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Spatial Derivatives of Flow Quantities Behind Curved Shocks of All Strengths

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SUMMARY

Uniformly valid equations are developed for calculating spatial derivatives of flow quantities behind a curved shock of any strength. For steady two-dimensional inviscid flow, explicit formulas in terms of shock curvature are derived. Those factors which yield the equations indeterminate as the shock strength approaches 0 cancel analytically, resulting in equations which are valid for shocks of all strengths.

An example of an application of the method is outlined for the problem of shock coalescence in which asymmetric effects are included through derivatives in the circumferential direction. The solution of the coalescence problem requires values for spatial derivatives of the flow variables behind a resulting shock which is often very weak.

INTRODUCTION

In many flow-field computational problems, it often becomes necessary to calculate spatial derivatives of the flow quantities. Numerically determining these derivatives by finite difference or other approximations often leads to great inaccuracies in the solution. Analytical methods of calculating spatial derivatives of flow variables behind curved shocks are developed in references 1 to 3. These methods involve taking the tangential derivative of the standard shock equations along the shock, thus resulting in expressions relating the derivatives behind the shock and the curvature of the shock. This approach works well and has many applications except when the shock strength approaches 0 and the equations become indeterminate.

Included in this paper is a method of determining, for these derivatives, expressions which are shown to remain finite as the shock strength approaches 0. An equation for the jump in the derivative of a flow variable across the shock is derived by using the flow equations in section A of part I. It is the ratios in this derived equation which cause difficulties as the shock approaches a Mach wave. In section B alternate expressions, which remain finite, are derived for these ratios by using the standard shock equations.

An application for the method may be found in reference 4 where the shock-coalescence problem with asymmetric effects is solved in a sonic boom propagation routine. The asymmetric effects are felt through cross derivatives of the flow variables. It is in solving part of the shock-coalescence problem that it becomes necessary to know the derivatives of the flow variables in both the axial and the radial directions behind the resulting shocks. In this application, the calculations are performed away from the disturbing body, and the shocks involved often become so weak as to cause numerical difficulties. In the method developed in reference 4, the approximation is made that one of the resulting shocks is so weak that the flow variables can be assumed to be continuous across the shock, thereby limiting the applicability of that method to cases in which this is true.

The method developed in part I removes the need for a weak-shock approximation in the coalescence solution and thus makes the procedure applicable to shocks of any strength. Part II of this paper outlines the procedure for incorporating the equations derived in this paper into the coalescence system.

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SYMBOLS

A,B	dummy variables
$\mathcal{A}$	right side of equation (11)
a	speed of sound
C_N	$= a^2 - (\bar{q} \cdot \hat{\nu})^2$; zero for characteristic
$\mathcal{D}$	defined beneath equation (7)
f_1,f_2,f_3,f_4	shock surfaces
f_3'',f_4''	curvature of shock surfaces
$\mathcal{G}$	defined beneath equation (8)
g	acceleration due to gravity
H	enthalpy
h	slipstream surface
$\hat{i}$	unit vector in axial direction
$\hat{j}$	unit vector in radial direction
$\mathcal{K}$	defined in equation (9)
k	shock curvature
m	mass (see eq. (27))
n	direction normal to streamline
p	pressure
q	velocity
R	gas constant
S	entropy
$S_1,S_2,\ldots,S_{20}$	ratios defined by using shock equations (see part I(B))

T temperature

$U_{\psi\psi}\big|_{\psi=0}$ second circumferential derivative of axial velocity in plane of symmetry

u,v velocity components in axial and radial directions, respectively

V_1 velocity

$V_{\psi\psi}\big|_{\psi=0}$ second circumferential derivative of radial velocity in plane of symmetry

$X(\psi),R(\psi)$ intersection of shock surfaces (fig. 2)

x,r,ψ axial, radial, and circumferential coordinates, respectively

z vertical coordinate in Cartesian coordinate system

α shock angle in general problem

β_3,β_4 shock angles of f_3 and f_4, respectively, in coalescence problem

γ ratio of specific heats

θ flow angle

λ arc length

ν direction normal to shock

$\hat{\nu}$ unit vector normal to shock

ρ density

σ direction along streamline

τ direction tangent to shock

$\hat{\tau}$ unit vector tangent to shock

Subscripts:

x,r first derivative with respect to x or r, respectively

ν derivative normal to shock

τ derivative tangent to shock

1,2 region ahead or behind shock, respectively

4,5 regions 4 and 5 in shock-coalescence problem

Mathematical notation:

'	first derivative with respect to x
"	second derivative with respect to x
-	vector quantity, when over symbol
$[\|\ \|]$	jump in variable across shock, for example, $[\|u\|] = u_2 - u_1$
$\rightarrow$	approaches

PART I

DERIVATION OF EQUATIONS FOR SPATIAL DERIVATIVES BEHIND SHOCK IN TERMS OF CHANGES IN SHOCK ANGLE

A - Development of Ratios Needed Across Shock and Explicit Formulas for Derivatives

There are essentially five steps involved in determining the formulas for the needed derivatives. For ease in following the procedure, the steps are outlined next. After the outline, the details of the derivation are given in each step.

(1) Resolve the flow equations into derivatives parallel to, $\partial/\partial\tau$, and perpendicular to, $\partial/\partial\nu$, the shock.

(2) Reduce these equations to one equation in terms of $\partial/\partial\nu$ of one variable, say u_2; that is,

$$C_{N_2}(\partial u_2/\partial\nu) = \text{R.H.S.}$$

where R.H.S. represents the right-hand side of the equation.

(3) Repeat steps (1) and (2) for the field ahead of the shock. Form a difference between the two equations:

$$C_{N_2}[|\partial u/\partial\nu|] + (\partial u_1/\partial\nu)[|C_N|] = [|\text{R.H.S.}|]$$

where $[|\ \ |]$ represents the jump across the shock.

(4) Solve for $[|\partial u/\partial\nu|]$; that is,

$$[|\partial u/\partial\nu|] = (-\partial u_1/\partial\nu)[|C_N|]\Big/C_{N_2} + [|\text{R.H.S.}|]\Big/C_{N_2}$$

The shock equations are used (see section B of part I) to show that the numerators of the ratios $[|C_N|]/C_{N_2}$ and $[|R.H.S.|]/C_{N_2}$ are divisible by C_{N_2}. Consequently, explicit formulas can be derived for those quotients which will be finite as the shock strength and therefore C_{N_2} approach 0.

(5) Derive explicit formulas for the derivatives behind the shock and the curvature of the streamline in terms of the slope and the rate of slope change of the shock and of the local flow field and the derivative ahead of the shock.

Step 1.- As shown in figure 1, α is defined to be the shock angle with respect to the x-axis and θ_j is the flow direction, where the subscript $j = 1$ indicates flow ahead of the shock and $j = 2$ indicates flow behind the shock. (The numbers (1) to (5) in the figures refer to regions 1 to 5.)

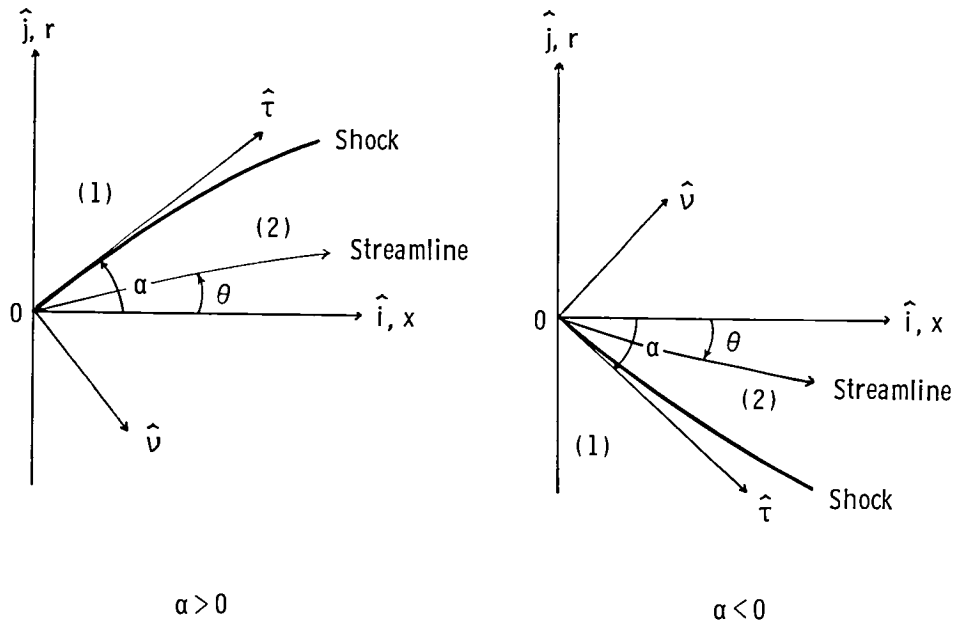

Figure 1.- Coordinate system.

With these definitions, the unit vectors tangent and normal to the shock become, respectively,

$$\hat{\tau} = \hat{i} \cos \alpha + \hat{j} \sin \alpha \tag{1}$$

$$\hat{\nu} = (\hat{i} \sin \alpha - \hat{j} \cos \alpha) \operatorname{sgn} \alpha \tag{2}$$

(where sgn represents the sign (signum) of the argument) and the first partial derivatives with respect to x and r in terms of τ and ν are, respectively,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau} \cos \alpha + \frac{\partial}{\partial \nu} \sin \alpha \operatorname{sgn} \alpha \tag{3}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \tau} \sin \alpha - \frac{\partial}{\partial \nu} \cos \alpha \operatorname{sgn} \alpha \tag{4}$$

The flow equations given in cylindrical coordinates x,r (see ref. 5) in the vertical plane of symmetry and rederived in reference 4 are

$$(a^2 - u^2)u_x - uv(v_x + u_r) + (a^2 - v^2)v_r = -\frac{a^2 v}{r} - gv \tag{5}$$

$$v_x - u_r = -\frac{1}{q}\frac{\partial H}{\partial n} + \frac{T}{q}\frac{\partial S}{\partial n} \tag{6}$$

where $q(\partial/\partial n) = u(\partial/\partial r) - v(\partial/\partial x)$, and $\partial/\partial n$ is the derivative normal to the streamline. Applying equations (3) and (4) to equation (5) and rearranging yields

$$\left[(a^2 - u^2) \sin \alpha + uv \cos \alpha\right]u_\nu \operatorname{sgn} \alpha - \left[uv \sin \alpha + (a^2 - v^2) \cos \alpha\right]v_\nu \operatorname{sgn} \alpha = -\mathcal{D} \tag{7}$$

where

$$\mathcal{D} = \frac{a^2 v}{r} + gv + \left[(a^2 - u^2) \cos \alpha - uv \sin \alpha\right]u_\tau + \left[-uv \cos \alpha + (a^2 - v^2) \sin \alpha\right]v_\tau$$

Likewise after applying equations (3) and (4) to equation (6) and rearranging, one obtains

$$(u_\nu \cos \alpha + v_\nu \sin \alpha) \operatorname{sgn} \alpha = -\mathcal{G} \tag{8}$$

where

$$\mathcal{G} = \frac{1}{q}\left(\frac{\partial H}{\partial n} - T\frac{\partial S}{\partial n}\right) - u_\tau \sin \alpha + v_\tau \cos \alpha$$

Step 2.- Elimination of v_ν from equations (7) and (8) yields the following single equation for u_ν:

$$\left[a^2 - (u \sin \alpha - v \cos \alpha)^2\right]\frac{\partial u}{\partial \nu} = -\mathcal{K} \operatorname{sgn} \alpha \tag{9}$$

or

$$\left[a^2 - (\bar{q} \cdot \hat{\nu})^2\right]\frac{\partial u}{\partial \nu} = -\mathcal{K} \operatorname{sgn} \alpha$$

where

$$\mathcal{K} = \mathcal{D} \sin \alpha + \mathcal{G}\left[uv \sin \alpha + (a^2 - v^2) \cos \alpha\right]$$

As the shock approaches the characteristic, a^2 approaches $(\bar{q} \cdot \hat{\nu})^2$ and hence the right-hand side $\mathcal{K}$ which goes to 0, becomes the characteristic equation. This explains why the equations for the derivatives become indeterminate as the shock strength approaches 0.

Step 3.- Equation (9) applies to the regions ahead of and behind the shock. Applying it to each and forming their difference gives

$$\left[a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2\right]\frac{\partial u_2}{\partial \nu} - \left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]\frac{\partial u_1}{\partial \nu} = -(\mathcal{K}_2 - \mathcal{K}_1) \tag{10}$$

If $[|\ |]$ = Jump, for example, $[|\mathcal{K}|] = \mathcal{K}_2 - \mathcal{K}_1$, then equation (10) may be rewritten as

$$\left[a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2\right]\left[\left|\frac{\partial u}{\partial \nu}\right|\right] + \frac{\partial u_1}{\partial \nu}\left[\left|a^2 - (\bar{q} \cdot \hat{\nu})^2\right|\right] = -[|\mathcal{K}|] \operatorname{sgn} \alpha$$

where use has been made of the fact that

$$[|AB|] = A_2B_2 - A_1B_1 = A_2B_2 - A_2B_1 + B_1A_2 - B_1A_1 = A_2[|B|] + B_1[|A|]$$

Step 4.- By solving for $[|\partial u/\partial \nu|]$, one obtains

$$\left[\left|\frac{\partial u}{\partial \nu}\right|\right] = -\frac{\left[\left|a^2 - (\bar{q} \cdot \hat{\nu}|)^2\right]\right.}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2}\frac{\partial u_1}{\partial \nu} - \frac{[|\mathcal{K}|] \operatorname{sgn} \alpha}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = \mathcal{A} \tag{11}$$

It is now necessary to show that the right-hand side of equation (11), or $\mathcal{A}$, remains finite as $a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2 \to 0$. If C_{N_2} is defined as $a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2$, then the first term of the right-hand side of equation (11) becomes

$$1 - \frac{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}{C_{N_2}} \tag{12}$$

To compute $[|\mathcal{K}|]/C_{N_2}$, expand $[|\mathcal{K}|]$ to get

$$[|\mathcal{K}|] = [|\mathcal{D}|] \sin\alpha + [|\mathcal{G}|][u_2 v_2 \sin\alpha + (a_2^2 - v_2^2)\cos\alpha]$$

$$+ \mathcal{G}_1([|uv|] \sin\alpha + [|a^2 - v^2|] \cos\alpha) \tag{13}$$

Expanding further gives

$$\frac{[|\mathcal{D}|]}{C_{N_2}} = \frac{1}{r}\frac{[|a^2 v|]}{C_{N_2}} + \frac{g[|v|]}{C_{N_2}} + \left(\frac{[|a^2 - u^2|]}{C_{N_2}} \cos\alpha - \frac{[|uv|]}{C_{N_2}} \sin\alpha\right) u_{2_\tau}$$

$$+ [(a_1^2 - u_1^2)\cos\alpha - u_1 v_1 \sin\alpha]\frac{[|u_\tau|]}{C_{N_2}}$$

$$-\left(\frac{[|uv|]}{C_{N_2}} \cos\alpha - \frac{[|a^2 - v^2|]}{C_{N_2}} \sin\alpha\right) v_{2_\tau}$$

$$- [v_1 u_1 \cos\alpha - (a_1^2 - v_1^2)\sin\alpha]\frac{[|v_\tau|]}{C_{N_2}} \tag{14}$$

and

$$\frac{[|\mathcal{G}|]}{C_{N_2}} = \frac{[|(1/q)/(\partial H/\partial n)|]}{C_{N_2}} - \frac{[|T(\partial S/\partial n)|]}{C_{N_2}} - \sin\alpha\,\frac{[|u_\tau|]}{C_{N_2}} + \cos\alpha\,\frac{[|v_\tau|]}{C_{N_2}} \tag{15}$$

Therefore, to evaluate $[|\mathcal{K}|]/c_{N_2}$, it is necessary to evaluate and show that numerous ratios remain finite as $c_{N_2} \to 0$. These evaluations are provided later by the shock equations in part I(B) of this section as indicated by the notation $S_{\#}$ beside the ratio. These ratios are

$$\frac{[|v|]}{c_{N_2}} = S_7$$

$$\frac{[|uv|]}{c_{N_2}} = \frac{[|u|]}{c_{N_2}} v_2 + \frac{[|v|]}{c_{N_2}} u_1 = S_6 v_2 + S_7 u_1$$

$$\frac{[|a^2 - v^2|]}{c_{N_2}} = S_{14}$$

$$\frac{[|a^2 - u^2|]}{c_{N_2}} = S_{15}$$

$$\frac{[|a^2 v|]}{c_{N_2}} = S_{16}$$

$$\frac{[|(1/q)(\partial H/\partial n)|]}{c_{N_2}} = S_{17}$$

$$\frac{[|(T/q)(\partial S/\partial n)|]}{c_{N_2}} = S_{20}$$

$$\frac{[|v_\tau|]}{c_{N_2}} = S_{13} S_3$$

and

$$\frac{[|u_\tau|]}{c_{N_2}} = S_{12} S_3$$

To evaluate expression (12), the first term on the right-hand side of equation (11), $\dfrac{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}{C_{N_2}}$ is shown to remain finite by S_3.

With the definitions of the aforementioned ratios and the determination that they remain finite, then $\mathcal{A}$, the right-hand side of equation (11), is now defined and is finite for all shock strengths.

Step 5.- The explicit formulas for the derivatives in region 2 behind the shock are now developed. All flow quantities and derivatives in region 1 are known and all flow quantities (not derivatives) in region 2 are known.

From equation (43), in the equations developed after this section,

$$\left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\tau} = S_2 k\left[a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2\right] \operatorname{sgn} \alpha \tag{16}$$

and from equation (46),

$$\left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\nu} = S_{11}\left[a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2\right]$$

By definition,

$$\left[\left|\frac{\partial q}{\partial \tau}\right|\right] = \hat{i}\,\frac{\partial[|u|]}{\partial \tau} + \hat{j}\,\frac{\partial[|v|]}{\partial \tau}$$

Also by definition,

$$\left[\left|\frac{\partial q}{\partial \tau}\right|\right] = \frac{\partial q_2}{\partial \tau} - \frac{\partial q_1}{\partial \tau} = \frac{\partial \bar{q}_2}{\partial \tau} \cdot \tau\hat{\tau} + \frac{\partial \bar{q}_2}{\partial \tau} \cdot \nu\hat{\nu} - \frac{\partial \bar{q}_1}{\partial \tau} \cdot \tau\hat{\tau} - \frac{\partial \bar{q}_1}{\partial \tau} \cdot \nu\hat{\nu}$$

$$= \hat{i}\left(\left[\left|\frac{\partial q}{\partial \tau}\right|\right] \cdot \hat{\tau} \cos \alpha + \left[\left|\frac{\partial q}{\partial \tau}\right|\right] \cdot \hat{\nu} \sin \alpha\right)$$

$$+ \hat{j}\left(\left[\left|\frac{\partial q}{\partial \tau}\right|\right] \cdot \hat{\tau} \sin \alpha - \left[\left|\frac{\partial q}{\partial \tau}\right|\right] \cdot \hat{\nu} \cos \alpha\right)$$

By equating the $\hat{i}$ and $\hat{j}$ components in the aforementioned definitions, the following relationships are obtained. From the $\hat{i}$ component,

$$\frac{\partial[|u|]}{\partial\tau} = \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \hat{\tau} \cos\alpha + \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \hat{\nu} \sin\alpha$$

$$\frac{\partial u_2}{\partial\tau} = \frac{\partial u_1}{\partial\tau} + \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \hat{\tau} \cos\alpha + \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \hat{\nu} \sin\alpha$$

By using equations (16) and (17), this becomes

$$\frac{\partial u_2}{\partial\tau} = \frac{\partial u_1}{\partial\tau} + S_2 k C_{N_2} \cos\alpha \operatorname{sgn}\alpha + S_{11} C_{N_2} \sin\alpha \tag{18}$$

Likewise from the $\hat{j}$ component,

$$\frac{\partial[|v|]}{\partial\tau} = \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \tau \sin\alpha - \left[\left|\frac{\partial q}{\partial\tau}\right|\right] \cdot \nu \cos\alpha$$

and thus

$$\frac{\partial v_2}{\partial\tau} = \frac{\partial v_1}{\partial\tau} + S_2 k C_{N_2} \sin\alpha \operatorname{sgn}\alpha - S_{11} C_{N_2} \cos\alpha \tag{19}$$

From equation (11), one obtains

$$\frac{\partial u_2}{\partial\nu} = \frac{\partial u_1}{\partial\nu} + \mathcal{A} \tag{20}$$

and from equation (8),

$$\frac{\partial v_2}{\partial\nu} = -\frac{1}{\sin\alpha}\left(\mathcal{G}_2 \operatorname{sgn}\alpha + \cos\alpha \frac{\partial u_2}{\partial\nu}\right) \tag{21}$$

By using the derivatives in terms of τ and ν, the following relationships may be defined directly:

$$\frac{\partial u_2}{\partial x} = \frac{\partial u_2}{\partial\tau} \cos\alpha + \frac{\partial u_2}{\partial\nu} \sin\alpha \operatorname{sgn}\alpha \tag{22}$$

$$\frac{\partial u_2}{\partial r} = \frac{\partial u_2}{\partial \tau} \sin \alpha - \frac{\partial u_2}{\partial \nu} \cos \alpha \operatorname{sgn} \alpha \tag{23}$$

$$\frac{\partial v_2}{\partial x} = \frac{\partial v_2}{\partial \tau} \cos \alpha + \frac{\partial v_2}{\partial \nu} \sin \alpha \operatorname{sgn} \alpha \tag{24}$$

$$\frac{\partial v_2}{\partial r} = \frac{\partial v_2}{\partial \tau} \sin \alpha + \frac{\partial v_2}{\partial \nu} \cos \alpha \operatorname{sgn} \alpha \tag{25}$$

where these four derivatives are functions of the shock curvature k.

B - Definitions of Ratios S_1 to S_{20} by Using Shock Equations

The jump in the normal derivative across the shock is given by equation (11). The evaluation of this equation is defined in terms of ratios S_1 to S_{20}. In this section the standard shock equations will be used to define the ratios and to show that they remain finite as the shock approaches a characteristic.

By rewriting the shock conditions as derived in reference 2 in terms of the jump notation $[|\ |]$, the r momentum equation becomes

$$[|\bar{q} \cdot \hat{\tau}|] = 0 \tag{26}$$

the continuity equation becomes

$$[|\rho\bar{q} \cdot \hat{\nu}|] = 0 \qquad \rho_1\bar{q}_1 \cdot \hat{\nu} = m = \rho_2\bar{q}_2 \cdot \hat{\nu} \tag{27}$$

the x momentum equation becomes

$$[|\rho(\bar{q} \cdot \hat{\nu})^2|] + [|p|] = 0 \qquad m[|\bar{q} \cdot \hat{\nu}|] + [|p|] = 0 \tag{28}$$

and the energy equation becomes

$$[|(\bar{q} \cdot \hat{\nu})^2|] + \frac{2\gamma}{\gamma - 1} R[|T|] = 0 \tag{29}$$

By using the equation of state

$$p = \rho RT \tag{30}$$

and equations (27) and (28) to eliminate p and ρ, the following equation is obtained:

$$[|\bar{q} \cdot \hat{\nu}|] + R\left[\left|\frac{T}{\bar{q} \cdot \hat{\nu}}\right|\right] = 0 \tag{31}$$

Expanding $[|T/(\bar{q} \cdot \hat{\nu})|]$ and rearranging gives

$$\left[\left|\frac{T}{\bar{q} \cdot \hat{\nu}}\right|\right] = \frac{T_2}{\bar{q}_2 \cdot \hat{\nu}} - \frac{T_1}{\bar{q}_2 \cdot \hat{\nu}} + \frac{T_1}{\bar{q}_2 \cdot \hat{\nu}} - \frac{T_1}{\bar{q}_1 \cdot \hat{\nu}} = \frac{[|T|]}{\bar{q}_2 \cdot \hat{\nu}} - \frac{T_1}{(\bar{q}_2 \cdot \hat{\nu})(\bar{q}_1 \cdot \hat{\nu})}[|\bar{q} \cdot \hat{\nu}|]$$

Therefore, equation (31) may be written as

$$[|\bar{q} \cdot \hat{\nu}|][(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu}) - RT_1] + R[|T|](\bar{q}_1 \cdot \hat{\nu}) = 0 \tag{32}$$

Equation (29) may be rewritten as

$$(\bar{q}_1 \cdot \hat{\nu} + \bar{q}_2 \cdot \hat{\nu})[|\bar{q} \cdot \hat{\nu}|] + \frac{2\gamma}{\gamma - 1} R[|T|] = 0 \tag{33}$$

Eliminating $[|T|]$ from equations (32) and (33) gives

$$[|\bar{q} \cdot \hat{\nu}|]\left[\frac{2\gamma}{\gamma - 1}(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu}) - \frac{2\gamma}{\gamma - 1} RT_1 - (\bar{q}_1 \cdot \hat{\nu})^2 - (\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu})\right] = 0$$

and since $[|\bar{q} \cdot \hat{\nu}|] \neq 0$,

$$2\gamma RT_1 = (\gamma + 1)(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu}) - (\bar{q}_1 \cdot \hat{\nu})^2(\gamma - 1) \tag{34}$$

or

$$(\gamma + 1)(\bar{q}_1 \cdot \hat{\nu})[|\bar{q} \cdot \hat{\nu}|] = 2[\gamma RT_1 - (\bar{q}_1 \cdot \hat{\nu})^2] \tag{35}$$

which becomes

$$(\gamma + 1)(\bar{q}_1 \cdot \hat{\nu})[|\bar{q} \cdot \hat{\nu}|] = 2a_1^2\left[1 - \frac{(\bar{q}_1 \cdot \hat{\nu})^2}{a_1^2}\right]$$

or

$$\frac{[|\bar{q} \cdot \hat{\nu}|]}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2} = \frac{2}{(\gamma + 1)(\bar{q}_1 \cdot \hat{\nu})} = S_1 \tag{36}$$

Equation (32) may be rewritten with

$$\left[\left|\frac{T}{\bar{q} \cdot \hat{\nu}}\right|\right] = -\frac{[|\bar{q} \cdot \hat{\nu}|]T_2}{(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu})} + \frac{[|T|]}{\bar{q}_1 \cdot \hat{\nu}}$$

to obtain

$$2\gamma RT_2 = 2(\bar{q}_2 \cdot \hat{\nu})^2 - (\gamma + 1)[|\bar{q} \cdot \hat{\nu}|](\bar{q}_2 \cdot \hat{\nu})$$

or

$$\frac{[|\bar{q} \cdot \hat{\nu}|]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = \frac{-2}{(\gamma + 1)(\bar{q}_2 \cdot \hat{\nu})} = S_2 \tag{37}$$

From equations (36) and (37), one obtains

$$\frac{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = -\frac{\bar{q}_1 \cdot \hat{\nu}}{\bar{q}_2 \cdot \hat{\nu}} = S_3 \tag{38}$$

From equations (33) and (37), one may obtain

$$\frac{R[|T|]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = -\frac{\gamma - 1}{\gamma(\gamma + 1)}\left(\frac{\bar{q}_1 \cdot \hat{\nu}}{\bar{q}_2 \cdot \hat{\nu}} + 1\right) = S_4 \tag{39}$$

and from equations (28) and (37),

$$\frac{[|p|]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = \frac{2m}{(\gamma + 1)(\bar{q}_2 \cdot \hat{\nu})} = \frac{2\rho_2}{(\gamma + 1)(\bar{q}_2 \cdot \hat{\nu})} = S_5 \tag{40}$$

Note that

$$[|\bar{q} \cdot \tau|] = [|u|] \cos \alpha + [|v|] \sin \alpha = 0$$

$$[|\bar{q} \cdot \nu|] = ([|u|] \sin \alpha - [|v|] \cos \alpha) \operatorname{sgn} \alpha$$

$$[|u|] = [|\bar{q} \cdot \tau|] \cos \alpha \operatorname{sgn} \alpha + [|\bar{q} \cdot \nu|] \sin \alpha$$

$$[|v|] = [|\bar{q} \cdot \tau|] \sin \alpha - [|\bar{q} \cdot \nu|] \cos \alpha \operatorname{sgn} \alpha$$

where the first expressions defined in $[|u|]$ and $[|v|]$ go to 0. Therefore, from equation (37),

$$\frac{[|u|]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = S_2 \sin \alpha = S_6 \tag{41}$$

$$\frac{[|v|]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = -S_2 \cos \alpha \operatorname{sgn} \alpha = S_7 \tag{42}$$

The ratios S_1 to S_7 are all used for the ν derivatives behind the shock. From the aforementioned expressions, it is seen that these ratios all remain finite as the shock strength approaches 0 $\left(\text{i.e., } a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2 \to 0\right)$.

The tangential derivatives of $[|\ |]$ across the shock will now be developed. If τ is defined as the arc length along the shock, then, from equation (26),

$$\frac{\partial}{\partial \tau}[|\bar{q} \cdot \hat{\tau}|] = 0 = \left[\left|\frac{\partial}{\partial \tau} \bar{q}\right|\right] \cdot \hat{\tau} + [|\bar{q}|] \cdot \frac{\partial \hat{\tau}}{\partial \tau} = \left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\tau} \mp k\hat{\nu}[|\bar{q}|]$$

since

$$\frac{\partial \hat{\tau}}{\partial \tau} = \mp k\hat{\nu} \qquad \frac{\partial \hat{\nu}}{\partial \tau} = \pm k\hat{\tau}$$

where k is the curvature of the shock. Therefore,

$$\left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\tau} = \pm[|\bar{q} \cdot \hat{\nu}|]k$$

and

$$\frac{\left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\tau}}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = \pm S_2 k \text{ sgn } \alpha = S_8 \tag{43}$$

From equation (37),

$$\frac{\partial}{\partial \tau}[|\bar{q} \cdot \hat{\nu}|] = \left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\nu} \pm [|\bar{q}|] \cdot k\hat{\tau}$$

and since $[|\bar{q}|] \cdot \hat{\tau} = 0$, then

$$\frac{\partial}{\partial \tau}[|\bar{q} \cdot \hat{\nu}|] = \left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\nu}$$

Taking the $\partial/\partial\tau$ of equation (36) yields

$$\left[\left|\frac{\partial \bar{q}}{\partial \tau}\right|\right] \cdot \hat{\nu} = -\frac{2}{\gamma + 1} \frac{1}{(\bar{q}_1 \cdot \hat{\nu})^2} \frac{\partial(\bar{q}_1 \cdot \hat{\nu})}{\partial \tau}\left[a_1^2 - (q_1 \cdot \hat{\nu})^2\right]$$

$$+ \frac{2}{\gamma + 1} \frac{1}{\bar{q}_1 \cdot \hat{\nu}} \frac{\partial}{\partial \tau}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]$$

and dividing by $a_1^2 - (\bar{q} \cdot \hat{\nu})^2$ gives

$$\frac{\left[\left|\frac{\partial q}{\partial \tau}\right|\right] \cdot \hat{\nu}}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2} = -\frac{2}{\gamma + 1} \frac{1}{(\bar{q}_1 \cdot \hat{\nu})^2} \frac{\partial(\bar{q}_1 \cdot \hat{\nu})}{\partial \tau}$$

$$+ \frac{2}{\gamma + 1} \frac{1}{\bar{q}_1 \cdot \hat{\nu}} \frac{\frac{\partial}{\partial \tau}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2} = S_9 \tag{44}$$

To show that

$$\frac{\frac{\partial}{\partial\tau}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}$$

remains finite as $a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2 \to 0$, the following substitutions will be made. Denote λ_1 as the arc length along the characteristic, in the same sense of τ along the shock, and α_1 as the angle of the characteristic with the x-axis. Then,

$$\frac{\partial}{\partial\lambda_1}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right] = 0$$

$$\frac{\partial}{\partial\lambda_1} = \cos\alpha_1 \frac{\partial}{\partial x} + \sin\alpha_1 \frac{\partial}{\partial r}$$

and

$$\frac{\partial}{\partial\tau} = \cos\alpha \frac{\partial}{\partial x} + \sin\alpha \frac{\partial}{\partial r}$$

$$\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\lambda_1} = (\cos\alpha - \cos\alpha_1)\frac{\partial}{\partial x} + (\sin\alpha - \sin\alpha_1)\frac{\partial}{\partial r}$$

Recall that as $\alpha \to \alpha_1$, $a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2 \to 0$. The equation

$$\cos\alpha - \cos\alpha_1 = \frac{1}{\cos\alpha + \cos\alpha_1}(\cos^2\alpha - \cos^2\alpha_1)$$

can be rewritten as

$$\cos\alpha - \cos\alpha_1 = \frac{1}{\cos\alpha + \cos\alpha_1}(\sin^2\alpha_1 - \sin^2\alpha)$$

Multiplying and dividing by q_1^2 gives

$$\cos\alpha - \cos\alpha_1 = \frac{1}{q_1^2(\cos\alpha + \cos\alpha_1)}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]$$

since $q_1 \sin\alpha_1 = a_1$ and $q_1 \sin\alpha = \bar{q}_1 \cdot \hat{\nu}$. Similarly,

$$\sin\alpha - \sin\alpha_1 = \frac{1}{q_1^2(\sin\alpha + \sin\alpha_1)}\left[(\bar{q}_1 \cdot \hat{\nu})^2 - a_1^2\right]$$

Therefore,

$$\frac{\partial}{\partial\tau} - \frac{\partial}{\partial\lambda} = \frac{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}{q_1^2(\cos\alpha + \cos\alpha_1)}\frac{\partial}{\partial x} + \frac{(\bar{q}_1 \cdot \hat{\nu})^2 - a_1^2}{q_1^2(\sin\alpha + \sin\alpha_1)}\frac{\partial}{\partial r}$$

By substituting this value into the second expression of equation (44), the following results are obtained:

$$\frac{\frac{\partial}{\partial\tau}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2} = \frac{\frac{\partial}{\partial\tau}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right] - \frac{\partial}{\partial\lambda_1}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2}$$

$$= \frac{\frac{\partial}{\partial x}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{q_1^2(\cos\alpha + \cos\alpha_1)} - \frac{\frac{\partial}{\partial r}\left[a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2\right]}{q_1^2(\sin\alpha + \sin\alpha_1)} = S_{10} \tag{45}$$

Then, S_{10} remains finite as $\alpha \to \alpha_1$. The term $\partial(\bar{q}_1 \cdot \hat{\nu})/\partial\tau$ in equation (44) may be expressed as

$$\frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial\tau} = \frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial x}\cos\alpha + \frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial r}\sin\alpha$$

where

$$\frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial x} = \frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial \tau} \frac{\partial \tau}{\partial x} = \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} + \frac{\partial \hat{\nu}}{\partial \tau} \cdot \bar{q}\right) \sec \alpha$$

$$= \left[\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} + \operatorname{sgn} \alpha \, (k\hat{\tau} \cdot \bar{q})\right] \sec \alpha$$

and

$$\frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial r} = \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} + \frac{\partial \hat{\nu}}{\partial \tau} \cdot \bar{q}\right) \csc \alpha = \left[\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} + \operatorname{sgn} \alpha \, (k\hat{\tau} \cdot \bar{q})\right] \csc \alpha$$

Therefore,

$$\frac{\partial(\bar{q} \cdot \hat{\nu})}{\partial \tau} = 2\left[\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} + \operatorname{sgn} \alpha \, (k\hat{\tau} \cdot \bar{q})\right]$$

Equation (44) becomes

$$\frac{\left[\,|\partial \bar{q}/\partial \tau|\,\right] \cdot \hat{\nu}}{a_1^2 - (\bar{q}_1 \cdot \hat{\nu})^2} = -\frac{4}{\gamma + 1} \frac{1}{(\bar{q}_1 \cdot \hat{\nu})^2}\left[\left(\frac{\partial \bar{q}_1}{\partial \tau} \cdot \hat{\nu}\right) + (k\bar{q}_1 \cdot \hat{\tau}) \operatorname{sgn} \alpha\right]$$

$$+ \frac{2}{(\gamma + 1)(\bar{q}_1 \cdot \hat{\nu})^2} S_{10} = S_{11} \tag{46}$$

where S_{11} remains finite.

By definition,

$$\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\tau} = u_\tau \cos \alpha + v_\tau \sin \alpha$$

$$\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu} = (u_\tau \sin \alpha - v_\tau \cos \alpha) \operatorname{sgn} \alpha$$

Thus,

$$u_\tau = \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\tau}\right) \cos \alpha + \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu}\right) \sin \alpha \operatorname{sgn} \alpha$$

$$v_\tau = \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\tau}\right) \sin \alpha - \left(\frac{\partial \bar{q}}{\partial \tau} \cdot \hat{\nu}\right) \cos \alpha \operatorname{sgn} \alpha$$

and

$$\frac{[|u_\tau|]}{C_{N_1}} = \frac{[|\partial \bar{q}/\partial \tau|] \cdot \hat{\tau}}{C_{N_1}} \cos \alpha + \frac{[|\partial \bar{q}/\partial \tau|] \cdot \hat{\nu}}{C_{N_1}} \sin \alpha \operatorname{sgn} \alpha$$

$$= S_8 \cos \alpha + S_{11} \sin \alpha \operatorname{sgn} \alpha = S_{12} \tag{47}$$

Similarly,

$$\frac{[|v_\tau|]}{C_{N_1}} = S_8 \sin \alpha - S_{11} \cos \alpha \operatorname{sgn} \alpha = S_{13} \tag{48}$$

$$\frac{[|a^2|]}{C_{N_2}} = \frac{a_2^2 - a_1^2}{C_{N_2}} = \frac{\gamma R[|T|]}{C_{N_2}} = \gamma S_4$$

$$\frac{[|v^2|]}{C_{N_2}} = \frac{(v_1 + v_2)[|v|]}{C_{N_2}} = (v_1 + v_2)S_7$$

$$\frac{[|a^2 - v^2|]}{C_{N_2}} = \frac{[|a^2|] - [|v^2|]}{C_{N_2}} = \gamma S_4 - (v_1 - v_2)S_7 = S_{14} \tag{49}$$

$$\frac{[|a^2 - u^2|]}{C_{N_2}} = \gamma S_4 - (u_1 + u_2)S_6 = S_{15} \tag{50}$$

and

$$\frac{[|a^2 v|]}{C_{N_2}} = \frac{[|a^2|]v_2}{C_{N_2}} + \frac{a_1^2[|v|]}{C_{N_2}} = \gamma S_4 v_2 + a_1^2 S_7 = S_{16} \tag{51}$$

By definition,

$$[|H|] = 0 \qquad \left(\frac{\partial H}{\partial \sigma}\right)_{1,2} = 0$$

$$\frac{\partial H}{\partial n} = \frac{\partial H/\partial \tau}{\sin(\alpha - \theta)}$$

in regions 1 and 2, and

$$\left[\left|\frac{\partial H}{\partial n} \sin(\alpha - \theta)\right|\right] = 0$$

$$\sin(\alpha - \theta) = (\bar{q} \cdot \hat{\nu})/q$$

Therefore,

$$\left[\left|\frac{1}{q} \frac{\partial H}{\partial n}(\bar{q} \cdot \hat{\nu})\right|\right] = 0$$

Expanding the previous equation gives

$$\left[\left|\frac{1}{q} \frac{\partial H}{\partial n}\right|\right](\bar{q} \cdot \hat{\nu})_2 + \frac{1}{q}\left(\frac{\partial H}{\partial n}\right)_1 [|\bar{q} \cdot \hat{\nu}|] = 0$$

When dividing by C_{N_2}, one obtains

$$\frac{\left[\left|\frac{1}{q} \frac{\partial H}{\partial n}\right|\right]}{a_2^2 - (\bar{q}_2 \cdot \hat{\nu})^2} = - \frac{\frac{1}{q_1} \frac{\partial H_1}{\partial n}}{\bar{q}_2 \cdot \hat{\nu}} \frac{[|\bar{q} \cdot \hat{\nu}|]}{C_{N_2}} = \frac{\frac{1}{q_1} \frac{\partial H_1}{\partial n}}{\bar{q}_2 \cdot \hat{\nu}} S_2 = S_{17} \tag{52}$$

Also by definition in regions 1 and 2,

$$\frac{\partial S}{\partial \sigma} = 0$$

and

$$\frac{\partial S}{\partial n} = \frac{\partial S/\partial \tau}{\sin(\alpha - \theta)} = \frac{q}{\bar{q} \cdot \hat{\nu}} \frac{\partial S}{\partial \tau}$$

Thus,

$$\frac{1}{q} \frac{\partial S}{\partial n}(\bar{q} \cdot \hat{\nu}) = \frac{\partial S}{\partial \tau}$$

$$\left[\left|\frac{T}{q} \frac{\partial S}{\partial n}\right|\right] = \left[\left|\frac{T(\partial S/\partial \tau)}{\bar{q} \cdot \hat{\nu}}\right|\right]$$

Expanding $T \frac{\partial S}{\partial \tau}$ by using the expression for entropy in reference 6 gives

$$\left[\left|T \frac{\partial S}{\partial \tau}\right|\right] = \frac{\gamma}{\gamma - 1} R\left[\left|\frac{\partial T}{\partial \tau}\right|\right] - \left[\left|\frac{1}{\rho} \frac{\partial p}{\partial \tau}\right|\right]$$

$$\left[\left|\frac{T}{\bar{q} \cdot \hat{\nu}} \frac{\partial S}{\partial \tau}\right|\right] = \frac{\gamma}{\gamma - 1} R\left[\left|\frac{1}{\bar{q} \cdot \hat{\nu}} \frac{\partial T}{\partial \tau}\right|\right] - \frac{1}{\rho_1(\bar{q}_1 \cdot \hat{\nu})}\left[\left|\frac{\partial p}{\partial \tau}\right|\right]$$

After dividing by C_{N_2}, $\left[\left|\frac{1}{\bar{q} \cdot \hat{\nu}} \frac{\partial T}{\partial \tau}\right|\right] \Big/ C_{N_2}$ and $\left[\left|\frac{\partial p}{\partial \tau}\right|\right] \Big/ C_{N_2}$ must be evaluated.

To examine the factor $[|\partial p/\partial \tau|]/C_{N_2}$, recall from equation (28) that

$$[|p|] + m[|\bar{q} \cdot \hat{\nu}|] = 0$$

Taking the tangential derivative and dividing by C_{N_2} gives

$$[|p|]_\tau + m_\tau[|\bar{q} \cdot \hat{\nu}|] + m[|\bar{q} \cdot \hat{\nu}|]_\tau = 0$$

$$\frac{[|p_\tau|]}{C_{N_2}} = -m_\tau \frac{[|\bar{q} \cdot \hat{\nu}|]}{C_{N_2}} - \frac{m[|\bar{q} \cdot \hat{\nu}|]_\tau}{C_{N_2}}$$

$$= -m_\tau \frac{[|\bar{q} \cdot \hat{\nu}|]}{C_{N_2}} - m \frac{[|\bar{q}|]_\tau \cdot \hat{\nu}}{C_{N_2}}$$

Thus,

$$\frac{[|p_\tau|]}{C_{N_2}} = -m_\tau S_2 - mS_{11} = S_{18} \tag{53}$$

The factor $\left[\left|\frac{1}{q \cdot \nu}\frac{\partial T}{\partial \tau}\right|\right]$ shall be examined by first obtaining an expression for $R[|T|]_\tau$ from the energy equation (eq. (29)):

$$R[|T|]_\tau = -\frac{\gamma - 1}{2\gamma}[|\bar{q} \cdot \hat{\nu})^2|]_\tau$$

$$= -\frac{\gamma - 1}{\gamma}[|(\bar{q} \cdot \hat{\nu})(\bar{q} \cdot \hat{\nu})_\tau|]$$

$$= -\frac{\gamma - 1}{\gamma}\left[(\bar{q}_2 \cdot \hat{\nu})\left([|\bar{q}|]_\tau \cdot \hat{\nu}\right) + (\bar{q}_1 \cdot \hat{\nu})_\tau[|\bar{q} \cdot \hat{\nu}|]\right]$$

After dividing by C_{N_2}, this becomes

$$\frac{R[|T|]_\tau}{C_{N_2}} = -\frac{\gamma - 1}{\gamma}[(\bar{q}_2 \cdot \hat{\nu})S_{11} + (\bar{q}_1 \cdot \hat{\nu})_\tau S_2]$$

By applying the expansion formula for $[|AB|]$ (shown below eq. (10)) to $R[|T_\tau/(\bar{q} \cdot \hat{\nu})|]$, one obtains

$$R\left[\left|\frac{T_\tau}{\bar{q} \cdot \hat{\nu}}\right|\right] = \frac{R[|T_\tau|]}{\bar{q}_2 \cdot \hat{\nu}} + RT_{1_\tau}\left[\left|\frac{1}{\bar{q} \cdot \hat{\nu}}\right|\right]$$

$$= \frac{R[|T_\tau|]}{\bar{q}_2 \cdot \hat{\nu}} + RT_{1_\tau}\frac{-[|\bar{q} \cdot \hat{\nu}|]}{(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu})}$$

Therefore, after substituting for $R[|T_\tau|]/(\bar{q}_2 \cdot \hat{\nu})$,

$$\frac{R[|T_\tau/(\bar{q} \cdot \hat{\nu})|]}{C_{N_2}} = -\frac{\gamma - 1}{\gamma}\left[S_{11} + \frac{(\bar{q}_1 \cdot \hat{\nu})_\tau}{\bar{q}_2 \cdot \hat{\nu}} S_2 - \frac{RT_{1_\tau}}{(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu})} S_2\right]$$

$$= -\frac{\gamma - 1}{\gamma}\left[S_{11} + S_2 \frac{(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_1 \cdot \hat{\nu})_\tau - RT_{1_\tau}}{(\bar{q}_1 \cdot \hat{\nu})(\bar{q}_2 \cdot \hat{\nu})}\right] = S_{19} \tag{54}$$

Returning to the evaluation of $\left[\left|\frac{T}{q}\frac{\partial S}{\partial n}\right|\right] \Big/ C_{N_2}$ gives

$$\frac{\left[\left|\frac{T}{q}\frac{\partial S}{\partial n}\right|\right]}{C_{N_2}} = \frac{\left[\left|T\frac{\partial S}{\partial \tau}\Big/(\bar{q} \cdot \hat{\nu})\right|\right]}{C_{N_2}} = \frac{-\frac{1}{m}\left[\left|\frac{\partial p}{\partial \tau}\right|\right] + \frac{\gamma}{\gamma - 1}\left[\left|RT_\tau/(\bar{q} \cdot \hat{\nu})\right|\right]}{C_{N_2}}$$

$$= -\frac{1}{m} S_{18} + \frac{\gamma}{\gamma - 1} S_{19} = S_{20} \tag{55}$$

The quantities S_1 to S_{20} remain finite as C_{N_2} approaches 0 and thus provide uniformly valid expressions for the ratios needed to evaluate equation (11) and therefore $\partial u_2/\partial v$. The solution process for the derivatives behind the shock continues with step 5 where the spatial derivatives are determined as functions of the curvature of the shock.

An application for this system of equations will be shown in part II, in which the equations for shock coalescence with asymmetric effects are developed.

PART II

APPLICATION OF METHOD TO SHOCK COALESCENCE WITH ASYMMETRIC EFFECTS

The coalescence system is briefly described here. (See ref. 2 for a detailed derivation of the governing equations of the system.) Two shock surfaces $f_1(r,\phi)$ and $f_2(r,\phi)$ coalesce to form a resultant shock surface $f_3(r,\phi)$, a contact surface $h(x,\phi)$, and a weak (isentropic) shock or expansion or the opposite family $f_4(r,\phi)$. (See fig. 2.) The problem is first solved axisymmetrically, and at some distance away from the body it may be treated as a two-dimensional problem, as represented in figure 3. By assuming that all conditions in regions 1, 2, and 3 are known and that the point of coalescence is known, the unknowns for the system are u_4, v_4, p_4, T_4, u_5, v_5, p_5, T_5, β_3, and β_4, where β is the shock angle with respect to the upstream flow. The equations for the system are four shock equations across f_3, four shock equations across f_4 (assuming that f_4 is a shock), and matching pressure and flow direction at the slipstream surface h.

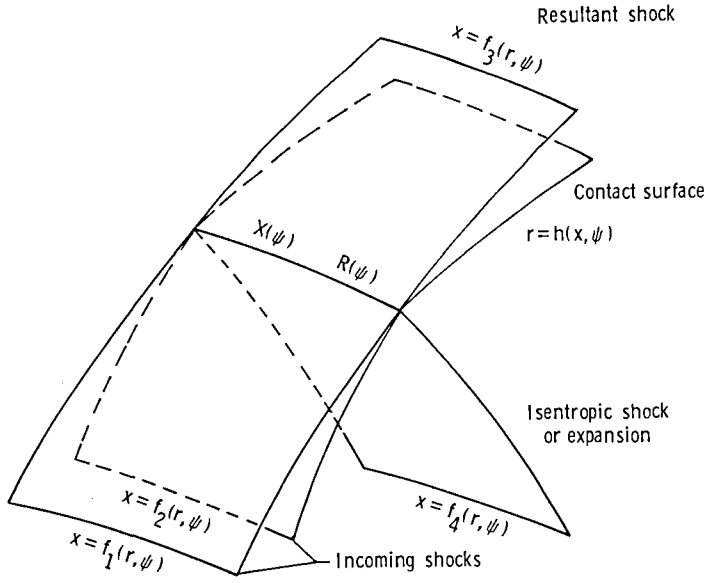

Figure 2.- Three-dimensional shock coalescence.

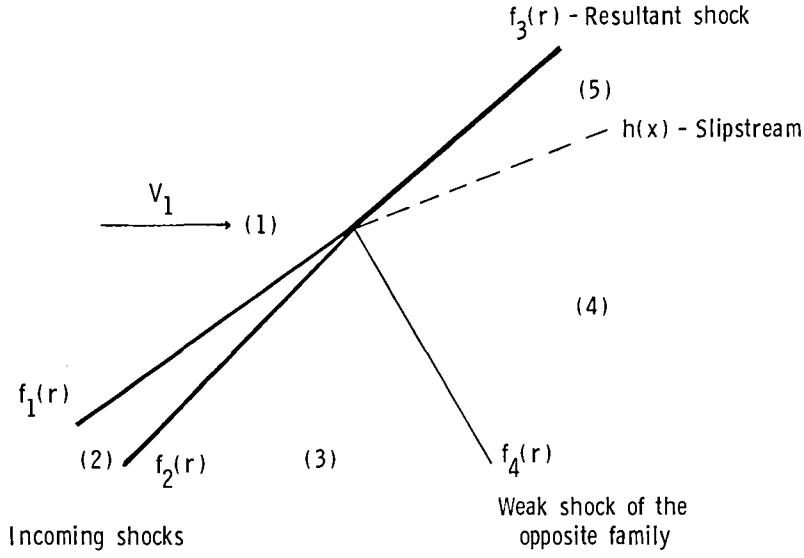

Figure 3.- Two-dimensional representation of shock coalescence.

The system is closed with 10 unknowns and 10 equations. To derive the asymmetric system of governing equations, the system is treated as five intersecting surfaces along $X(\phi),R(\phi)$. (See fig. 2.) The governing equations are the five intersection equations, five shock equations at f_3 (continuity, energy, and three momentum), five shock equations at f_4, and three slipstream equations at h. The second derivative with respect to ϕ (circumferential direction) is taken for each of the governing equations along the corresponding surface. The equations are then reduced to the $\phi = 0$ plane resulting in 18 unknowns of the form $U_{\phi\phi}\big|_{\phi=0}$, $V_{\phi\phi}\big|_{\phi=0}$, and so forth. The asymmetric system would be closed with 18 unknowns and 18 equations, except that it becomes necessary in deriving the asymmetric equations to know the spatial derivatives of u, v, p, and T in both regions 4 and 5.

The method derived in part I of this paper is applied and equations (22) to (25) provide a method for obtaining u_{4_x}, v_{4_x}, u_{4_r}, v_{4_r}, u_{5_x}, v_{5_x}, u_{5_r}, and v_{5_r} in terms of f_3'' and f_4'', respectively, which are unknown.

The Euler equations in the plane of symmetry, $\phi = 0$, are valid in regions 4 and 5 and are now applied to find relations for p_x and p_r in those regions. Using the Euler equations for p_x and p_r gives

$$p_{4_x} = \frac{1}{\bar{T}_4}\left(-p_4 u_4 u_{4_x} - p_4 v_4 u_{4_r}\right) \tag{56}$$

$$p_{5_x} = \frac{1}{\bar{T}_5}\left(-p_5 u_5 u_{5_x} - p_5 v_5 u_{5_r}\right) \tag{57}$$

$$p_{4_r} = \frac{1}{\bar{T}_4}\left(-p_4 u_4 v_{4_x} - p_4 v_4 v_{4_r} - g p_4\right) \tag{58}$$

$$p_{5_r} = \frac{1}{\bar{T}_5}\left(-p_5 u_5 v_{5_x} - p_5 v_5 v_{5_r} - g p_5\right) \tag{59}$$

The three equations valid at the slipstream surface h (the tangential derivative along h of slipstream conditions) are

$$p_{4_x} + p_{4_r} h_x = p_{5_x} + p_{5_r} h_x \tag{60}$$

$$u_{4_x} h_x + u_4 h_{xx} - v_{4_x} + u_{4_r} h_x^2 - v_{4_r} h_x = 0 \tag{61}$$

$$u_{5_x} h_x + u_5 h_{xx} - v_{5_x} + u_{5_r} h_x^2 - v_{5_r} h_x = 0 \tag{62}$$

There are 15 equations which may be solved for the 15 unknowns u_{4_x}, u_{4_r}, v_{4_x}, v_{4_r}, u_{5_x}, u_{5_r}, v_{5_x}, v_{5_r}, p_{4_x}, p_{4_r}, p_{5_x}, p_{5_r}, f_3'', f_4'', and h_{xx}.

To determine the derivatives for temperature T, the enthalpy relationships are used, where enthalpy H is defined as

$$H = \frac{\gamma}{\gamma - 1} RT + \frac{q^2}{2} + gz$$

By definition, $\partial H/\partial\sigma = 0$ where

$$q \frac{\partial}{\partial\sigma} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial r}$$

and from the shock equations derived in the previous section,

$$\frac{\left[\left|\frac{1}{q}\frac{\partial H}{\partial n}\right|\right]}{C_{N_2}} = S_{17}$$

Therefore,

$$\frac{\partial H_2}{\partial n} = q_2\left(C_{N_2} S_{17} + \frac{1}{q_1}\frac{\partial H_1}{\partial n}\right)$$

From the definition of enthalpy,

$$\frac{\partial H_2}{\partial n} = \frac{\gamma}{\gamma - 1}\frac{\partial T_2}{\partial n} + \frac{1}{2}\frac{\partial\left(u_2^2 + v_2^2\right)}{\partial n} = \frac{\gamma}{\gamma - 1}\frac{\partial T_2}{\partial n} + u_2\frac{\partial u_2}{\partial n} + v_2\frac{\partial v_2}{\partial n}$$

Thus,

$$\frac{\partial T_2}{\partial n} = \frac{\gamma - 1}{\gamma}\left(\frac{\partial H_2}{\partial n} - u_2\frac{\partial u_2}{\partial n} - v_2\frac{\partial v_2}{\partial n}\right)$$

$$= \frac{\gamma - 1}{\gamma}\left(\frac{\partial H_2}{\partial n} - \frac{u_2^2}{q_2}\frac{\partial u_2}{\partial r} + \frac{u_2 v_2}{q_2}\frac{\partial u_2}{\partial x} - \frac{u_2 v_2}{q_2}\frac{\partial v_2}{\partial r} + \frac{v_2^2}{q_2}\frac{\partial v_2}{\partial x}\right) \tag{63}$$

Along the streamline,

$$\frac{\partial T_2}{\partial\sigma} = \frac{\gamma - 1}{\gamma}\left(-u_2\frac{\partial u_2}{\partial\sigma} - v_2\frac{\partial v_2}{\partial\sigma}\right)$$

$$= \frac{\gamma - 1}{\gamma}\left(-\frac{u_2^2}{q_2}\frac{\partial u_2}{\partial x} - \frac{u_2 v_2}{q_2}\frac{\partial u_2}{\partial r} - \frac{u_2 v_2}{q_2}\frac{\partial v_2}{\partial x} - \frac{v_2^2}{q_2}\frac{\partial v_2}{\partial r}\right) \tag{64}$$

From the normal and streamline derivatives of T, the following relationships are determined:

$$\frac{\partial T_2}{\partial x} = \frac{u_2}{q_2}\frac{\partial T_2}{\partial \sigma} - \frac{v_2}{q_2}\frac{\partial T_2}{\partial n} \tag{65}$$

$$\frac{\partial T_2}{\partial r} = \frac{v_2}{q_2}\frac{\partial T_2}{\partial r} + \frac{u_2}{q_2}\frac{\partial T_2}{\partial n} \tag{66}$$

Equations (65) and (66) applied in regions 4 and 5 provide the four additional equations needed to determine T_{4_x}, T_{4_r}, T_{5_x}, and T_{5_r}.

For the curvature k_2 of the streamline $\psi = r - g_2(x)$ behind the shock, note that

$$k_2 = \frac{g_2''}{\left\{1 + \left[g_2'(x)\right]^2\right\}^{3/2}}$$

where $g_2'(x) = v_2/u_2$ or $v_2 = u_2\, g'(x)$. The derivative of this equation along the streamline $r = g_2(x)$ yields

$$v_{2_x} + v_{2_r} g_2' = u_2 g_2'' + \left(u_{2_x} + u_{2_r} g_2'\right) g_2'$$

Therefore,

$$k_2 = \frac{1}{u_2\left[1 + (g_2')^2\right]^{3/2}}\left[\left(v_{2_x} + v_{2_r} g_2'\right) - \left(u_{2_x} + u_{2_r} g_2'\right) g_2'\right]$$

$$= \frac{u_2^2 v_{2_x} + u_2 v_2\left(v_{2_r} - u_{2_x}\right) - u_{2_r} v_2^2}{(u_2^2 + v_2^2)^{3/2}} \tag{67}$$

All quantities on the R.H.S. of equation (67) are known behind the shock from equations (23) to (25) and (56) to (62). Thus, the curvature of a given streamline behind the shock may be found by using equation (67).

CONCLUDING REMARKS

A set of uniformly valid explicit formulas for spatial derivatives of flow quantities behind a shock wave have been developed in terms of the shock curvature. The equations for jump conditions across a shock have been rearranged so that those quantities which become indeterminate as the shock strength approaches 0 have been eliminated analytically. The resulting formulas are therefore valid for shocks of any strength.

In the solution of shock coalescence with asymmetric effects as applied to sonic boom propagation, the need for spatial derivatives behind the shocks arises. Shock strengths in sonic boom problems may vary from moderate to very weak. This problem then has need of equations which are valid for finite shocks and also for shocks approaching zero strength. An outline is provided to show the application of the uniformly valid formulas developed in this paper to the shock-coalescence problem.

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16. Abstract

Explicit formulas in terms of shock curvature are developed for spatial derivatives of flow quantities behind a curved shock for two-dimensional inviscid steady flow. Factors which yield the equations indeterminate as the shock strength approaches 0 have been canceled analytically so that formulas are valid for shocks of any strength. An application for the method is shown in the solution of shock coalescence when nonaxisymmetric effects are felt through derivatives in the circumferential direction. The solution of this problem requires flow derivatives behind the shock in both the axial and radial direction.

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