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THE GÖRTLER VORTEX INSTABILITY MECHANISM IN THREE-DIMENSIONAL BOUNDARY LAYERS

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Abstract

It is well known that the two-dimensional boundary layer on a concave wall is centrifugally unstable with respect to vortices aligned with the basic flow for sufficiently high values of the Görtler number. However, in most situations of practical interest the basic flow is three-dimensional and previous theoretical investigations do not apply. In this paper the linear stability of the flow over an infinitely long swept wall of variable curvature is considered. If there is no pressure gradient in the boundary layer it is shown that the instability problem can always be related to an equivalent two-dimensional calculation. However, in general, this is not the case and even for small values of the crossflow velocity field dramatic differences between the two and three-dimensional problems emerge. In particular, it is shown that when the relative size of the crossflow and chordwise flow is $O(R^{-1/2})$, where R is the Reynolds number of the flow, the most unstable mode is time-dependent. When the size of the crossflow is further increased, the vortices in the neutral location have their axes locally perpendicular to the vortex lines of the basic flow. In this regime the eigenfunctions associated with the instability become essentially 'centre modes' of the Orr-Sommerfeld equation destabilized by centrifugal effects. The critical Görtler number for such modes can be predicted by a large wavenumber asymptotic analysis; the results suggest that for order unity values of the ratio of the crossflow and chordwise velocity fields, the Görtler instability mechanism is almost certainly not operational.

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1. INTRODUCTION

In recent years there has been much interest in the effect of boundary layer growth on the instability mechanisms to which boundary layers are susceptible. Our concern here is with the Görtler vortex instability mechanism which is known to occur in two-dimensional boundary layer flows over concave walls. The latter problem has been discussed by Görtler (1940), Hämmerlin (1955, 1956), Smith (1955), Floryan and Saric (1979) and Hall (1982a,b, 1983). In the latter three papers it was shown that the parallel flow approximation used previously to reduce the linear stability equations to ordinary differential equations is justifiable only for vortices of small wavelength. In this case the effect of boundary layer growth can be taken care of in a self-consistent manner and an asymptotic form for the right-hand branch of the neutral curve can be derived. At order one values of the vortex wavelength the parallel flow approximation is not valid and the full linearized partial differential equations must be solved numerically. The reader is referred to the paper by Hall (1983) for a discussion of the results and implications of such a calculation, it suffices here to say that the physically unacceptable results of previous calculations in this wavenumber regime are shown to be a direct consequence of the parallel flow approximation.

There are many practical situations where Görtler vortices are thought to occur but in most of these the basic boundary layer flow is three-dimensional. If, for example, we are concerned with the flow on the concave region of a swept laminar flow wing or that over a turbine blade, then the three-dimensional nature of the basic flow is important and cannot be neglected; thus previous theoretical stability calculations do not apply. In order to determine the effect of three-dimensionality on Görtler vortices for such flows we shall in this paper investigate the instability of the boundary layer on the concave part of an infinitely long swept cylinder.

The basic boundary layer flow associated with this flow is easily calculated using the method of Sears (1948). The parameters which characterize the flow are α, the angle of yaw, R the Reynolds number and δ the curvature parameter. If $\alpha = 0$, we know that steady Görtler vortices occur for $O(1)$ values of $R^{1/2}\delta$ and are aligned with the streamlines of the basic flow. The aim of this paper is to determine how the structure of the instability changes when α is increased from zero. The present analysis is restricted to small values of the vortex wavelength, but our experience with the two-dimensional problem would suggest that the results which we obtain are useful at $O(1)$ values of the wavelength. However, the differences between small wavelength Görtler vortices in two- and three-dimensional boundary layers are so fundamental that we expect that vortices with $O(1)$ wavelength cannot even occur in a three-dimensional boundary.

The relative size of the crossflow and chordwise flow is the crucial factor in determining the structure of Görtler vortices in three-dimensional boundary layers. Suppose that λ is a typical value for the ratio of the latter velocities in the boundary layer. We show that, when λ is increased from zero, the first crucial change in the structure of the instability occurs when $\lambda \sim R^{-1/2}$. In this regime the vortices become time-dependent and meander as they develop in the chordwise direction. Surprisingly, the orientation of the most dangerous mode is always determined by the vortex lines of the basic flow. We show that at the neutral location the vortex boundaries align themselves so as to be locally perpendicular to the vortex lines of the basic flow. Thus, in general, unlike the two-dimensional case, the vortices are not locally parallel to the flow direction. At larger values of λ further dramatic changes in the vortex structure emerge, until finally the eigenfunctions develop into 'centre modes' of the Orr-Sommerfeld equation made unstable by centrifugal effects. We are able to find the development of the vortices up to the regime $\lambda \sim R^{-1/8}$ and it appears that for $\lambda \gg R^{-1/8}$ the Görtler vortex instability mechanism is either not present or cannot be described by asymptotic means. Thus, we are able to describe the evolution of Görtler vortices for $0 < \lambda < O(R^{-1/8})$ but it turns out that by writing $\lambda = \bar{\lambda} R^{-1/2}$ all the regimes of interest can be obtained by first letting $R \to \infty$, with $\bar{\lambda}$ fixed and then taking further limits on $\bar{\lambda}$. This is the procedure which we will adopt in this paper.

We find that the above results do not apply when the basic three-dimensional boundary layer has zero pressure gradient. In this case, the stability equations can be solved for $O(1)$ values of α and, in fact, can be reduced to an equivalent two-dimensional problem. The vortices are then aligned with the streamlines of the basic flow again and the most dangerous modes are steady.

The procedure adopted in the rest of this paper is as follows. In Section 2 we formulate the stability equations for the flow over a slightly yawed infinite cylinder and solve these equations asymptotically for small vortex wavelengths. In Section 3 we discuss the special case of flows with zero pressure gradient and in Section 4 we discuss our results and their practical implications.

2. GORTLER VORTICES IN A SLIGHTLY THREE-DIMENSIONAL BOUNDARY LAYER

We consider the stability of the boundary layer flow over the cylinder $y = 0$, $-\infty < z < \infty$. The z axis is taken to be a generator of the cylinder and y measures distance normal to the surface. The x coordinate measures distance along the curved surface which is taken to have variable curvature $\frac{1}{a}\kappa\left(\frac{x}{\ell}\right)$ where a and ℓ are length scales. The Reynolds number R, the curvature parameter δ and the Görtler number G are defined by

$$R = \frac{U_0 \ell}{\nu}, \tag{2.1a}$$

$$\delta = \frac{\ell}{a}, \tag{2.1b}$$

$$G = 2R_E^{1/2}\,\delta, \tag{2.1c}$$

where U_0 is a typical velocity in the x direction. We assume that the Reynolds number is large, whilst δ is taken to be small. More precisely, we consider the limit $\delta \to 0$, with G held fixed. The basic three-dimensional boundary layer flow is assumed to be of the form

$$\underset{\sim}{u} = U_0\left(\overline{u}(X,Y), R^{-1/2}\,\overline{v}(X,Y),\ \lambda\overline{w}(X,Y)\right)(1 + O(R^{-1/2}),$$

where

$$X = \frac{x}{\ell}, \quad Y = \frac{y}{\ell}R^{1/2},$$

and λ is to be specified shortly. Following the procedure of Sears (1948) $\overline{u}$ and $\overline{v}$ are found by integrating numerically the two-dimensional boundary layer equations. $\lambda\overline{w}$ is then calculated from the z momentum equation. At this stage we need only insist that the basic flow be independent of the spanwise coordinate, later we shall indicate the relevance of our calculation to some particular boundary layer profiles of practical importance.

The appropriate scaling for λ, which, fixes the angle of yaw of the cylinder, can be found by taking $\lambda \sim R^{-\gamma}$, $\gamma > 0$ and varying γ until the Görtler instability mechanism driven by $\overline{u}$ is modified by the crossflow at zeroth-order. It is known from the two-dimensional problem that the characteristic wavelength of the vortices is of the same order of magnitude as the boundary layer thickness, so that the convective terms $U_0\,\overline{u}\frac{\partial}{\partial x}$ and $U_0\,\lambda\overline{w}\frac{\partial}{\partial z}$ will be of comparable order when $R \to \infty$ if $\lambda \sim R^{-1/2}$. We write

$$\lambda = \overline{\lambda}R^{-1/2}$$

to effectively restrict our attention to flows over cylinders yawed at an angle $\alpha \sim O(R^{-1/2})$ to the oncoming flow. At a later stage we can consider the limits $\overline{\lambda} \to 0$, $\overline{\lambda} \to \infty$ in order to recover some information about the regimes $\lambda \ll R^{-1/2}$ and $\lambda \gg R^{-1/2}$ respectively. We define the variable Z by

$$Z = R^{1/2}\, z/\ell$$

and perturb the basic flow by writing

$$\underline{u} = U_0\big(\overline{u} + U(t,X,Y)E,\ \overline{v}R^{-1/2} + V(t,X,Y)R^{-1/2}\,E,\ \overline{\lambda}R^{-1/2}\,\overline{w}$$

$$+ W(t,X,Y)R^{-1/2}\,E\big)\big(1 + O(R^{-1/2})\big), \qquad (2.1c)$$

where t is a time variable scaled on ℓ/U_0 whilst

$$E = \exp(iaZ).$$

We then substitute the above expression for $\underline{u}$ into the Navier-Stokes equations and linearize to obtain

$$U_t + \bar{u}U_X + U\bar{u}_X + \bar{v}U_Y + V\bar{u}_Y + \bar{\lambda}\,\bar{w}iaU = U_{YY} - a^2 U, \tag{2.2a}$$

$$V_t + \bar{u}V_X + U\bar{v}_X + \bar{v}V_Y + V\bar{v}_Y + \bar{\lambda}\,\bar{w}iaV + G\kappa(X)\bar{u}U = -P_Y + V_{YY} - a^2 V, \tag{2.2b}$$

$$W_t + \bar{u}W_X + \bar{\lambda}\,U\bar{w}_X + \bar{v}W_Y + \bar{\lambda}\,V\bar{w}_Y + \bar{\lambda}\,\bar{w}iaW = -iaP + W_{YY} - a^2 W, \tag{2.2c}$$

$$U_X + V_Y + iaW = 0. \tag{2.2d}$$

Here P is the nondimensional pressure perturbation corresponding to the disturbed velocity field and we have neglected terms of relative order $R^{-1/2}$. The equations (2.2) reduce to the corresponding two-dimensional equations when $\bar{\lambda}=0$. We shall see later that if $\bar{u} = \bar{w}$ we can solve the three-dimensional stability problem even for $\lambda = 0(1)$. In fact, in this case the three-dimensional problem can always be reduced to an equivalent two-dimensional problem so that the results of Hall (1982a, 1983), hereafter referred to as I, II, can be used. For the remainder of this section we assume that $\bar{u} \neq \bar{w}$ and determine how the crossflow modifies the instability.

It is clear that if the wavenumber a is $0(1)$ then (2.2) must be solved numerically in the manner described in II, but we must allow for a possible time dependence of the instability. Our experience in I, II showed that it is only in the limit $a \to \infty$ that the usual results of stability theory apply and, in fact, for $a \sim O(1)$ there is no such thing as a unique neutral curve. However, we found that the asymptotic results for $a >> 1$ gave useful information about the instability for $a \sim O(1)$, so we shall concentrate here on the further limit $a \to \infty$.

It was shown in I that for $a >> 1$, the vortices concentrate themselves in an internal viscous layer of thickness $a^{-1/2}$. In the neutral location this layer is in the position where $\bar{u}(X,Y)$ most violates Rayleigh's criterion. If the vortices are not locally neutrally stable, this is not the case and we shall show that for sufficiently large crossflow velocities the position of the viscous layer where the vortices are concentrated is never fixed by Rayleigh's criterion.

We are now in a position to determine the asymptotic solution of (2.2) for large values of the nondimensional wavenumber a. Before doing so, we must specify the relative size of $\bar{\lambda}$, which determines the magnitude of the crossflow velocity field. The aim of our calculation is to determine the effect of increasing $\bar{\lambda}$ on the Görtler vortex instability mechanism, so it is convenient for us to take the limit $a \to \infty$, $\bar{\lambda} \sim a^J$ for $J \geqslant 0$. Afterwards we shall indicate for what regimes of the curvature and flow velocities these further limits apply. It turns out that there are three major regimes for J which enable us to describe how a Görtler vortex eigenfunction evolves into essentially an Orr-Sommerfeld eigenfunction when the crossflow is increased. These three regimes correspond to (a) $a \sim \bar{\lambda}^2$, (b) $a \sim \bar{\lambda}^{3/5}$, and (c) $\alpha \sim \bar{\lambda}^{1/3}$, respectively. We shall see that for $\bar{\lambda} >> 1$, the first significant difference between the two- and three-dimensional problem emerges when $a \sim \bar{\lambda}^2$.

If a is decreased to $O(\bar{\lambda}^{-3/5})$ the mechanism which enables the vortices to be concentrated in an internal viscous layer changes. Thus for the two-dimensional problem the decay of the vortices is due to the fact that the local Görtler number has a maximum at the centre of the layer whereas for the three-dimensional problem with $a \sim \bar{\lambda}^{-3/5}$, the decay of the vortices is facilitated by convective effects. A further decrease in a to $O(\bar{\lambda}^{-1/3})$ shows that the Görtler vortices at this stage are essentially Orr-Sommerfeld eigenfunctions destabilized by centrifugal effect. Moreover, for a $a \sim \lambda^{1/3}$ a neutral curve with both left and right hand branches and a minimum Görtler number is obtained, this suggests that for a $a << \bar{\lambda}^{-1/3}$, instability is almost certainly impossible at finite values of G. We now describe the three regimes described above:

a. <u>The limit $a \to \infty$ with $\bar{\lambda} \sim O(a)^{1/2}$, $G \sim O(a^4)$.</u>

It is known from the two-dimensional problem that for $a >> 1$ the flow is neutrally stable for $G \sim a^4$. We anticipate that this will also be the case for $\bar{\lambda} \sim a^{1/2}$ and therefore write

$$G = g_0 a^4 + g_1 a^3 + \cdots, \tag{2.3}$$

whilst $\bar{\lambda}$ and $\frac{\partial}{\partial t}$ are replaced by $\hat{\lambda} a^{1/2}$ and $i\sigma$ with

$$\sigma = a^{3/2} \sigma_0 + a\sigma_1 + \cdots. \tag{2.4}$$

The aim of our calculation is to determine the frequency σ and Görtler number G such that the flow is locally neutrally stable at X. However, in order to show how non-neutral disturbances evolve with X, we allow for growth or decay in X by writing

$$U = \sum_0^\infty a^{n/2} U_n(X,\eta)\exp\{i\sigma t + a^2 \int^X \beta_0(X) + a^{-1/2} \beta_1(X) +,\cdots dX\} , \quad (2.5)$$

together with similar expansions for V/a^2, $W/a^{3/2}$, $P/a^{5/2}$. The variable η is defined by

$$\eta = \{Y - \overline{Y}(X)\} a^{1/2}, \quad (2.6)$$

where $Y = \overline{Y}(X)$ is the location of the viscous layer where the vortices are concentrated. The thickness of this layer was shown in I to be $O(a^{-1/2})$ and it corresponds to a second-order turning point of a WKB solution of (2.2). The growth rate functions $\beta_i(X)$, $i=0,\cdots$, appearing in (2.5) are in general, complex quantities and for given G and σ will clearly vary with X. The velocity components of the basic flow expand locally around $Y = \overline{Y}(X)$ as

$$\overline{u} = u_0(X) + a^{-1/2} u_1(X)\eta + a^{-1} u_2(X)\eta^2 +,\cdots,$$

together with similar expansions for $\overline{v}$ and $\overline{w}$. It is a routine matter to substitute the above expansion into (2.2) and by successively equating like powers of $a^{-1/2}$ we generate a sequence of equations to determine (U_n,V_n,W_n,P_n) for $n = 0,1,\cdots$. The system corresponding to $n = 0$ is

$$(u_0 \beta_0 + 1)U_0 = -V_0 u_1, \quad (2.7a)$$

$$(u_0 \beta_0 + 1)V_0 = -g_0\kappa u_0 U_0, \quad (2.7b)$$

$$W_0 = iV_{0\eta}, \tag{2.7c}$$

$$P_0 = [1 + u_0\,\beta_0]\,iW_0. \tag{2.7d}$$

At this order the crossflow has no effect on the expansion procedure and (2.7a,b) have a solution if

$$(u_0\,\beta_0 + 1)^2 = g_0\,u_0\,u_1\,\kappa, \tag{2.8}$$

so that instability occurs if $g_0\,u_0\,u_1\,\kappa > 1$ and the growth rate β_0 is real. The functions V_0, W_0 and P_0 can then be expressed in terms of U_0 using (2.7a,b,c). At next order the system of equations which determines (U_1, V_1, W_1, P_1) is found to be

$$[u_0\,\beta_0 + 1]U_1 + u_1\,V_1 = -\eta[u_1\,\beta_0\,U_0 + 2u_2\,V_0]$$
$$- i\hat{\lambda}w_0\,U_0 - i\sigma_0\,U_0 - u_0\,\beta_1\,U_0,$$

$$[u_0\,\beta_0 + 1]V_1 + g_0\,\kappa u_0\,U_1 = -\eta[u_1\,\beta_0\,V_0 + g_0\,\kappa u_1\,U_0]$$
$$- i\hat{\lambda}w_0\,V_0 - i\sigma_0\,V_0 - u_0\,\beta_1\,V_0,$$

$$W_1 - i\,V_{1\eta} = i\beta_0\,U_0,$$

$$P_1 - iW_1[1 + u_0\,\beta_0] = iW_0\,u_1\,\beta_0\,\eta - \hat{\lambda}w_0\,W_0 - \sigma_0\,W_0 + iu_0\,\beta_1\,U_0.$$

The above system has a consistent solution if

$$i\sigma_0 + u_0\,\beta_1 + i\hat{\lambda} w_0 = 0, \tag{2.8a}$$

and

$$2\beta_0\, u_1[1 + \beta_0\, u_0] = g_0\, \kappa[2u_0\, u_2 + u_1^2]. \tag{2.8b}$$

The first of these equations shows that β_1 is purely imaginary and for a given value of σ_0 is determined by

$$\beta_{1i} = \frac{-\hat{\lambda} w_0 - \sigma_0}{u_0}, \tag{2.8c}$$

so that the orientation of the vortices varies with X. The second of the above constraints fixes $Y = \overline{Y}(x)$ as the location of the layer. In fact if g_0 is given we see from (2.7), (2.8b) that at any value of X the growth rate and $\overline{Y}$ are determined by solving

$$F(X,Y) = 0, \qquad F_Y(X,Y) = 0, \qquad Y = \overline{Y}(X),$$

where

$$F = (\overline{u}\beta_0 + 1)^2 - g_0\, \overline{u}\, \overline{u}_Y\, \kappa.$$

Thus, at any value of X the position of the viscous layer adjusts to produce a local maximum (in Y) for β_0. In the neutral case, $\beta_0 = 0$ and the position of the viscous layer corresponds to $\frac{\partial}{\partial Y}(\overline{u}\,\overline{u}_Y) = 0$, which is, of course, where Rayleigh's criterion is most violated. We note that at this order $[U_0, V_0, W_0, P_0]$ and $[U_1, V_1, W_1, P_1]$ remain undetermined, so that it is necessary to proceed further. In fact the next order system of equations can be solved if a consistency condition is satisfied, and this leads to the required differential equation for U_0. We thus obtain

$$\frac{\partial^2 U_0}{\partial\eta^2} - \frac{2\eta\{\beta_1 u_1 + w_1 \hat{\lambda} i\} U_0}{\{3 + u_0 \beta_0\}}$$
$$+ \frac{\{g_2 u_0 u_1 \kappa - 2(1 + u_0 \beta_0)(i\sigma_1 + u_0 \beta_2)\} U_0}{\{1 + u_0 \beta_0\}\{3 + u_0 \beta_0\}} - h\eta^2 U_0 = 0, \qquad (2.9)$$

where

$$h = \frac{F_{YY}(X,\overline{Y})}{2\{1 + u_0 \beta_0\}\{3 + u_0 \beta_0\}} .$$

We are interested in the solutions of (2.9) which decay when $\eta \to \pm\infty$ and after some manipulation we find that the most rapidly growing solution (in the X direction) can be written

$$U = \exp - h^{1/2} \left\{\eta + \frac{i[\beta_{1i} u_1 + w_1 \hat{\lambda}]}{h[3 + u_0 \beta_0]}\right\}^2 . \qquad (2.10)$$

Here we have used the fact that β_{1r} is zero and the eigenrelation corresponding to this eigensolution is

$$\frac{\{g_2 u_0 u_1 \kappa - 2[1 + u_0 \beta_0] u_0 \beta_{2r}\}}{\{1 + u_0 \beta_0\}\{3 + u_0 \beta_0\}} - \frac{[\beta_{1i} u_1 + w_1 \hat{\lambda}]^2}{h[3 + u_0 \beta_0]^2} = \sqrt{h} ,$$
$$(2.11)$$
$$\sigma_1 + u_0 \beta_{2i} = 0.$$

These equations determine β_{2r}, β_{2i} as functions of X as the disturbance develops in the X direction. If we are interested in the neutral case, we can set $\beta_0 = \beta_{2r} = 0$ in (2.7), (2.11) to obtain the following two-term asymptotic expansion of the Görtler number,

$$G = \frac{1}{\kappa\, u_0\, u_1}\left\{a^4 + 3a^3\left[\sqrt{h_0} + \frac{\{\beta_{1i}\, u_1 + w_1\, \hat{\lambda}\}^2}{9h_0}\right] + \cdots\right\}, \qquad (2.12a)$$

where h_0 represents h evaluated with $\beta_0 = 0$ whilst β_{1i} is determined by (2.8c). The three-dimensionality of the basic flow affects the Görtler number through the squared term in the curly brackets and therefore has a stabilizing influence on the basic flow with $\hat{\lambda} = 0$.

However, if the frequency σ_0 is chosen such that

$$u_1\, \beta_{1i} + w_1\, \hat{\lambda} = 0, \qquad (2.12b)$$

the Görtler number, correct to $O(a^3)$, reduces to its value for the flow with $\hat{\lambda} = 0$. Thus, even at this early stage, the importance of the modes which satisfy (2.12b) when they are locally neutrally stable is apparent. The constraint (2.12b) requires that the boundaries between neutral Görtler vortices be locally orthogonal to the vortex lines of the basic flow.

We further note that steady Görtler vortices $(\sigma_0 = 0,\ \beta_{1i} \neq 0)$ and Görtler vortices propagating in the spanwise direction $(\sigma_0 \neq 0,\ \beta_{1i} = 0)$ are also possible. The former modes are locally parallel to the streamlines of the basic flow and are more stable than the oblique modes satisfying (2.12b). We shall see in the remainder of this section that it is the modes corresponding to (2.12b) which are important for $\bar{\lambda} \gg a^{1/2}$, indeed, for $\bar{\lambda}$ sufficiently large, the first-order term in (2.12a) can even be modified by three-dimensional effects.

b. <u>The limit $a \to \infty$ with $\bar{\lambda} \sim O(a^{5/3})$, $G \sim O(a^4)$</u>

We have seen above that for $\bar{\lambda} \sim O(a^{1/2})$ the decay of the vortices away from the centre of the viscous layer is facilitated by the local maximum of the effective Görtler number at that level. We shall now show how this decay is facilitated by convective effects when $\bar{\lambda}$ is increased further. In order to simplify the details of our expansion procedure we shall concentrate our attention on determining the neutral locations of the most dangerous Görtler vortex modes. We have seen above that even for $\bar{\lambda} \sim a^{1/2}$ the most dangerous modes are locally perpendicular to the vortex lines of the basic flow, and it is the development of these modes which we consider below.

Suppose then that we concern ourselves with the mode which is locally neutrally stable at $X = X_n$ with a viscous layer located at $Y = \bar{Y}(X_n)$. The crucial property of the most dangerous three-dimensional modes discussed in (a) is that at the viscous layer they cause the operator $\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} + \bar{\lambda}\, ai\bar{w}$ to expand locally as $s\bar{\lambda}i(Y - \bar{Y})^2$ where s is a constant. If $\bar{\lambda}$ is taken to be $O(a)$, this causes a term proportional to $i\eta^2$ to appear in the equation corresponding to (2.9). This term, which is of course due to convective effects, enhances the decay of the vortex away from the viscous layer. In fact, if $\bar{\lambda} = \hat{\lambda}a^{5/3}$, the decay of the vortex away from the viscous layer is entirely due to convective effects. However, at this stage the expansion procedure of (a) must be significantly altered because the thickness of the viscous layer now decreases to $O(a^{-2/3})$. We therefore define the variable η by

$$\eta = \{Y - \bar{Y}\}a^{2/3},$$

and in order to obtain some information about non-neutral disturbances, it is convenient to work in a $a^{-2/3}$ neighbourhood of X_n. It is of course possible to describe non-neutral modes for $|X - X_n| \gg a^{-2/3}$ but it turns out that the Y variation of the disturbance is then on more than one scale so that the appropriate expansion procedure becomes much more tedious, for that reason we shall here restrict our attention only to $|X - X_n| \sim a^{-2/3}$. Thus, we define

$$\tilde{X} = (X - X_n)a^{2/3}$$

so that near (X_n, Y_n) we can write

$$\kappa(X) = \kappa_0 + \tilde{X}\kappa_1 a^{-2/3} + \tilde{X}^2 \kappa_2 a^{-4/3} + \cdots$$

$$\bar{u} = u_{00} + \tilde{X}a^{-2/3}u_{10} + \eta a^{-2/3}u_{01} + \tilde{X}^2 a^{-4/3}u_{20} + \tilde{X}\eta a^{-4/3}u_{11} + \eta^2 a^{-4/3} u_{02} + \cdots$$

together with a similar expansion for $\bar{w}$. The time scale for the instability now becomes $O(a^{-8/3})$, so σ is expanded as

$$\sigma = \sigma_0 a^{8/3} + \sigma_1 a^2 + \sigma_2 a^{4/3} + \cdots$$

whilst the Görtler number G now expands as

$$G = g_0 a^4 + g_1 a^{10/3} + \cdots.$$

Finally, we expand the X velocity component U as

$$U = \{U_0(\tilde{X},\eta) + a^{-2/3} U_1(\tilde{X},\eta) + \cdots\}$$

$$\exp\{i\sigma t + a^2 \int^{\tilde{X}} \beta_0(\tilde{X}) + a^{-2/3} \beta_1(\tilde{X}) + \cdots d\tilde{X}\}$$

together with similar expansions for V/a^2, $W/a^{5/3}$, $P/a^{8/3}$.

It now remains for us to substitute the above equations in the disturbance equations (2.2) and to equate like powers of the small parameter $a^{-2/3}$. However, before doing so, it is convenient to note that the operator $\partial/\partial t + \bar{u}(\partial/\partial X) + i\hat{\lambda}\ a^{8/3}\ \bar{w}$ expands as

$$\begin{aligned}\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial X} + \hat{\lambda}\ a^{8/3}\ i\bar{w} &= a^{4/3}\{i\sigma_2 + u_{00}\,\beta_2 + \beta_1(u_{10}\,\tilde{X} + u_{01}\,\eta) \\ &\quad + \beta_0(u_{20}\,\tilde{X}^2 + u_{02}\,\eta^2 + u_{11}\,\tilde{X}\eta) \\ &\quad + i\hat{\lambda}(w_{20}\,\tilde{X}^2 + w_{02}\,\eta^2 + w_{11}\,\tilde{X}\eta)\} + \cdots \\ &= a^{4/3}\,M + \cdots. \end{aligned} \qquad (2.13)$$

Here we have assumed that β_0, β_1 are purely imaginary and satisfy

$$i\sigma_0 + \beta_0\,u_{00} + i\hat{\lambda}w_{00} = 0, \qquad (2.14a)$$

$$i\sigma_1 + \beta_1\,u_{00} + i\hat{\lambda}w_{10}\,\tilde{X} + \beta_0\,u_{10}\,\tilde{X} = 0, \qquad (2.14b)$$

$$\beta_0\,u_{01} + i\hat{\lambda}w_{01} = 0. \qquad (2.14c)$$

The equations (2.14a,c) determine β_0 and σ_0, whilst (2.14b) determines β_1. The second-order term in the expansion of σ, namely σ_1, remains undetermined at this order; this is also the case at next order. Thus, σ_1, σ_2 can apparently be chosen so that there is a continuum of three-dimensional modes having the asymptotic structure described above.

It is now a routine matter to substitute the above expansions in (2.2) and equate like powers of $a^{-2/3}$. The zeroth-order system gives

$$V_0 = g_0 \, \kappa_0 \, u_{00} \, U_0,$$

together with the eigenrelation

$$g_0 \, \kappa_0 \, u_{00} \, u_{01} = 1.$$

At next order we obtain a pair of linear equations for (U_1, V_1) which have a consistent solution if

$$\frac{\partial^2 U_0}{\partial \eta^2} + [\alpha_0 + i\gamma_0]U_0 + [\alpha_1 + i\gamma_1]\tilde{X}U_0 + [\alpha_2 + i\gamma_2]\eta U_0$$

$$+ i\hat{\lambda}\gamma_3 \, \tilde{X}^2 \, U_0 + i\hat{\lambda}\gamma_4 \, \tilde{X}\eta \, U_0 + i\hat{\lambda}\gamma_5 \, \eta^2 \, U_0 = 0, \qquad (2.15c)$$

where the real constants α_0, α_1, etc. are defined by

$$\alpha_0 = \frac{u_{00} \, g_1 \, \kappa_0}{3} - \frac{2}{3} u_{00} \, \beta_{2r}, \qquad (2.16a)$$

$$\gamma_0 = -\frac{2}{3}[\sigma_2 + u_{00} \, \beta_{2i}] \qquad (2.16b)$$

$$\alpha_1 = \frac{1}{3} g_0 \kappa_0[u_{10} u_{01} + u_{11} u_{00}] + \frac{1}{3} g_0 \kappa_1 u_{00}, \tag{2.16c}$$

$$\gamma_1 = \frac{2}{3} \frac{\sigma_1 u_{10}}{u_{00}}, \tag{2.16d}$$

$$\alpha_2 = \frac{1}{3} g_0 \kappa_0[u_{10}^2 + 2u_{02} u_{00}], \tag{2.16e}$$

$$\gamma_2 = \frac{2}{3} \sigma_1 \frac{u_{01}}{u_{00}}. \tag{2.16f}$$

$$\gamma_3 = -\frac{2}{3} [w_{20} - \frac{u_{10}}{u_{00}} (w_{10} - \frac{w_{01}}{u_{01}}) - u_{20} \frac{w_{01}}{u_{01}}], \tag{2.16g}$$

$$\gamma_4 = -\frac{2}{3}[w_{11} - \frac{u_{11}}{u_{01}} w_{01} - \frac{u_{01}}{u_{00}} (w_{10} - \frac{u_{10} w_{01}}{u_{01}})], \tag{2.16h}$$

$$\gamma_5 = -\frac{2}{3} [w_{02} - \frac{u_{02}}{u_{01}} w_{01}]. \tag{2.16i}$$

Thus the vertical structure of the disturbance is determined by (2.15) and the most dangerous eigensolution is

$$U_0 = \exp - \frac{\sqrt{-i\hat{\lambda}\gamma_5}}{2} \{\eta - \frac{\alpha_2 + i\gamma_2}{2(i\hat{\lambda}\gamma_5)} + \frac{\gamma_4 \tilde{X} i\hat{\lambda}}{2(i\hat{\lambda}\gamma_5)}\}^2$$

whilst the corresponding eigenrelation becomes

$$\{\alpha_0 + i\gamma_0\} + \{\alpha_1 + i\gamma_1\}\tilde{X} + i\hat{\lambda}\gamma_3 \tilde{X}^2 - \frac{\{\alpha_2 + i\gamma_2 + i\hat{\lambda}\gamma_4 \tilde{X}\}^2}{4i\hat{\lambda}\gamma_5} - \sqrt{-i\hat{\lambda}\gamma_5} = 0.$$

We have assumed that the disturbances are neutral at $\tilde{X} = 0$ so that by taking $\beta_{2r} = \tilde{X} = 0$ the neutral value of g_1 is determined as

$$\frac{u_{00}\,\kappa_0\,g_1}{3} = \sqrt{\frac{\hat{\lambda}|\gamma_5|}{2} + \frac{\alpha_2\,\gamma_2}{2\hat{\lambda}\gamma_5}}\ .$$

Note that we have not needed to use the condition that $\bar{u}\,\bar{u}_Y$ has a maximum at $Y = \bar{Y}$ in the above calculation. However, since we are interested in the most dangerous modes we now impose this condition in order to produce a minimum value for g_0. In this case $\alpha_2 = 0$ and the above expression for g_1 simplifies to

$$g_1 = \sqrt{\frac{\hat{\lambda}|\gamma_5|}{2}}\left\{\frac{3}{u_{00}\,\kappa_0}\right\}$$

in which case the flow is locally neutrally stable at X_n if G expands as

$$G = \frac{1}{\kappa(X_n)u_{00}\,u_{01}}\left\{a^4 + 3a^{10/3}\sqrt{\frac{\hat{\lambda}|\gamma_5|}{2}}\,u_{01} + \cdots\right\}. \qquad (2.17)$$

Thus, the first correction term from the two-dimensional result is now $O(a^{10/3})$ and always has a stabilizing effect. If we replace $\hat{\lambda}$ by $\frac{\bar{\lambda}}{a^{5/3}}$ in (2.17), we obtain an asymptotic expansion of G in terms of $\bar{\lambda}$ and a which remains valid until the two terms shown in (2.17) are comparable. This occurs when $\bar{\lambda}$ is formally $O(a^3)$ and this is the regime we now consider. We shall show that at this stage the neutral value of the Görtler number can be shifted by an $O(1)$ amount from its two-dimensional value. Moreover, we shall see that at this stage the eigenfunctions essentially reduce to Orr-Sommerfeld ´centre modes´ destabilized by centrifugal effects.

c. <u>The limit $a \to \infty$ with $\bar{\lambda} \sim O(a^3)$, $G \sim O(a^4)$.</u>

We first note that if we wish to retain the critical layer structure of (b), with $\bar{\lambda}$ now replaced by $\hat{\lambda}\, a^3$, we must formally take $\partial/\partial X \sim O(a^4)$ in (2.2). It then follows from the continuity and momentum equations that this is possible only if $W \sim a^3$, $V \sim a^2$, $P \sim a^3$, with U again taken to be $O(1)$. Let us again assume that the location of the critical layer at the neutral value of $X = X_N$ is $Y = \bar{Y}$ and define stretched variables η and $\tilde{X}$ by

$$\eta = \{Y - \bar{Y}\}a, \qquad \tilde{X} = \{X - X_n\}a,$$

so that the vortices have their depth and wavelength comparable. The disturbance then expands as

$$U = \{U_0(\eta,\tilde{X}) + a^{-1}\, U_1(\eta,\tilde{X}) + \cdots\} \exp\{i\sigma t + a^3 \int^{\tilde{X}} \beta_0(\tilde{X}) + a^{-1}\,\beta_1(\tilde{X}) + \cdots\, d\tilde{X}\}$$

together with similar expansions for V/a^2, W/a^3, and P/a^3. We retain the expansions of κ, $\bar{u}$, $\bar{w}$ given in (b) but with $a^{-2/3}$ replaced by a^{-1}. Finally, the frequency and Görtler number expand as

$$\sigma = a^4\, \sigma_0 + a^3\, \sigma_1 + \cdots,$$

$$G = a^4\, g_0 + a^3\, g_1 + \cdots.$$

The eigenrelations (2.14a,b,c) are again taken to be satisfied so that $\partial/\partial t + \bar{u}(\partial/\partial X) + \hat{\lambda}\, i a^4\, \bar{w}$ expands as $a^2 M$ with M defined by (2.13).

We first consider the equation of continuity, which at order a^4, a^3 yields

$$\beta_0 U_0 + iW_0 = 0,$$

$$\beta_0 U_1 + iW_1 = -\beta_1 U_0 - \frac{\partial V_0}{\partial \eta} .$$

The second of these equations can be written in the form

$$i\tilde{W}_1 = -\beta_1 U_0 - \frac{\partial V_0}{\partial \eta} , \tag{2.18a}$$

if we first define $W_1 = i\beta_0 U_1 + \tilde{W}_1$. The three momentum equations then yield

$$LU_0 = u_{01} V_0, \tag{2.18b}$$

$$LV_0 = \frac{\partial P_0}{\partial \eta} + g_0 \kappa_0 u_{00} U_0, \tag{2.18c}$$

$$L\tilde{W}_1 = iP_0 + \hat{\lambda} V_0 \{2\gamma_6 \eta + \gamma_7 \tilde{X}\} , \tag{2.18d}$$

where $\gamma_6 = 2\{w_{02} - \frac{u_{02}}{u_{01}} w_{01}\}$, $\gamma_7 = w_{11} - \frac{u_{11}}{u_{01}} w_{01}$, and $L \equiv \frac{\partial^2}{\partial \eta} - 1 - M$ with M defined by (2.13). The system (2.18), together with the conditions U_0, V_0, $\tilde{W}_1 \to 0$ when $|\eta| \to \infty$, specifies an eigenvalue problem for $\beta_{2r} + i\beta_{2i}$ if $\hat{\lambda}$, σ_2, and $\tilde{X}$ are given. For simplicity, we consider only the neutral case $\beta_{2r} = 0$ so that we can set $\tilde{X} = \beta_{2r} = 0$ in (2.18). Furthermore, we will discuss the solution of (2.18) only for $\sigma_1 = \sigma_2 = 0$, we do not prove that this will lead to the most dangerous modes for all $\hat{\lambda}$, but it is certainly the case for $\hat{\lambda} << 1$. The system (2.18) can then be simplified to

$$\left\{\frac{\partial^2}{\partial\eta^2} - 1 - u_{00}\,\beta_2 - i\hat{\lambda}\gamma_6\,\eta^2\right\}\left\{\frac{\partial^2}{\partial\eta^2} - 1\right\} V_0 + 2i\hat{\lambda}\gamma_6\,V_0 = -\,g_0\,\kappa_0\,u_{00}\,U_0, \quad (2.19a)$$

$$\left\{\frac{\partial^2}{\partial\eta^2} - 1 - u_{00}\,\beta_2 - i\hat{\lambda}\gamma_6\,\eta^2\right\}U_0 = u_{01}\,V_0. \quad (2.19b)$$

Here $\beta_{2r} = 0$ and β_{2i}, g_0 must be chosen such that (2.19a,b) are satisfied, and with U_0, $V_0 \to 0$, $\eta \to \pm\infty$. It is clear from (2.19) that $g_0(-\hat{\lambda}) = g_0(\hat{\lambda})$, $\beta_2(-\hat{\lambda}) = \overline{\beta}_2(\hat{\lambda})$ so without any loss of generality we shall from now on assume that $\hat{\lambda}\gamma_6 > 0$. It is convenient at this stage to note that the eigenrelation for g_0 can be written

$$g_0\,\kappa_0\,u_{00}\,u_{01} = \mathscr{N}(\gamma_6\,\hat{\lambda}), \quad (2.19c)$$

where $\mathscr{N}(\gamma_6\,\hat{\lambda})$ must be calculated numerically. However, if we let $\hat{\lambda} \to 0$ in (2.19), we obtain

$$\mathscr{N}(\gamma_6\,\hat{\lambda}) \to 1 + 0(\hat{\lambda})^{2/3},$$

which is consistent with the results of (b). Thus for $a \gg 0(\hat{\lambda}^{1/3})$

$$g_0\,\kappa_0\,u_{00}\,u_{01} \sim 1,$$

which is, of course, the two-dimensional result. When $\gamma_6\,\hat{\lambda} \to \infty$, it can be shown that the functions U_0 and V_0 have a double boundary layer structure corresponding to $\eta = 0(1)$ and $\eta = 0(\hat{\lambda}\gamma_6)^{-1/4}$. In this limit we write

$$\mathscr{N} = c(\hat{\lambda}\gamma_6)^{3/2} + \cdots$$

$$u_{00}\,\beta_2 = id(\hat{\lambda}\gamma_6)^{1/2} + \cdots$$

and define $\zeta = (\hat{\lambda}\gamma_6)^{+1/4}\,\eta$. In the inner region where $\zeta = 0(1)$, we write

$$\tilde{U}_0 = \tilde{U}_0^0(\zeta) + 0(\hat{\lambda}\gamma_6)^{-1/2},$$

$$\tilde{V}_0 = \frac{(\hat{\lambda}\gamma_6)^{1/2}}{u_{01}}\,\tilde{V}_0^0(\zeta) + 0(1),$$

and then $\tilde{U}_0$, $\tilde{V}_0$ satisfy

$$\left\{\frac{\partial^2}{\partial\zeta^2} - id - i\zeta^2\right\}\frac{\partial^2}{\partial\zeta^2}\tilde{V}_0^0 + 2i\tilde{V}_0^0 = -c\,\tilde{U}_0^0,$$
$$\left\{\frac{\partial^2}{\partial\zeta^2} - id - i\zeta^2\right\}\tilde{U}_0^0 = \tilde{V}_0^0. \tag{2.20}$$

In the outer region, where $\eta \sim 0(1)$, the first approximation to (2.19) reduces to a second-order differential equation for V_0 independent of U_0 and a linear equation which involves U_0 and V_0. The solution of these equations in terms of Bessel functions shows that the condition U_0, $V_0 \to 0$ when $\eta \to \infty$ leads to the following matching conditions on (2.20):

$$V_0^0 \sim \frac{1}{\zeta}, \quad \tilde{U}_0^0 \sim \frac{i}{\zeta^3}, \quad \zeta \to \infty. \tag{2.21}$$

The most dangerous modes of (2.20) are even in ζ so we solve (2.20) subject to (2.21) and

$$\tilde{U}_0^{0\prime} = \tilde{U}_0^{0\prime\prime\prime} = \tilde{V}_0^{0\prime} = 0, \zeta = 0. \tag{2.22}$$

The eigenvalues c,d associated with (2.20), (2.21), and (2.22) were found by a shooting procedure using (2.21) and the two other independent but exponentially decaying solutions of (2.20) which can be found for $|\zeta| >> 1$. The three independent solutions were integrated from a suitably large value of ζ to $\zeta = 0$ where they were combined to satisfy (2.22). The least stable mode corresponds to

$$c = 4.71, \quad d = -2.89.$$

For intermediate values of $\gamma_6 \hat{\lambda}$, $\mathscr{N}$ and β_{2i} were calculated numerically by solving (2.19) using a shooting procedure with the three independent exponentially decaying solutions of these equations. Figures 1 and 2 illustrates the results of such a calculation for the least stable mode which is an even function of η. In these figures we have also shown the one-term asymptotic approximations to the neutral curves valid for $\gamma_6 \hat{\lambda} >> 1$. The monotonic increase of the function $\mathscr{N}$ has a profound and unexpected influence on the nature of the neutral curve. We shall now show that for $\bar{\lambda} >> 1$ a neutral curve with a minimum value for the Görtler number is predicted by our calculations.

In order to see why this is the case we first note that (2.19c) written in terms of G and $\bar{\lambda}$ gives

$$G\kappa_0 \, u_{00} \, u_{01} = a^4 \mathscr{N}\left(\frac{\gamma_6 \bar{\lambda}}{a^3}\right), \tag{2.23}$$

and the previously described asymptotic structure for $\mathscr{N}$ implies that

$$G\kappa_0 \, u_{00} \, u_{01} \sim a^4, \qquad \bar{\lambda}^{-1/3} << a,$$

$$G\kappa_0 \, u_{00} \, u_{01} \sim \frac{4.71(\bar{\lambda}\gamma_6)^{3/2}}{a^{1/2}}, \quad \bar{\lambda}^{-1/3} >> a.$$

Thus if $\bar{\lambda}$ is held fixed, the Görtler number decreases or increases with the wavenumber, depending on the size of $a/\bar{\lambda}^{1/3}$. Hence, there will be a minimum value of G at some $O(1)$ value of $a/\bar{\lambda}^{1/3}$. In fact, it is easily shown from (2.23) that this occurs when

$$\mathcal{N}'(\gamma_6 \hat{\lambda}) = \frac{4}{3\gamma_6 \hat{\lambda}} \mathcal{N}(\gamma_6 \hat{\lambda}),$$

and our calculations show that this occurs where $\gamma_6 \hat{\lambda} = 3.18$ and the minimum value of G is given by

$$G \kappa_0 u_{00} u_{01} = 3.88(\gamma_6 \bar{\lambda})^{4/3},$$

and then

$$a = .68(\gamma_6 \bar{\lambda})^{1/3}.$$

In Figure 3 we have shown the dependence of $G \kappa_0 u_{00} u_{01}$ on a implied by (2.23) for several values of $\bar{\lambda}\gamma_6$. We note that for a particular three-dimensional boundary layer the quantities κ_0, u_{00}, u_{01},γ_6 must be calculated for a particular choice of $(X_n, \bar{Y})$ and then (2.23) and Figure 1 can be used to generate the neutral curve for the required values of $\bar{\lambda}$. The crucial result shown in Figure 3 is that for $|\bar{\lambda}| \gg 1$ a minimum Görtler number exists for a disturbance concentrated at a fixed height above the wall. The eigenfunctions associated with some of the results of Figures 1 and 2 are shown in Figures 4 and 5. We note that the vortices spread further away from the wall when $\hat{\lambda}\gamma_6$ increases. Suppose then that we have a three-dimensional boundary layer flow with $\bar{\lambda} \gg 1$ but fixed. For any given height $\bar{Y}$ of the critical layer we have shown that neutral disturbances occur for $G > O(\bar{\lambda}^{4/3})$ with $a \sim \bar{\lambda}^{1/3}$. This means that the Görtler mechanism in a three-dimensional boundary layer with $\bar{\lambda} >> 1$ is not operational at $O(1)$ values of the Görtler number. Since the crossflow on a laminar flow wing is certainly large compared to $R^{-1/2}$, with R scaled on the chordwise velocity, we have effectively shown that the Görtler mechanism is almost certainly unimportant in such a flow. Thus, it appears likely that crossflow instabilities associated with inflection points in the basic flow velocity component in particular directions will be most important in determining the state of the boundary layer.

It is interesting to note that if U_0 is set equal to zero in (2.19), then the equation for V_0 reduces to the rescaled Orr-Sommerfeld equation for a two-dimensional Tollmien-Schlichting wave, having its critical layer located where the velocity field has a local maximum. Such ´centre modes´ are, of course, stable and have a wave-packet structure in the Y direction similar to that discussed by Tatsumi and Gotoh (1969). Thus, we interpret the neutral modes found above as Tollmien-Schlichting waves destabilized by centrifugal effects.

3. THE DEGENERATE CASE OF FLOWS WITH ZERO PRESSURE GRADIENT

The analysis of Section 2 does not apply if $\bar{u}$ and $\bar{w}$ are linearly related as is the case when the pressure gradient in the boundary layer vanishes. Suppose that this is the case and the basic flow is written in the form

$$\underset{\sim}{u} = \left(\bar{u}(X,Y), R^{-1/2}\,\bar{v}(X,Y), c\bar{u}(X,Y)\right)\left(1 + O(R^{-1}\ln R)\right),$$

where c is a constant and X,Y are as defined in Section 2, whilst $\bar{u}$ and $\bar{v}$ are determined in terms of the Blasius function $F\left(\frac{Y}{\sqrt{X}}\right)$. We perturb the above flow such that the disturbance velocity field is periodic in the X and Z directions, but with the wavefronts parallel to the basic flow. Thus we write the perturbation velocity field $\underset{\sim}{u}'$ as

$$\frac{\underset{\sim}{u}'}{U_0} = \left(UE,\ R^{-1/2}\,VE,\ cUE + R^{-1/2}\,\tilde{W}E(1 + c^2)^{1/2}\right)\left(1 + O(R^{-1/2})\right)$$

where $E = \exp i[aZ - cR^{1/2}\,Xa]$. The $O(1)$ contribution to the Z component of the disturbance velocity field has been inserted in order that the component of the disturbance velocity field in a direction perpendicular to the basic flow direction is $O(R^{-1/2})$. We also assume that the perturbation is independent of time and the pressure perturbation $P(X,Y)$ corresponding to the disturbed flow is scaled on $R^{-1}\,\rho\,U_0^2$. In order to determine the equations satisfied by U, V, $\tilde{W}$ and P, it is convenient to use the X,Y momentum equations and the Z momentum equation subtracted from the X momentum equation multiplied by c. The particular X-dependence of the perturbation we have chosen means that $\partial/\partial X$ in these equations and the continuity equation should be replaced by $-cR^{1/2}\,ia + \frac{\partial}{\partial X}$. After some manipulation it can be shown that the equations satisfied by U, V, $\tilde{W}$ and P are

$$U_X + V_Y + ia\tilde{W}\sqrt{1 + c^2} = 0$$

$$U\bar{u}_X + \bar{u}U_X + \bar{v}V_Y + V\bar{u}_Y = U_{YY} - a^2U[1 + c^2]$$

$$\bar{u}V_X + U\bar{v}_X + \bar{v}V_Y + V\bar{v}_Y + G\kappa(X)\bar{u}U = -P_Y + V_{YY} - a^2\ V[1 + c^2]$$

$$\bar{u}\tilde{W}_X + \bar{v}\tilde{W}_Y = -iaP[1 + c^2]^{1/2} + \tilde{W}_{YY} - a^2\ \tilde{W}[1 + c^2].$$

The above equations are in fact identical to those appropriate to the two-dimensional boundary layer flow $(\bar{u}, R^{-1/2}\bar{v}, 0)$ if we identify $\sqrt{1 + c^2}\ a$ as the 'total' wavenumber of the perturbation. Hence, the two-dimensional results of I, II are applicable and no further calculation is necessary. We note, however, that in this case, the vortices are aligned with the basic flow and that the effective Görtler number is determined by the component of the basic flow normal to the leading edge.

4. CONCLUSION

Let us first consider the implications of the results of the last section which are appropriate to boundary layers with zero pressure gradient. Suppose then that we have a Blasius boundary layer over a wall of variable curvature with the leading edge perpendicular to the flow direction at infinity. The Görtler vortex instability mechanism described in I, II is operational in such a flow and the manner in which the instability develops in the streamwise direction depends on the Görtler number G and the wavenumber a. If the wavenumber a is not large, this development also has a strong dependence on the form of the initial disturbance and the concept of a unique neutral curve is not tenable. Suppose next that the leading edge is inclined at an angle α to the oncoming flow. The basic flow again consists of a Blasius boundary layer with the velocity field in the plane of the wall parallel to the flow direction at infinity. We choose to look for a Görtler vortex instability with the vortices parallel to the flow at infinity and define the Görtler number with respect to the component of the flow velocity at infinity perpendicular to the leading edge. The disturbance equations which we found in Section 3 are identical to those appropriate to the two-dimensional flow obtained by neglecting the crossflow. Thus, when the leading edge is turned through an angle α, the vortices remain parallel to their original direction. However, their development in the streamwise direction will vary with α, since the instability is governed by the Görtler number scaled on the velocity component perpendicular to the leading edge.

Let us now turn to the results of Section 3 which correspond to the flow over a wall swept at angle $\alpha \sim O(R^{-1/2})$ to the oncoming flow. The results we have found are quite general and it was not necessary for us to be specific about the form of the basic boundary layer flow. The results apply to, for example, the situation where the zero sweep solution is given by the Falkner-Skan solutions or to the case when the basic three-dimensional flow is given by the Sowerby-Loos solution (see Rosehead (1963)) which is, of course, relevant to the flow over a rotating blade.

The results of Section 2 were obtained by formally taking the crossflow velocity field to be $O(R^{-1/2})$ and then considering various asymptotic limits involving a, G and $\overline{\lambda}$. We have concentrated our attention on the limit $\overline{\lambda} \to \infty$, since we are interested in finding the structure of the instability for crossflow velocity field large compared to $R^{-1/2}$. We have shown that for a three-dimensional boundary layer, the most dangerous modes are time-dependent and the orientation of the vortices in the neutral location is fixed by the vortex lines of the basic flow and not by the streamlines, as we would expect based on the results for the two-dimensional problem. The most surprising result we have found is that for the three-dimensional problem with $\overline{\lambda} >> 1$, the large wavenumber asymptotics developed in I produces a neutral curve with distinct left- and right-hand branches and a minimum Görtler number. Each such neutral curve corresponds to a disturbance with its critical layer fixed at a given height above the wall.

In essence we have shown that, since the minimum value of the Görtler number on any such curve is $O(\overline{\lambda}^{4/3})$, the Görtler vortex instability mechanism is almost certainly unimportant in the concave region of laminar flow swept wings where the angle of sweep is large compared to $R^{-1/2}$. Therefore it would appear that crossflow or Tollmien-Schlichting wave instabilities (which occur for Reynolds number typically of order 100) are more likely to lead to the breakdown of laminar flow.

We can, in fact, be more precise about the angle of sweep at which the Görtler vortex mechanism ceases to be important. This can be done by noting that the Görtler number was initially defined in terms of the two independent parameters δ, and R, so that $G >> 1$ implies that

$$\delta >> R^{-1/2}.$$

Thus, the analysis of Section 2c can be repeated by writing the disturbance equations in terms of R, δ and $\overline{\lambda}$ but with

$$\delta \sim R^{-p}, \qquad p > 0$$

$$\overline{\lambda} \sim R^{+q}, \qquad q > 0.$$

The whole expansion procedure can then be reformulated in terms of R as the large parameter, and we can determine for what range of values of p and q the exact structure of Section 2c is recovered. Such a calculation shows that the results of Section 2c remain valid for

$$q < 3/8,$$

so that the critical Görtler number is asymptotically large whenever the crossflow is $O(R^{-1/8})$. There seems little doubt, therefore, that for $O(1)$ angles of sweep, the Görtler vortex mechanism is not significant in the Reynolds number regimes where crossflow and Tollmien-Schlichting instabilities occur.

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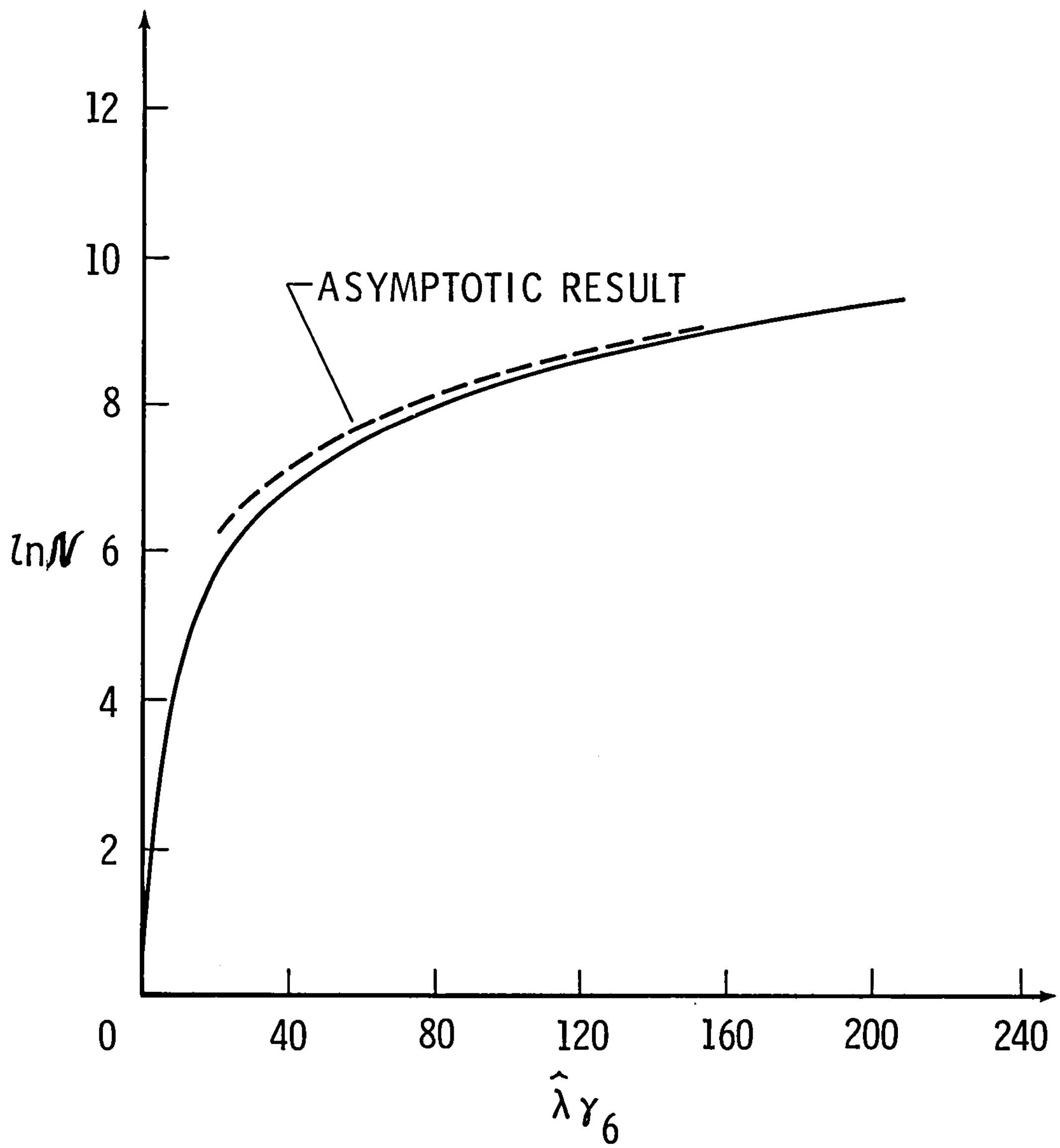

Figure 1. The dependence of $\mathcal{N}$ on $\hat{\lambda}\gamma_6$, the dotted curve corresponds to the one-term asymptotic expansion of $\mathcal{N}$ valid for $\hat{\lambda}\gamma_6 \gg 1$.

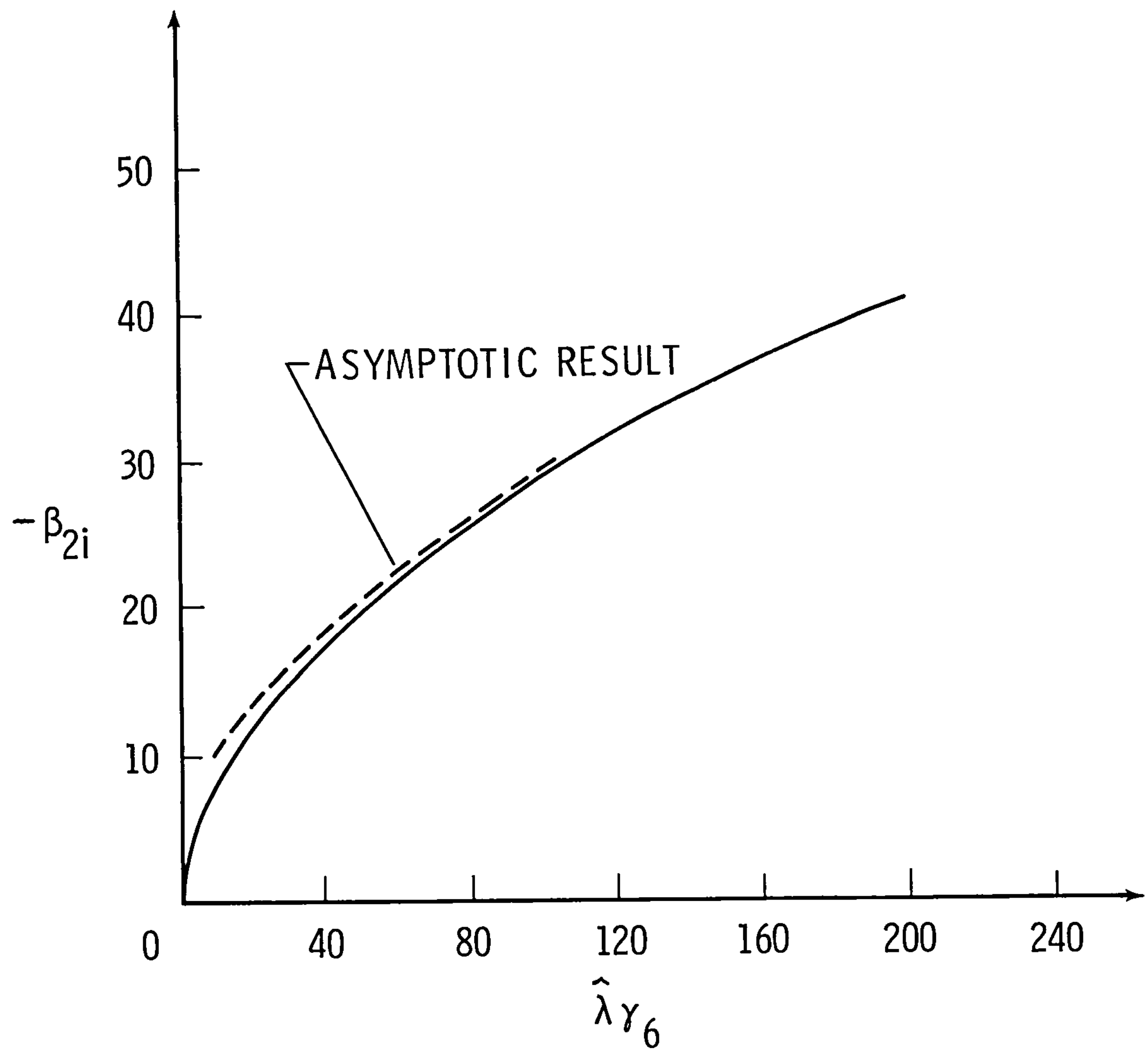

Figure 2. The dependence of β_{2i} on $\hat{\lambda}\gamma_6$, the dotted curve corresponds to the one-term asymptotic expansion of β_{2i} valid for $\hat{\lambda}\gamma_6 \gg 1$.

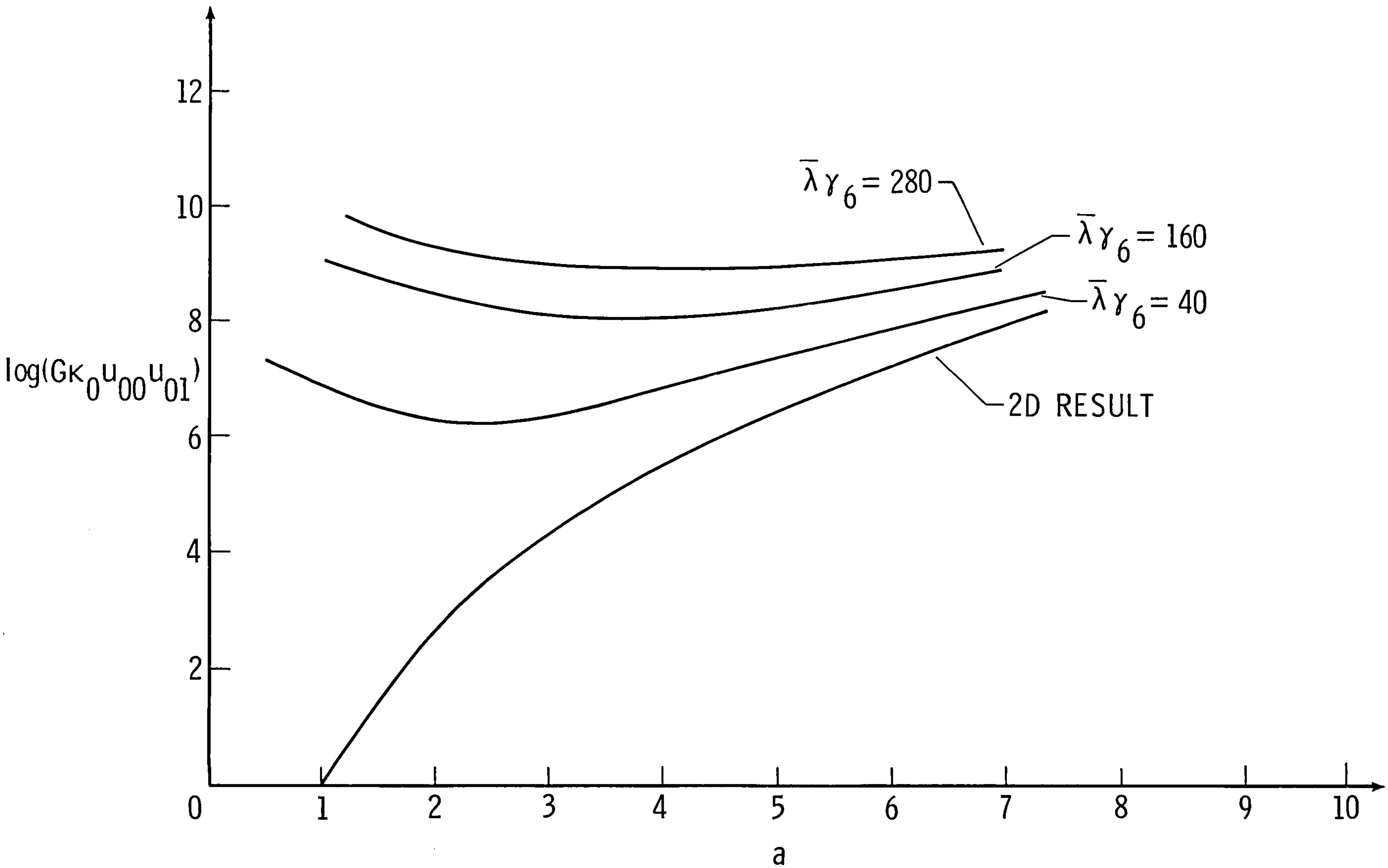

Figure 3. The neutral curves for $\bar{\lambda} \sim a^{1/3}$ for different values of $\bar{\lambda}$.

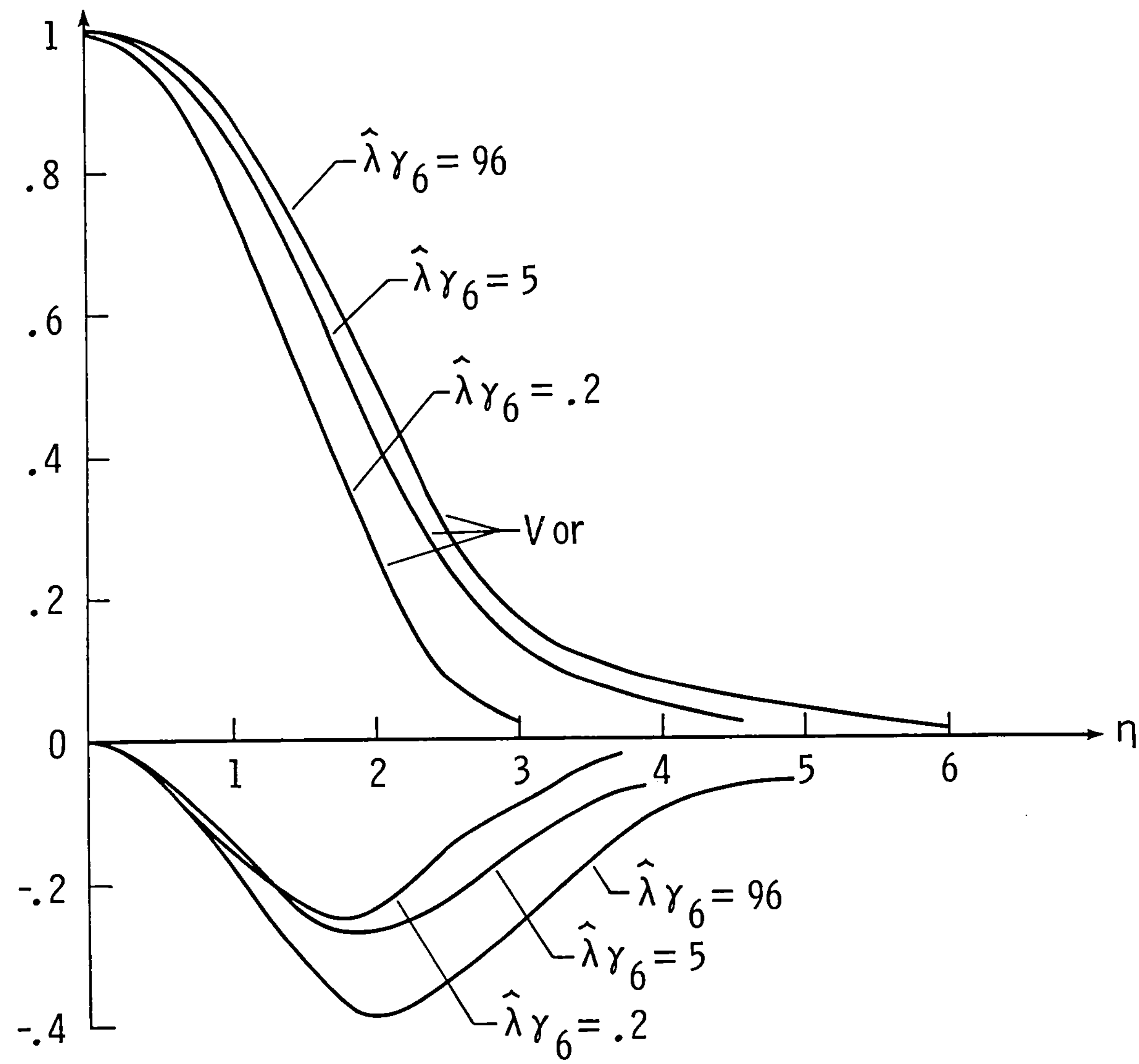

Figure 4. The eigenfunctions V_0 for different values of $\hat{\lambda}\gamma_6$, the functions are scaled such that $V_0(0) = 1$.

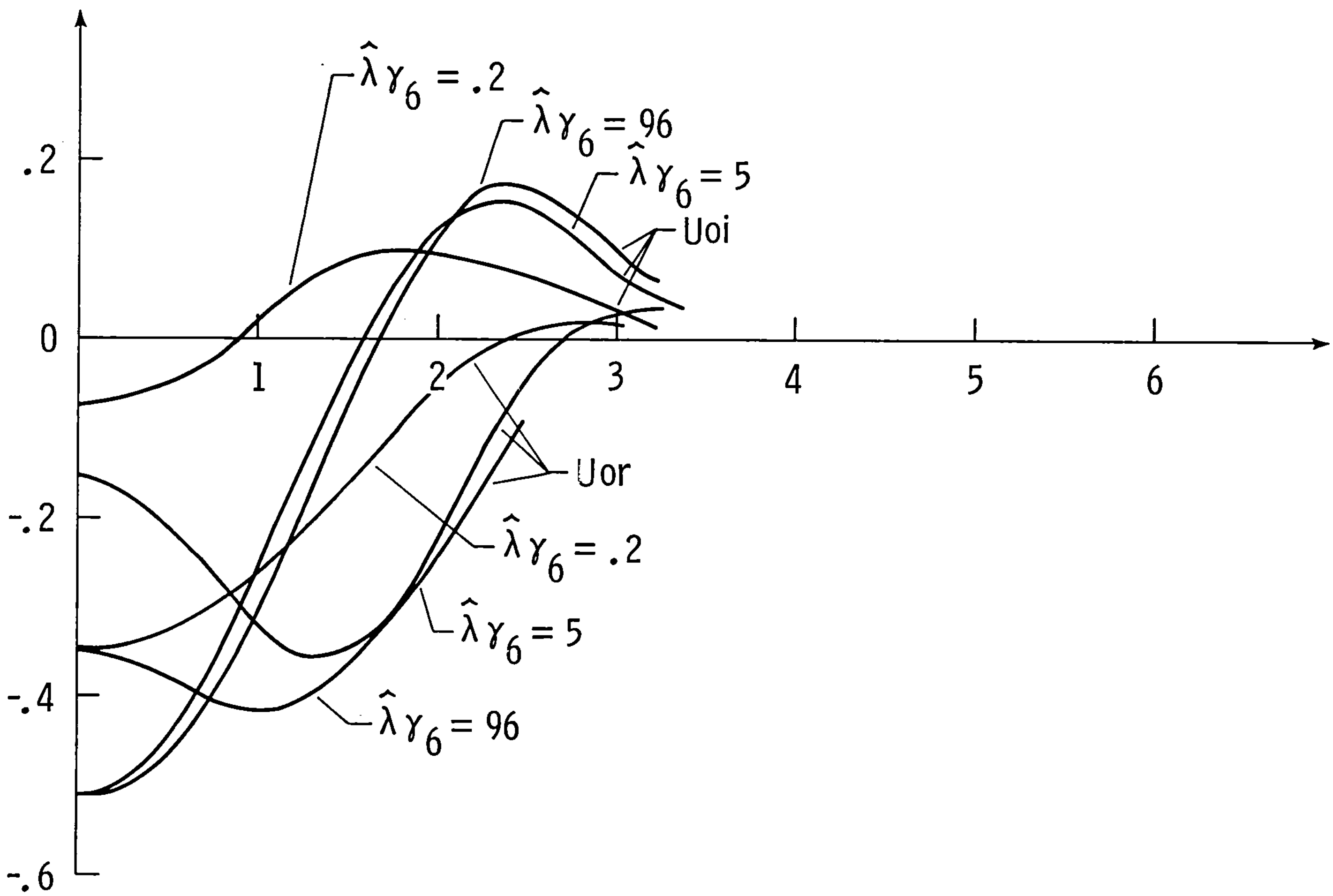

Figure 5. The eigenfunctions U_0 for diffferent values of $\hat{\lambda}\gamma_6$.

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16. Abstract

It is well known that the two-dimensional boundary layer on a concave wall is centrifugally unstable with respect to vortices aligned with the basic flow for sufficiently high values of the Görtler number. However, in most situations of practical interest the basic flow is three-dimensional and previous theoretical investigations do not apply. In this paper the linear stability of the flow over an infinitely long swept wall of variable curvature is considered. If there is no pressure gradient in the boundary layer it is shown that the instability problem can always be related to an equivalent two-dimensional calculation. However, in general, this is not the case and even for small values of the crossflow velocity field dramatic differences between the two and three-dimensional problems emerge. In particular, it is shown that when the relative size of the crossflow and chordwise flow is $O(R^{-1/2})$, where R is the Reynolds number of the flow, the most unstable mode is time-dependent. When the size of the crossflow is further increased, the vortices in the neutral location have their axes locally perpendicular to the vortex lines of the basic flow. In this regime the eigenfunctions associated with the instability become essentially ´centre modes´ of the Orr-Sommerfeld equation destabilized by centrifugal effects. The critical Görtler number for such modes can be predicted by a large wavenumber asymptotic analysis; the results suggest that for order unity values of the ratio of the crossflow and chordwise velocity fields, the Görtler instability mechanism is almost certainly not operational.

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