} \end{Bmatrix} \tag{7}$$

where

$$[T_1] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \tag{8}$$

with $m = \cos\theta$ and $n = \sin\theta$.

The relations between the total strains in the two systems is

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = [T_2] \begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \varepsilon_{z'} \\ \gamma_{y'z'} \\ \gamma_{x'z'} \\ \gamma_{x'y'} \end{Bmatrix} , \tag{9}$$

where

$$[T_2] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \tag{10}$$

The thermal strains transform like the total strains, i.e.,

$$\begin{Bmatrix} \alpha_1\Delta T \\ \alpha_2\Delta T \\ \alpha_3\Delta T \\ 0 \\ 0 \\ 0 \end{Bmatrix} = [T_2] \begin{Bmatrix} \alpha_{x'}\Delta T \\ \alpha_{y'}\Delta T \\ \alpha_{z'}\Delta T \\ \alpha_{y'z'}\Delta T \\ \alpha_{x'z'}\Delta T \\ \alpha_{x'y'}\Delta T \end{Bmatrix} \tag{11}$$

where $\alpha_{x'}$ ... $\alpha_{x'y'}$ are coefficients of thermal expansion in the x'-y'-z' system. The coefficients of thermal expansion in the x'-y'-z' system are given explicitly by inverting eq. 11. The result is

$$\alpha_{x'} = m^2\alpha_1 + n^2\alpha_2; \quad \alpha_{y'} = n^2\alpha_1 + m^2\alpha_2, \quad \alpha_{z'} = \alpha_3$$
$$\alpha_{y'z'} = 0 = \alpha_{x'z'}; \quad \alpha_{x'y'} = 2mn(\alpha_1-\alpha_2). \tag{12}$$

Substituting the transformed stresses and strains into the constitutive equations, eqs. 1 and 4, gives

$$\begin{Bmatrix} \varepsilon_{x'} - \alpha_{x'}\Delta T \\ \varepsilon_{y'} - \alpha_{y'}\Delta T \\ \varepsilon_{z'} - \alpha_{z'}\Delta T \\ \gamma_{y'z'} \\ \gamma_{x'z'} \\ \gamma_{x'y'} - \alpha_{x'y'}\Delta T \end{Bmatrix} = [\bar{S}] \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \sigma_{z'} \\ \tau_{y'z'} \\ \tau_{x'z'} \\ \tau_{x'y'} \end{Bmatrix} \tag{13}$$

where

$$[\bar{S}] = [T_2]^{-1}\,[S][T_1]\ , \tag{14}$$

$[\bar{S}]$ being referred to as the transformed compliance. The transformed compliance matrix is of the form

$$[\overline{S}] = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & \overline{S}_{26} \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & \overline{S}_{36} \\ 0 & 0 & 0 & \overline{S}_{44} & \overline{S}_{45} & 0 \\ 0 & 0 & 0 & \overline{S}_{45} & \overline{S}_{55} & 0 \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{36} & 0 & 0 & \overline{S}_{66} \end{bmatrix} \tag{15}$$

with

$$\begin{aligned}
\overline{S}_{11} &= m^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22} \\
\overline{S}_{12} &= \overline{S}_{21} = (m^4 + n^4) S_{12} + m^2 n^2 (S_{11} + S_{22} - S_{66}) \\
\overline{S}_{13} &= \overline{S}_{31} = -(n^2 S_{23} + m^2 S_{13}) \\
\overline{S}_{16} &= \overline{S}_{61} = m^3 n (2S_{11} - 2S_{12} - S_{66}) + mn^3 (2S_{12} - 2S_{22} + S_{66}) \\
\overline{S}_{22} &= n^4 S_{11} + m^2 n^2 (2S_{12} + S_{66}) + n^4 S_{22} \\
\overline{S}_{23} &= \overline{S}_{32} = -(m^2 S_{23} + n^2 S_{13}) \\
\overline{S}_{26} &= \overline{S}_{62} = mn[m^2 (2S_{22} - S_{66} + 2S_{12}) + n^2 (S_{66} - 2S_{11} - 2S_{12})] \\
\overline{S}_{33} &= S_{33} \\
\overline{S}_{36} &= \overline{S}_{63} = 2mn (S_{13} - S_{23}) \\
\overline{S}_{44} &= m^2 S_{44} + n^2 S_{55} \\
\overline{S}_{45} &= \overline{S}_{54} = mn (S_{44} - S_{55}) \\
\overline{S}_{55} &= n^2 S_{44} + m^2 S_{55} \\
\overline{S}_{66} &= 4m^2 n^2 (S_{11} + S_{22} + 2S_{12} - S_{66}) + S_{66} \, .
\end{aligned} \tag{16}$$

The inverse relation is given by

$$\begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \sigma_{z'} \\ \tau_{y'z'} \\ \tau_{x'z'} \\ \tau_{x'y'} \end{Bmatrix} = [\overline{C}] \begin{Bmatrix} \varepsilon_{x'} - \alpha_{x'}\Delta T \\ \varepsilon_{y'} - \alpha_{y'}\Delta T \\ \varepsilon_{z'} - \alpha_{z'}\Delta T \\ \gamma_{y'z'} \\ \gamma_{x'y'} \\ \gamma_{x'y'} - \alpha_{x'y'}\Delta T \end{Bmatrix} \tag{17}$$

with

$$[\overline{C}] = [T_1]^{-1}[C][T_2] \tag{18}$$

The matrix $[\overline{C}]$ is referred to as the transformed stiffness matrix. It is of the form

$$[\overline{C}] = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66} \end{bmatrix} \tag{19}$$

The transformed $\overline{C}_{ij}$ components are given in terms of the principal lamina stiffness coefficients as;

$$
\begin{aligned}
\bar{C}_{11} &= m^4 C_1 + m^2 n^2 (C_{12} + 2C_{66}) + n^4 C_{22} \\
\bar{C}_{12} &= \bar{C}_{21} = n^2 m^2 (C_{11} + C_{22} - 4C_{66}) + (n^4 + m^4) C_{12} \\
\bar{C}_{13} &= \bar{C}_{31} = m^2 C_{13} + n^2 C_{23} \\
\bar{C}_{16} &= \bar{C}_{61} = mn[m^2 (C_{11} - C_{12} - 2C_{66}) + n^2 (C_{12} - C_{22} + 2C_{66})] \\
\bar{C}_{22} &= n^4 C_{11} + 2n^2 m^2 (C_{12} + 2C_{66}) + m^4 C_{22} \\
\bar{C}_{23} &= \bar{C}_{32} = n^2 C_{13} + m^2 C_{23} \\
\bar{C}_{26} &= \bar{C}_{62}\ mn[n^2 (C_{11} - C_{12} - 2C_{66}) + m^2 (C_{12} - C_{22} + 2C_{66})] \\
\bar{C}_{33} &= C_{33} \\
\bar{C}_{36} &= \bar{C}_{63} = mn(C_{13} - C_{23}) \\
\bar{C}_{44} &= m^2 C_{44} + n^2 C_{55} \\
\bar{C}_{45} &= \bar{C}_{54} = mn(C_{55} - C_{44}) \\
\bar{C}_{55} &= n^2 C_{44} + m^2 C_{55} \\
\bar{C}_{66} &= m^2 n^2 (C_{11} - 2C_{12} + C_{22}) + C_{66} (m^2 - n^2)^2
\end{aligned}
\tag{20}
$$

Finally, the stresses and strains can be transformed to the r-θ-x system. The familiar tensor transformation is used, i.e.,

$$\sigma'_{ij} = a_{ik} a_{j\ell} \sigma_{k\ell} \ . \tag{21}$$

The quantity a_{mn} is the direction cosine between the +m axis in the x'-y'-z' system and the +n axis in the r-θ-x system. The stresses on the left of eq. 21 are the stresses on the x'-y'-z' system while the stresses on the right are the stresses on the r-θ-x system. The following table shows the specific values of a_{mn} in the transformation.

Values of a_{mn}

	r	θ	x
x'	$a_{11} = 0$	$a_{12} = 0$	$a_{13} = 1$
y'	$a_{21} = 0$	$a_{22} = 1$	$a_{23} = 0$
z'	$a_{31} = -1$	$a_{32} = 0$	$a_{33} = 0$

(22)

Using eqs. 21 and 22,

$$\begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \sigma_{z'} \\ \tau_{y'z'} \\ \tau_{x'z'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ -\tau_{r\theta} \\ -\tau_{xr} \\ \tau_{x\theta} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{x'} - \alpha_{x'}\Delta T \\ \varepsilon_{y'} - \alpha_{y'}\Delta T \\ \varepsilon_{z'} - \alpha_{z'}\Delta T \\ \gamma_{y'z'} \\ \gamma_{x'z'} \\ \gamma_{x'y'} - \alpha_{x'y'}\Delta T \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x - \alpha_x\Delta T \\ \varepsilon_\theta - \alpha_\theta\Delta T \\ \varepsilon_r - \alpha_r\Delta T \\ -\gamma_{r\theta} \\ -\gamma_{xr} \\ \gamma_{x\theta} - \alpha_{x\theta}\Delta T \end{Bmatrix} \quad (23,24)$$

Because $\overline{C}_{44}$, $\overline{C}_{45}$, $\overline{C}_{55}$ and $\overline{S}_{44}$, $\overline{S}_{45}$, $\overline{S}_{55}$ are the only nonzero terms in the 4th and 5th row and column of the $[\overline{C}]$ and $[\overline{S}]$ matrix, respectively, the constitutive equations in the r-θ-x system can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} = [\overline{C}] \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_r - \alpha_r \Delta T \\ \gamma_{r\theta} \\ \gamma_{xr} \\ \gamma_{x\theta} - \alpha_{x\theta} \Delta T \end{Bmatrix} \tag{25}$$

and

$$\begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_r - \alpha_r \Delta T \\ \gamma_{r\theta} \\ \gamma_{xr} \\ \gamma_{x\theta} - \alpha_{x\theta} \Delta T \end{Bmatrix} = [\overline{S}] \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{xr} \\ \tau_{x\theta} \end{Bmatrix} \tag{26}$$

With eqs. 25 and 26, the constitutive behavior of the tube is expressed in terms of the r-θ-x coordinate system.

Equilibrium

The stress equilibrium equations can be derived if the equilibrium of forces which act on a small element isolated from a cylindrical body are considered. Summing forces in the x, θ and r directions results in the following stress equilibrium equations:

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{xr}}{\partial x} + F_r = 0 \tag{27.a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta x}}{\partial x} + \frac{2}{r}\tau_{r\theta} + F_\theta = 0 \tag{27.b}$$

$$\frac{\partial \tau_{xr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta x}}{\partial \theta} + \frac{\partial \sigma_x}{\partial x} + \frac{1}{r}\tau_{xr} + F_x = 0 \ , \tag{27.c}$$

where F_r, F_θ and F_x are the body forces in the r, θ and x directions. In the present work these forces will be zero.

Strain-Displacement Relations

The strain-displacement relations in polar coordinates can be determined in a variety of ways. These relations are

$$\varepsilon_r = \frac{\partial w}{\partial r}, \ \varepsilon_\theta = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} + w\right], \ \varepsilon_x = \frac{\partial u}{\partial x}$$

$$\gamma_{r\theta} = \frac{1}{r}\left[\frac{\partial w}{\partial \theta} - v + r\frac{\partial v}{\partial r}\right] \tag{28}$$

$$\gamma_{xr} = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial x}$$

$$\gamma_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} \ ,$$

where w, v and u are the displacement components in the e_r, e_θ and e_x polar coordinate directions. This is a notation commonly used in the area of elasticity. It should be noted that the engineering shear strain is being used here. In most general elasticity

problems, displacements are a function of x, θ and r and can be expressed as:

$$u = u(x,\theta,r)$$
$$v = v(x,\theta,r) \tag{29}$$
$$w = w(x,\theta,r) \ .$$

Compatibility Equations

For completeness, the compatibility equations for the strains are recorded. In many cases these equations are automatically satisfied. For thoroughness, however, they should always be checked. The compatibility equations are

$$\frac{\partial^2 \varepsilon_r}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial r^2} - \frac{\partial^2 \gamma_{rx}}{\partial r \partial x} = 0 \tag{30.a}$$

$$\frac{\partial^2 \varepsilon_\theta}{\partial x^2} - \frac{1}{r}\frac{\partial^2 \gamma_{\theta x}}{\partial \theta \partial x} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_x}{\partial \theta^2} + \frac{1}{r}\frac{\partial \varepsilon_x}{\partial r} - \frac{1}{r}\frac{\partial \gamma_{rx}}{\partial x} = 0 \tag{30.b}$$

$$\frac{\partial^2 \varepsilon_r}{\partial \theta^2} - r\frac{\partial \varepsilon_r}{\partial r} - \frac{\partial^2 (r\gamma_{r\theta})}{\partial r \partial \theta} + \frac{\partial}{\partial r}\left(r^2 \frac{\partial \varepsilon_\theta}{\partial r}\right) = 0 \tag{30.c}$$

$$\frac{\partial^2 \gamma_{r\theta}}{\partial x^2} - r\frac{\partial^2}{\partial r \partial x}\left(\frac{1}{r}\gamma_{\theta x}\right) - \frac{1}{r}\frac{\partial^2 \gamma_{rx}}{\partial \theta \partial x} + 2\frac{\partial^2}{\partial r \partial \theta}\left(\frac{1}{r}\varepsilon_x\right) = 0 \tag{30.d}$$

$$\frac{2}{r}\frac{\partial^2 \varepsilon_r}{\partial \theta \partial x} - \frac{\partial^2}{\partial r \partial \theta}\left(\frac{1}{r}\gamma_{rx}\right) + \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\gamma_{\theta x})\right] - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial x}(r^2\gamma_{r\theta}) = 0 \tag{30.e}$$

$$\frac{\partial^2 \gamma_{rx}}{\partial \theta^2} - 2r\frac{\partial \varepsilon_r}{\partial x} - \frac{\partial^2}{\partial r \partial \theta}(r\gamma_{x\theta}) + 2r\frac{\partial^2}{\partial r \partial x}(r\varepsilon_\theta) - \frac{\partial^2}{\partial \theta \partial x}(r\gamma_{r\theta}) = 0 \tag{30.f}$$

Boundary Conditions

To complete the statement of the problem the boundary conditions on all tube surfaces must be specified. The boundary value problem may be categorized into two types. In the first type the displacements u, v and w are prescribed on the body's surfaces and the stresses, strains, and displacements within the body are to be determined. In the second type, stresses (or tractions) are prescribed on the body's surfaces and the stresses, strains, and displacements within the body are to be determined. More often in elasticity problems one is faced with mixed boundary value problem, a problem for which stresses are prescribed for part of the body's surface, and displacements on the remainder. For the problem at hand, the boundary conditions will be generally specified as

on the cylindrical surfaces, i.e., @ $r = r_o$ and $r = r_i$

either σ_r or w is specified, and

either τ_{rx} or u is specified, and

either $\tau_{r\theta}$ or v is specified.

on the cylinder's ends, i.e., @ $x = x_{end}$

either σ_x or u is specified, and

either $\tau_{x\theta}$ or v is specified, and

either τ_{xr} or w is specified.

However, it is possible to specify integrated conditions at the ends of the tube, i.e.,

$$\int_A \sigma_x \, dA = \text{specified}$$

$$\int_A r\tau_{x\theta} \, dA = \text{specified}$$

$$\begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \qquad @\ x = x_{end}$$

etc.

Here A is the annular cross-sectional area of the layer.

For the most part, solutions to the general field equations and the specified boundary conditions do not exist. Hence, physical assumptions are made which simplify the mathematics. The mathematically simplier problem is then solved and the resulting answers interpreted in the context of the physical assumptions. The approximations for determining the residual thermal stresses in composite tubes will be stated later.

Multiple Layers

Laminated composite tubes are most often constructed of multiple layers. Each layer is governed by the field equations discussed in the previous sections. Since these layers are part of the tube, certain conditions should be satisfied at each layer interface. For the layers within the laminate, these interface conditions replace boundary conditions on the cylindrical surfaces. The interface conditions between the kth and $(k-1)^{st}$ layers are that all tractions acting on the surface are continuous and all displacements are continuous. These conditions can be expressed as:

$$u^k = u^{k-1},\ v^k = v^{k-1},\ w^k = w^{k-1}$$

$$\sigma_r^k = \sigma_r^{k-1},\ \tau_{r\theta}^k = \tau_{r\theta}^{k-1},\ \tau_{rx}^k = \tau_{rx}^{k-1}\ . \qquad @\ r = r_{interface}.$$

At the end of each layer

either σ_x or u is specified

either $\tau_{x\theta}$ or v is specified $\qquad @\ x = x_{end}.$

either τ_{xr} or w is specified.

As with the single layer, it is possible to specify integrated conditions at the end of the multiple layer tube, i.e.,

$$\int_A \sigma_x\, dA = \text{specified}$$

$$\int_A r\tau_{x\theta}\, dA = \text{specified} \qquad @\ x = x_{end}.$$

.

.

.

etc.

Here A is the annular cross-sectional area of the multiple layers.

Assumptions and Simplifications for Computing Residual Thermal Stresses

In the following sections we shall discuss the assumptions and simplifications used in determining residual thermal stresses in multi-

layer cross-ply composite tubes.

1) Axisymmetric simplification

If the tube's material properties and temperature distribution are independent of θ, then the displacements u, v, and w are not dependent on θ. Hence

$$\begin{aligned} u(r,\theta,x) &= u(r,x) \\ v(r,\theta,x) &= v(r,x) \\ w(r,\theta,x) &= w(r,x) \end{aligned} \tag{31}$$

In addition

$$\frac{\partial}{\partial\theta} \equiv 0. \tag{32}$$

As a result, the strains will be independent of θ. Through Hooke's Law, the stresses will then be independent of θ. Thus

$$\begin{aligned} &\sigma_r = \sigma_r(r,x),\ \sigma_\theta = \sigma_\theta(r,x),\ \sigma_x = \sigma_x(r,x) \\ &\tau_{r\theta} = \tau_{r\theta}(r,x),\ \tau_{xr} = \tau_{xr}(r,x),\ \tau_{x\theta} = \tau_{x\theta}(r,x) \\ &\varepsilon_r = \varepsilon_r(r,x),\ \varepsilon_\theta = \varepsilon_\theta(r,x),\ \varepsilon_x = \varepsilon_x(r,x) \\ &\gamma_{r\theta} = \gamma_{r\theta}(r,x),\ \gamma_{xr} = \gamma_{xr}(r,x),\ \gamma_{x\theta} = \gamma_{x\theta}(r,x)\ . \end{aligned} \tag{33}$$

The stress-equilibrium, strain-displacement, and compatibility equations simplify to:

- simplified stress equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta x}}{\partial x} + \frac{2}{r}\tau_{r\theta} = 0 \qquad (34)$$

$$\frac{\partial \tau_{xr}}{\partial r} + \frac{\partial \sigma_x}{\partial x} + \frac{1}{r}\tau_{xr} = 0$$

- simplified strain-displacement equations.

$$\varepsilon_r = \frac{\partial w}{\partial r}, \quad \varepsilon_\theta = \frac{w}{r}, \quad \varepsilon_x = \frac{\partial u}{\partial x}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} \qquad (35)$$

$$\gamma_{xr} = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial x}$$

$$\gamma_{x\theta} = \frac{\partial v}{\partial x}$$

- simplified compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial r^2} - \frac{\partial^2 \gamma_{rx}}{\partial r \partial x} = 0 \tag{36.a}$$

$$\frac{\partial^2 \varepsilon_\theta}{\partial x^2} + \frac{1}{r}\frac{\partial \varepsilon_x}{\partial r} - \frac{1}{r}\frac{\partial \gamma_{rx}}{\partial x} = 0 \tag{36.b}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varepsilon_\theta}{\partial r}\right) - r\frac{\partial \varepsilon_r}{\partial r} = 0 \tag{36.c}$$

$$\frac{\partial^2 \gamma_{r\theta}}{\partial x^2} - r\frac{\partial^2}{\partial r \partial x}\left(\frac{1}{r}\gamma_{\theta x}\right) = 0 \tag{36.d}$$

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\gamma_{\theta x}\right)\right] - \frac{1}{r^2}\frac{\partial^2}{\partial r \partial x}\left(r^2 \gamma_{r\theta}\right) = 0 \tag{36.e}$$

$$\frac{\partial \varepsilon_r}{\partial x} - \frac{\partial^2}{\partial r \partial x}\left(r\varepsilon_\theta\right) = 0 \tag{36.f}$$

2) Cross-ply assumption

Only cross-ply laminated cylinders are considered in the present study. Then within each layer there is no coupling between extensional and shear stresses. The cylindrical lamina constitutive relations for a layer are reduced to the following:

- simplified strain-stress relations

$$\left\{\begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_r\end{Bmatrix} - \begin{Bmatrix}\alpha_x\\ \alpha_\theta\\ \alpha_r\end{Bmatrix}\Delta T\right\} = \begin{bmatrix}\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13}\\ & \bar{S}_{22} & \bar{S}_{23}\\ & & \bar{S}_{33}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_r\end{Bmatrix}$$

$$\begin{Bmatrix}\gamma_{r\theta}\\ \gamma_{xr}\\ \gamma_{x\theta}\end{Bmatrix} = \begin{bmatrix}\bar{S}_{44} & 0 & 0\\ & \bar{S}_{55} & 0\\ & & \bar{S}_{66}\end{bmatrix}\begin{Bmatrix}\tau_{r\theta}\\ \tau_{xr}\\ \tau_{x\theta}\end{Bmatrix}. \tag{37}$$

- simplified stress-strain relations

$$\begin{Bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_r\end{Bmatrix} = \begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13}\\ & \bar{C}_{22} & \bar{C}_{23}\\ & & \bar{C}_{33}\end{bmatrix}\left\{\begin{Bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_r\end{Bmatrix} - \begin{Bmatrix}\alpha_x\\ \alpha_\theta\\ \alpha_r\end{Bmatrix}\Delta T\right\}$$

$$\begin{Bmatrix}\tau_{r\theta}\\ \tau_{xr}\\ \tau_{x\theta}\end{Bmatrix} = \begin{bmatrix}\bar{C}_{44} & 0 & 0\\ & \bar{C}_{55} & 0\\ & & \bar{C}_{66}\end{bmatrix}\begin{Bmatrix}\gamma_{r\theta}\\ \gamma_{xr}\\ \gamma_{x\theta}\end{Bmatrix} \tag{38}$$

In a single orthotropic layer or in a multiple-layer cross-ply tube subjected to a spatially uniform temperature distribution, there are no hoop displacements induced by thermal effects, i.e.,

$$v \equiv 0. \tag{39}$$

To summarize, the field equations which govern the residual thermal stress state in the individual layers in cross-ply composite tubes in a spatially uniform temperature field are:

- equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}(\sigma_r - \sigma_\theta) = 0 \tag{40.a}$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{\theta x}}{\partial x} + \frac{2}{r}\tau_{r\theta} = 0 \tag{40.b}$$

$$\frac{\partial \tau_{xr}}{\partial r} + \frac{\partial \sigma_x}{\partial x} + \frac{1}{r}\tau_{xr} = 0 \tag{40.c}$$

- strain-displacement

$$\varepsilon_r = \frac{\partial w}{\partial r}, \quad \varepsilon_\theta = \frac{w}{r}, \quad \varepsilon_x = \frac{\partial u}{\partial x}$$

$$\gamma_{r\theta} = 0 \tag{41}$$

$$\gamma_{xr} = \frac{\partial u}{\partial r} + \frac{\partial w}{\partial x}$$

$$\gamma_{x\theta} = 0$$

- compatibility

$$\frac{\partial^2 \varepsilon_r}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial r^2} - \frac{\partial^2 \gamma_{rx}}{\partial r \partial x} = 0$$

$$\frac{\partial^2 \varepsilon_\theta}{\partial x^2} + \frac{1}{r}\frac{\partial \varepsilon_x}{\partial r} - \frac{1}{r}\frac{\partial \gamma_{rx}}{\partial x} = 0$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varepsilon_\theta}{\partial r}\right) - r\frac{\partial \varepsilon_r}{\partial r} = 0 \qquad (42)$$

$0 = 0$ (automatically satisfied)

$0 = 0$ (automatically satisfied)

$$\frac{\partial \varepsilon_r}{\partial x} - \frac{\partial^2}{\partial r \partial x}(r\varepsilon_\theta) = 0$$

- constitutive equations

$$\left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \end{Bmatrix} - \begin{Bmatrix} \alpha_x \\ \alpha_\theta \\ \alpha_r \end{Bmatrix} \Delta T \right\} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} \\ & \bar{S}_{22} & \bar{S}_{23} \\ & & \bar{S}_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \end{Bmatrix}$$

$$\gamma_{r\theta} = 0 = \tau_{r\theta}, \ \gamma_{xr} = \bar{S}_{55}\,\tau_{xr}, \ \gamma_{x\theta} = 0 = \tau_{x\theta}, \qquad (43)$$

or

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \end{Bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} \\ & \overline{C}_{22} & \overline{C}_{23} \\ & & \overline{C}_{33} \end{bmatrix} \left\{ \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \end{Bmatrix} - \begin{Bmatrix} \alpha_x \\ \alpha_\theta \\ \alpha_r \end{Bmatrix} \Delta T \right\} \tag{44}$$

$$\tau_{r\theta} = 0 = \gamma_{r\theta}, \quad \tau_{xr} = \overline{C}_{55}\gamma_{xr}, \quad \tau_{x\theta} = 0 = \gamma_{x\theta} .$$

The above system of governing partial differential equations, and associated boundary conditions, are still quite difficult to solve for a general axisymmetric situation. The most difficult aspect of the equations as they relate to tubes is the ability to deal with finite-length geometries, i.e., dealing with boundary conditions at the ends of the tube. Further assumptions can be made, however, and particular problems can be solved. With a proper selection of particular problems, the solution to the more general problem can be approximated and perhaps bounded. This is the approach to the problem of interest here. Three related and solveable problems are examined with the idea that the residual stresses calculated from the three simplier problems closely approximate and indeed bound the stresses for a more general situation.

The three solutions are

i) plane-strain solution,

ii) plane-stress solution,

iii) generalized plane-strain solution.

For certain stress components, the first two solutions represent bounds to the problem. The generalized plane strain solution is the most accurate representation of away-from-end response. The solutions are discussed separately below.

Plane-Strain Solution

In the plane-strain analysis it is assumed that the tube is restrained from axial motion. Consequently, the axial displacement, u, is equal to zero and from the strain-displacement relations

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0 \ .$$

In addition, it will be assumed that none of the other variables in the problem vary with x. As a result, the lamina constitutive relations, eq. 44, become

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \end{Bmatrix} = [\overline{C}] \begin{Bmatrix} -\alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_r - \alpha_r \Delta T \end{Bmatrix} \tag{45}$$

$$\tau_{xr} = \overline{C}_{55} \gamma_{xr} .$$

When σ_θ and σ_r from eq. 45 are substituted into the first equilibrium equation, eq. 40a, the result is

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = \overline{C}_{23} \frac{d\varepsilon_\theta}{dr} + \overline{C}_{33} \frac{d\varepsilon_r}{dr} \tag{46}$$

$$+ \frac{1}{r} \{(\overline{C}_{23} - \overline{C}_{22})\varepsilon_\theta + (\overline{C}_{33} - \overline{C}_{23})\varepsilon_r\} = \frac{1}{r} \sum ,$$

where:

$$\sum = \{(\overline{C}_{23} - \overline{C}_{22})\alpha_\theta + (C_{33} - C_{23})\alpha_r + (\overline{C}_{13} - \overline{C}_{12})\alpha_x\}\Delta T \ . \qquad (47)$$

From the strain-displacement relations, eqs. 41, substitution for ε_θ and ε_r in eq. 46 leads to a nonhomogenous ordinary differential equation for w(r). That equation is

$$\overline{C}_{33}\left[\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right] - \overline{C}_{22}\frac{w}{r^2} = \frac{1}{r}\sum \ . \qquad (48)$$

The solution to eq. 48 can be obtained by making the following variable changes:

$$r = e^t.$$

This transforms eq. 48 into the following form:

$$\overline{C}_{33}\frac{d^2w}{dt^2} - \overline{C}_{22}w = e^t \sum \ .$$

The above ordinary differential equation has the following homogeneous solution:

$$w_h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \ ,$$

where,

$$\lambda_{1,2} = \pm\sqrt{\frac{\overline{C}_{22}}{\overline{C}_{33}}} \ .$$

To find the particular solution it is assumed

$$w_p = Be^t \ .$$

This yields the following total solution

$$w(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} e^t \ .$$

In the r domain

$$w(r) = A_1 r^{\lambda_1} + A_2 r^{\lambda_2} + \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} r \ . \tag{49}$$

In the case when $\overline{C}_{33} = \overline{C}_{22}$ the last term of eq. 49 should be replaced by

$$\frac{\sum}{2\overline{C}_{22}} r \ln r, \text{ and } \lambda_{1,2} = \pm 1 \ .$$

In addition, if $\alpha_r = \alpha_\theta$ and $\overline{C}_{12} = \overline{C}_{13}$, as in the case of 0° lamina which assumed to be transversely isotropic, then eq. 49 takes the following form

$$w(r) = A_1 r + A_2/r \ . \tag{50}$$

The strains ε_r and ε_θ can be determined from the strain-displacement relations. The strains are

$$\varepsilon_r = \frac{dw}{dr} = \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} + \sum_{\ell=1}^{2} A_\ell \lambda_\ell r^{\lambda_\ell - 1}$$

$$\varepsilon_\theta = \frac{1}{r} w = \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} + \sum_{\ell=1}^{2} A_\ell \, r^{\lambda_\ell - 1} \quad . \tag{51}$$

The stresses can be determined by the constitutive relations, namely

$$\sigma_i = \frac{C_{i3} + C_{i2}}{\overline{C}_{33} - \overline{C}_{22}} \sum - C_{ij}\alpha_j \, \Delta T + \sum_{\ell=1}^{2} (C_{i2} + \lambda_\ell C_{i3}) \, A_\ell \, r^{\lambda_\ell - 1} \quad , \tag{52}$$

where

$i = x, \theta, r$	for stresses and coefficients of thermal expansion,
$i = j = 1,2,3$	for stiffness in the same order,

and the double subscript j indicates summation from 1 to 3. For a transversly isotropic 0° lamina, the strains are

$$\varepsilon_r = A_1 - \frac{A_2}{r^2}$$

$$\varepsilon_\theta = A_1 + \frac{A_2}{r^2} \tag{53}$$

and the stresses are

$$\sigma_i = -\overline{C}_{ij}\alpha_j\Delta T + A_1(\overline{C}_{i2} + \overline{C}_{i3}) + A_2(\overline{C}_{i2} - \overline{C}_{i3})\frac{1}{r^2}, \tag{54}$$

i and j having the meanings of eq. 52.

Since σ_x is not considered to be a function of x, the third equilibrium equation, eq. 40c, results in

$$\tau_{xr} = Br^{-1} . \tag{55}$$

For a single layer the constants A_1 and A_2 are determined by the condition that $\sigma_r = 0$ at the tube's inner and outer boundary, r_i and r_o, respectively. With no traction τ_{rx} applied to the inner or outer boundary, B of eq. 55 must be zero. Thus only radial, houp, and axial stresses are generated.

Multiple Layer Plane-Strain Solution

Since the plane-strain solution constrains u to be zero, continuity of u at the interface between the (k-1)st and the kth layers is automatically satisfied. The axisymmetry of the problem and the uniform temperature distribution forces the tangential displacement, v, to be zero for each layer. Therefore, the continuity of v is automatically satisfied. The continuity on w leads to

$$w^k(r) = w^{k-1}(r) \text{ @ } r = r_{interface} .$$

The stress τ_{rx} is zero throughout since there is no applied traction to cause it to be nonzero. Thus the B in eq. 55 is zero for each layer. Continuity of the only nonzero stress, σ_r, at the interface results in

$$\sigma_r^k(r) = \sigma_r^{k-1}(r) \text{ @ } r = r_{interface} \cdot$$

Enforcing $w^k = w^{k-1}$ at the N-1 interfaces of an N layer cylinder yields N-1 equations to solve for the N A_1's and N A_2's of eq. 49 or 50, whichever is applicable. Enforcing $\sigma_r^k = \sigma_r^{k-1}$ at each of the N-1 interfaces gives N-1 more equations from which to solve for the A_1's and A_2's. Enforcing the traction-free condition on σ_r at the inner and outer radii gives two more equations. Thus all conditions are satisfied in the multilayer tube and the N A_1's and N A_2's can be determined. From eqs. 52 or 54 the stresses can be computed.

Plane-Stress Solution

In the plane-stress elasticity solution attention is focused on a thin axial slice of thickness ℓ. It is assumed that $\ell << R$ and the stress vector in the axial direction is equal to zero. In addition, it is assumed that all stress components are independent of x. Since the axisymmetry assumption leads to the stresses being independent of θ, the plane stress assumption limits the stresses, and strains, to be functions only of r. In addition, except for the axial displacement, the displacements are functions only of r. The assumptions employed in plane-stress are summarized below.

- stresses

$$\sigma_x = \tau_{xr} = \tau_{x\theta} = \tau_{r\theta} = 0$$
$$\sigma_r = \sigma_r(r)$$
$$\sigma_\theta = \sigma_\theta(r)$$

- displacements

$$u = u(x,r)$$
$$w = w(r)$$
$$v = 0.$$

For the individual layers in a cross-ply tube, Hooke's relations for the above case may be written in the following reduce stiffness form:

$$\begin{Bmatrix} \sigma_\theta \\ \\ \sigma_r \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{22} & \overline{Q}_{23} \\ \\ \text{sym} & \overline{Q}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_\theta - \alpha_\theta \Delta T \\ \\ \varepsilon_r - \alpha_r \Delta T \end{Bmatrix} \tag{56}$$

where the $\overline{Q}_{ij}$'s are the reduced stiffness coefficients which can be written as:

$$\overline{Q}_{ij} = \overline{C}_{ij} - \frac{\overline{C}_{1i}\overline{C}_{1j}}{\overline{C}_{11}} \qquad i,j = 2,3 \ .$$

The strain-displacement relations can be substituted into the above relations and the stresses written in terms of the displacement w as

follows:

$$\sigma_\theta = \overline{Q}_{22}\left(\frac{w}{r} - \alpha_\theta \Delta T\right) + \overline{Q}_{23}\left(\frac{dw}{dr} - \alpha_r \Delta T\right)$$

$$\sigma_r = \overline{Q}_{23}\left(\frac{w}{r} - \alpha_\theta \Delta T\right) + \overline{Q}_{33}\left(\frac{dw}{dr} - \alpha_r \Delta T\right) . \tag{57}$$

When σ_θ and σ_r are substituted into the first equilibrium equation, eq. 40a, the result is

$$\overline{Q}_{33}\frac{d^2w}{dr^2} + \overline{Q}_{33}\frac{1}{r}\frac{dw}{dr} - \overline{Q}_{22}\frac{1}{r^2}w = \frac{1}{r}\sum , \tag{58}$$

where

$$\sum = \left[(\overline{Q}_{23} - \overline{Q}_{22})\,\alpha_\theta + (\overline{Q}_{33} - \overline{Q}_{23})\,\alpha_r\right]\Delta T .$$

The solution to eq. 58 can be obtained by the procedure outlined for the plane-strain case. The result is:

$$w(r) = A_1 r^{\lambda_1} + A_2 r^{\lambda_2} + \frac{\sum}{\overline{Q}_{33} - \overline{Q}_{22}}\, r , \tag{59}$$

where

$$\lambda_1, \lambda_2 = \pm\sqrt{\frac{\overline{Q}_{22}}{\overline{Q}_{33}}} .$$

As for the plane-strain case, if $\overline{Q}_{22} = \overline{Q}_{33}$ the last term in eq. 59 must be replaced by:

$$\frac{\sum}{2\overline{Q}_{33}} r \ln r, \quad \text{and} \quad \lambda_\ell = \pm 1$$

In addition if $\alpha_\theta = \alpha_r$, as will be the case for 0° lamina if we assume transversely isotropic condition, then eq. 59 becomes

$$w(r) = A_1 r + \frac{A_2}{r} . \tag{60}$$

The strains ε_θ and ε_r are obtained by substitution of the displacement w into the strain-displacement relations to get

$$\begin{aligned} \varepsilon_\theta &= \frac{1}{r} w = \frac{\sum}{\overline{Q}_{33} - \overline{Q}_{22}} + \sum_{\ell=1}^{2} A_\ell r^{\lambda_\ell - 1} \\ \varepsilon_r &= \frac{dw}{dr} = \frac{\sum}{\overline{Q}_{33} - \overline{Q}_{22}} + \sum_{\ell=1}^{2} A_\ell \lambda_\ell r^{\lambda_\ell - 1} . \end{aligned} \tag{61}$$

The residual stress components are determined from the constitutive relation to be

$$\sigma_i = \left(\frac{\overline{Q}_{2i} + \overline{Q}_{i3}}{\overline{Q}_{33} - \overline{Q}_{22}} \right) \sum - \overline{Q}_{ij} \alpha_j \Delta T + \sum_{\ell=1}^{2} (\overline{Q}_{2i} + \lambda_\ell \overline{Q}_{i3}) A_\ell r^{\lambda_\ell - 1} \tag{62}$$

where

$i = \theta, r$ for stresses and coefficients of thermal expansion,

$i = j = 2,3$ for stiffness in the same order,

and the double subscript j indicates summation from 2 to 3.

For the transversly isotropic case, the strains are given by

$$\varepsilon_r = A_1 - \frac{A_2}{r^2}$$
$$\varepsilon_\theta = A_1 + \frac{A_2}{r^2} \tag{63}$$

and the stresses are

$$\sigma_i = -\overline{Q}_{ij}\alpha_j\Delta T + A_1(\overline{Q}_{i2} + \overline{Q}_{i3}) + A_2(\overline{Q}_{i2} - \overline{Q}_{i3})\frac{1}{r^2}\,. \tag{64}$$

The boundary conditions on the ends of the cylinder are that all stresses are zero there, a condition imposed by the plane stress assumption. The constants A_1 and A_2 are determined so as to satisfy the condition that $\sigma_r = 0$ on the inner and outer radii. Only radial, σ_r, and hoop, σ_θ, stresses are generated.

Multiple-Layer Plane-Stress Solutions

The plane stress solution for multiple layers is unrealistic in that it actually models concentric tubes which can slide axially relative to one another. An axial restraint would require nonzero stresses in the direction of the cylinder axis, a condition not allowed. Thus the cylinders have no choice but to slide relative to one another. Thus $u^k \neq u^{k-1}$ at the interfaces. Since $v \equiv 0$ through, $v^k = v^{k-1}$ at all interfaces. Enforcing $w^k = w^{k-1}$ at N-1 interfaces of a N layer cylinder yields N-1 equations to solve for the N A_1's and N A_2's. The τ_{rx} stress is zero throughout by the plane stress assumption. Hence $\tau^k_{rx} = \tau^{k-1}_{rx}$ at each interface. The stress $\tau_{r\theta}$ is zero throughout also. Enforcing

$\sigma_r^k = \sigma_r^{k-1}$ leads to N-1 more equations for the N A_1's and N A_2's. Using $\sigma_r = 0$ at the inner and outer radii gives the final two equations from which to solve for the N A_1's and N A_2's. Once all the A_1's and A_2's are found, eqs. 62 or 64 can be used to find the stresses in a particular layer. As with a single layer only hoop and radial stresses are generated.

Generalized Plane-Strain

In the generalized plane-strain analysis it is assumed that the tube is very long. Interest centers on the portion of the tube away from the ends. It is assumed that in this region none of the strains, and consequently none of the stresses, vary with x. Because of the axisymmetric nature of the problem none of the strains, nor displacements, vary with θ either. By using the first two compatibility equations, eq. 36, it can be shown ε_x = a constant. Thus for the generalized plane-strain analysis

$$\varepsilon_x = \varepsilon^0, \text{ a constant}, \quad \varepsilon_\theta = \varepsilon_\theta(r), \quad \varepsilon_r = \varepsilon_r(r)$$
$$v = \gamma_{r\theta} = \gamma_{x\theta} = 0$$
$$u = u(r,x) \qquad w = w(r) \ .$$

From Hooke's Law

$$\tau_{r\theta} = \tau_{x\theta} = 0 \ .$$

Hooke's relations for the nonzero stresses may be written as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} \\ & \bar{C}_{22} & \bar{C}_{23} \\ & & \bar{C}_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon^0 - \alpha_x \Delta T \\ \varepsilon_\theta - \alpha_\theta \Delta T \\ \varepsilon_r - \alpha_r \Delta T \end{Bmatrix} \tag{65}$$

$$\tau_{xr} = \bar{C}_{55}\, \gamma_{xr} \; .$$

Substituting the stress components σ_θ and σ_r into the first equilibrium equation gives

$$\bar{C}_{33} \left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) - \bar{C}_{22}\frac{1}{r^2} w = \frac{\sum}{r} \tag{66}$$

where

$$\sum = \bar{\sum} + (\bar{C}_{12} - \bar{C}_{13})\, \varepsilon^0$$

and

$$\bar{\sum} = \{(\bar{C}_{23} - \bar{C}_{22})\, \alpha_\theta + (\bar{C}_{33} - \bar{C}_{23})\, \alpha_r + (\bar{C}_{13} - \bar{C}_{12})\, \alpha_x\}\, \Delta T \; .$$

It should be noted that eq. 66 is identical to eq. 48 except for the nomenclature. Equation 66 can be solved by the procedure outlined previously. The resulting solution is

$$w(r) = A_1 r^{\lambda_1} + A_2 r^{\lambda_2} + \frac{\sum}{\bar{C}_{33} - \bar{C}_{22}}\, r \quad , \tag{67}$$

where

$$\lambda_{1,2} = \pm\sqrt{\frac{\bar{C}_{22}}{\bar{C}_{33}}} \; .$$

In the case that $\overline{C}_{33} = \overline{C}_{22}$ the last term in eq. 67 has to be replaced by:

$$\frac{\sum}{2\overline{C}_{33}} r \ln r \quad \text{and} \quad \lambda_{1,2} = \pm 1$$

In addition if $\alpha_r = \alpha_\theta$ and $\overline{C}_{12} = \overline{C}_{13}$, as will be the case for 0° lamina if it assumed to be transversely isotropic, the radial displacement w reduces to:

$$w(r) = A_1 r + A_2/r \tag{68}$$

The strain is obtained by substitution of w(r) into the strain-displacement relation to get:

$$\varepsilon_r = \frac{dw}{dr} = \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} + \sum_{\ell=1}^{2} A_\ell \lambda_\ell r^{\lambda_\ell - 1}$$

$$\varepsilon_\theta = \frac{1}{r} w = \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}} + \sum_{\ell=1}^{2} A_\ell r^{\lambda_\ell - 1} \tag{69}$$

$$\varepsilon_x = \varepsilon^0$$

The stresses components for the layer are determined from the constitutive relation to get

$$\sigma_i = \{\overline{C}_{1i}\varepsilon^0 + \frac{\overline{C}_{i3} + \overline{C}_{i2}}{\overline{C}_{33} - \overline{C}_{22}} \sum - \overline{C}_{ij}\alpha_j\Delta T + \sum_{\ell=1}^{2} (\overline{C}_{i2} + \lambda_\ell \overline{C}_{i3})A_\ell r^{\lambda_\ell - 1}\} , \quad (70)$$

where

$i = x,\theta,r$ for stresses and coefficients of thermal expansion,

$i = j = 1,2,3$ for stiffness in the same order,

and the double subscript j indicates summation from 1 to 3.

For the transversly isotropic case,

$$\varepsilon_r = A_1 - \frac{A_2}{r^2}$$
$$\varepsilon_\theta = A_1 + \frac{A_2}{r^2} \quad (71)$$

and the stresses are

$$\sigma_i = \overline{C}_{1i}\varepsilon^0 - \overline{C}_{ij}\alpha_j\Delta T + A_1(\overline{C}_{2i} + \overline{C}_{3i}) + A_2(\overline{C}_{2i} - \overline{C}_{3i})\frac{1}{r^2} . \quad (72)$$

Since σ_x is not a function of x, eq. 40c yields

$$\tau_{xr} = Br^{-1} . \quad (73)$$

The method for determining the constants ε^0, A_1, and A_2 for a single layer is slightly different for the generalized plane strain case than it was for the two other solutions. The conditions $\sigma_r = 0$ at the inner and outer radii still are enforced. However, to determine ε^0, an integrated condition must be used. This condition states that despite thermal effects, there is no net axial force in the tube. That condition is written as

$$\int_A \sigma_x dA = 0.$$

Here A is the annular cross-sectional area of the tube. Since $\tau_{xr} = 0$ at the inner and outer boundary, B of eq. 73 is zero.

Multiple Layer Generalized Plane-Strain Solution

Since ε_x is a constant in each layer, i.e., ε^0, enforcing $u^k = u^{k-1}$ at the interfaces leads to the conclusion that ε_x is same in all layers. Call this strain ε^0. It is an unknown which must be solved for. Since $v \equiv 0$ throughout, $v^k = v^{k-1}$ at each interface is automatically satisfied. Enforcing $w^k = w^{k-1}$ at the N-1 interfaces of a N layer tube leads to N-1 equations from which to find the N A_1's, N A_2's, and ε^0. Since τ_{rx} is zero on the inner and outer radii, it will be zero throughout and so $\tau^k_{rx} = \tau^{k-1}_{rx}$ is automatically satisfied at each interface. Enforcing $\sigma^k_r = \sigma^{k-1}_r$ at the N-1 interfaces provides N-1 more equations. Requiring σ_r to be zero at the inner and outer radii provides 2 more conditions for the 2N+1 unknowns A_1's, A_2's, and ε^0. Using an integrated condition that

$$\int_A \sigma_x \, dA = 0 \tag{74}$$

provides the final equation from which to solve for the unknown constants. Here A is the annular cross-sectional area of all N layers. This integral condition is physically interpreted to mean that under thermal loading only, there is no net axial force acting on the tube

cross-section. Since the stresses are independent of x in this analysis, the integral condition actually states the absence of a net mechanical load at every cross-section. In each individual layer, however, there are axial stresses, in addition to hoop and radial stresses.

Details of Multiple Layer Solution

To illustrate the details of obtaining a multiple layer solution, the necessary steps for the generalized plane-strain analysis of a two layer cross-ply tube are presented in this section.

Assume that the inner layer has its fibers running axially (0°) and the layer is considered transversely isotropic in the r-θ plane. The outer layer has its fibers running circumferentially (90°). This lamina can be considered transversely isotropic in the r-x plane but that will not affect the analysis.

As stated earlier, for each layer the only displacement explicitly involved in the analysis is the radial displacement. However, to apply the boundary conditions both the radial and axial stresses are involved. For the inner layer, referred to here as layer no. 1, the displacement is

$$w^1(r) = A_1^1 \quad r + A_2^1 /r \tag{75a}$$

and the radial and axial stresses are

$$\sigma_r{}^1 = C_{13}\varepsilon^o + A_1^1(C_{23} + C_{33}) + A_2^1(C_{23} - C_{33})\frac{1}{r^2} - (C_{13}\alpha_x + C_{23}\alpha_\theta + C_{33}\alpha_r)\Delta T \tag{75b}$$

$$\sigma_x{}^1 = C_{11}\varepsilon^o + A_1^1(C_{12} + C_{13}) + A_2^1(C_{12} - C_{13})\frac{1}{r^2} - (C_{11}\alpha_x + C_{12}\alpha_\theta + C_{13}\alpha_r)\Delta T\ . \tag{75c}$$

For the outer layer, here layer no. 2, the radial displacement is given by

$$w^2(r) = A_1^2 r^\lambda + A_2^2 r^{-\lambda} + \frac{\sum}{\overline{C}_{33} - \overline{C}_{22}}\, r, \quad \lambda = \sqrt{\frac{\overline{C}_{22}}{\overline{C}_{33}}} \tag{76a}$$

The stresses are

$$\sigma_r{}^2 = \overline{C}_{13}\varepsilon^o + A_1^2(\overline{C}_{23} + \lambda\overline{C}_{33})r^{\lambda-1} + A_2^2(\overline{C}_{23} - \lambda\overline{C}_{33})r^{-(\lambda+1)} - (\overline{C}_{13}\alpha_x + \overline{C}_{23}\alpha_\theta + \overline{C}_{33}\alpha_r)\Delta T + \frac{\overline{C}_{23} + \overline{C}_{33}}{\overline{C}_{33} - \overline{C}_{22}}\sum \tag{76b}$$

$$\sigma_x{}^2 = \overline{C}_{11}\varepsilon^o + A_1^2(\overline{C}_{12} + \lambda\overline{C}_{13})r^{\lambda-1} + A_2^2(\overline{C}_{12} - \lambda\overline{C}_{13})r^{-(\lambda+1)} - (\overline{C}_{11}\alpha_x + \overline{C}_{12}\alpha_\theta + \overline{C}_{13}\alpha_r)\Delta T + \frac{\overline{C}_{12} + \overline{C}_{13}}{\overline{C}_{33} - \overline{C}_{22}}\sum \tag{76c}$$

where the bar over the stiffness coefficients, C_{ij}, designate a 90° lamina and unbarred coefficients a 0° lamina.

If the interface radius is denoted as r_1, with the inner and outer radius being denoted by r_i and r_o, the boundary conditions which must be

enforced on the cylindrical surfaces are:

$$\sigma_r^{\,1}(r_i) = 0 \tag{77a}$$

$$\sigma_r^{\,1}(r_1) = \sigma_r^{\,2}(r_1) \tag{77b}$$

$$\sigma_r^{\,2}(r_o) = 0 \tag{77c}$$

$$w^1(r_1) = w^2(r_1) \ . \tag{77d}$$

Across a cross-section, the net axial force being zero, eq. 74, translates into

$$2\pi \int_{r_i}^{r_1} \sigma_x^1(r)\ r dr + 2\pi \int_{r_1}^{r_o} \sigma_x^2(r)\ r dr = 0 \ . \tag{77e}$$

Substituting the stresses and displacements given by eqs. 75 and 76 into 77a-d, carrying out the integration of 77e, and arranging into matrix form leads to the system of linear algebraic equations given on the next two pages. Solving these for $A_1^{\,1}$, $A_2^{\,1}$, $A_1^{\,2}$, $A_2^{\,2}$ and ε^o, the stresses can be computed in each layer by back substitution into the proper equation.

Laminated Shell Theory (LST)

The degree of sophistication employed in the analysis of shells depends to a great extent on the type of assumptions made regarding the changes in shell curvature and twist. In the present work the Donnell shell approximation on curvature and twist changes were employed. Under such approximations all terms of changes in curvature and twist, except

$$
\begin{bmatrix}
(C_{23}+C_{33}) & (C_{23}-C_{33})\dfrac{1}{r_i^{\,2}} & 0 & 0 & C_{13} \\[2ex]
(C_{23}+C_{33}) & (C_{23}-C_{33})\dfrac{1}{r_1^{\,2}} & -(\bar{C}_{23}+\lambda\bar{C}_{33})\, r_1^{\,-\beta_2} & -(\bar{C}_{23}-\lambda\bar{C}_{33})\, r_1^{\,-\beta_1} & (C_{13}-\bar{C}_{13}) - \dfrac{(\bar{C}_{23}+\bar{C}_{33})(\bar{C}_{12}-\bar{C}_{13})}{(\bar{C}_{33}-\bar{C}_{22})} \\[2ex]
0 & 0 & (\bar{C}_{23}+\lambda\bar{C}_{33})\, r_1^{\,-\beta_2} & \bar{C}_{23}-\lambda\bar{C}_{33})\, r^{-\beta_1} & \bar{C}_{13} + \dfrac{(\bar{C}_{23}+\bar{C}_{33})(\bar{C}_{12}-\bar{C}_{13})}{(\bar{C}_{33}-\bar{C}_{22})} \\[2ex]
r_1 & \dfrac{1}{r_1} & -\, r_1^{\,\lambda} & -\, r_1^{\,-\lambda} & \dfrac{\bar{C}_{12}-\bar{C}_{13}}{\bar{C}_{33}-\bar{C}_{22}}\, r_1 \\[2ex]
(C_{12}+C_{13})\dfrac{r_1^{\,2}-r_i^{\,2}}{2} & 0 & (\bar{C}_{12}+\lambda\bar{C}_{13})\dfrac{r_o^{\,\beta_1}-r_1^{\,\beta_1}}{\beta_1} & (\bar{C}_{12}-\lambda\bar{C}_{13})\dfrac{r_o^{\,\beta_2}-r_1^{\,\beta_2}}{\beta_2} & C_{11}\left[\dfrac{r_1^{\,2}-r_i^{\,2}}{2}\right] + \left[\bar{C}_{11} + \dfrac{\bar{C}_{12}^{\,2}-\bar{C}_{13}^{\,2}}{\bar{C}_{33}-\bar{C}_{22}}\right]\dfrac{r_o^{\,2}-r_1^{\,2}}{2}
\end{bmatrix}
\begin{Bmatrix} A_1^{\,1} \\ A_2^{\,1} \\ A_1^{\,2} \\ A_2^{\,2} \\ \varepsilon^{o} \end{Bmatrix}
$$

where $\beta_1 = \lambda + 1$

$\beta_2 = 1 - \lambda$

and α_1 = thermal expansion coefficient in the fiber direction

α_2 = thermal expansion coefficient transverse to the fiber

$$
= \begin{bmatrix}
[C_{13}\alpha_1 + (C_{23} + C_{33})\,\alpha_2]\,\Delta T \\[2ex]
[C_{13}\alpha_1 + (C_{23} + C_{33})\,\alpha_2]\,\Delta T + \dfrac{\bar{C}_{23} + \bar{C}_{33}}{\bar{C}_{33} - \bar{C}_{22}}\,\bar{\Sigma} - [(\bar{C}_{13} + \bar{C}_{33})\,\alpha_2 + \bar{C}_{23}\alpha_1]\,\Delta T \\[2ex]
[(\bar{C}_{13} + \bar{C}_{33})\,\alpha_2 + \bar{C}_{23}\alpha_1]\,\Delta T - \dfrac{\bar{C}_{23} + \bar{C}_{33}}{\bar{C}_{33} - \bar{C}_{22}}\,\bar{\Sigma} \\[2ex]
\dfrac{\Sigma}{\bar{C}_{33} - \bar{C}_{22}}\,r_1 \\[2ex]
\{[C_{11}\alpha_1 + (C_{12} + C_{13})\alpha_2]\,\Delta T\}\,\dfrac{r_1^2 - r_i^2}{2} + \left\{[(\bar{C}_{11} + \bar{C}_{13})\,\alpha_2 + \bar{C}_{12}\alpha_1]\,\Delta T - \dfrac{(\bar{C}_{12} + \bar{C}_{13})}{(\bar{C}_{33} - \bar{C}_{22})}\,\bar{\Sigma}\right\}\,\dfrac{r_o^2 - r_1^2}{2}
\end{bmatrix}
$$

those involving second derivatives of W, are neglected [15]. The above assumptions lead to the following form of curvature changes.

$$\chi_x = \frac{\partial^2 W}{\partial x^2} \tag{78a}$$

$$\chi_\theta = \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} \tag{78b}$$

$$\chi_{x\theta} = \frac{1}{R}\frac{\partial^2 W}{\partial x \partial \theta} \tag{78c}$$

where R designates the radial distance to the laminated shell mid-plane. The above first order approximation gives good results for homogeneous, elastic quasi-shallow shells ($10 < R/t$). It remains to be seen if the above assumption on curvature changes will yield agreement with a more exact solution such as plane elasticity. The body analyzed was axisymmetric and under the influence of constant temperature, therefore all curvature changes reduced to zero.

$$\chi_x = \chi_\theta = \chi_{x\theta} = 0 \tag{79}$$

The cross-ply laminated shell constitutive relations are given by the following expressions

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_\theta^o \\ \gamma_{x\theta}^o \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ & a_{22} & 0 \\ \text{Sym.} & & a_{66} \end{bmatrix} \begin{Bmatrix} N_x + N_x^T \\ N_\theta + N_\theta^T \\ N_{x\theta} + 0 \end{Bmatrix} \tag{80}$$

and

$$\begin{Bmatrix} M_x \\ M_\theta \\ M_{x\theta} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_\theta^o \\ \gamma_{xy}^o \end{Bmatrix} - \begin{Bmatrix} M_x^T \\ M_\theta^T \\ M_{x\theta}^T \end{Bmatrix} \tag{81}$$

where $\varepsilon_x{}^o$, $\varepsilon_y{}^o$ and $\gamma_{xy}{}^o$ are the strains at the shell's midsurface. The membrane force and moment resultants, the N's and M's, are defined by,

$$[N_x, N_\theta, N_{x\theta}] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_\theta, \tau_{x\theta}](1 + \frac{z}{R})\, dz \tag{82a}$$

$$[M_x, M_\theta, M_{x\theta}] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_\theta, \tau_{x\theta}](1 + \frac{z}{R})\, zdz \tag{82b}$$

$$[N_x{}^T, N_\theta{}^T, N_{x\theta}{}^T] = \int_{-h/2}^{h/2} [\overline{Q}_{1j}, \overline{Q}_{2j}, \overline{Q}_{3j}]\alpha_j \Delta T(1 + \frac{z}{R})\, dz \tag{83a}$$

$$[M_x{}^T, M_\theta{}^T, M_{x\theta}{}^T] = \int_{-h/2}^{h/2} [\overline{Q}_{1j}, \overline{Q}_{2j}, \overline{Q}_{3j}]\alpha_j \Delta T(1 + \frac{z}{R})\, zdz. \tag{83b}$$

The a_{ij} coefficients are the elements of the inverted A_{ij} matrix. The A_{ij} and B_{ij} matrices are defined by,

$$[A_{ij}, B_{ij}] = \int_{-h/2}^{h/2} \overline{Q}_{ij}[1, z]dz\ , \tag{84}$$

where

$$\overline{Q}_{ij} = \overline{C}_{ij} - \frac{\overline{C}_{i3}\overline{C}_{j3}}{\overline{C}_{33}} \qquad i,j = 1,2,6 \ . \tag{85}$$

Since we are only interested in the thermoelastic shell problem,

$$N_x = N_\theta = N_{x\theta} = 0 \ .$$

In addition, in the present investigation it was assumed that $z/R << 1$. Hence, the z/R term in eqs. 82 and 83 were neglected. The laminate midsurface strains are obtained by substituting N_x^T and N_θ^T into eq. 80. The residual stresses in the kth layer are then calculated through the lamina constitutive relations,

$$\begin{aligned} \sigma_x &= \overline{Q}_{11}(\varepsilon_1{}^0-\alpha_1\Delta T) + \overline{Q}_{12}(\varepsilon_2{}^0-\alpha_2\Delta T) \\ \sigma_\theta &= \overline{Q}_{12}(\varepsilon_1{}^0-\alpha_1\Delta T) + \overline{Q}_{22}(\varepsilon_2{}^0-\alpha_2\Delta T) \ . \end{aligned} \tag{86}$$

It should be noted that the M's are not zero and can be calculated by eq. 81 since the M^T's are known.

RESULTS

In the following section the numerical results of the study are presented. The discussion will focus on a set of selected cases that illustrate particularly significant findings in the current study.

The analysis was conducted using T300/934 and GY70/934 graphite-epoxy composite systems. The T300 graphite fibers represent a high strength-low modulus fiber and the GY70 fibers are low strength-high modulus fibers. The material properties or both composite systems are given in Appendix A. The curing and service temperatures were taken as 250 and -250°F, respectively. This yielded a maximum temperature difference of 500°F. Results are presented for all three elasticity solution methods and the results are compared to the laminated shell theory (LST).

Only 4 layer tubes are considered. The next 12 figures illustrate the effects of changing the location through the wall thickness of the two T300/934 0° (axial) and the two T300/934 90° (circumferential) layers, e.g., $[0/90/0/90]_t$ vs. $[90/0/90/0]_t$. The stress levels are plotted as a function of location through the wall thickness. The ratio of the inner radius to the wall thickness were chosen to be 12.5. This represented the geometry of tubes being investigated in another phase of this overall study. Following this, the next twelve figures illustrate the effect of changing the number of 0° and 90° layers in a tube e.g., $(0/90_3)_t$ vs. $(0/90/0/90)_t$.

After the various 4-layer stacking arrangements are discussed, the results for a 4 layer T300/934 tube with all layers circumferential (90_4) are illustrated. Following that, the comparison between LST stress predictions and the plane strain elasticity solution is illustra-

ted. This will illustrate the reason for pursuing the elasticity solution in this study. Finally, the effect of material properties on residual stress is demonstrated by varying the 0° and 90° laminae material properties in a $(0/90/0/90)_t$ laminated tube. Using two types of fibers, T300 and GY70, the benefit of laminate hybridizing on stress levels is shown.

The axial residual stress, σ_x, in a $(0/90/0/90)_t$ tube operating at 500°F below its curing temperature is given in Fig. 3. The apparent lack of agreement between the plane-strain, generalized plane-strain, and LST analyses can be explained by the different assumptions employed in each analysis. (The axial stress in the plane stress analysis is zero). In the plane-strain analysis it is assumed that the tube is restrained at its ends from axial motion (i.e., u=0 all along the tube). Therefore, a net axial force is required at the tube ends to meet this condition. Consequently, the net axial stress across a tube cross-section is not zero as may be seen from the sum of area under the plane-strain axial stress cure. On the other hand, in the generalized plane-strain analysis it is assumed that the tube is unrestrained at its ends. In addition, it was required that the net axial stress across a tube cross-section is equal to zero. Away from the ends of a finite length tube the generalized plane-strain assumption closely resembles the LST analysis in the x direction. Therefore, these solution approaches are expected to give close agreement as indeed they do. It should also be noted that the axial stress in the 90° laminae (hoop fiber) is tensile and of considerable magnitude (≈ 12,000 psi). Such a stress can lead to circumferential cracking in those plies.

Figure 4 illustrates the hoop stress in the $(0/90/0/900)_t$ tube. As can be seen, the layers with their fiber in the axial direction are

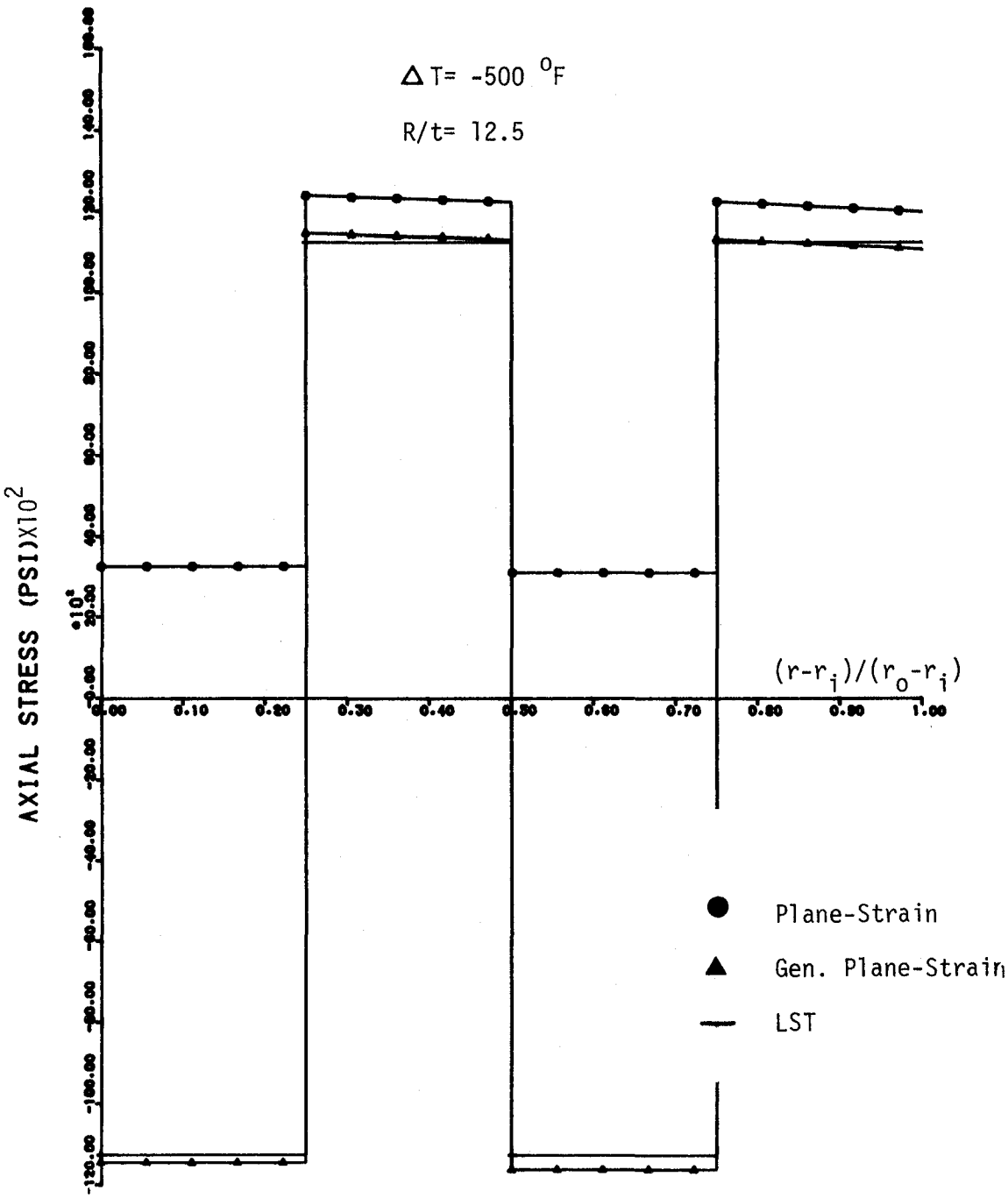


Fig. 3. Residual Axial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube

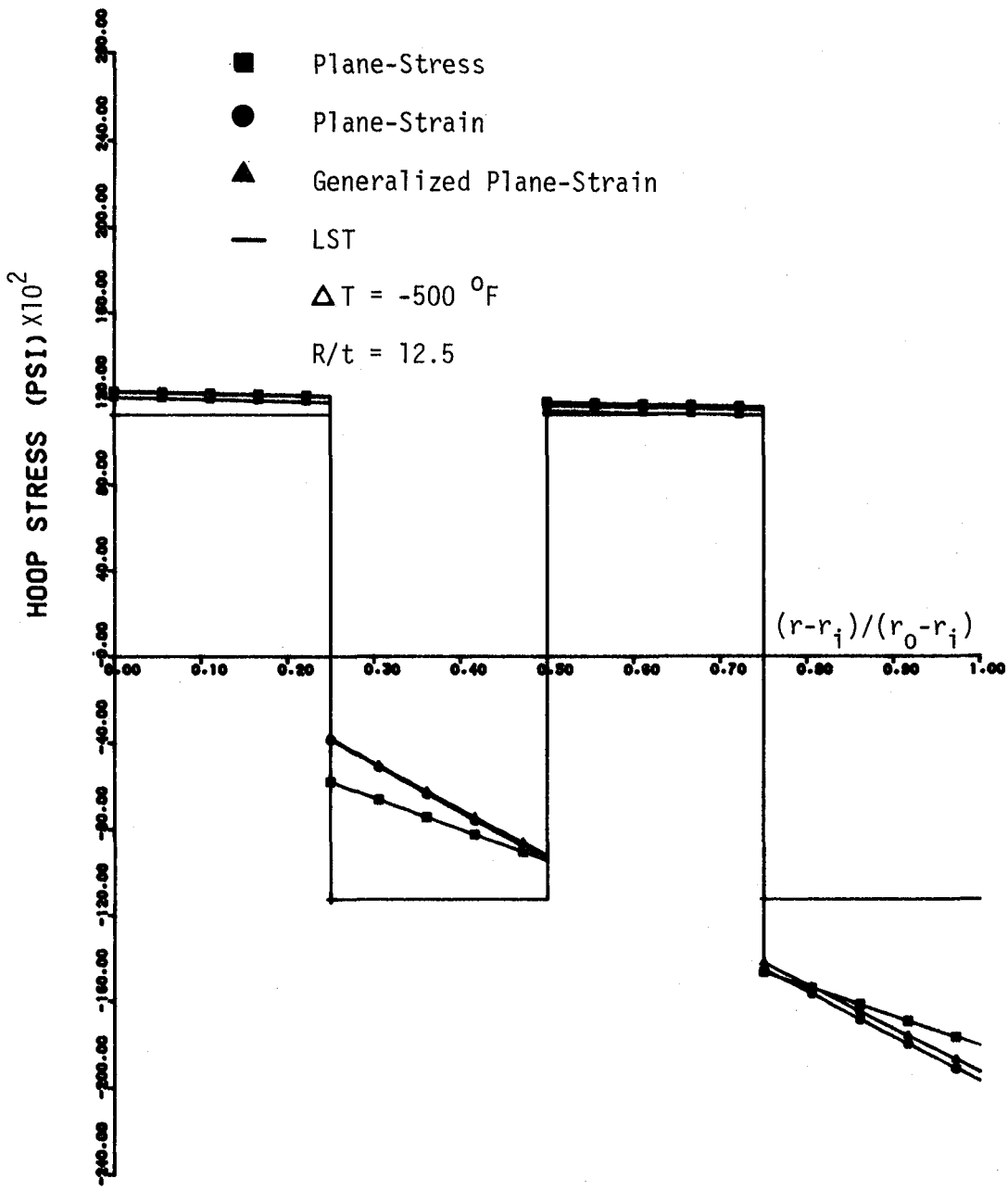


Fig. 4. Residual Hoop Stress in a $(0/90/0/90)_t$ Laminated Composite Tube

under considerable hoop tensile stress (≈ 12,000 psi). This represents tension in the matrix direction at a level that will undoubtedly cause cracking parallel to the fibers. Conversely, the layers with their fibers in the circumferential direction are under compression in the fiber direction. This should cause no problems as far as cracking or other damage. There is not much difference in the three elasticity solutions. There is a difference between the LST and the three elasticity solutions. For the case shown in Fig. 4, the elasticity solutions and the LST solution differ by 40% at the outer radius. That the three elasticity solutions are similar is interesting in its own right. The closeness of the three solutions demonstrates that the hoop stress is independent of assumptions regarding the stress or strain state in the axial direction. Recall, in Fig. 3, the plane stress, plane strain, and generalized plane strain were quite different. All elasticity solutions predict a gradient in the hoop stress in the 90° laminae but very little gradient in the 0° lamina. As was shown by the elasticity analyses, the hoop stress in the 90° lamina varies as a function of $r^{\lambda-1}$ and $r^{-(\lambda+1)}$, where $\lambda = \sqrt{C_{22}/C_{33}}$ for the plane-strain and generalized plane-strain solutions and $\lambda = \sqrt{Q_{22}/Q_{33}}$ for the plane stress analysis. As for the 0° lamina, assuming a transverse isotropic condition, the hoop stress will vary as function of $1/r^2$.
Since $\lambda \approx 3.7$ in the T300/934 material, the hoop stress in the 90° lamina will vary with r much more rapidly than in the 0° lamina.

Figure 5 shows the radial stress distribution in the $(0/90/0/90)_t$ tube. The stress is everywhere tension but the magnitude is quite low compared to the hoop stress or the through-the-thickness strength of a lamina. Again, the three elasticity solutions are quite close to one another.

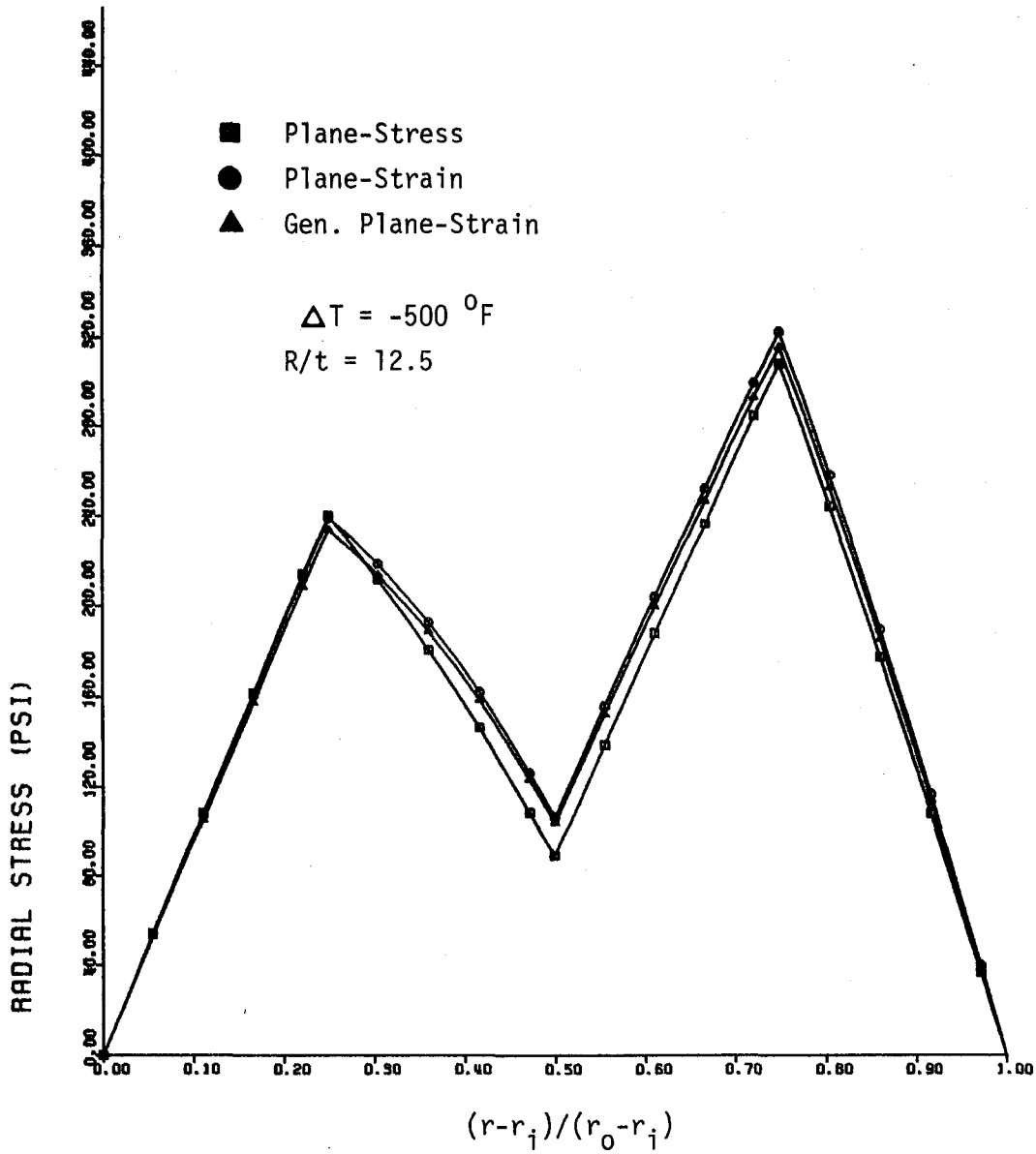


Fig. 5. Residual Radial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube

Figures 6-14 show other stacking arrangements with two 0° layers and two 90° layers. It appears from these figures that the maximum axial and hoop tensile stress is unaltered by stacking arrangement. The peak compressive stress in the hoop direction does vary with stacking arrangement as does the radial stress distribution. Clustering the 0°'s and/or 90°'s, Figs. 9-14, does little to alter the peak tensile stress in the matrix direction.

Changing one of the 90° lamina to be a 0° lamina doesn't alter the tensile axial stress in the single 90° lamina, Fig. 15, but it does significantly increase the compressive hoop stress in that 90° lamina, Fig. 16. The tensile matrix stress in the 0° lamina is still about 12,000 psi. For a $(90/0_3)_t$ tube, the radial stress, as shown in Fig. 17, is everywhere compressive. Repositioning the single 0° lamina, Figs. 18-20, has little effect on the peak axial or hoop stresses.

The effects of having more 90° lamina than 0° lamina are shown in Figs. 21-26. Still the peak hoop tensile stress in the matrix direction is about 12,000 psi. The peak compressive stress in the fiber direction is greater than in other cases. It is interesting to note that LST predicts the wrong sign for the hoop stress at the inner region of the cluster of 90° lamina.

The residual stresses in a $(90_4)_t$ laminated tube are presented in Figs. 27 and 28 as can be seen these stresses are not zero as might intuitively be expected. The residual stresses arises due to thermal expansion mismatch in the radial and hoop directions. For the $(90_4)_t$ case, the plane stress and generalized plane-strain solution predict no axial stress. However, the plane strain solution predicts an axial

stress which is practically constant with r and has a value of ≈ -20500 psi.

In order to demonstrate the effect of tube aspect ratio (R/t) on residual stress magnitude, the maximum hoop and radial stresses are plotted versus R/t ratios in Fig. 29 and 30 for a $(0,90,0,90)_t$ tube. In Fig. 29 the maximum hoop stresses in the 0 and 90 degree laminae calculated by generalized plane-strain elasticity are normalized by the value determined from laminated shell theory. As indicated by the plot, only at very large aspect ratios does LST give close agreement with an elasticity analysis. This is particularly true for the 90 degree lamina. The residual radial stress approaches a small value (10 psi) as the tube aspect ratio (R/t) increases. However, it should be noted that even at small aspect ratios the residual radial stress is less than one percent of the hoop stress.

Finally, Fig. 31 demonstrates the effect of laminated tube hybridization. It appears that the use of such hybridization can be beneficial in hoop stress reduction. Specifically, the tensile hoop stress in the 0° plies, of a 4-ply GY70/934 tube, was reduced approximately 20 percent by substituting T300 fibers for the 90° laminae. The above process maintains high axial stiffness while reducing the tensile hoop stress. An opposite effect is noted when the GY70 fibers in the 0° laminae are replaced by T300 fibers (i.e., increasing the hoop stress in the 0° laminae). The above phenomenon demonstrates the usefulness of hybrid composite tubes in order to maximize longitudinal stiffness and minimize residual hoop stress.

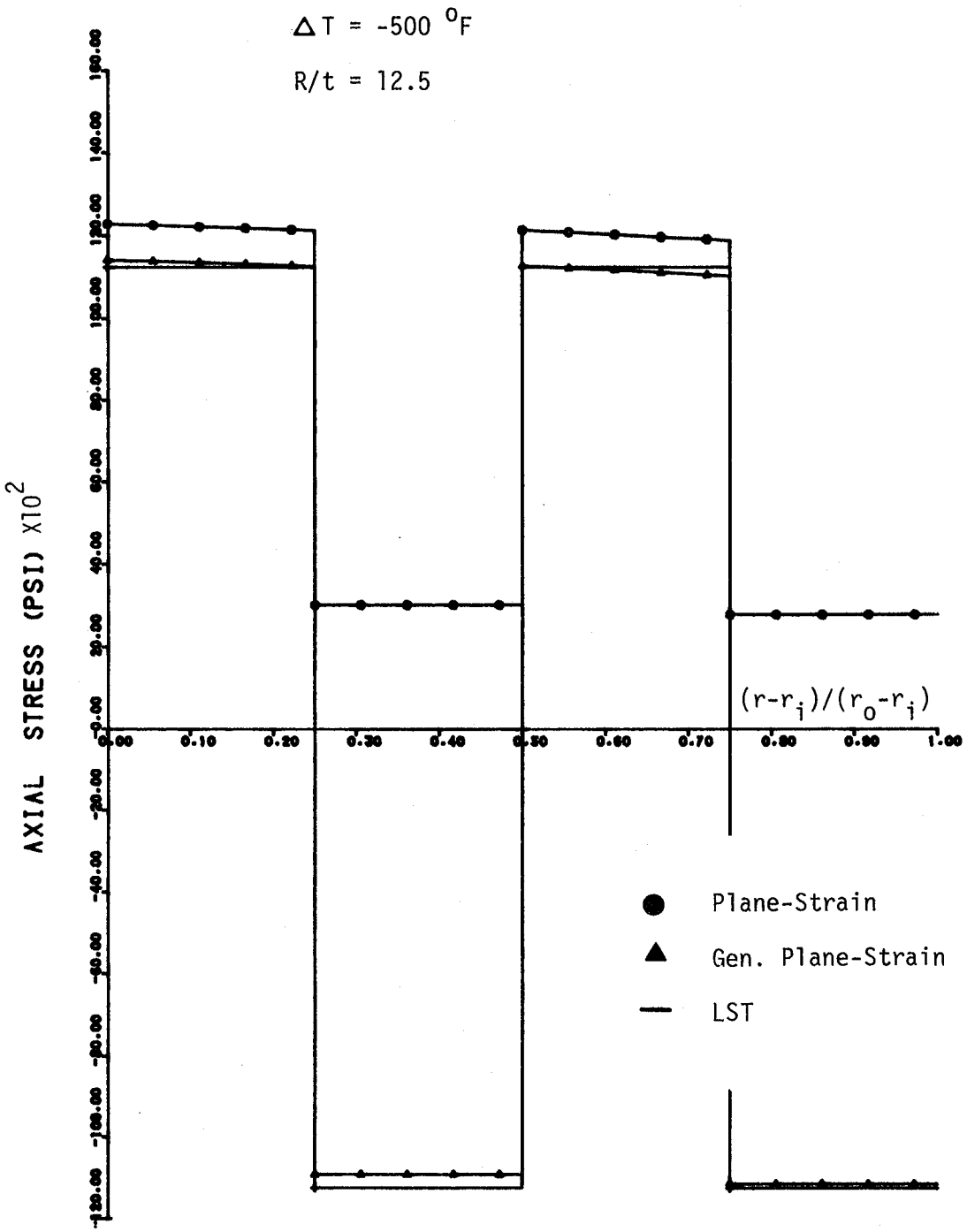


Fig. 6. Residual Axial Stress in a $(90/0/90/0)_t$ Laminated Composite Tube

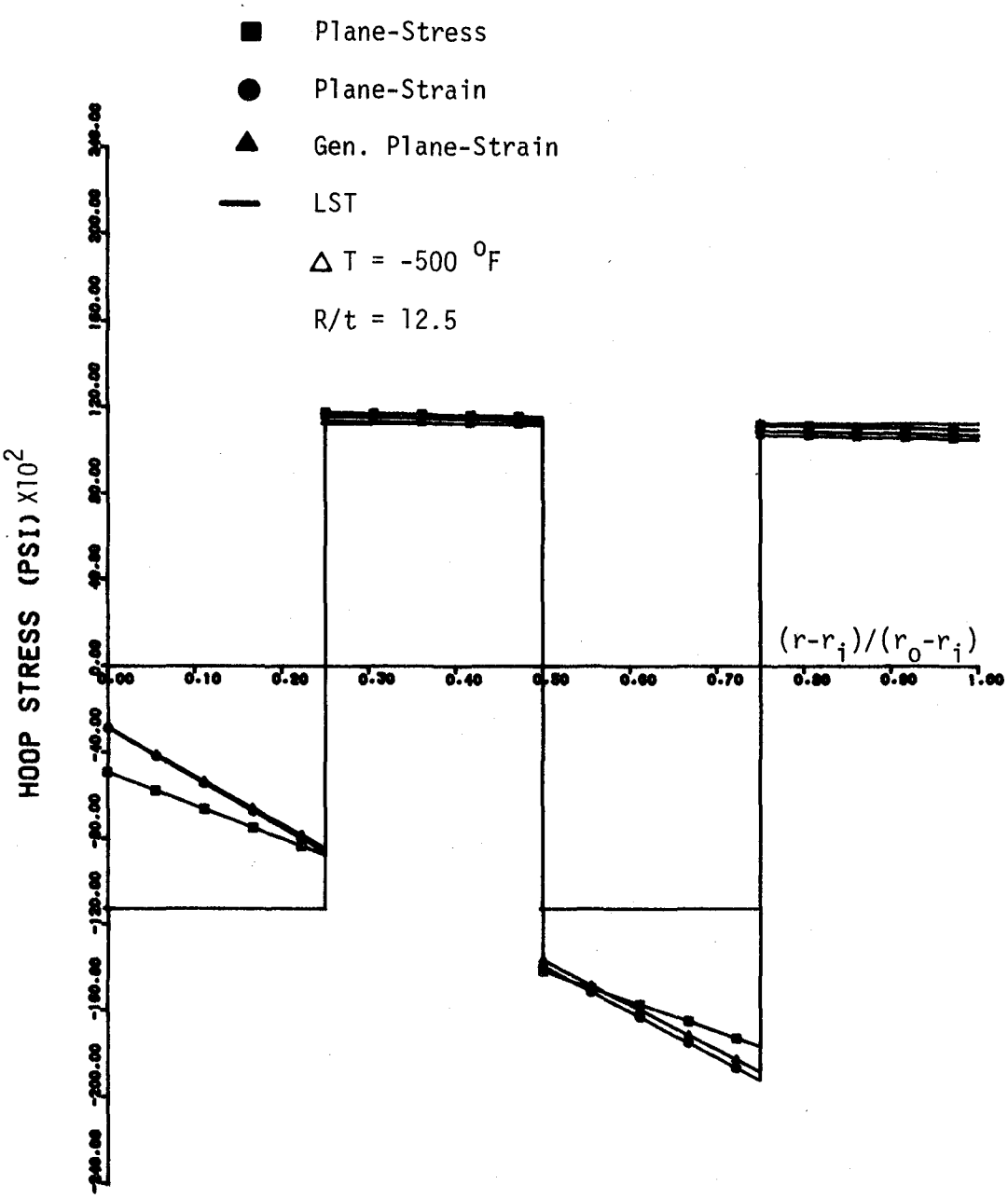


Fig. 7. Residual Hoop Stress in a $(90/0/90/0)_t$ Laminated Composite Tube

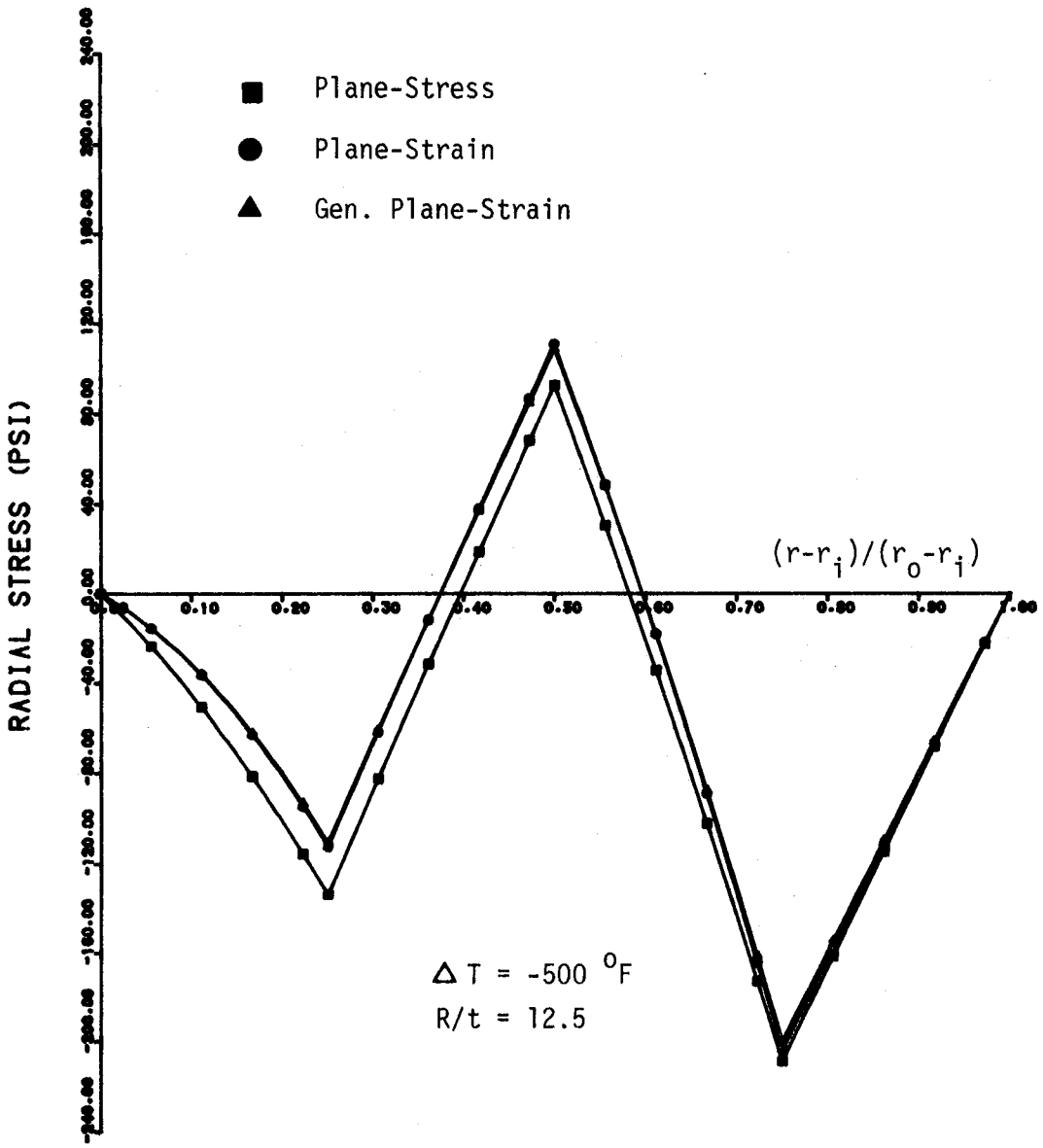


Fig. 8. Residual Radial Stress in a $(90/0/90/0)_t$ Laminated Composite Tube

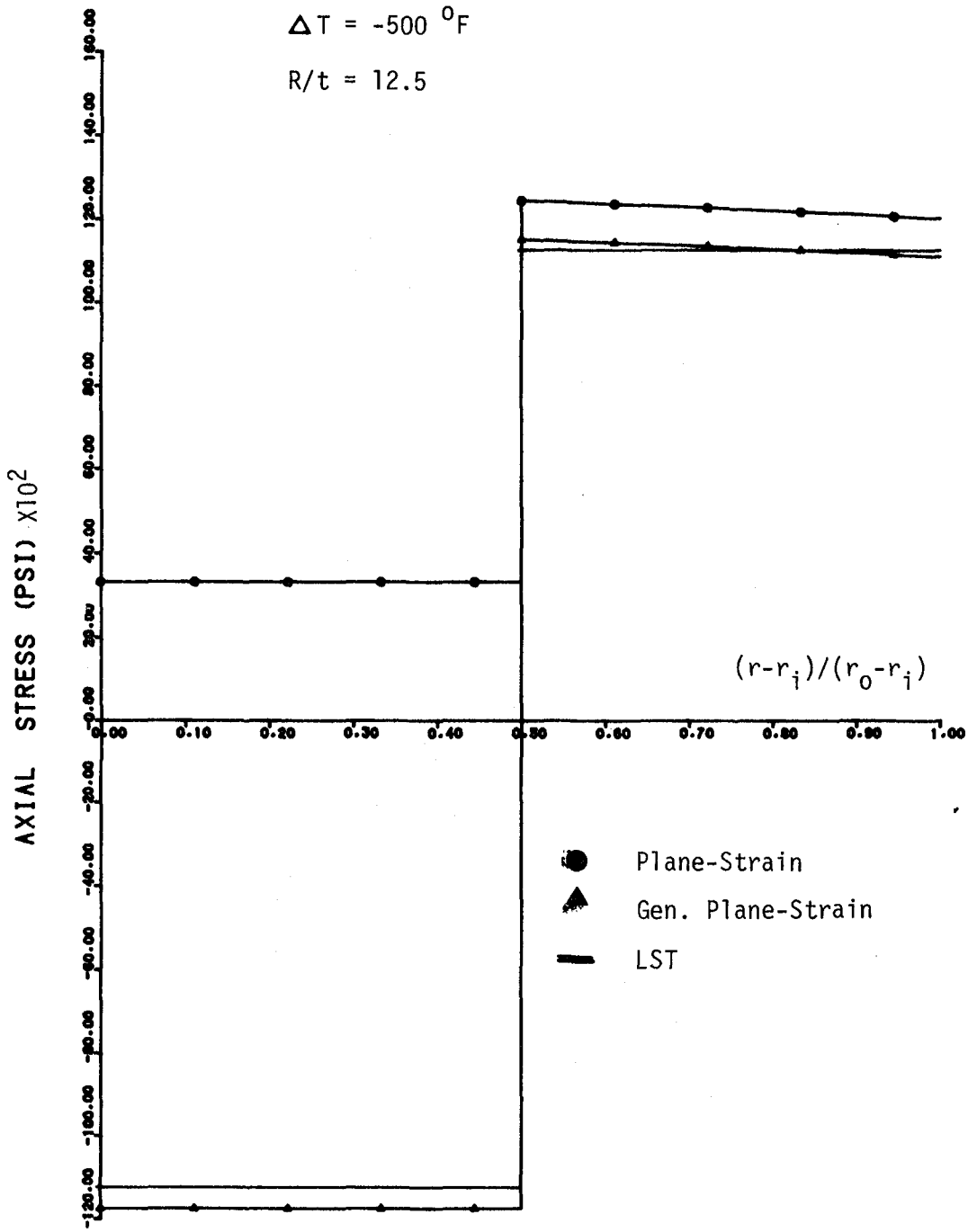


Fig. 9. Residual Axial Stress in a $(0_2/90_2)_t$ Laminated Composite Tube

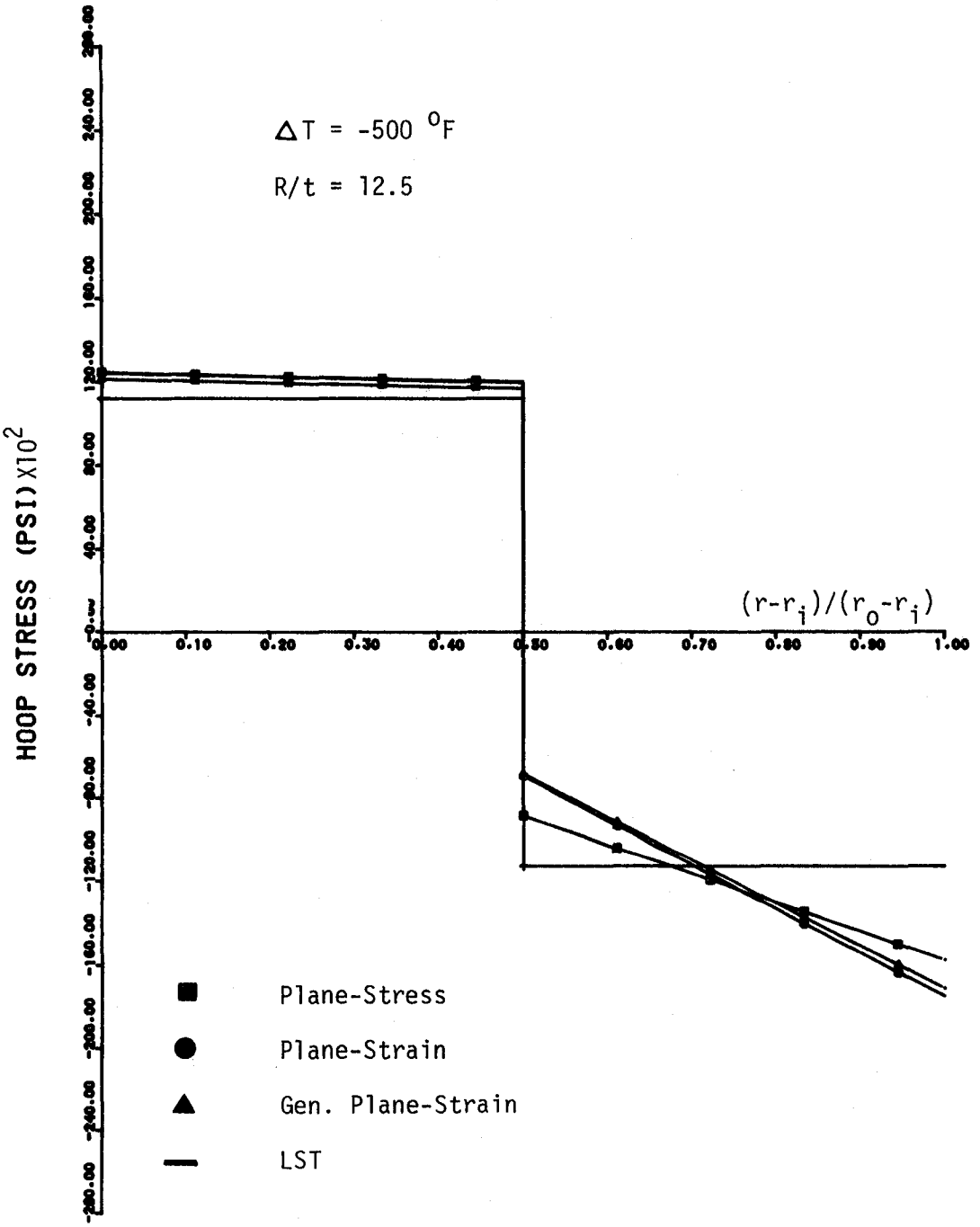


Fig.10. Residual Hoop Stress in a $(0_2/90_2)_t$ Laminated Composite Tube

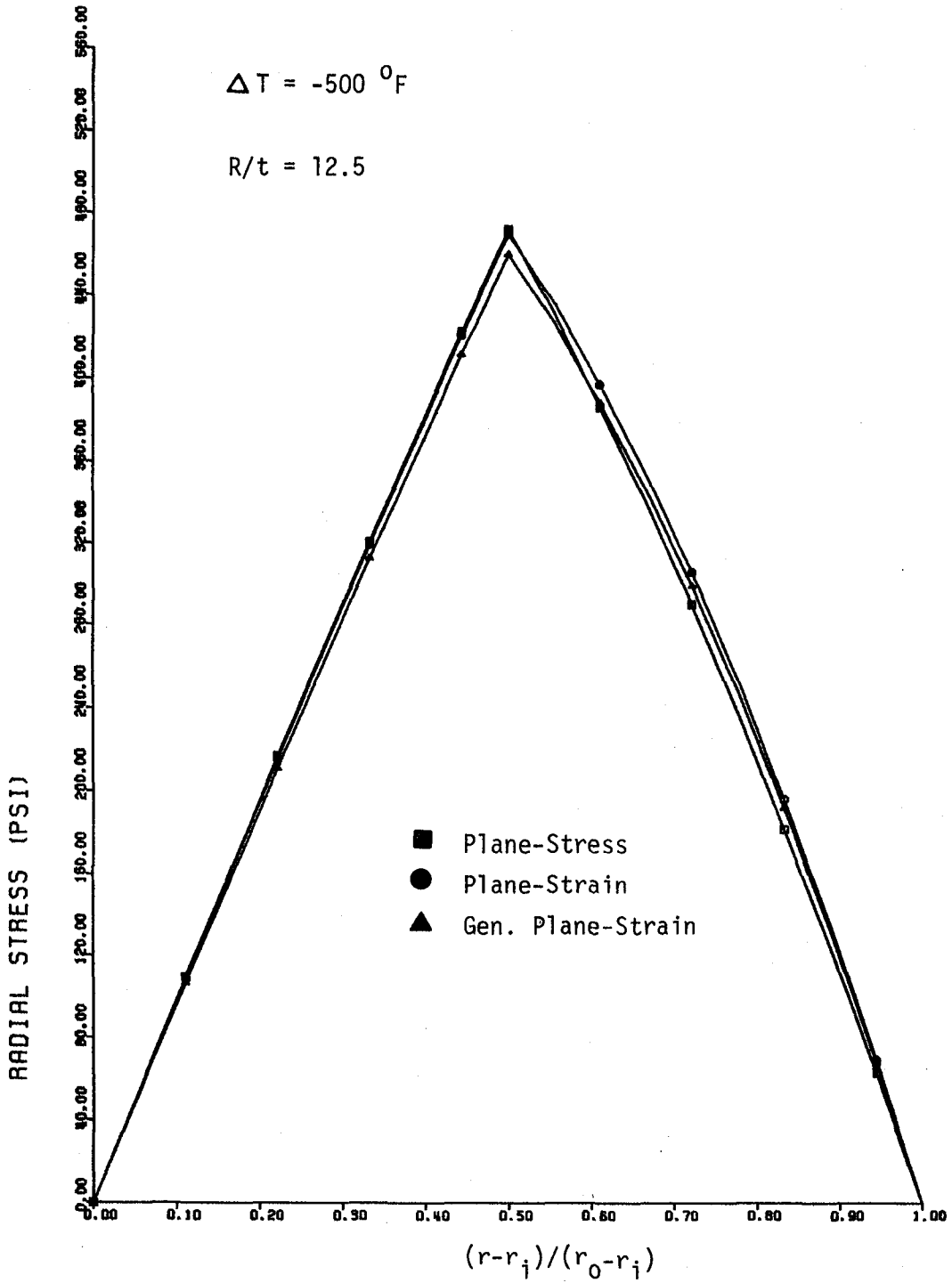


Fig.11. Residual Radial Stress in a $(0_2/90_2)_t$ Laminated Composite Tube

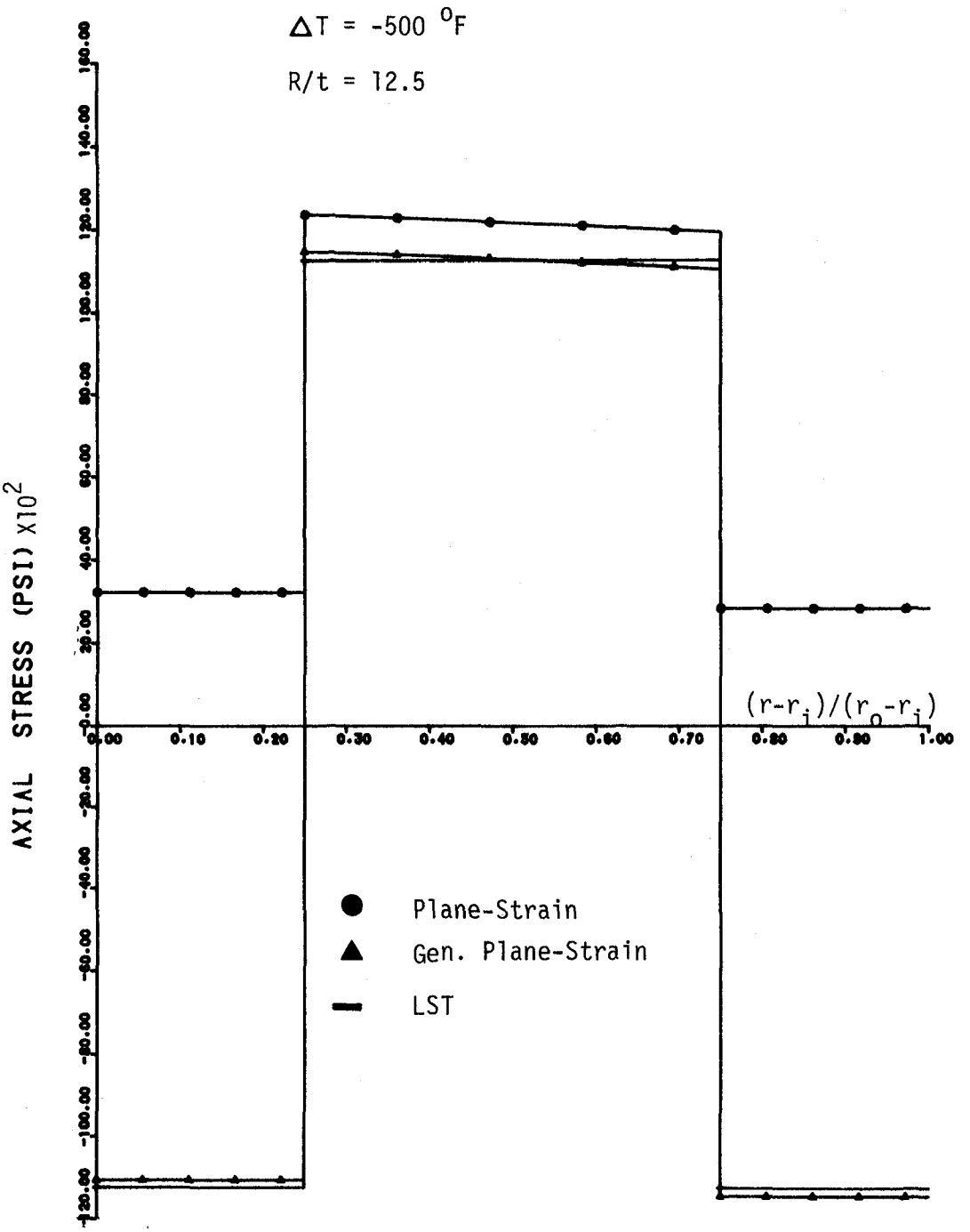


Fig. 12. Residual Axial Stress in a $(0/90_2/0)_t$ Laminated Composite Tube

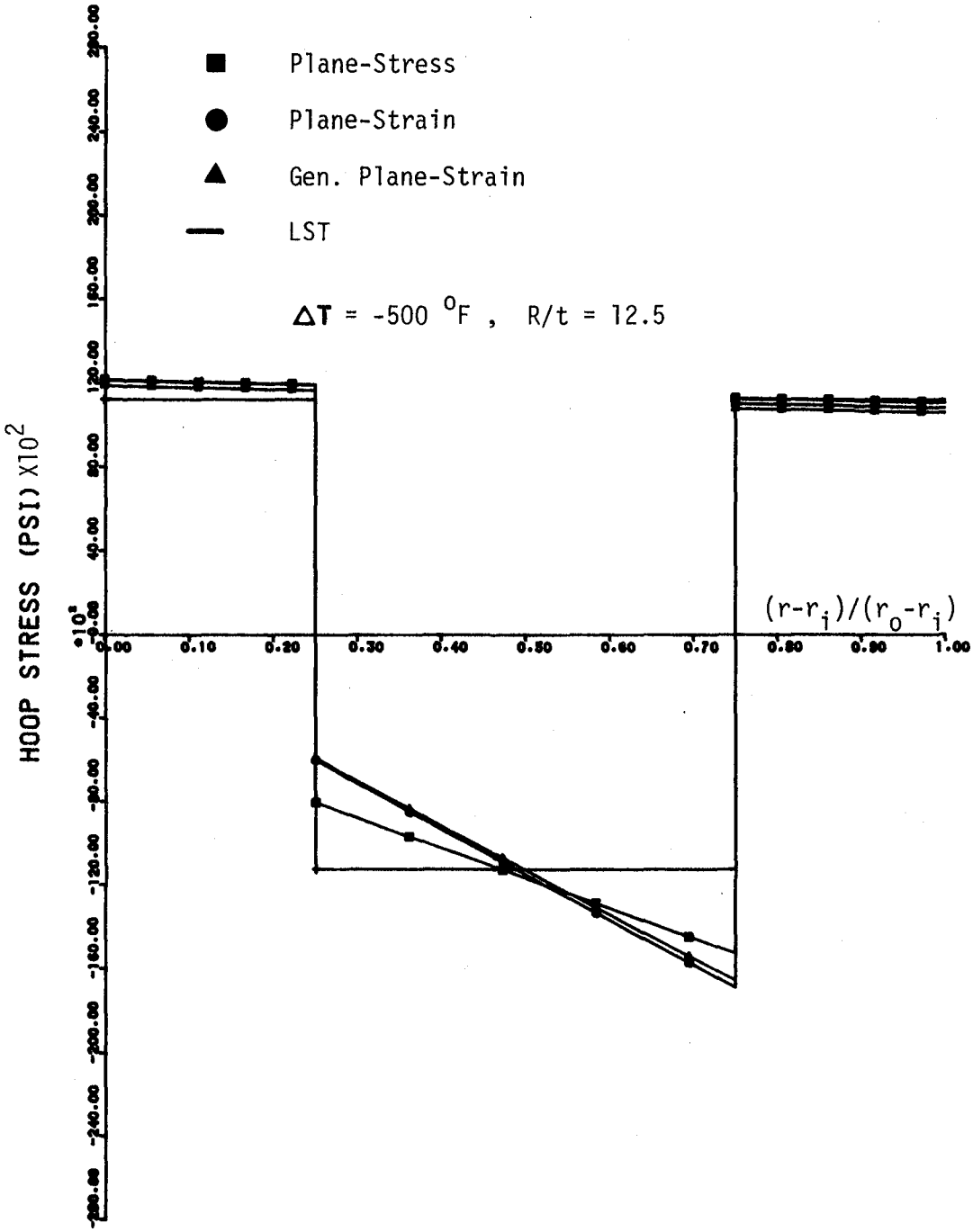


Fig. 13 . Residual Hoop Stress in a $(0/90_2/0)_t$ Laminated Composite Tube

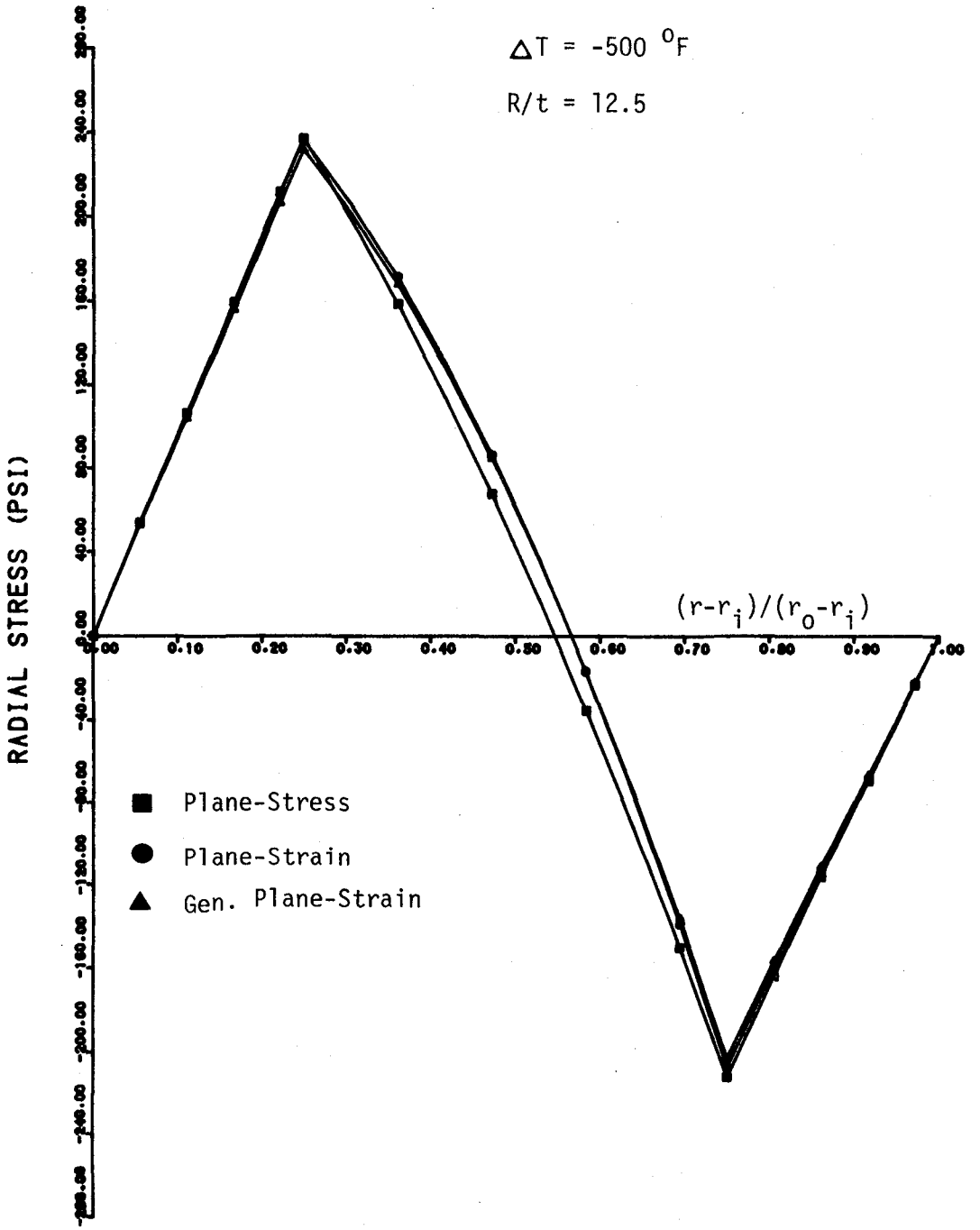


Fig. 14 . Residual Radial Stress in a $(0/90_2/0)_t$ Laminated Composite Tube

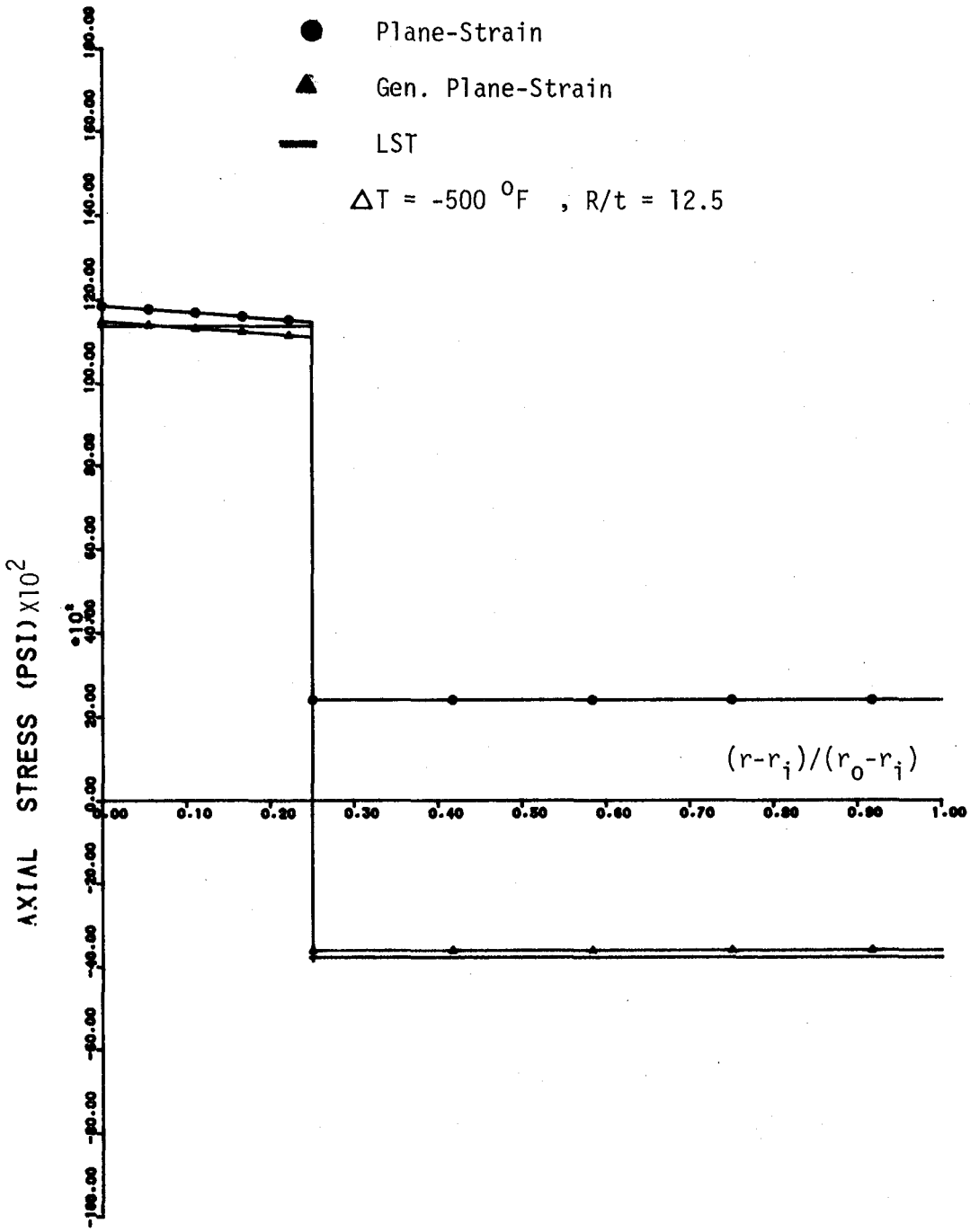


Fig. 15. Residual Axial Stress in a $(90/0_3)_t$ Laminated Composite Tube

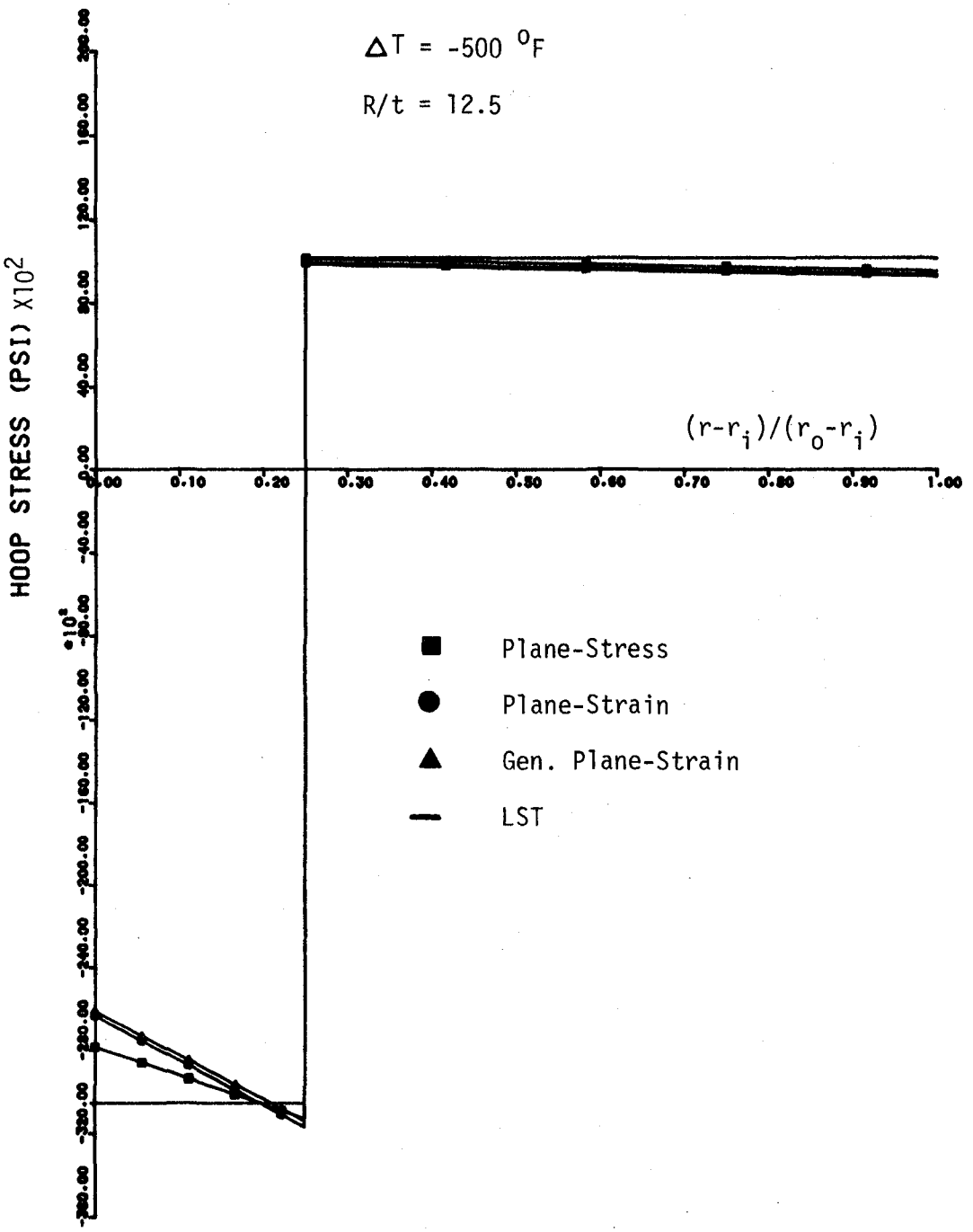


Fig. 16. Residual Hoop Stress in a $(90/0_3)_t$ Laminated Composite Tube

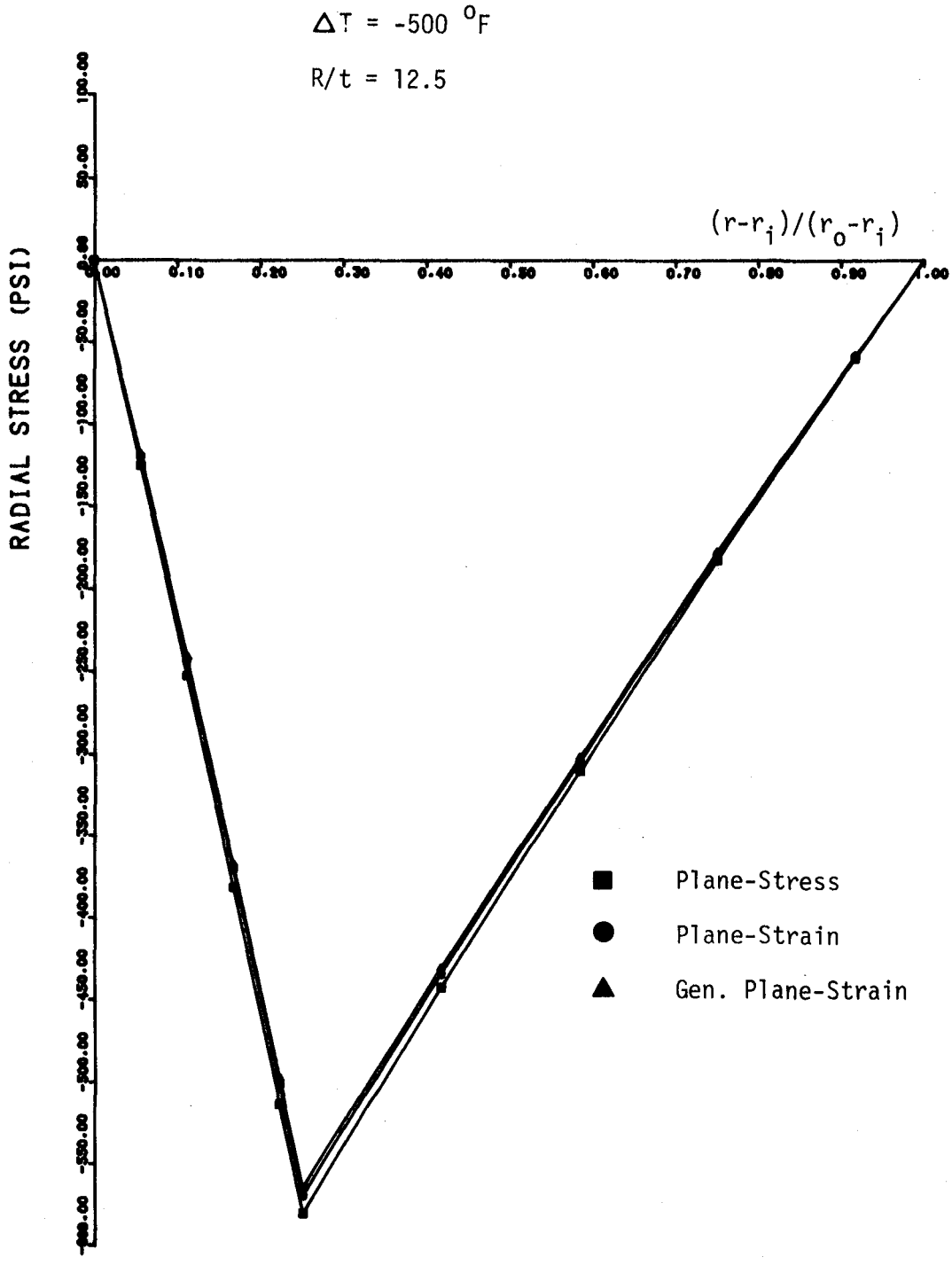


Fig. 17 . Residual Radial Stress in a $(90/0_3)_t$ Laminated Composite Tube

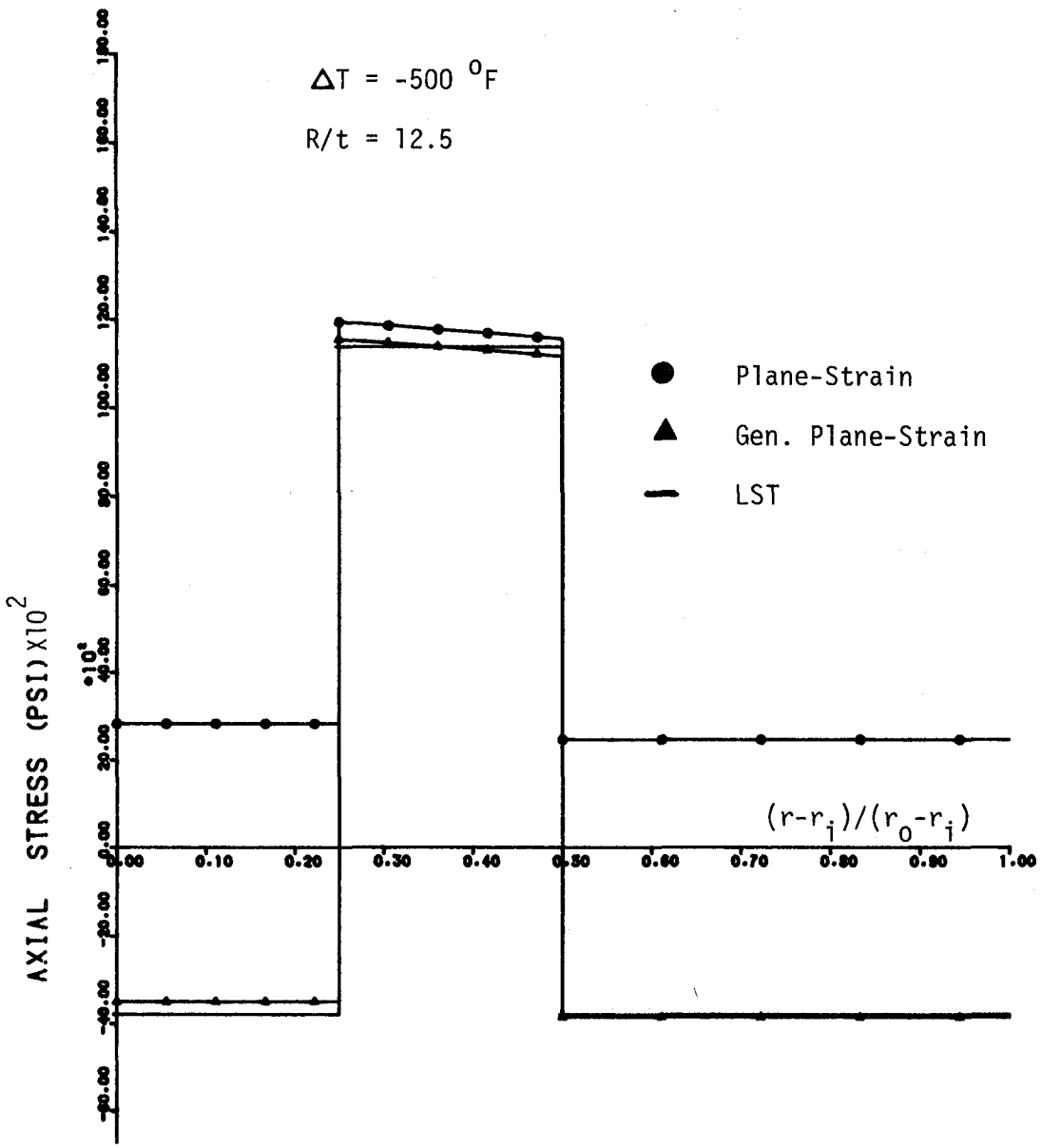


Fig. 18. Residual Axial Stress in a $(0/90/0_2)_t$ Laminated Composite Tube

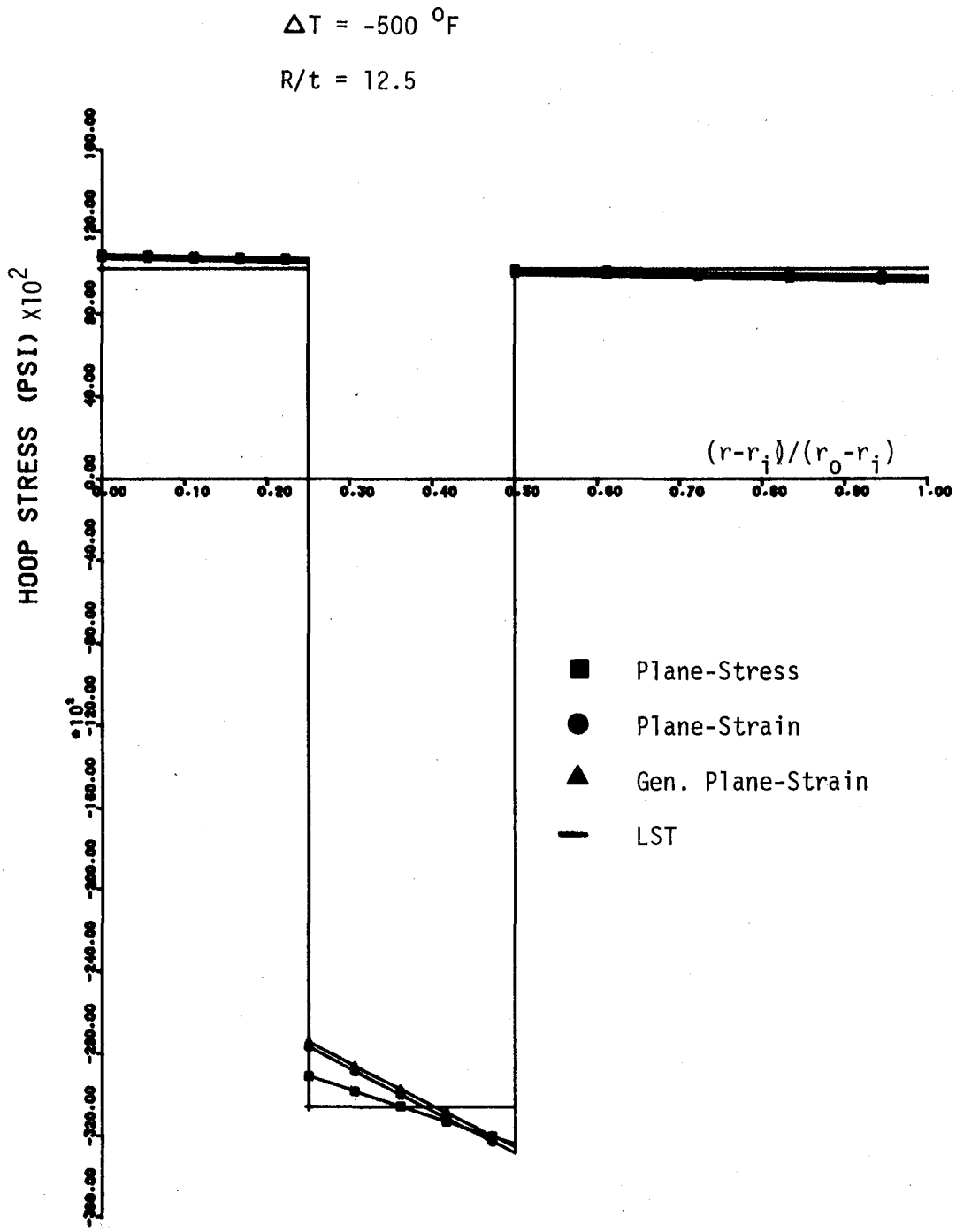


Fig. 19. Residual Hoop Stress in a $(0/90/0_2)_t$ Laminated Composite Tube

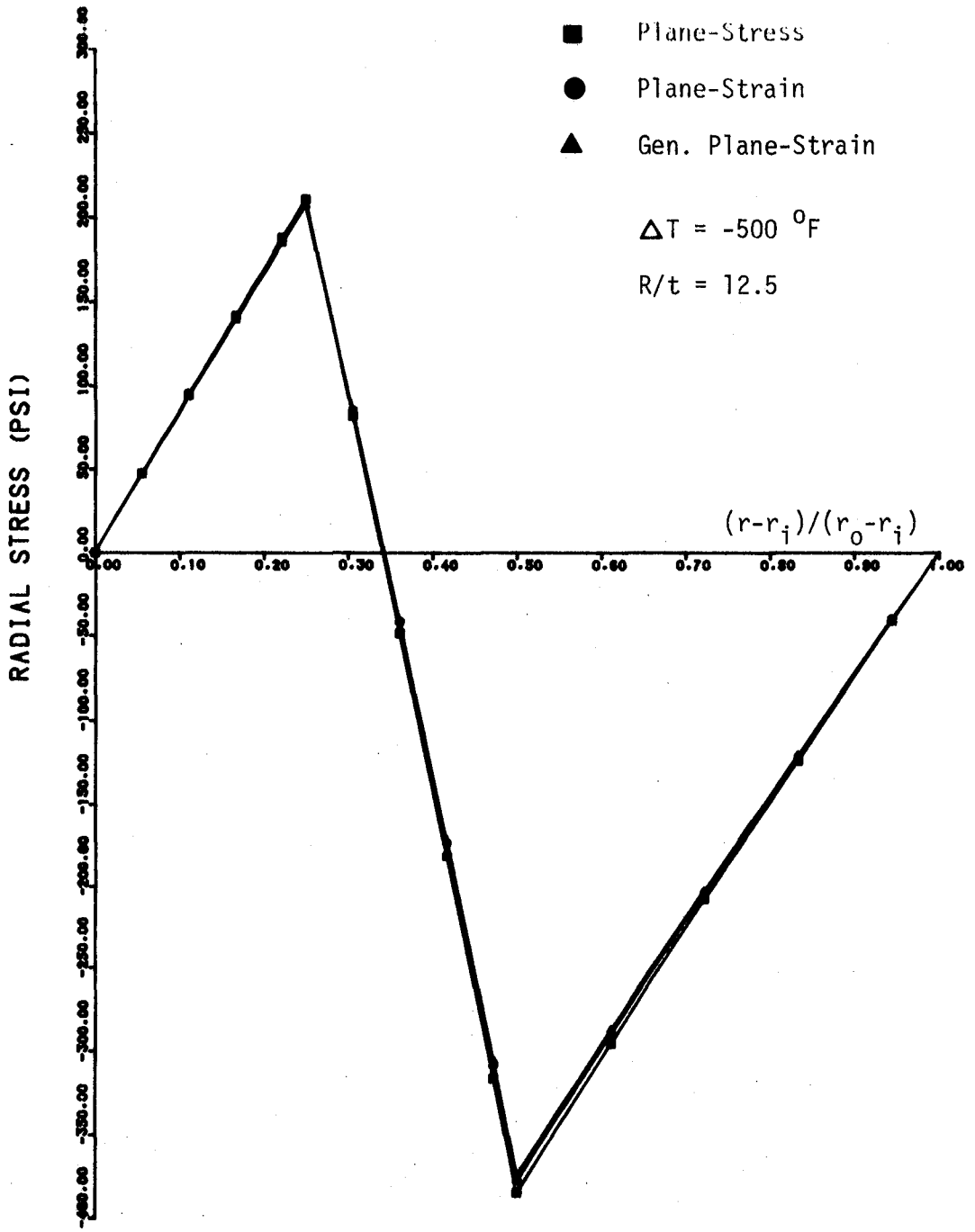


Fig. 20. Residual Radial Stress in a $(0/90/0_2)_t$ Laminated Composite Tube

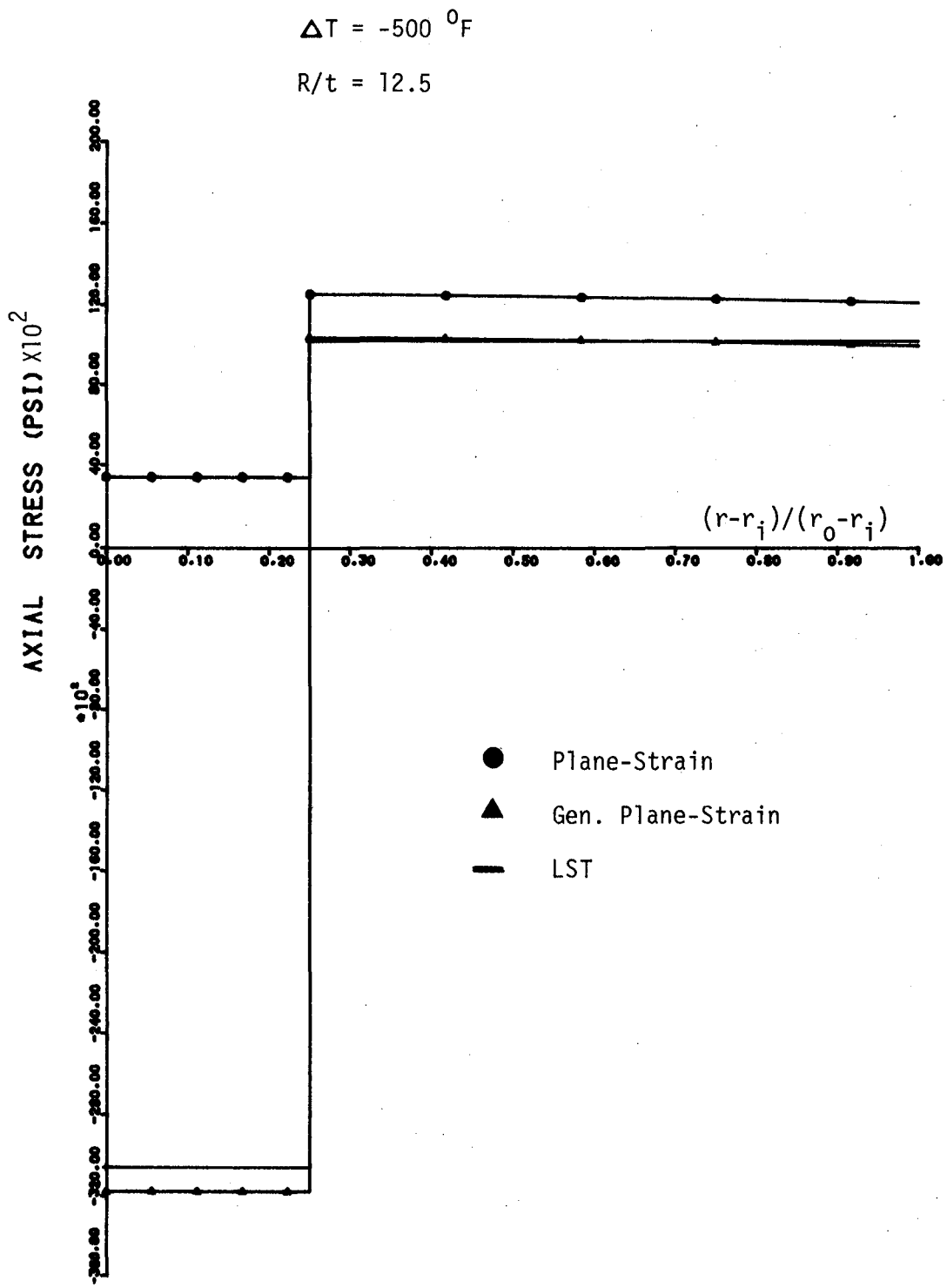


Fig. 21 . Residual Axial Stress in a $(0/90_3)_t$ Laminated Composite Tube

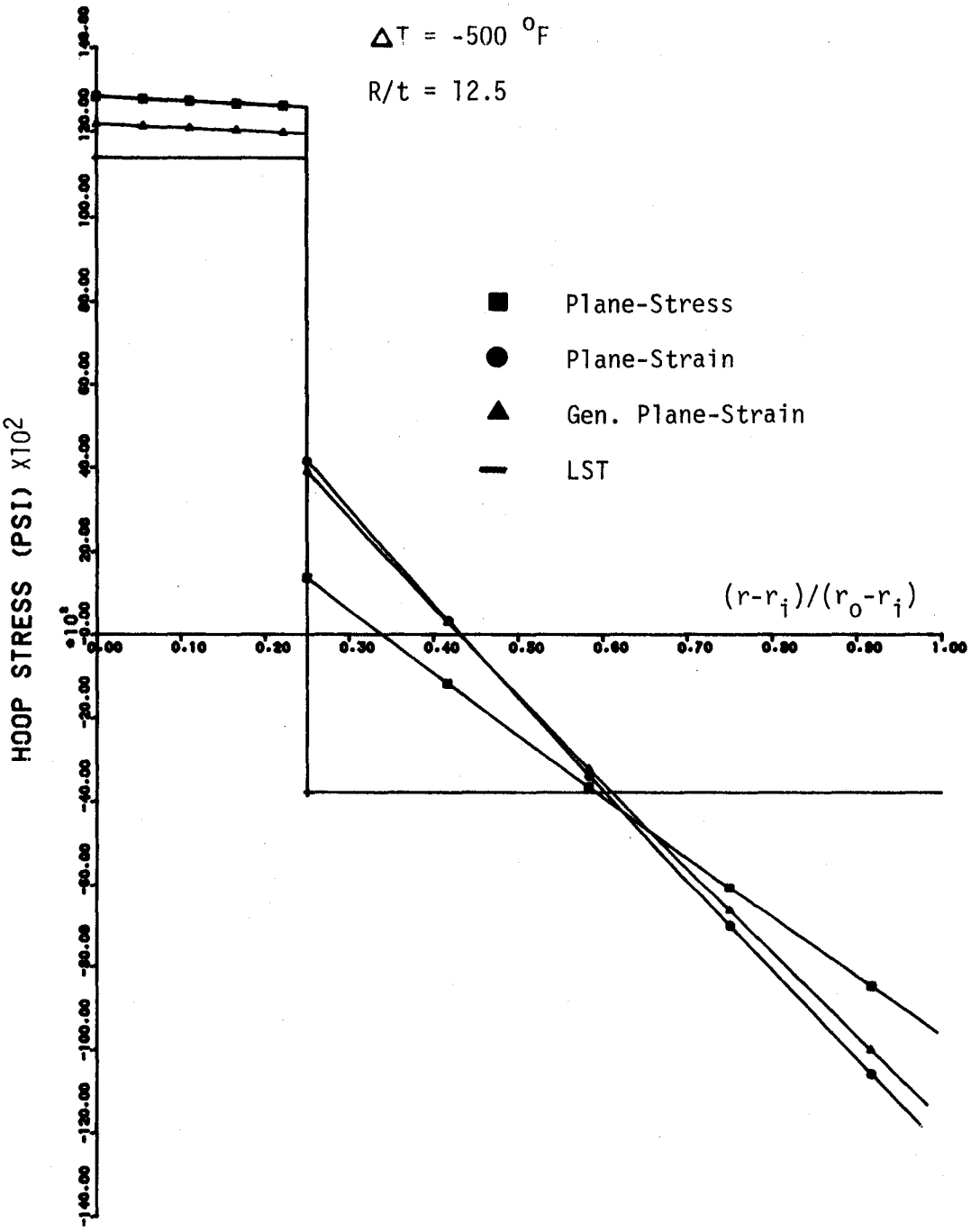


Fig. 22. Residual Hoop Stress in a $(0/90_3)_t$ Laminated Composite Tube

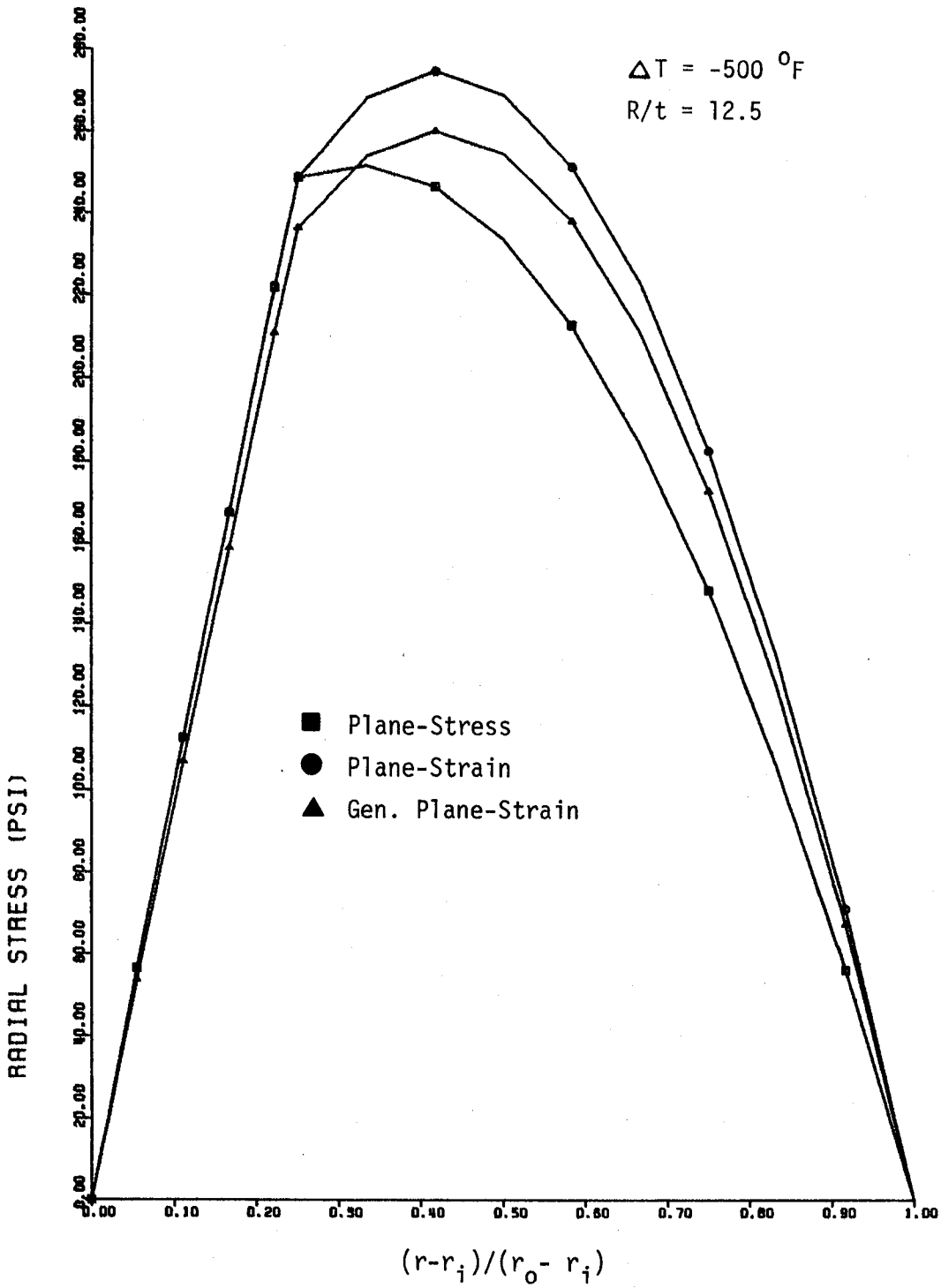


Fig. 23 . Residual Radial Stress in a $(0/90_3)_t$ Laminated Composite Tube

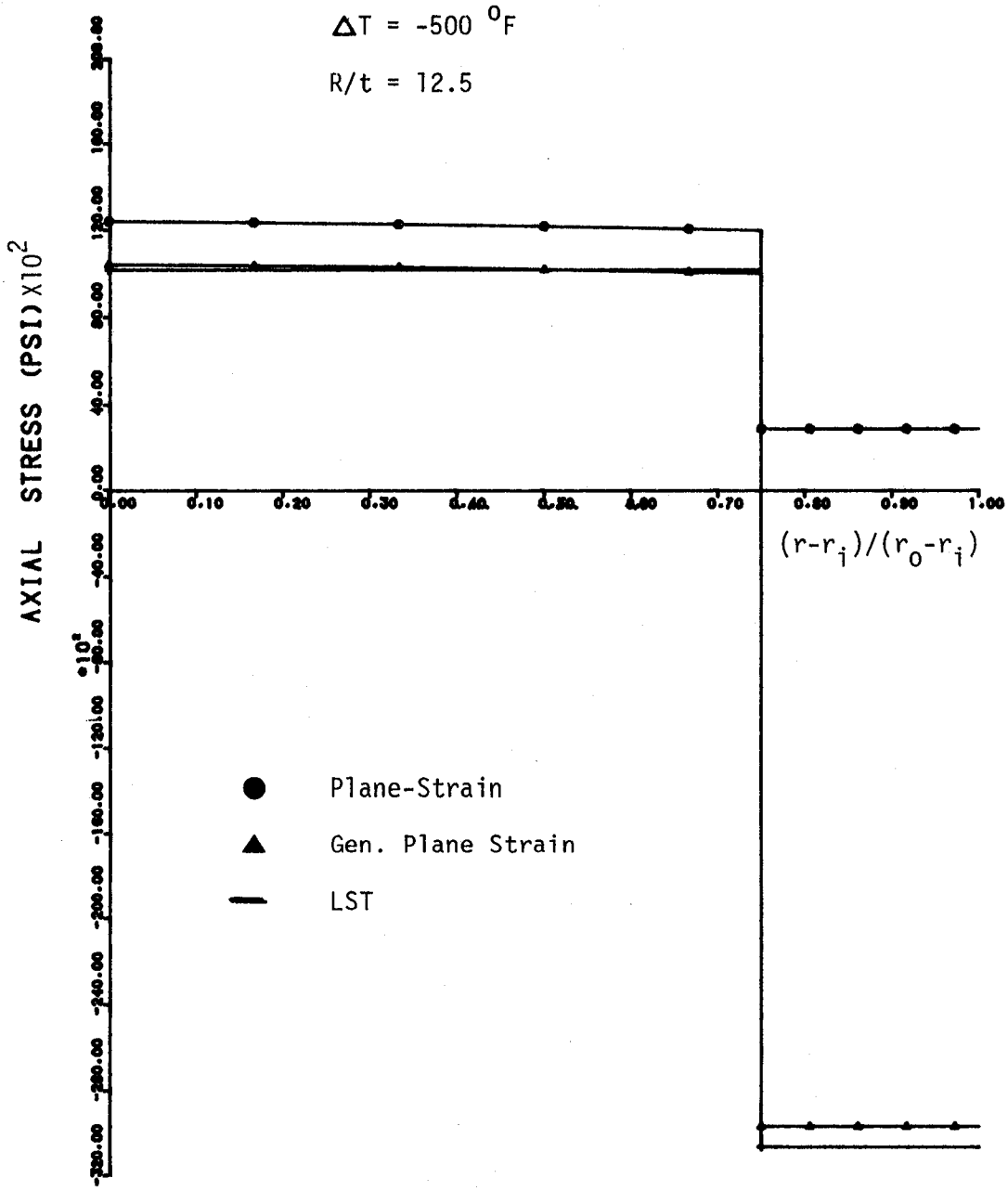


Fig. 24. Residual Axial Stress in a $(90_3/0)_t$ Laminated Composite Tube

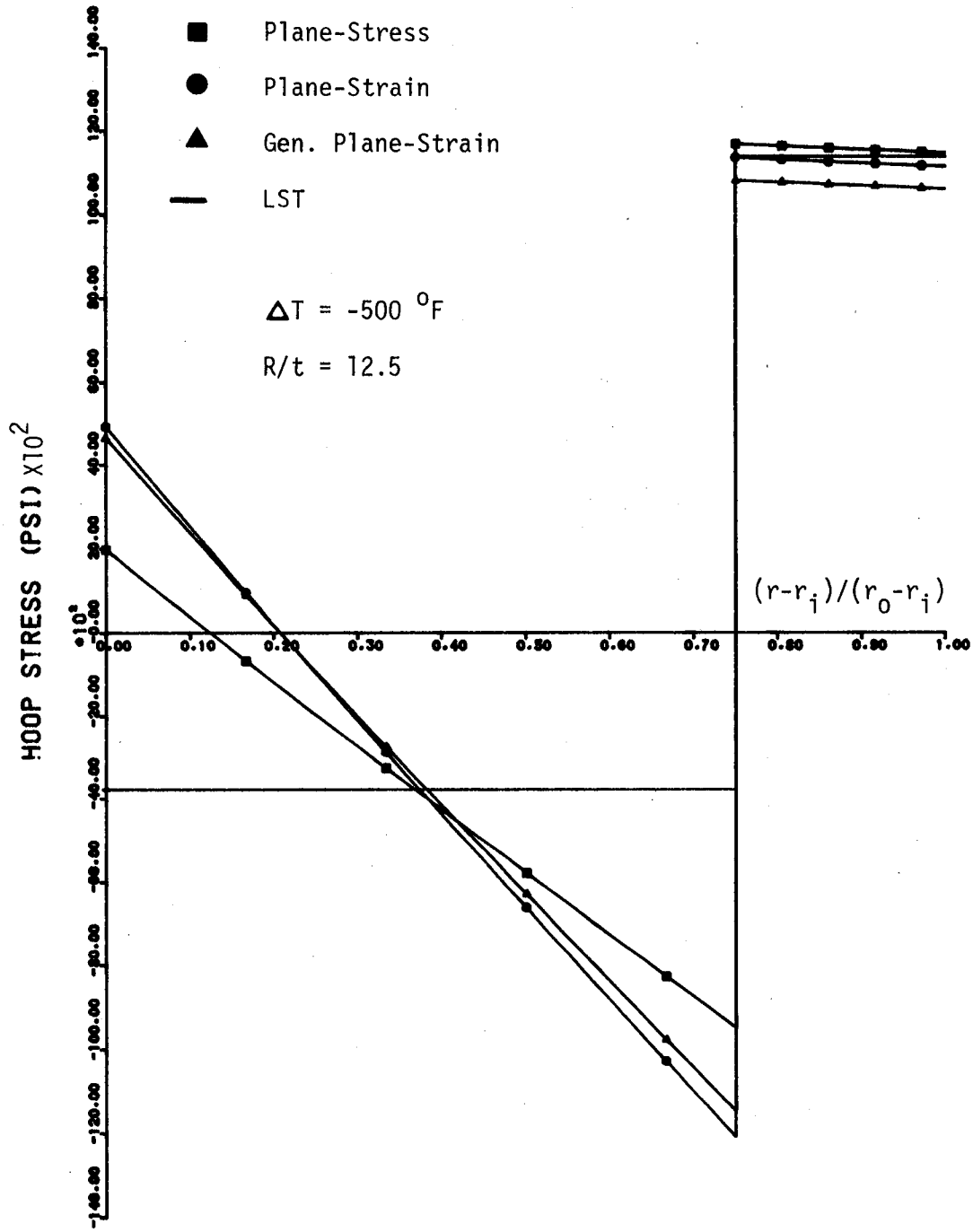


Fig. 25. Residual Hoop Stress in a $(90_3/0)_t$ Laminated Composite Tube

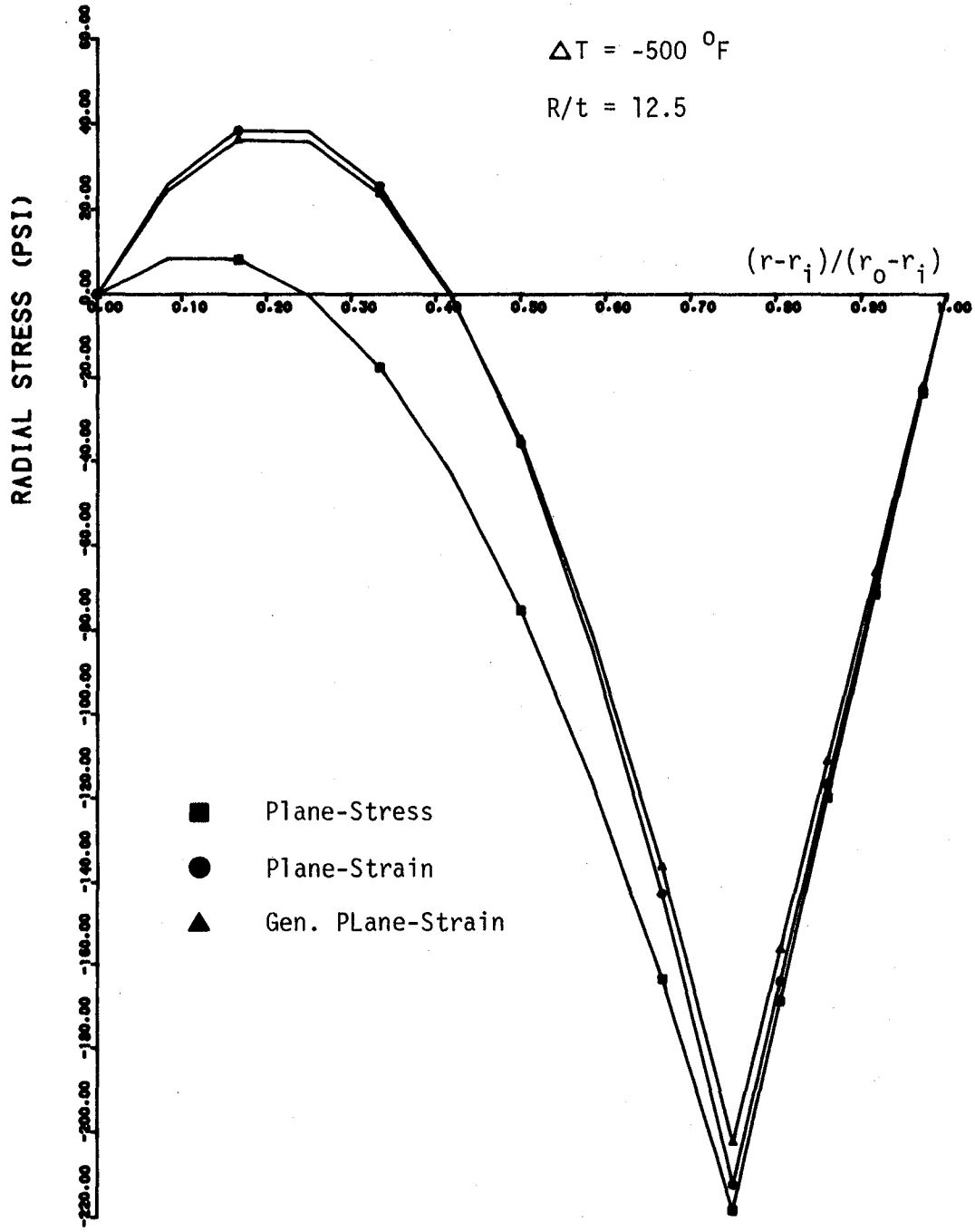


Fig. 26. Residual Radial Stress in a $(90_3/0)_t$ Laminated Composite Tube

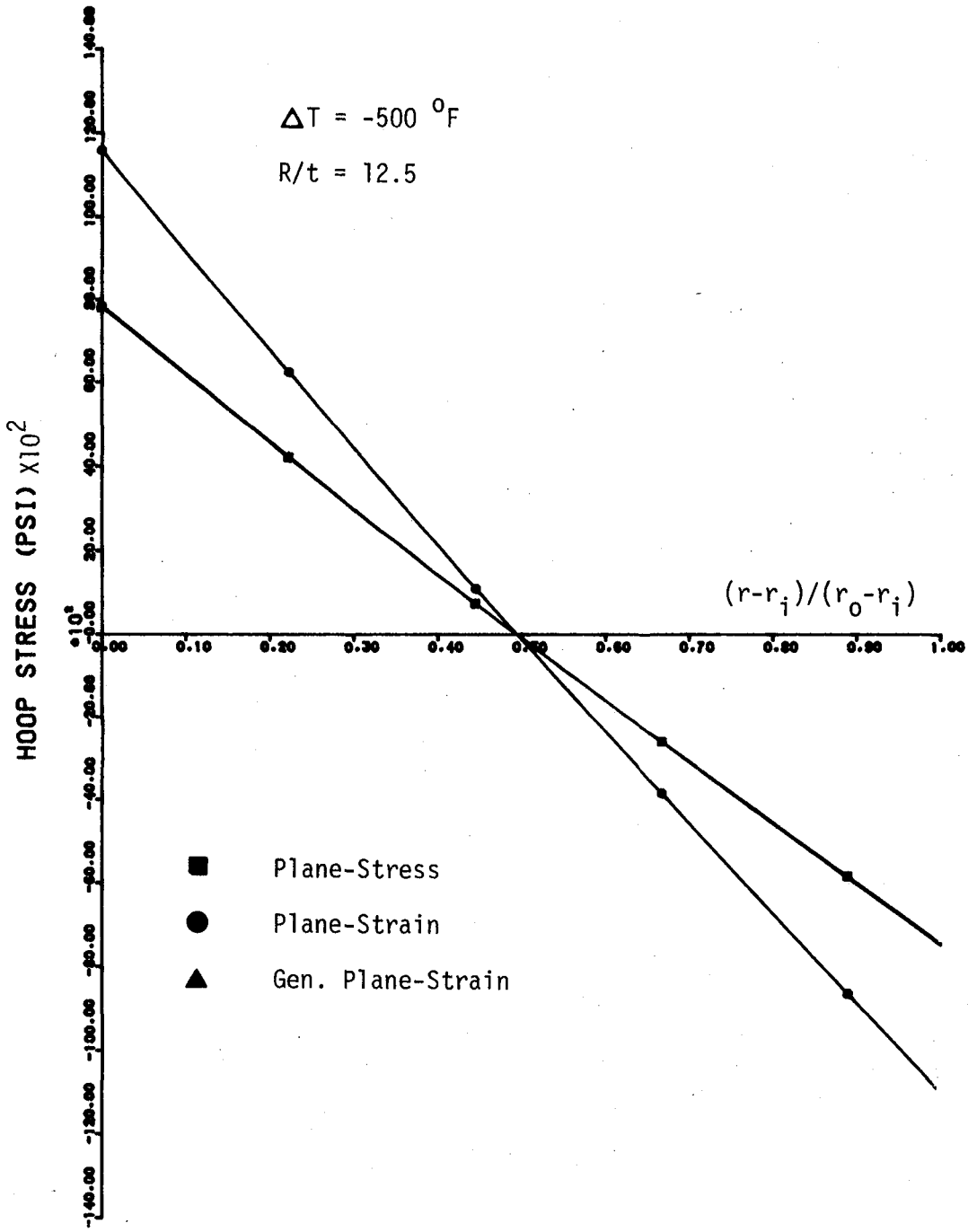


Fig. 27. Residual Hoop Stress in a $(90_4)_t$ Laminated Composite Tube

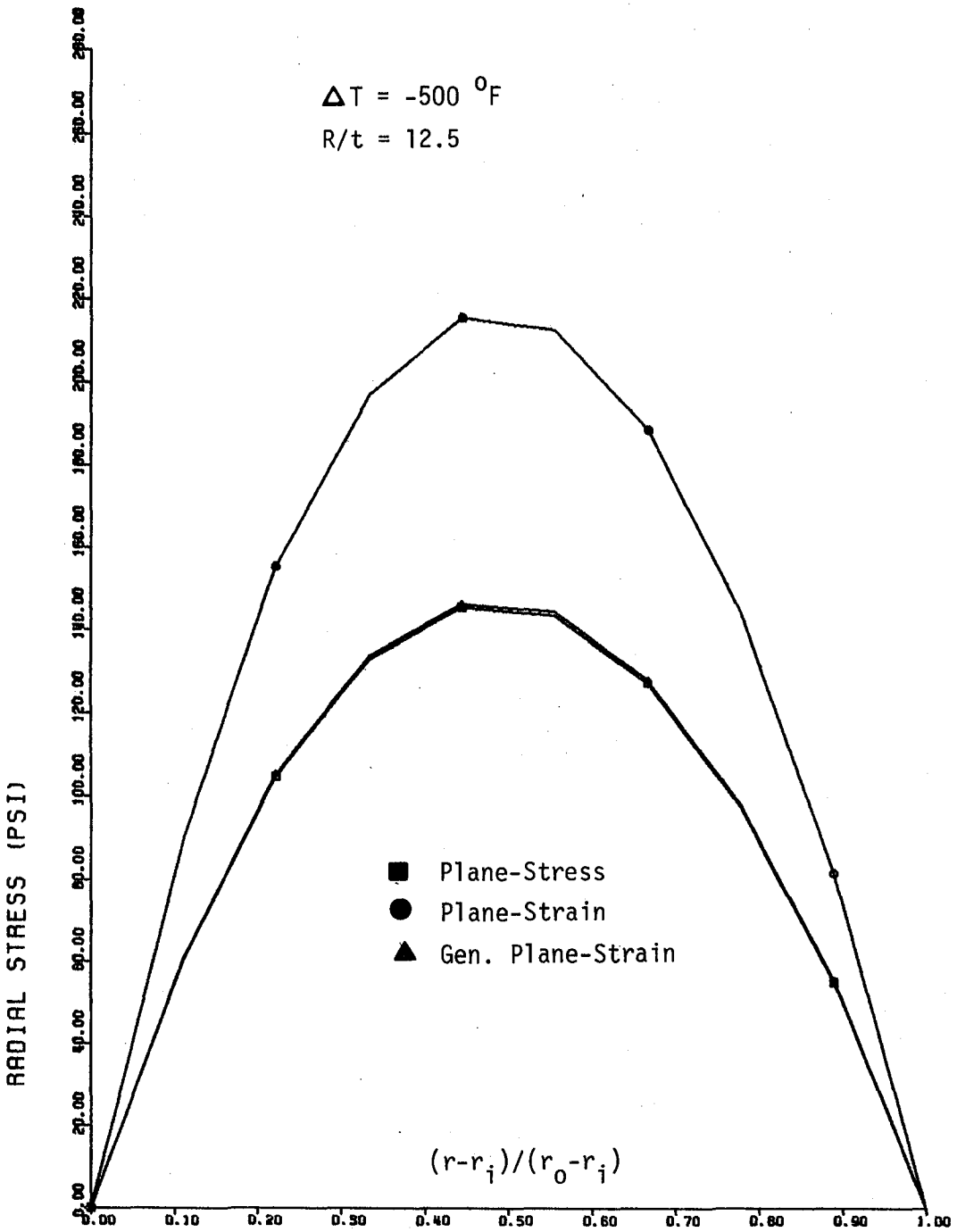


Fig. 28. Residual Radial Stress in a $(90_4)_t$ Laminated Composite Tube

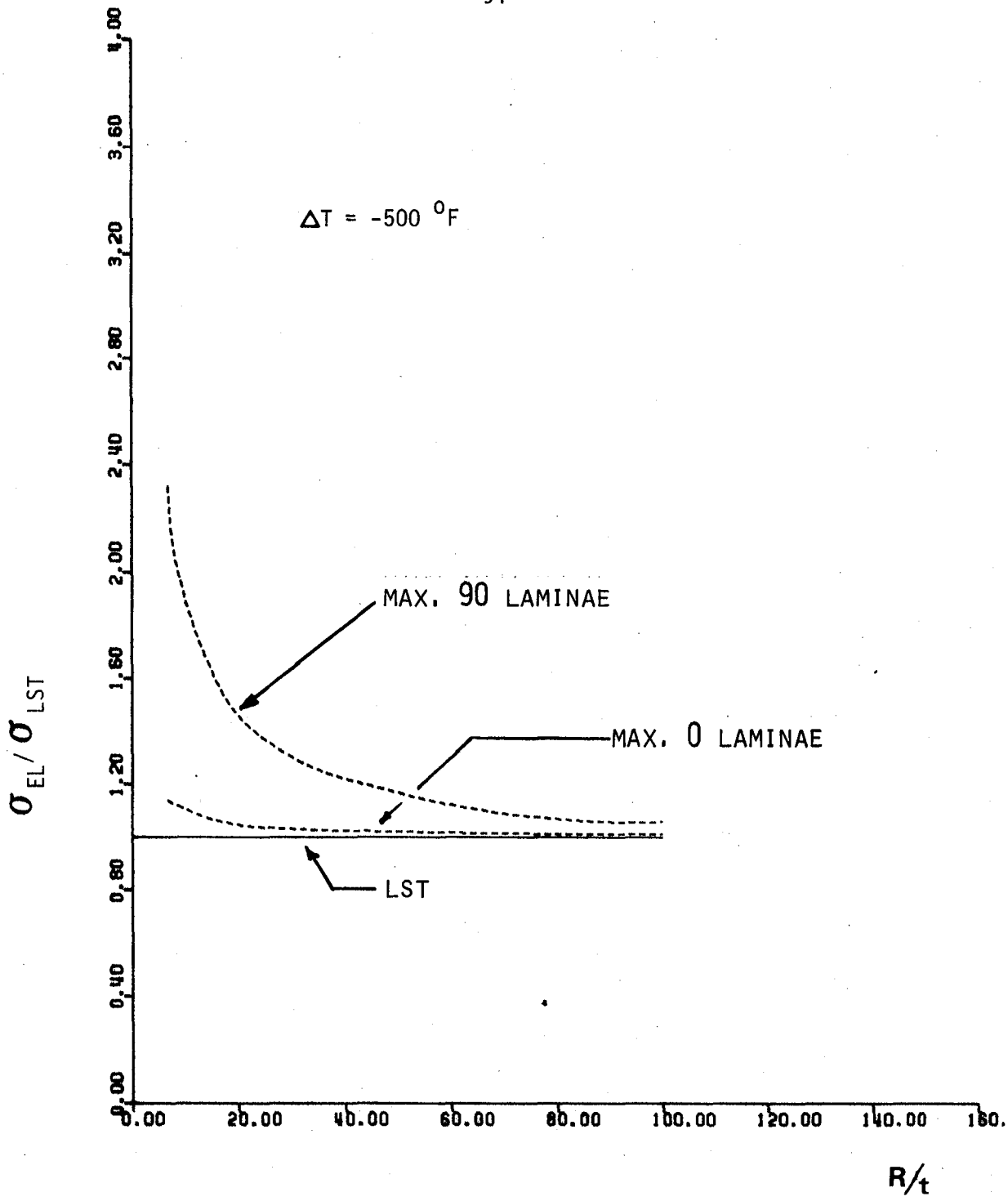


Fig. 29 . Normalized Residual Hoop Stress in a $(0/90/0/90)_t$ Laminated Composite Tube vs. R/t Ratios.

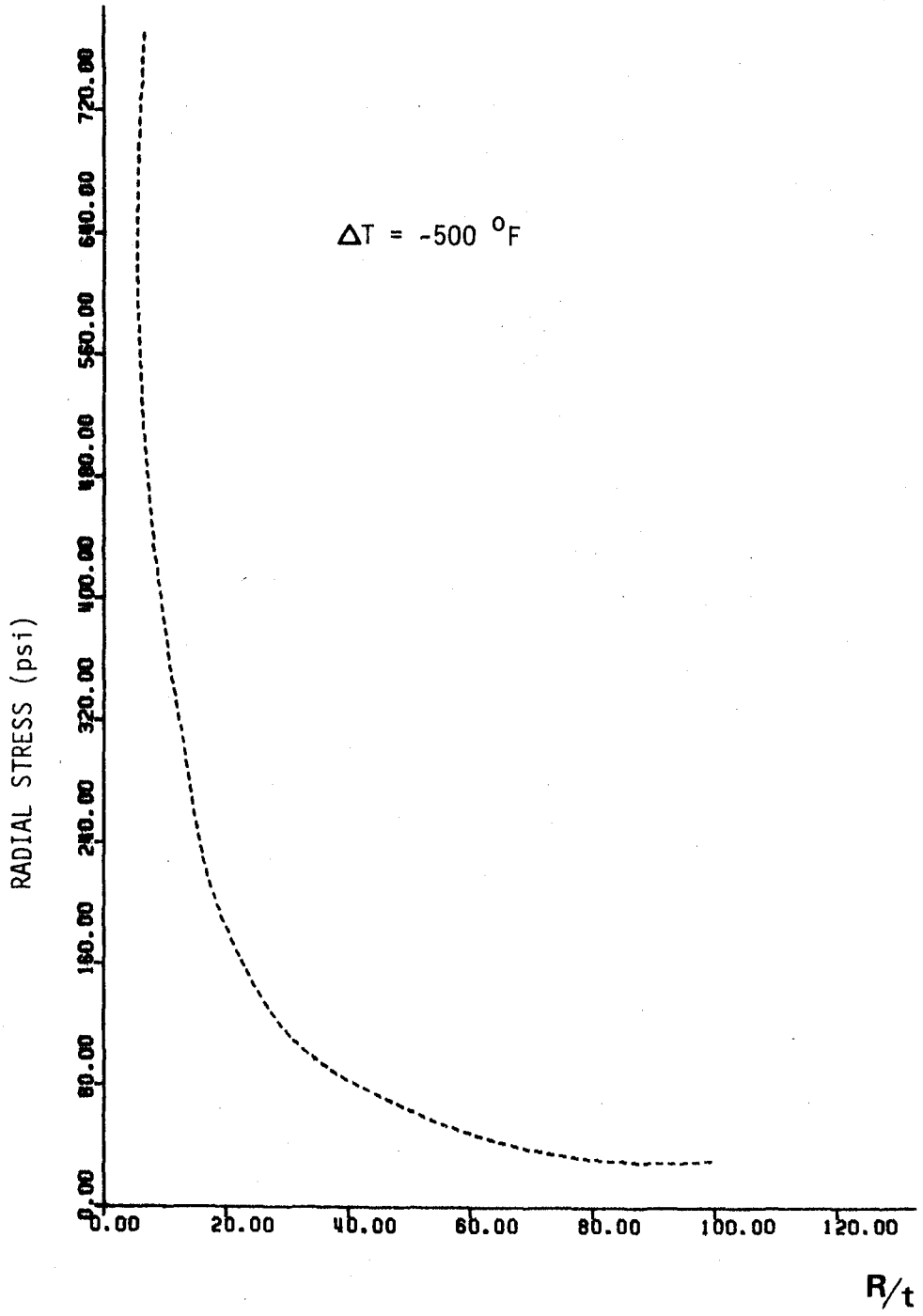


Fig. 30. Residual Radial Stress in a $(0/90/0/90)_t$ Laminated Composite Tube vs. R/t Ratios.

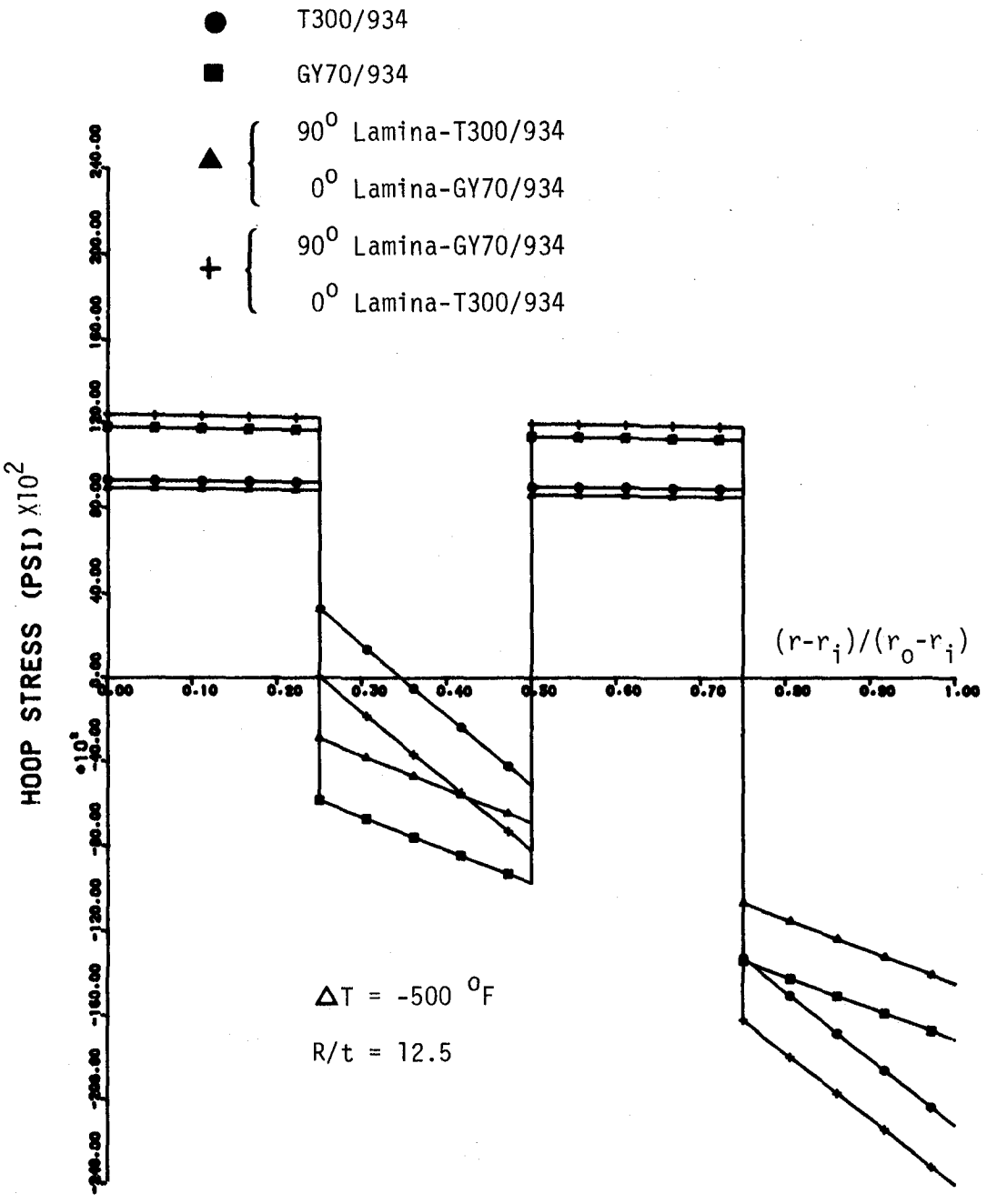


Fig. 31. Residual Hoop Stress in a $(0/90/0/90)_t$ Hybrid Laminated Composite Tube

CONCLUSION

In the present section the conclusions that can be drawn from the investigation are summarized below.

(1) It was shown that the laminated composite tube thermomechanics problem can be treated accurately by laminated shell theory provided that the tube's R/t ratio is relatively large (R/t > 10). However, at smaller R/t ratios considerable deviation is observed between LST and plane elasticity solutions, particularly in the hoop stress for a 90° lamina.

(2) The residual radial stress was shown to be negligible as compared with the other stress components (i.e., hoop and axial).

(3) The data presented allows one to conclude that varying the stacking sequence while keeping the number of 0° and 90° laminae constant has little effect on the level of the axial stress in these layers. The radial stresses are affected. However, varying the number of 0° and 90° laminae in the laminated tube will effect both these stresses.

(4) It was shown that tube hybridization may contribute to both a decrease or increase in the hoop stress. This effect depends upon the placement of particular fibers in given directions.

(5) All plane elasticity solutions agreed well with each other as far as the calculated hoop and radial stresses. However, the plane-strain and generalized plane-strain show the best agreement with each other.

(6) The generalized plane-strain and LST analyses showed close agreement on the residual axial stress. However, the plane-strain analysis failed to give similar results. This was attributed to the kinematic constraints assumed in this analysis.

REFERENCES

1. R. E. Sherrer, "Filament Wound Cylinders with Axial-Symmetric Loads, Jo. Composite Materials, Vol. 1 (1967), p. 344.

2. N. J. Pagano, J. C. Halpin, and J. M. Whitney, "Tension Buckling of Anisotropic Cylinders," Jo. Composite Materials, Vol. 2 No. 2 (April 1968), p. 154.

3. N. J. Pagano and J. M. Whitney, "Geometric Design of Composite Cylindrical Characterization Specimens," Jo. Composite Materials, Vol. 4 (July 1970), p. 360.

4. Tu-Lung Weng, "Thermal Stresses in Anisotropic Hollow Cylinder," Jo. of Basic Engineering, June 1965, p. 391.

5. S. P. Timoshenko and J. N. Goodier, "Theory of Elasticity," McGraw-Hill, Third Edition, 1970, New York.

6. J. M. Whitney and J. C. Halpin, "Analysis of Laminated Anisotropic Tubes under Combined Loading," Jo. Composite Materials, Vol. 2, No. 3 (July 1968), p. 360.

7. Y. C. Pao, "On Higher-Order Theory for Thermoelastic Analysis of Heterogeneous Orthotropic Cylindrical Shells, Developments of Theoretical and Applied Mechanics," Vol. xx (1972), p. 787.

8. Y. C. Pao and K. Peterson, "Optimal Study of Interface Axial Shear Stresses in Two-Layer Axisymmetric Shell Induced During Curing Treatment," Jo. Composite Materials, Vol. 16 (May 1982), p. 204.

9. Y. Stavsky and I. Smolash, "Thermoelasticity of Heterogenous Orthotropic Cylindrical Shells, Int. Jo. Solids and Structures, Vol. 6 (1970), p. 1211.

10. J. M. Whitney, N. J. Pagano and R. B. Pipes, "Design and Fabrication of Tubular Specimens for Composite Characterization,

11. R. R. Rizzo and A. A. Vicario, "A Finite Element Analysis of Laminated Anisotropic Tubes," Jo. Composite Materials, Vol. 4 (July 1970), p. 344.

12. J. M. Whitney, "On the Use of Shell Theory for Determining Stresses in Composite Cylinder," Jo. Composite Materials, Vol. 5, (July 1971), p. 340.

13. J. A. Zukas, Effects of Transverse Normal and Shear Strains in Orthotropic Shells," AIAA Journal, Vol. 12, No. 12 (Dec. 1974), p. 1753.

14. R. M. Jones, Mechanics of Composite Materials, New York, McGraw-Hill, 1975.

15. L. R. Calcote, The Analysis of Laminated Composite Structures, New York, Van Nostrand Reinhold Co., 1969.

APPENDIX A

The following material properties of T300/934 and GY70/934 graphite-epoxy composites were used in the study.

T300/934 graphite-epoxy

Moduli (Msi)

$E_1 = 21.3$

$E_2 = 1.32$

$G_{12} = 0.67$

Poisson's Ratios

$\nu_{12} = \nu_{13} = 0.30$

$\nu_{23} = 0.49$

thermal expansion

$\alpha_1 = -\ 0.04E\text{-}6$ in./in. °F

$\alpha_2 = 18.70E\text{-}6$ in./in. °F

GY70/934 graphite-epoxy

moduli (Msi)

$E_{11} = 44.0$

$E_{22} = 0.975$

$G_{12} = 0.71$

Poisson's Ratios

$\nu_{12} = \nu_{13} = 0.129$

$\nu_{23} = 0.49$

Thermal Expansion

$\alpha_1 = -.578E\text{-}6$ in./in.^{0}F

$\alpha_2 = 14.42E\text{-}6$ in./in.^{0}F

BIBLIOGRAPHIC DATA SHEET	1. Report No. CCMS-84-04	2. VPI-E-84-16	3. Recipient's Accession No.
4. Title and Subtitle RESIDUAL STRESSES IN CROSS-PLY COMPOSITE TUBES			5. Report Date April, 1984 6.
7. Author(s) David Cohen and M. W. Hyer			8. Performing Organization Rept. No. VPI-E-84-16
9. Performing Organization Name and Address Virginia Polytechnic Institute & State University Engineering Science and Mechanics Blacksburg, Virginia 24061			10. Project/Task/Work Unit No. 11. Contract/Grant No. NAG-1-343
12. Sponsoring Organization Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, Virginia 23665			13. Type of Report & Period Covered 14.

15. Supplementary Notes

16. Abstracts

The residual thermal stresses in 4-layer cross-ply tubes are studied. The tubes considered had a small radius to wall-thickness ratios and so elasticity solutions were used. The residual thermal stress problem was considered to be axisymmetric and three elasticity solutions were derived and the results compared with the results using classical lamination theory. The comparison illustrates the limitations of classical lamination theory. The three elasticity solutions derived were: plane stress, plane strain, and generalized plane strain, the latter being the most realistic. The study shows that residual stresses in both the hoop and axial direction can be significant. Stacking arrangement effects the residual stress to some extent, as do the material properties of the individual lamina. The benefits of hybrid construction are briefly discussed.

17. Key Words and Document Analysis. 17a. Descriptors

composite tubes, graphite-epoxy tubes, residual stresses, orthotropic elasticity

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group 24-01, 39-02, 39-07

18. Availability Statement	19. Security Class (This Report)	21. No. of Pages
Distribution Unlimited	UNCLASSIFIED	97
	20. Security Class (This Page	22. Price

VIRGINIA TECH CENTER FOR COMPOSITE MATERIALS AND STRUCTURES

The Center for Composite Materials and Structures is a coordinating organization for research and educational activity at Virginia Tech. The Center was formed in 1982 to encourage and promote continued advances in composite materials and composite structures. Those advances will be made from the base of individual accomplishments of the thirty-four founding members who represent ten different departments in two colleges.

The Center functions by means of an Administrative Board which is elected yearly. The general purposes of the Center include:

- collection and dissemination of information about composites activities at Virginia Tech,
- contact point for other organizations and individuals,
- mechanism for collective educational and research pursuits,
- forum and mechanism for internal interactions at Virginia Tech.

The Center for Composite Materials and Structures is supported by a vigorous program of activity at Virginia Tech that has developed since 1963. Research expenditures for investigations of composite materials and structures total well over five million dollars with yearly expenditures presently approaching two million dollars.

Research is conducted in a wide variety of areas including design and analysis of composite materials and composite structures, chemistry of materials and surfaces, characterization of material properties, development of new material systems, and relations between damage and response of composites. Extensive laboratories are available for mechanical testing, nondestructive testing and evaluation, stress analysis, polymer synthesis and characterization, material surface characterization, component fabrication and other specialties.

Educational activities include eight formal courses offered at the undergraduate and graduate levels dealing with the physics, chemistry, mechanics, and design of composite materials and structures. As of 1982, some 33 Doctoral and 37 Master's students have completed graduate programs and several hundred Bachelor-level students have been trained in various aspects of composite materials and structures. A significant number of graduates are now active in industry and government.

Various Center faculty are internationally recognized for their leadership in composite materials and composite structures through books, lectures, workshops, professional society activities, and research papers.

FOUNDING MEMBERS OF THE CENTER

Aerospace and Ocean Engineering
Raphael T. Haftka
William L. Hallauer, Jr.
Eric R. Johnson

Chemical Engineering
Donald G. Baird

Chemistry
James E. McGrath
Thomas C. Ward
James P. Wightman

Civil Engineering
Raymond H. Plaut

Electrical Engineering
Ioannis M. Besieris
Richard O. Claus

Engineering Science and Mechanics
Hal F. Brinson
John C. Duke, Jr.
Daniel Frederick
Robert A. Heller
Edmund G. Henneke, II
Carl T. Herakovich
Michael W. Hyer
Robert M. Jones
Manohar P. Kamat
Alfred C. Loos
Don H. Morris
Daniel Post
Junuthula N. Reddy
Kenneth L. Reifsnider
Wayne W. Stinchcomb

Industrial Engineering and Operations Research
Joel A. Nachlas

Materials Engineering
David W. Dwight
D. P. H. Hasselman
Charles R. Houska
M. R. Louthan, Jr.

Mathematics
Werner E. Kohler

Mechanical Engineering
Norman S. Eiss, Jr.
Charles E. Knight
S. W. Zewari

Inquiries should be directed to:

Center for Composite Materials & Structures
College of Engineering
Virginia Tech
Blacksburg, VA 24061
Phone: (703) 961-4969

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

UT PROSIM