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A NUMERICAL SCHEME FOR THE IDENTIFICATION
OF HYBRID SYSTEMS DESCRIBING THE VIBRATION
OF FLEXIBLE BEAMS WITH TIP BODIES

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A Numerical Scheme for the Identification of Hybrid Systems Describing the Vibration of Flexible Beams with Tip Bodies*

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Abstract

A cubic spline based Galerkin-like method is developed for the identification of a class of hybrid systems which describe the transverse vibration of flexible beams with attached tip bodies. The identification problem is formulated as a least squares fit to data subject to the system dynamics given by a coupled system of ordinary and partial differential equations recast as an abstract evolution equation (AEE) in an appropriate infinite dimensional Hilbert space. Projecting the AEE into spline-based subspaces leads naturally to a sequence of approximating finite dimensional identification problems. The solutions to these problems are shown to exist, are relatively easily computed, and are shown to, in some sense, converge to solutions to the original identification problem. Numerical results for a variety of examples are discussed.

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1. INTRODUCTION

In this paper we develop an approximation scheme for the identification of systems describing the planar transverse vibration of beams with attached tip bodies. Standard models from the theory of elasticity for the vibration of structures of this type involve hybrid systems of coupled partial and ordinary differential equations which describe the motion of the beam and tip bodies respectively. The approximation scheme is based upon the formulation of the identification problem as a least squares fit to data subject to the dynamical equations recast as an abstract evolution equation in an infinite dimensional Hilbert space. Using a cubic spline based Galerkin method, a sequence of successively higher (but finite) dimensional state approximations are constructed. This leads naturally to a sequence of approximating identification problems, each of which is shown to have a solution that can readily be computed using standard numerical techniques. Results from linear semigroup theory and the theory of evolution operators are used to demonstrate convergence of the state approximation. This in turn is used to argue that solutions to the finite dimensional identification problems, in some sense, approximate solutions to the original identification problem. Our effort here is similar in spirit to the approach taken in [1], [2], [3], [7], and [11] wherein approximation schemes for the estimation of parameters in beam equations with standard boundary conditions (i.e., clamped, simply supported, free, etc.) are developed. Our work is based to a large extent on the ideas suggested in the short note by Burns and Cliff [5].

Although our general approach is applicable to a broad class of problems (see Section 4), to illustrate our method we consider a beam, clamped at one end and cantilevered at the other with an attached tip body. In Section 2 the derivation of the equations of motion for the beam/tip body system is outlined, the equivalent abstract evolution equation is derived and the identification problem is formulated. In Section 3 the approximation scheme is constructed and convergence results are discussed. Numerical results for several examples are presented in Section 4.

Our notation is, for the most part, standard. The usual Sobolev spaces of real-valued functions on the interval [a, b] whose k^{th} derivatives are L^2 are denoted by $H^k(a, b)$. These spaces are assumed to be endowed with the

usual Sobolev inner products $\langle\cdot, \cdot\rangle_k$ and their induced norms $|\cdot|_k$. For Z a normed linear space with norm $|\cdot|_Z$ and $f : [0, T] \to Z$ we say that $f \in L^2([0, T], Z)$ if $\int_0^T |f(t)|_Z^2 \, dt < \infty$. Similarly f will be said to be an element of $C([0, T], Z)$ if the map $t \to f(t)$ from $[0, T]$ into Z is ℓ time continuously differentiable on $(0, T)$. Finally for a function of one or more real variables, the symbol $D_\theta f$ ($D_\theta^k f$) will be used to denote the 1st (k^{th}) derivative of f with respect to the independent variable θ. If f is a function of a single variable only, the subscript may be omitted. On occasion, the short-hand notation $D_\theta f(\theta_0)$ or $Df(\theta_0)$ will be used in place of $D_\theta f|_{\theta_0}$ or $Df|_{\theta_0}$ to denote the derivative of f evaluated at θ_0.

2. THE PARTIAL DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS, THEIR ABSTRACT FORMULATION, AND THE IDENTIFICATION PROBLEM

We consider (see Fig. 2.1) an inextensible beam of length ℓ, having spatially dependent linear mass density ρ and flexural stiffness EI. The tip body is assumed to be of mass m, have mass center at a distance c from the end of the beam directed at an angle δ measured from the extension of the longitudinal axis of the beam and having moment of inertia J about its center of mass.

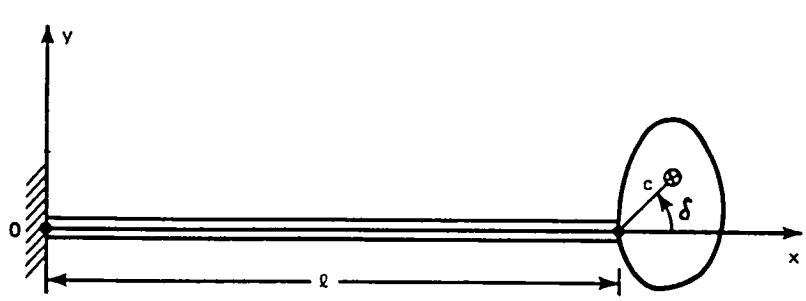

Figure 2.1.

Assuming small deformations ($|u(t, x)| << \ell$), using the standard Euler-Bernoulli theory (neglecting rotatory inertia and shear deformations) and elementary Newtonian mechanics the equation describing the verticle displacement $u(t, x)$ of the beam at position $x \in [0, \ell]$ at time $t > 0$

$$\rho D_t^2 u = -D_x^2 EID_x^2 u + D_x \tau D_x u + f \tag{2.1}$$

is obtained where $\tau(t, x)$ is the internal tension resulting from loads directed parallel to the beam's longitudinal axis and $f(t, x)$ describes effects due to lateral or transverse loading and/or rigid-body rotations (see [6], [14], [17]).

If we let S denote the shear force and M the bending moment then using the standard moment equilibrium equation for a beam under tension

$$S = \tau D_x u - D_x M,$$

the basic bending moment-curvature relationship from the Euler-Bernoulli theory

$$M = EID_x^2 u,$$

and the equations for the translational motion of the tip body we obtain the first boundary condition at $x = \ell$

$$\begin{aligned} mD_t^2 u(t, \ell) + mc \cos \delta \, D_t^2 D_x u(t, \ell) &= g_1(t) + D_x EI(\ell) D_x^2 u(t, \ell) \\ &\quad - \tau(t, \ell) D_x u(t, \ell), \end{aligned} \tag{2.2}$$

where g_1 describes the net translational effects on the tip body's center of mass which result from externally applied lateral loads and moments (see [18]). The second boundary conditon at $x = \ell$, derived from the equations for the rotational motion of the tip body is given by

$$\begin{aligned} JD_t^2 D_x u(t, \ell) &= - c \cos \delta \, D_x \, EI(\ell) D_x^2 u(t, \ell) - EI(\ell) D_x^2 u(t, \ell) \\ &\quad - c \sin \delta \, \tau(t, \ell), + g_2(t), \end{aligned} \tag{2.3}$$

where g_2 is defined analogously to g_1 with regard to rotational effects (see [18]).

The boundary condtions at the clamped end, $x = 0$, are of course, given by

$$u(t, 0) = 0 \qquad D_x u(t, 0) = 0, \tag{2.4}$$

while the temporal boundary conditions (initial conditions) are of the form

$$u(0, x) = \phi(x) \qquad D_t u(0, x) = \psi(x). \tag{2.5}$$

The equations (2.1) and (2.2) as they are written above are, in fact, nonlinear. Indeed, the internal tension $\tau(t, x)$ is the sum of any externally applied loads $\sigma(t, x)$ which are directed parallel to the longitudinal axis and the axially directed force $mc \sin \delta \, D_t^2 D_x u(t, \ell)$ which results from the angular acceleration of the tip body (see [18]). Discarding the nonlinear terms in (2.1) and (2.2) as second order effects and choosing $w_1(t) = D_t u(t, \ell)$, $w_2(t) = D_t D_x u(t, \ell)$, $w_3(t, x) = D_x^2 u(t, x)$, and $w_4(t, x) = D_t u(t, x)$ we rewrite (2.1), (2.2), (2.3), and (2.5) in state space form as

$$D_t \vec{w}(t, x) = \begin{bmatrix} \alpha D_x EI(\ell) w_3(t, \ell) + \beta EI(\ell) w_3(t, \ell) - \frac{1}{m} \sigma(t, \ell) \int_0^\ell w_3(t, \theta) d\theta \\ -\beta D_x EI(\ell) w_3(t, \ell) - \gamma EI(\ell) w_3(t, \ell) \\ D_x^2 w_4(t, x) \\ -\frac{1}{\rho} D_x^2 EI w_3(t, x) + \frac{1}{\rho} D_x \sigma(t, x) \int_0^x w_3(t, \theta) d\theta \end{bmatrix}$$

$$+ \begin{bmatrix} \mu\sigma(t, \ell) + \frac{1}{m} g_1(t) - \beta g_2(t) \\ \lambda\sigma(t, \ell) + \gamma g_2(t) \\ 0 \\ \frac{1}{\rho} f(t, x) \end{bmatrix} \tag{2.6}$$

$$\vec{w}(0,\ x) = \begin{bmatrix} \psi(\ell) \\ D_x\psi(\ell) \\ D_x^2\phi(x) \\ \psi(x) \end{bmatrix} \tag{2.7}$$

where

$$\alpha = \frac{J + mc^2}{mJ + m^2c^2 \sin^2 \delta},$$

$$\beta = \frac{c \cos \delta}{J + mc^2 \sin^2 \delta},$$

$$\gamma = \frac{1}{J + mc^2 \sin^2 \delta},$$

$$\mu = -\frac{mc^2 \cos \delta \sin \delta}{J + mc^2 \sin^2 \delta},$$

$$\lambda = -\frac{c \sin \delta}{J + mc^2 \sin^2 \delta},$$

and

$$\vec{w}(t,\ x) = \left(w_1(t),\ w_2(t),\ w_3(t,\ x),\ w_4(t,\ x)\right)^T.$$

Recalling (2.4) displacement, $u(t,\ x)$ is recovered from $\vec{w}(t,\ x)$ by

$$u(t,\ x) = \int_0^x \int_0^\theta w_3(t,\ \tau)\ d\tau\ d\theta.$$

The identification problem which we shall consider involves the estimation of the flexural stiffness EI, the mass density ρ, the externally applied forces

and moments in the form of σ, f, g_1, g_2, and the initial conditions ϕ and ψ. Although (laying identifiability questions aside) our approximation and convergence results would be applicable to inverse problems involving the estimation of any or all of the parameters in (2.6) and (2.7), for ease of exposition, we assume that the rigid-body mass properties m, J, c, and δ of the tip body are known a priori. The identification problem is formulated as a least-squares fit to data. Our approach is based upon recasting (2.6) and (2.7) in terms of an abstract evolution equation.

Let Q be a subset of R^L and assume that the unknown temporally and/or spatially varying functions EI, ρ, σ, f, g_1, g_2, ϕ, and ψ appearing in (2.6) and (2.7) which are to be identified have been parameterized by $q \varepsilon Q$ (i.e., $EI(x) = EI(x; q)$, $\rho(x) = \rho(x; q)$, $\sigma(t, x) = \sigma(t, x; q)$, etc.). We require and assume throughout that the following assumptions hold:

A1: Q is a compact subset of R^L.

A2: The mappings $q \to EI(q)$ and $q \to \rho(q)$ are continuous from Q into $H^2(0, \ell)$ and Q into $H^1(0, \ell)$, respectively, and there exist positive constants m_{EI}, m_ρ, M_{EI}, M_ρ such that $m_{EI} \leq EI(q) \leq M_{EI}$, $m_\rho < \rho(q) < M_\rho$ for all $q \varepsilon Q$.

A3: The mappings $q \to \phi(q)$ and $q \to \psi(q)$ are continuous from Q into $H^2(0, \ell)$ and Q into $H^0(0, \ell)$, respectively.

A4: There exists a $T > 0$ such that the mapping $t \to \sigma(t, \cdot; q)$ is an element of $C^1([0, T], H^1(0, \ell))$ and the mapping $q \to \sigma(t, \cdot; q)$ is continuous from Q into $H^1(0, \ell)$ for each $t \varepsilon [0, T]$.

A5: The function f satisfies:

(i) The mapping $t \to f(t, \cdot; q)$ is an element of $L^2([0, T], H^0(0, \ell))$ for each $q \varepsilon Q$.

(ii) The mapping $q \to f(t, \cdot; q)$ is continuous from Q into $H^0(0, \ell)$ for each $t \varepsilon [0, T]$.

(iii) There exists $K_f \varepsilon L^2(0, T)$ independent of $q \varepsilon Q$ for which $\left| f(t, \cdot; q) \right|_0 \leq K_f(t)$ for all $q \varepsilon Q$ and $t \varepsilon [0, T]$.

A6: The functions g_i, $i = 1, 2$ satisfy

(i) $g_i \varepsilon L^2(0, T)$ for each $q \varepsilon Q$.

(ii) The mappings $q \to g_i(t; q)$ are continuous from Q into R for each $t \varepsilon [0, T]$.

(iii) There exist $K_{g_i} \varepsilon L^2(0, T)$ independent of $q \varepsilon Q$ for which $|g_i(t; q)| \leq K_{g_i}(t)$ for all $q \varepsilon Q$ and $t \varepsilon [0, T]$.

Let $Z = R^2 \times H^0(0, \ell) \times H^0(0, \ell)$ and for each $q \varepsilon Q$ let Z_q denote the Hilbert space $\{Z, \langle \cdot, \cdot \rangle_q\}$ where

$$\langle (r_1, u_1, v_1), (r_2, u_2, v_2) \rangle_q = r_1^T W r_2 + \int_0^\ell EI(q) u_1 u_2 + \int_0^\ell \rho(q) v_1 v_2$$

with

$$W = \begin{bmatrix} m & mc \cos \delta \\ mc \cos \delta & J + mc^2 \end{bmatrix}.$$

The definition of $\langle \cdot, \cdot \rangle_q$ is motivated by an energy expression. Indeed the sum of the kinetic and strain energies for the system described by (2.6) and (2.7) is given by

$$\frac{1}{2} [w_1(t), w_2(t)] W \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + \frac{1}{2} \int_0^\ell \rho w_4(t, x)^2 \, dx + \frac{1}{2} \int_0^\ell EI w_3(t, x)^2 \, dx$$

(see [18]).

Define $A_0(q) : D_0 \subset Z_q \to Z_q$ by

$$D_0 = \{(r, u, v) \varepsilon Z: \ u, v \varepsilon H^2(0, \ell), \ v(0) = Dv(0) = 0, \ r = (v(\ell), Dv(\ell))^T\},$$

$$\begin{aligned} A_0(q)\big((v(\ell), Dv(\ell))^T, u, v\big) = \big(& (\alpha DEI(\ell; q) u(\ell) + \beta EI(\ell; q) u(\ell), \\ & - \beta DEI(\ell; q) u(\ell) - \gamma EI(\ell; q) u(\ell))^T, D^2 v, \\ & - \frac{1}{\rho(q)} D^2 EI(q) u \big). \end{aligned} \tag{2.8}$$

For each $t\varepsilon[0, T]$ define $B(t; q): \; Z_q \to Z_q$ by

$$B(t; q)(r, u, v) = \left((-\frac{1}{m}\sigma(t, \ell; q)\int_0^{\ell} u, 0)^T, 0, \frac{1}{\rho(q)} D_x\sigma(q)\int_0^{\cdot} u\right)$$

and $A(t; q): D_0 \subset Z_q \to Z_q$ by

$$A(t; q) = A_0(q) + B(t; q).$$

For each $q\varepsilon Q$ it can be argued that the operator $A_0(q)$ is densely defined and dissipative (in fact conservative, i.e. $< A_0(q)\; z,z >_q = 0,\; z\varepsilon D_0$). Moreover, it can be shown that it is skew self adjoint (i.e., $A_0(q)^* = -A_0(q)$) and therefore that it is closed and maximal dissipative. This in turn implies that $A_0(q)$ is the infinitesimal generator of a C_0 semigroup of contractions $\{S_0(t; q): t \geq 0\}$ on Z_q (see [10], Theorems 4.4 and 4.5). It is in fact the case that Stone's theorem ([20], pg. 345) implies that $S_0(t; q)$ is defined for $t < 0$ and that $\{S_0(t; q): -\infty < t < \infty\}$ is a C_0 group of unitary operators on Z_q.

The operators $B(t; q)$ are bounded (uniformly in t and q for $t\varepsilon(0, T)$ and $q\varepsilon Q$) from which it follows that $\{A(t; q)\}_{t\varepsilon[0,T]}$ is a stable family of infinitesimal generators of C_0 semigroups $\{S_t(\tau; q): \tau \geq 0\}$ on Z_q with stability constants 1 and $K = \sup_{\substack{t\varepsilon[0,T] \\ q\varepsilon Q}} |B(t; q)|_q$ (see [12], Section 5.2). Since D_0 is independent of t and $t \to \sigma(t, \cdot, q)\varepsilon C^1((0, T), H^1(0, \ell))$ for each $q\varepsilon Q$, the homogeneous initial value problem

$$D_t z(t) = A(t; q)z(t) \qquad 0 \leq s < t \leq T \tag{2.9}$$

$$z(s) = z_0 \tag{2.10}$$

with $z_0\varepsilon Z$ has a unique evolution system $\{U(t, s; q): 0 \leq s \leq t \leq T\}$ associated with it which satisfies

(i) $\quad |U(t, s; q)|_q \leq e^{K(t-s)}$

(ii) $\quad U(t, s; q)D_0 \subset D_0$

(iii) $U(t, s; q)z$ is strongly continuously differentiable in Z_q for all $z \varepsilon D_0$ with

$D_t U(t, s; q)z = A(t; q)U(t, s; q)z$

and

$D_s U(t, s; q)z = -U(t, s; q)A(s; q)z$

for $0 \leq s \leq t \leq T$.

If $z_0 \varepsilon D_0$, $z(t) = U(t, s; q)z_0$ is the unique solution to (2.9) and (2.10).

For each $t \varepsilon [0, T]$ and $q \varepsilon Q$ let $F(t; q) \varepsilon Z$ and $z_0(q) \varepsilon Z$ be given by

$$F(t; q) = \Big((\mu\sigma(t, \ell; q) + \frac{1}{m} g_1(t; q) - \beta g_2(t; q), \lambda\sigma(t, \ell; q) + \gamma g_2(t; q))^T, 0, \frac{1}{\rho(q)} f(t, \cdot; q) \Big)$$

and

$$z_0(q) = \big(r(q), D^2\phi(q), \psi(q) \big)$$

respectively where $r(q) \varepsilon R^2$ and consider

$$D_t z(t) = A(t;q)z(t) + F(t; q) \tag{2.11}$$

$$z(0) = z_0(q). \tag{2.12}$$

Writing formally

$$z(t; q) = U(t, 0; q)z_0(q) + \int_0^t U(t, \tau; q)F(\tau; q)\, d\tau, \tag{2.13}$$

assumptions A3-A6 imply that the function $t \to z(t; q)$ is well defined and continuous from $[0, T]$ into Zq. If, in addition, $t \to F(t; q) \varepsilon C^1([0, T], Z_q)$, $\psi(q) \varepsilon H^2(0, \ell)$, $\psi(0; q) = D\psi(0, q) = 0$ and $r(q) = (\psi(\ell; q), D\psi(\ell; q))^T$ (i.e., $z_0(q) \varepsilon D_0$) then $z(t; q)$ as given by (2.13) is the unique classical solution to (2.11), (2.12) in the sense that $z(\cdot; q) \varepsilon C^1([0, T], Z_q)$, $z(t; q) \varepsilon D_0$, $0 \leq t \leq T$ and (2.11), (2.12) is satisfied in Z_q (see [12], Section 5.5). Under assumptions A3-A6 only, however, a classical solution to (2.11), (2.12) does not, in general, exist. In this case, $z(t; q)$ as given by (2.13) is known as a mild or generalized solution to (2.11), (2.12) in that it

is the limit of classical solutions to sequences of problems of the form (2.11), (2.12) for which a unique classical solution does exist. (See [12]).

In light of the above remarks we use (2.13) to formulate the identification problem. For each $x \varepsilon [0, \ell]$ define the operators $C(x): Z \to R$ by

$$C(x)(r, u, v) = \int_0^x \int_0^\tau u(\sigma)\, d\sigma\, d\tau. \tag{2.14}$$

We assume that we have been provided with displacement measurements, $\{u(t_i, x_j)\}_{\substack{i=1,\nu \\ j=1,\mu}}$, $t_i \varepsilon [0, T]$, $i = 1, 2, \ldots \nu$, $x_j \varepsilon [0, \ell]$ $j = 1, 2, \ldots \mu$, taken from the actual system and state the identification problem as

(ID) Find $q \varepsilon Q$ which minimizes

$$J(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| C(x_j) z(t_i; q) - u(t_i, x_j) \right|^2$$

where $z(t; q)$ is given by (2.13).

The infinite dimensionality of the constraints, (2.13), of course necessitates the use of some form of approximation in solving problem (ID). We develop one such scheme in the next section.

3. APPROXIMATION AND CONVERGENCE RESULTS

Our approximation scheme is based upon the formulation of a sequence of approximating identification problems in which the underlying state equations are finite dimensional semi-discrete approximations to (2.13). The approximating evolution equations are constructed using a standard cubic spline based Galerkin approach to effect the spatial discretization. It will be shown that each of the approximating identification problems has a solution, and via convergence of the states, that the resulting sequence of solutions admits a subsequence which converges to a solution to problem (ID).

Working abstractly at first, for each $N = 1, 2, \ldots$ and each $q \varepsilon Q$ let Z_q^N be a finite dimensional subspace of Z_q which is contained in D_0. Let P_q^N denote

the orthogonal projection of Z_q onto Z_q^N with respect to the $\langle\cdot, \cdot\rangle_q$ inner product. Define the linear operators $A_0^N(q)$: $Z_q^N \to Z_q^N$, $B^N(t; q): Z_q^N \to Z_q^N$, and $A^N(t; q): Z_q^N \to Z_q^N$ by $A_0^N(q) = P_q^N A_0(q)$, $B^N(t; q) = P_q^N B(t; q)$, and $A^N(t; q) = P_q^N A(t; q) = A_0^N(q) + B^N(t; q)$, respectively. The finite dimensionality of Z_q^N implies of course, that each of these operators is bounded, although not necessarily uniformly in N.

Since $A_0(q)$ is conservative, the $B(t; q)$ are bounded uniformly for $t \varepsilon [0, T]$ and $q \varepsilon Q$ and the P_q^N are orthogonal projections, it follows that the $A_0^N(q)$ are conservative and that the $B^N(t; q)$ are bounded uniformly in N as well. Indeed for $z^N \varepsilon Z_q^N$ we have

$$\langle A_0^N(q)z^N, z^N\rangle_q = \langle P_q^N A_0(q)z^N, z^N\rangle_q = \langle A_0(q)z^N, z^N\rangle_q = 0$$

and

$$\left|B^N(t; q)z^N\right|_q = \left|P_q^N B(t; q)z^N\right|_q \leq \left|B(t; q)z^N\right|_q \leq K\left|z^N\right|_q.$$

This in turn implies that the $A_0^N(q)$ are infinitestimal generators of C_0 semigroups of contractions, $\{S_0^N(t; q): t \geq 0\}$ on Z_q^N and that the initial value problems

$$D_t z^N(t) = A^N(t; q)z^N(t) \tag{3.1}$$

$$z^N(0) = z_0^N \tag{3.2}$$

have unique evolution systems $\{U^N(t, s; q): 0 \leq s \leq t \leq T\}$ associated with them which satisfy

(i) $\left|U^N(t, s; q)\right|_q \leq e^{K(t-s)}$

(ii) $D_t U^N(t, s; q)z^N = A^N(t; q)U^N(t, s; q)z^N$ for all $z^N \varepsilon Z_q^N$, $0 \leq s \leq t \leq T$.

The two parameter families $U^N(t, s; q)$ are the solution operators for the initial value problems (3.1) and (3.2). We note that since for each N, Z_q^N is finite dimensional, once a suitable basis has been chosen, the initial value problem (3.1) and (3.2) can be written in matrix form with $U^N(t, s; q)$ then being represetned by the corresponding principal fundamental matrix solution.

For each $q\varepsilon Q$ and N = 1, 2, ... we define the function $z^N(\cdot; q):[0, T] \to Z_q^N$ by

$$z^N(t; q) = U^N(t, 0; q)P_q^N z_0(q) + \int_0^t U^N(t, \tau; q)P_q^N F(\tau; q)\, d\tau \tag{3.3}$$

and state the approximating identification problem as

(IDN) Find $q\varepsilon Q$ which minimizes

$$J^N(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| C(x_j)z^N(t_i; q) - u(t_i, x_j)\right|^2$$

where $z^N(t; q)$ is given by (3.3) and the operators $C(x)$ are as they were defined in (2.14).

Once a basis for Z_q^N has been chosen, problem (IDN) takes the form of a least squares minimization problem subject to a linear non-autonomous, non-homogeneous matrix ordinary differential equation which can be solved (assuming for the moment that a solution exists) using standard techniques and readily available software.

In terms of the abstract formulation above, our general convergence results are summarized in the following two theorems.

<u>Theorem 3.1</u>

Suppose $\{q^N\}$ is a sequence in Q with $q^N \to q^*\varepsilon Q$ as $N \to \infty$. Suppose further that

(1) $P_q^N \to I$ strongly in Z_q uniformly in q for $q \varepsilon Q$ as $N \to \infty$.

(2) $\lim_{N\to\infty} \left| U^N(t, s; q^N) P^N_{q^N} z - P^N_{q^N} U(t, s; q^*) z \right|_{q^N} = 0$ uniformly in t, s for $0 \leq s \leq t \leq T$.

Then $\lim_{N\to\infty} \left| z^N(t; q^N) - z(t; q^*) \right|_{q^N} \to 0$ for each $t \varepsilon [0, T]$.

Proof

$$\begin{aligned}
\left| z^N(t; q^N) - z(t; q^*) \right|_{q^N} \leq & \left| U^N(t, 0; q^N) P^N_{q^N} \left(z_0(q^N) - z_0(q^*) \right) \right|_{q^N} \\
& + \left| U^N(t, 0; q^N) P^N_{q^N} z_0(q^*) - P^N_{q^N} U(t, 0; q^*) z_0(q^*) \right|_{q^N} \\
& + \left| (P^N_{q^N} - I) U(t, 0; q^*) z_0(q^*) \right|_{q^N} \\
& + \int_0^t \left| U^N(t, \tau; q^N) P^N_{q^N} \right|_{q^N} \left| F(\tau; q^N) - F(\tau; q^*) \right|_{q^N} d\tau \\
& + \int_0^t \left| \left(U^N(t, \tau; q^N) P^N_{q^N} - P^N_{q^N} U(t, \tau; q^*) \right) F(\tau; q^*) \right|_{q^N} d\tau \\
& + \int_0^t \left| (P^N_{q^N} - I)\ U(t, \tau; q^*) F(\tau; q^*) \right|_{q^N} d\tau
\end{aligned}$$

The properties of $U^N(t, s; q)$, the fact that P_q^N is an orthogonal projection and assumption A3 imply that the first term above tends to zero as $N \to \infty$. Hypotheses (1) and (2) in the statement of the theorem imply that the third and second terms respectively tend to zero as $N \to \infty$. Similar arguments and the Lebesgue dominated convergence theorem can be used to argue that the last three terms tend to zero as well and the theorem is proven.

Theorem 3.2

Suppose that hypotheses (1) and (2) of Theorem 3.1 hold. Suppose further that for each $N = 1, 2, \ldots$ problem (IDN) has a solution denoted by $\bar{q}^N$. Then

the sequence $\{\bar{q}^N\}$ has a convergent subsequence, $\{\bar{q}^{N_k}\}$ with $\bar{q}^{N_k} \to \bar{q} \varepsilon Q$ $k \to \infty$. Moreover, $\bar{q}$ is a solution to problem (ID).

Proof

The existence of the convergent subsequence $\{\bar{q}^{N_k}\}$ is an immediate consequence of Assumption A1. Theorem 3.1, therefore, implies that $\left|z^N(t; \bar{q}^{N_k}) - z(t; \bar{q})\right|_{\bar{q}^{N_k}} \to 0$ as $k \to \infty$ for each $t \varepsilon [0, T]$. This in turn implies that $\left|C(x)z^{N_k}(t; \bar{q}^{N_k}) - C(x)z(t; \bar{q})\right| \to 0$ as $k \to \infty$ for each $x \varepsilon [0, \ell]$ and each $t \varepsilon [0, T]$. It then follows that for any $q \varepsilon Q$

$$J(\bar{q}) = \lim_{k\to\infty} J^{N_k}(\bar{q}^{N_k}) \leq \lim_{k\to\infty} J^{N_k}(q) = J(q)$$

and consequently that $\bar{q}$ is a solution to problem (ID). The final equality in the expression above follows from an application of Theorem 3.1 with the constant sequence $\{q\}$.

Remark

In the identification problem (ID) as stated in the previous section, the fit is based upon spatially sampled displacement measurements. We note, however, that the convergence results given in Theorems 3.1 and 3.2 remain valid for identification based upon spatially sampled slope measurements, spatially distributed displacement, slope or velocity observations, velocity data at $x = \ell$ or any combination thereof.

We next describe a particular realization involving cubic spline functions of the abstract ideas presented above and show that the resulting approximation system satisfies the hypotheses of Theorems 3.1 and 3.2

For each $N = 1, 2, \ldots$ let $\{B_j^N\}_{j=-1}^{N+1}$ denote the standard cubic B-splines on the interval $[0, \ell]$ corresponding to the partition $\Delta^N \equiv \{0, \frac{\ell}{N}, \frac{2\ell}{N}, \ldots \ell\}$

(see [13]) and let $\{\hat{B}_j^N\}_{j=1}^{N+1}$ denote the modified cubic B-splines which satisfy $\hat{B}_j^N(0) = D\hat{B}_j^N(0) = 0$, $j = 1, 2, \ldots N+1$. The $\{\hat{B}_j^N\}_{j=1}^{N+1}$ are given by

$$\hat{B}_1^N(x) = B_0^N(x) - 2\,B_1^N(x) - 2\,B_{-1}^N(x)$$

$$\hat{B}_j^N(x) = B_j^N(x) \quad j = 2, 3, \ldots N+1.$$

Let $S^3(\Delta^N) = \text{SPAN}\ \{B_j^N\}_{j=-1}^{N+1}$, $\hat{S}^3(\Delta^N) = \text{SPAN}\ \{\hat{B}_j^N\}_{j=1}^{N+1}$, $Z^N = \{(\hat{s}(\ell), D\hat{s}(\ell))^T, s, \hat{s})\varepsilon Z : s\varepsilon S^3(\Delta^N), \hat{s}\varepsilon\hat{S}^3(\Delta^N)\}$, and $Z_q^N = \{Z^N, \langle\cdot, \cdot\rangle_q\}$. Defining

$$\Phi_j^N = \left((0, 0)^T, B_j^N, 0\right) \quad j = -1, 2, \ldots, N+1 \tag{3.4}$$

$$\Psi_j^N = \left((\hat{B}_j^N(\ell), D\hat{B}_j^N(\ell))^T, 0, \hat{B}_j^N\right) \quad j = 1, 2, \ldots, N+1$$

we have that $Z^N = \text{SPAN}\ \{\Phi_j^N\}_{j=-1}^{N+1} + \text{SPAN}\ \{\Psi_j^N\}_{j=1}^{N+1}$, $\{\Phi_j^N\}_{j=-1}^{N+1} \cup \{\Psi_j^M\}_{j=1}^{N+1}$, is a basis for Z^N, and since $\hat{S}^3(\Delta^N) \subset S^3(\Delta^N) \subset H^2(0, \ell)$, that Z_q^N is a $2N + 4$ dimensional subspace of Z_q which is contained in D_0.

The vector representation ζ^N with respect to the basis (3.4) for $P_q^N z$, where z is an arbitrary element in Z_q can be computed using the standard normal equation characterization for P_q^N:

$$\langle P_q^N z - z, z^N\rangle_q = 0 \qquad z^N \varepsilon Z_q^N. \tag{3.5}$$

For $z = (r, u, v)\varepsilon\ Z_q$ we find that

$$\zeta^N = [M_q^N]^{-1}\ H_q^N(z)$$

where

$$M_q^N = \left[\begin{array}{c|c} C_q^N & 0 \\ \hline 0 & D_q^N \end{array}\right]$$

and

$$H_q^N(z) = \left[\begin{array}{c} u_q^N \\ \hline v_q^N \end{array}\right]$$

with

$$[C_q^N]_{i+2,j+2} = \int_0^\ell EI(q)B_i^N B_j^N \quad i,j = -1,\ 0,\ 1,\ \ldots,\ N+1$$

$$[D_q^N]_{i,j} = [\hat{B}_i^N(\ell),\ D\hat{B}_i^N(\ell)]\ W \begin{bmatrix} \hat{B}_j^N(\ell) \\ D\hat{B}_j^N(\ell) \end{bmatrix} + \int_0^\ell \rho(q)\hat{B}_i^N\hat{B}_j^N \quad i,j = 1,\ 2,\ \ldots,\ N+1$$

$$[u_q^N]_{i+2} = \int_0^\ell EI(q)uB_i^N \quad i = -1,\ 0,\ 1,\ \ldots\ N+1$$

and

$$[v_q^N]_i = r^T W \begin{bmatrix} \hat{B}_i^N(\ell) \\ D\hat{B}_i^N(\ell) \end{bmatrix} + \int_0^\ell \rho(q)v\hat{B}_i^N \quad i = 1,\ 2,\ \ldots,\ N+1.$$

The matrix representation $\bar{A}_0^N(q)$ for the operator $A_0^N(q)$ can be computed using (3.5) with $z = A_0(q)\tilde{z}^N$, $\tilde{z}^N$ an arbitary element in Z_q^N. We find

$$\bar{A}_0^N(q) = [M_q^N]^{-1}\ K_q^N$$

where

$$K_q^N = \left[\begin{array}{c|c} 0 & E_q^N \\ \hline -[E_q^N]^T & 0 \end{array}\right]$$

with

$$[E_q^N]_{i+2,j} = \int_0^\ell EI(q) B_i^N D^2 \hat{B}_j^N \quad i = -1,\ 0,\ 1,\ 2,\ \ldots,\ N+1,\ j = 1,\ 2,\ \ldots,\ N+1.$$

Similarly the matrix representation $\overline{B}^N(t;\ q)$ for the operator $B^N(t;\ q)$ is found to be

$$\overline{B}^N(t;\ q) = [M_q^N]^{-1}\ L_q^N(t)$$

where

$$L_q^N(t) = \left[\begin{array}{c|c} 0 & 0 \\ \hline G_q^N(t) & 0 \end{array}\right]$$

with

$$[G_q^N(t)]_{i,j+2} = -\int_0^\ell \sigma(t,\ \cdot;\ q)\ (D\hat{B}_i^N) \int_0^{\cdot} B_j^N) \qquad \begin{array}{l} i = 1,\ 2,\ \ldots,\ N+1 \\ j = -1,\ 0,\ 1,\ \ldots,\ N+1 \end{array}$$

It then immediately follows that the matrix representtion for $A^N(t;\ q)$ is given by

$$\overline{A}^N(t;\ q) = \overline{A}_0^N(q) + \overline{B}^N(t;\ q) = \left[\begin{array}{c|c} 0 & [C_q^N]^{-1} E_q^N \\ \hline [D_q^N]^{-1}[-[E_q^N]^T + G_q^N(t)] & 0 \end{array}\right].$$

If for each $t\varepsilon[0, T]$ and $q\varepsilon Q$ we set $F_N(t; q) = [M_q^N]^{-1} H_q^N(F(t; q))$ and $w_0^N(q) = [M_q^N]^{-1} H_q^N(z_0(q))$, with respect to the cubic splines, the evolution equation (3.3), in differentiated form, is given by the following 2N + 4 dimensional initial value problem

$$D_t w^N(t) = \bar{A}^N(t; q)\, w^N(t) + F^N(t; q) \tag{3.6}$$

$$w^N(0) = w_0^N(q). \tag{3.7}$$

The approximating identification problems take the form

Find $q\varepsilon Q$ which minimizes

$$J^N(q) = \sum_{i=1}^{\nu} \sum_{j=1}^{\mu} \left| \sum_{k=-1}^{N+1} w_{k+2}^N (t_i; q) s_k^N(x_j) - \hat{u}(t_i, x_j) \right|^2$$

where $w^N(t; q)$ is the solution to (3.6), (3.7) corresponding to $q\varepsilon Q$ and

$$s_j^N(x) = C(x)\, \Phi_j^N = \int_0^x \int_0^\tau B_j^N(\sigma)\, d\sigma\, d\tau \qquad j = -1, 0, 1, \ldots, N+1.$$

In order to demonstrate that the scheme described above satisfies the hypotheses of Theorems 3.1 and 3.2 the following approximation theoretic results for cubic splines will be required (see [15], Chapter 4). Let I^N denote the standard cubic spline interpolation operator on $[0, \ell]$ corresponding to the partition Δ^N. That is, for ϕ a function defined on the interval $[0, \ell]$, $I^N\phi$ is defined to be that element in $S^3(\Delta^N)$ which satisfies $(I^N\phi)(\frac{j\ell}{N}) = \phi(\frac{j\ell}{N})$, $j = 0, 1, 2, \ldots, N$ and $D(I^N\phi)(\frac{j\ell}{N}) = D\phi\,(\frac{j\ell}{N})$ $j = 0$ and N.

<u>Proposition 3.1</u> For $\phi\varepsilon H^2(0, \ell)$

$$\left| D^k(I^N\phi - \phi) \right|_0 \le C_k^1 N^{-2+k} \left| D^2\phi \right|_0 \qquad k = 0, 1$$

where C_k^1 is independent of ϕ and N.

<u>Proposition 3.2</u> For $\phi\varepsilon H^4(0, \ell)$

$$\left| D^k(I^N\phi - \phi) \right|_0 \le C_k^2 N^{-4+k} \left| D^4\phi \right|_0 \qquad k = 0, 1, 2$$

where C_k^2 is independent of ϕ and N.

Let

$$V = R^2 \times H^0\ (0,\ \ell) \text{ and let } V_q = \{V,\ \langle\langle\cdot,\ \cdot\rangle\rangle_q\},$$

$$\langle\langle(r_1,\ v_1),\ (r_2,\ v_2)\ \rangle\rangle_q \ = \ r_1^T W r_2 + \langle v_1,\ v_2\rangle_{\rho(q)}$$

where

$$\langle\phi,\ \psi\rangle_{\rho(q)} \ = \ \int_0^\ell \rho(q)\phi\psi.$$

Let

$$V^N \ = \ \{(\eta,\ \phi)\varepsilon V{:}\phi\varepsilon\hat{S}^3(\Delta^N),\ \eta \ = \ (\phi(\ell),\ D\phi(\ell))^T\}$$

and

$$V_q^N \ = \ \{V^N,\ \langle\langle\cdot,\ \cdot\rangle\rangle_q\}.$$

Let $P_1^N(q)$ denote the orthogonal projection of $H^0(0,\ \ell)$ onto $S^3(\Delta^N)$ with respect to the inner product $\langle\phi,\ \psi\rangle_{EI(q)} \ = \ \int_0^\ell EI(q)\phi\psi$ and $P_2^N(q)$ denote the orothogonal projection of V_q on to V_q^N. Adopting the convention that for $z = (r,\ u,\ v)\varepsilon Z_q$, $z^N = P_q^N z$ will be denoted by $(r^N,\ u^N,\ v^N)$, it is easily seen that $u^N = P_1^N(q)u$ and $(r^N,\ v^N) = P_2^N(q)(r,\ v)$.

<u>Lemma 1</u> Let $z = (r,\ u,\ v)\varepsilon D_0$. Then

(1) $\left|u^N - u\right|_2 \to 0$ as $N \to \infty$ uniformly in q for $q\varepsilon Q$.

(2) $\left|v^N - v\right|_2 \to 0$ as $N \to \infty$ uniformly in q for $q\varepsilon Q$.

<u>Proof</u>

To verify (1) we show that $\left|D^k(u^N - u)\right|_0 \to 0$ as $N \to \infty$, $k = 0,\ 1,\ 2$ uniformly in q. Recalling assumption A2, the fact that $z\varepsilon D_0$ implies $u\varepsilon H^2(0,\ \ell)$ and Proposition 3.1 above we have

$$\left|u^N - u\right|_0 = \left|P_1^N(q)u - u\right|_0 \le \frac{1}{m_{EI}} \left|P_1^N(q)u - u\right|_{EI(q)}$$

$$\le \frac{1}{m_{EI}} \left|I^N u - u\right|_{EI(q)} \le \frac{M_{EI}}{m_{EI}} \left|I^N u - u\right|_0$$

$$\le \frac{M_{EI}}{m_{EI}} C_0^1 N^{-2} \left|D^2 u\right|_0 \to 0 \quad \text{as } N \to \infty.$$

The convergence of the first derivative is argued using the Schmidt inequality ([15] Theorem 1.5),

$$\left|D(u^N - u)\right|_0 \le \left|D(u^N - I^N u\right|_0 + \left|D(I^N u - u)\right|_0$$

$$\le \tilde{K}N\left|u^N - I^N u\right|_0 + C_1^1 N^{-1}\left|D^2 u\right|_0$$

$$\le \tilde{K}N\left|u^N - u\right|_0 + \tilde{K}N\left|I^N u - u\right|_0 + C_1^1 N^{-1}\left|D^2 u\right|_0$$

$$\le \frac{\tilde{K}M_{EI}}{m_{EI}} C_0{}^1 N^{-1}\left|D^2 u\right|_0 + \tilde{K}\, C_0^1 N^{-1}\left|D^2 u\right|_0 + C_1^1 N^{-1}\left|D^2 u\right|_0 \to 0 \text{ as } N \to \infty.$$

For the second derivative, we first note that for $u \varepsilon H^2(0, \ell)$, the Schmidt inequality and the first integral relation ([15] pg. 52) imply

$$\left|D^2 P_1^N(q)u\right|_0^2 \le 2\left|D^2(P_1^N(q)u - I^N u)\right|_0^2 + 2\left|D^2 I^N u\right|_0^2$$

$$\le 2\tilde{K}N^4\left|P_1^N(q)u - I^N u\right|_0^2 + 2\left|D^2 u\right|_0^2 - 2\left|D^2(u - I^N u)\right|_0^2$$

$$\le 2\tilde{K}N^4\left|P_1^N(q)u - u\right|_0^2 + 2\tilde{K}N^4\left|I^N u - u\right|_0^2 + 2\left|D^2 u\right|_0^2$$

$$\le \hat{K}^2\left|D^2 u\right|_0^2 \tag{3.8}$$

where $\hat{K}$ is independent of u, N, and $q \varepsilon Q$. Now for $w \varepsilon H^4(0, \ell)$ with $w^N = P_1^N(q)w$,

Proposition 3.2 together with the Schmidt inequality imply that $\left|D^k(w^N - w)\right|_0 = O(N^{-4+k})$, $k = 0, 1, 2$. Therefore

$$\left|D^2(u^N - u)\right|_0 \leq \left|D^2(u^N - w^N)\right|_0 + \left|D^2(w^N - w)\right|_0 + \left|D^2(w - u)\right|_0$$

$$\leq \left|D^2(w^N - w)\right| + (1 + \hat{K})\left|D^2(w - u)\right|_0$$

$$\leq O(N^{-2}) + (1 + \hat{K})\left|D^2(w - u)\right|_0$$

where we have used (3.8) to bound $\left|D^2(u^N - w^N)\right|_0$. Since $H^4(0, \ell)$ is dense in $H^2(0, \ell)$ we can choose w and then N (since the $O(N^{-2})$ term depends upon $\left|D^4 w\right|_0$) to make the right hand side of the last inequality above arbitrarily small.

Turning next to statement (2) and recalling that $z \varepsilon D_0$ implies that $v \varepsilon H^2(0, \ell)$, $v(0) = Dv(0) = 0$ and $r = (v(\ell), Dv(\ell))^T$ we have

$$\left|v^N - v\right|_0^2 \leq \frac{1}{m_\rho} \left|\left|(r^N, v^N) - (r, v)\right|\right|_q^2$$

$$\leq \frac{1}{m_\rho} \left|\left|\left((\hat{I}^N v(\ell), D\hat{I}^N v(\ell))^T, \hat{I}^N v\right) - \left(v(\ell), Dv(\ell)\right)^T, v\right|\right|_q^2$$

$$= \frac{1}{m_\rho} \left|\hat{I}^N v - v\right|_{\rho(q)}^2 = \frac{1}{m_\rho} \left|I^N v - v\right|_{\rho(q)}^2$$

$$\leq \frac{M_\rho}{m_\rho} \left|I^N v - v\right|_0^2 \leq \frac{M_\rho}{m_\rho} (C_0^1)^2 N^{-4} \left|D^2 v\right|_0^2 \to 0$$

as $N \to \infty$ where for ϕ a function defined on $[o, \ell]$, $\hat{I}^N \phi$ denotes that element in $\hat{S}^3(\Delta^N)$ which satisfies $\hat{I}\Phi\left(\frac{j\ell}{N}\right) = \phi\left(\frac{j\ell}{N}\right)$, $j = 1, 2..N$ and $D\hat{I}^N \phi(\ell) = D\phi(\ell)$.

The convergence of the derivatives is verified in essentially the same manner as it was in the proof of Statement 1.

<u>Theorem 3.3</u>

For the cubic spline based scheme described above, hypothesis (1) of Theorem 3.1 is satisfied.

Proof

Let $z = (r, u, v) \varepsilon D_0$ and let $z^N = P_q^N z = (r^N, u^N, v^N)$. Then $r = (v(\ell), Dv(\ell))^T$, $r^N = (v^N(\ell), Dv^N(\ell))^T$, and

$$
\begin{aligned}
|P_q^N z - z|_q^2 &= |z^N - z|_q^2 \\
&= (r^N - r)^T W (r^N - r) + |u^N - u|_{EI(q)}^2 + |v^N - v|_{\rho(q)}^2 \\
&\leq ||W||_2 |v^N(\ell) - v(\ell)|^2 + ||W||_2 |Dv^N(\ell) - Dv(\ell)|^2 \\
&\quad + M_{EI} |u^N - u|_0^2 + M_\rho |v^N - v|_0^2 \\
&\leq \tilde{K}_1 |u^N - u|_2^2 + \tilde{K}_2 |v^N - v|_2^2
\end{aligned}
$$

where $||W||_2$ is the Euclidean matrix norm of W and $\tilde{K}_1$ and $\tilde{K}_2$ are constants which are independent of N, z, and $q \varepsilon Q$. Lemma 3.1 implies that the right hand side of the final inequality above tends toward zero as $N \to \infty$ and consequently $P_q^N z \to z$ as $N \to \infty$ for all $z \varepsilon D_0$. However, D_0 dense in Z_q (uniformly with respect to $q \varepsilon Q$) and the P_q^N uniformly bounded in N (being orthogonal projections) imply that $P_q^N z \to z$ as $N \to \infty$ for all $z \varepsilon Z_q$ uniformly in q for $q \varepsilon Q$ and the theorem is proven.

In order to verify hypothesis (2) of Theorem 3.1 we require the following lemma.

Lemma 3.2

Let $\{q^N\} \subset Q$ with $q^N \to q^* \varepsilon Q$ as $N \to \infty$. Let Z_q^N be the cubic spline based subspaces of Z_q defined above, P_q^N the orthogonal projections of Z_q onto Z_q^N, $A_0^N(q) = P_q^N A_o(q)$, and $\{S_0^N(t; q):t \geq 0\}$ the C_0 semigroups of contractions generated by the $A_0^N(q)$. Then

$$\left|S_0^N(t;\ q^N)P^N_{q^N}z - P^N_{q^N}S_0(t;\ q^*)z\right|_{q^N} \to 0$$

as $N \to \infty$ for each $z \varepsilon Z$, uniformly in t for $t \varepsilon [0,\ T]$.

Proof

Using a variation of the well known Trotter approximation theorem (see [4], Theorem 6.2) the desired conclusion will follow if we can show that

$$\left|R(\lambda,\ A_0^N(q^N))P^N_{q^N}z - P^N_{q^N}R(\lambda,\ A_0(q^*))z\right|_{q^N} \to 0$$

as $N \to \infty$ for each $z \varepsilon Z$ for some $\lambda > 0$ where $R(\lambda,\ A) = (\lambda - A)^{-1}$. However $\left|S^N(t;\ q)\right|_q \leq 1$ implies (see [9])

$$\left|R(\lambda,\ A_0^N(q^N))P^N_{q^N}z - P^N_{q^N}R(\lambda,\ A_0(q^*))z\right|_{q^N}$$

$$= \left|R(\lambda,\ A_0^N(q^N))(A_0^N(q^N)P^N_{q^N} - P^N_{q^N}A_0(q^*))R(\lambda,\ A_0(q^*))z\right|_{q^N}$$

$$\leq \frac{1}{\lambda}\left|(A_0^N(q^N)P^N_{q^N} - P^N_{q^N}A_0(q^*))R(\lambda,\ A_0(q^*))z\right|_{q^N}$$

$$= \left|(A_0^N(q^N)P^N_{q^N} - P^N_{q^N}A_0(q^*))y\right|_{q^N}$$

where we have chosen $\lambda = 1$ and $y = R(\lambda,\ A_0(q^*)z \varepsilon D_0$. Now

$$\left|(A_0^N(q^N)P^N_{q^N} - P^N_{q^N}A_0(q^*))y\right|_{q^N}$$

$$= \left|(P^N_{q^N}(A_0(q^N)P^N_{q^N} - P^N_{q^N}A_0(q^*))y\right|_{q^N}$$

$$\leq \left|(A_0(q^N) - A_0(q^*))P^N_{q^N}y\right|_{q^N} + \left|A_0(q^*)(P^N_{q^N} - I)y\right|_{q^N}$$

$$= T_1^N + T_2^N.$$

Recalling the definition of the operator $A_0(q)$, (2.8), the fact that $y \varepsilon D_0$ and the estimates given in Lemma 3.1, standard estimates yield $T_2^N \to 0$ as $N \to \infty$

while $q^N \to q^*$ as $N \to \infty$ and assumption A2 imply $T_1^N \to 0$ as $N \to \infty$.

Theorem 3.4

Hypothesis (2) of Theorem 3.1 is satisfied by the cubic spline scheme.

Proof

Since for $z \varepsilon Z$ and $0 \leq s \leq t \leq T$ we have

$$D_t U(t, s; q)z = A(t; q)U(t, s; q)z$$

$$= A_0(q)U(t, s; q)z + B(t; q)U(t, s; q)z$$

it follows that

$$U(t, s; q)z = S_0(t - s; q)z + \int_s^t S_0(t - \tau; q)B(\tau; q)U(\tau, s; q)z \, d\tau .$$

Similarly

$$U^N(t, s; q)P_q^N z = S_0^N (t - s; q)P_q^N z + \int_s^t S_0^N (t - \tau; q)B^N(\tau;q)U^N(\tau, s; q)P_q^N z \, d\tau .$$

Therefore, letting

$$\Delta^N(t, s) = \left| U^N(t, s; q^N)P_{q^N}^N z - P_{q^N}^N U(t, s; q^*)z \right|_{q^N},$$

we have

$$\Delta^N(t, s) \leq \left| S_0^N (t - s; q^N)P_{q^N}^N z - P_{q^N}^N S_0(t - s; q^*)z \right|_{q^N}$$

$$+ \int_s^t \left| S_0^N (t - \tau; q^N) \right|_{q^N} \left| B^N(\tau; q^N) \right|_{q^N} \Delta^N(\tau, s) \, d\tau$$

$$+ \int_s^t \left|S_0^N (t - \tau;\ q^N)\right|_{q^N} \left|B^N(\tau;\ q^N)\right|_{q^N} \left|(P^N_{q^N} - I)U(\tau,\ s;\ q^*)z\right|_{q^N} d\tau$$

$$+ \int_s^t \left|S_0^N (t - \tau;\ q^N)\right|_{q^N} \left|P^N_{q^N}\right|_{q^N} \left|(B(\tau;\ q^N) - B(\tau;\ q^*))U(\tau,\ s;\ q^*)z\right|_{q^N} d\tau$$

$$+ \int_s^t \left|(S_0^N (t - \tau;\ q^N)P^N_{q^N} - P^N_{q^N}S_0(t - \tau;\ q^*))B(\tau;\ q^*))U(\tau,\ s;\ q^*)z\right|_{q^N} d\tau$$

$$+ \int_s^t \left|(P^N_{q^N} - I)S_0(t - \tau;\ q^*)B(\tau;\ q^*))U(\tau,\ s;\ q^*)z\right|_{q^N} d\tau$$

or

$$\Delta^N(t,\ s) \ \leq\ \left|S_0^N (t - s;\ q^N)P^N_{q^N}z - P^N_{q^N}S_0(t - s;\ q^*)z\right|_{q^N}$$

$$+ K \int_s^t \Delta^N(\tau,\ s)\ d\tau + \int_s^t h^N(\tau;\ t,\ s)\ d\tau \tag{3.9}$$

where $K = \sup \left|B(t;\ q)\right|_q$. Lemma 3.2 implies that the first term on the right hand side of (3.9) tends toward zero as $N \to \infty$ uniformly in s and t for $0 \leq s \leq t \leq T$, while Lemmas 3.1 and 3.2, $q^N \to q^*$ as $N \to \infty$, assumption A2 and the boundedness and strong continuity of the operators imply $h^N(\tau;\ t,\ s) \to 0$ as $N \to \infty$ uniforming in τ, t, s for $0 \leq s \leq \tau \leq t \leq T$. Therefore (3.9) can be written as

$$\Delta^N(t,\ s) \leq \varepsilon^N + K \int_s^t \Delta^N(\tau,\ s)\ d\tau$$

with $\varepsilon^N \to 0$ as $N \to \infty$ uniformly in t and s. An application of the Gronwall inequality yields

$$\Delta^N(t,\ s) \leq \varepsilon^N e^{K(t - s)}$$

from which the theorem immediately follows.

Finally, for the cubic spline scheme, under our general assumptions, using either standard continuous dependence results for ordinary differential equations or the Trotter approximation theorem it can be argued that for each N = 1, 2, ... and $t \in [0,\ T]$, $z^N(t;\ q)$ given by (3.3) is continuous in q. Consequently $J^N(q)$ is continuous in q, which together with assumption A1 implies that problem (IDN) has a solution.

Having now demonstrated that the hypotheses of Theorems 3.1 and 3.2 are satisfied by the cubic spline scheme, we turn next to a discussion of examples and numerical results which provides an indication of how well the scheme performs in practice.

4. NUMERICAL RESULTS AND CONCLUDING REMARKS

In this section we present numerical results that were obtained by applying variations of the cubic spline based scheme discussed in the previous section to the identification of a variety of hybrid systems involving the vibration of beams with attached tip bodies. Since our primary objective was to demonstrate the feasibility of our scheme, we considered only relatively simple examples.

In the first example we consider a cantilevered beam with a tip (point) mass. The second example involves a free-free beam with an attached tip body at each end. A cantilevered beam with a tip body subject to an axially directed base acceleration is considered in the final example. Strictly speaking, the theory developed in the previous two sections applies directly only to the last example. However, the relatively minor modifications which are required to make our general method and the corresponding convergence results applicable to the other two examples should be immediately clear.

In general the observational data upon which the fits were based was obtained by generating solutions using fixed (or so called "true") values of the parameters with a Galerkin method and a finite number of the unforced system's natural mode shapes. Computing the modal frequencies and corresponding mode shapes for systems of the type considered here, in general requires the locating of zeros of transcendental equations involving the various beam and tip body parameters which appear in the problem. The resulting modal equations tend to be stiff and must be integrated using an appropriate method if a valid solution is to be obtained.

The approximating identification problems (IDN) are solved using the IMSL (see [8]) routine ZXSSQ. This routine is an interative Levenberg-Marquardt Newton's Method/Steepest descent hybrid algorithm for the minimization of the sum of squares of a system of functions of several variables. The required gradients and entries in the Jacobian matrix are computed numerically using finite difference approximations. The method required that we supply initial start-up values for the unknown parameters and a subroutine which evaluates $J^N(q)$ for a given value of q. The latter requirement necessitates the integration of the initial value problem (3.6) and (3.7). This is accomplished

using the IMSL routine DGEAR, a variable order ADAMS predictor-corrector method. It is interesting to note that uinlike the modal approximations, the spline equations did not require the use of the stiff option. This of course makes the spline schemes attractive from a computational point of view. We note also that due to the narrow support of the B-splines, the matrices which appear in the resulting Galerkin equations tend to be banded, thus facilitating efficient integration of the system of differential equations. The inner products which determine the entries in the mass and stiffness matrices, as well as the generalized Fourier coefficients required to project the non-homogeneous term and the initial conditions were computed using a composite two point Gauss-Legendre quadrature rule.

Example 4.1

We consider a cantileverd beam of lenth ℓ, constant stiffness EI and mass density ρ with a tip mass of mass m (see Figure 4.1). The differential equations and boundary conditions which describe the system are given by

$$\rho D_t^2 u = -EID_x^4 u + f$$

$$mD_t^2 u(t, \ell) = EID_x^3 u(t, \ell) + g$$

$$u(t, 0) = D_x u(t, 0) = D_x^2 u(t, \ell) = 0$$

$$u(0, x) = \phi(x) \quad D_t u(0, x) = \psi(x)$$

where we have assumed that no axially directed loading is present.

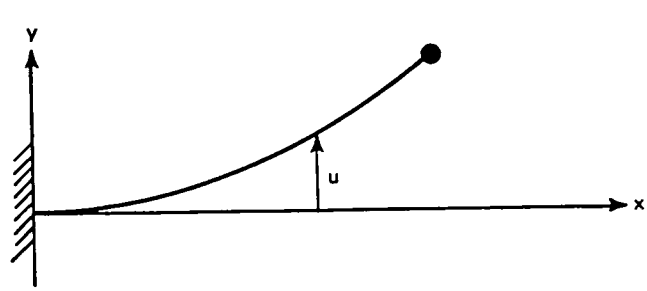

Figure 4.1.

We set $\ell = 1.0$, $EI = 1.0$, $\rho = 3.0$, $m = 1.5$, assumed that the system was initially at rest (i.e., $\phi = \psi = 0$) and excited the structure with an implusive lateral force at the end of the beam at time $t = 0$. The input disturbance was modeled as $f(t, x) = 20e^{-2t}e^{-20(1-x)}$ and $g(t) = 0$. Using the first three normal modes of the unforced system to generate observations in the form of displacement measurements at positions $x_j = 0.125(j+3)$, $j = 1, 2, \ldots, 5$, and times $t_j = 0.2\,j$, $j = 1, 2, \ldots, 10$ we identified EI and p. The start-up values for the Levenberg-Marquardt routine were taken to be $EI_0 = 0.7$ and $\rho_0 = 2.7$. The final converged values for $\overline{EI}^N$, $\overline{\rho}^N$ and the residual sum of squares $\overline{J}^N$ are given in Table 4.1 below.

Table 4.1.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	0.9976	3.0262	0.39×10^{-5}
3	0.9994	3.0382	0.46×10^{-5}
4	0.9951	3.0544	0.48×10^{-5}
5	0.9961	3.0409	0.40×10^{-5}
6	0.9995	2.9976	0.35×10^{-5}
True Values:	1.0	3.0	

It is clear that relatively accurate estimates of the parameters can be obtained using small values of N.

Based upon the scheme's performance on examples for which exact solutions were available, we feel that the somewhat eratic convergence exhibited in Table 4.1 is most likely a consequence of using approximate solutions to generate observations.

Example 4.2

In this example we identify the stiffness and mass density of a free-free beam of length ℓ which has a tip body (having different mass properties) attached to each end (see Fig. 4.2). The system was assumed to be initially at rest and then excited by a time varying, spatially distributed transverse load given by $f(t, x) = 10 \sin (2\pi t)e^{x}$.

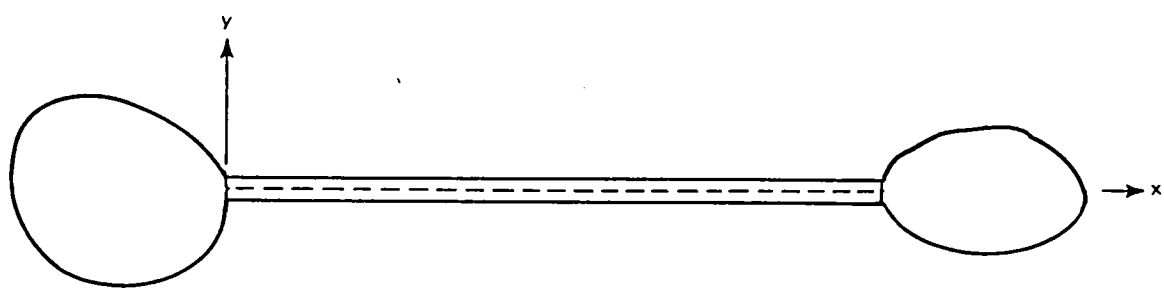

Figure 4.2.

The vibrations of the beam are described by the following partial differential equation and boundary conditions (see [16])

$$\rho D_t^2 u = -EID_x^4 u + f \qquad t > 0 \quad x \varepsilon (0, \ell)$$

$$D_t^2 u = -\alpha_1 EID_x^3 u + \beta_1 EID_x^2 u \qquad t > 0 \quad x = 0$$

$$D_t^2 D_x u = -\beta_1 EID_x^3 u + \gamma_1 EID_x^2 u \qquad t > 0 \quad x = 0 \tag{4.1}$$

$$D_t^2 u = \alpha_2 EID_x^3 u + \beta_2 EID_x^2 u \qquad t > 0 \quad x = \ell$$

$$D_t^2 D_x u = -\beta_2 EID_x^3 u - \gamma_2 EID_x^2 u \qquad t > 0 \quad x = \ell$$

$$u = 0 \qquad D_t u = 0 \qquad t = 0 \quad x \varepsilon [0, \ell]$$

with

$$\alpha_i = \frac{J_i + m_i c_i^2}{m_i (J_i + m_i c_i^2 \sin^2 \delta_i)},$$

$$\beta_i = \frac{c_i \cos \delta_i}{J_i + m_i c_i^2 \sin^2 \delta_i}$$

and

$$\gamma_i = \frac{1}{J_i + m_i c_i^2 \sin^2 \delta_i} \qquad i = 1, 2$$

where the mass properties for the tip bodies, m_i, J_i, c_i, and δ_i are as they were defined for the single tip body problem in Section 2.

Settling EI = 1.0, ρ = 3.0,

$$m_1 = 0.75, \; J_1 = 0.6, \; c_1 = 0.1, \; \delta_1 = \pi/6,$$

$$m_2 = 1.5, \; J_2 = 0.4, \; c_2 = 0.2, \; \delta_2 = \pi/3,$$

and ℓ = 1.0 we based our fit on velocity data at the ends of the beam only generated at times $t_j = 0.2\, j$, $j = 1, 2, \ldots, 10$ using the first six natural modes including the two which correspond to rigid body translation and rotation (see the remark following the proof of Theorem 3.2). We note that the system in (4.1) is free with respect to the intertial frame in which displacement is measured. A formula analogous to (2.14) therefore can not be used to recover displacement from the solutions to either the abstract or the approximating evolution equations.

The start-up values used were $EI_0 = 0.7$ and $\rho_0 = 2.7$. Our results are summarized in Table 4.2

Table 4.2

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	0.9957	3.0052	0.26×10^{-3}
3	0.9963	3.0065	0.18×10^{-3}
4	0.9982	3.0037	0.10×10^{-3}
5	1.0018	3.0008	0.71×10^{-4}
True Value:	1.0	3.0	

Example 4.3

For our final example, we consider a problem of the form discussed in Sections 2 and 3. The parameters to be estimated are the spatially invariant stiffness EI and mass density ρ. We assume that the system is initially at rest and is then acted upon by an impulsive lateral force at the end of the beam at time $t = 0$ and a piecewise constant base acceleration described by

$$f(t, x) = 20\, e^{-2t} e^{-20(1-x)}$$

and

$$a_0(t) = \begin{cases} 1.0 & 0 \le t \le 1.5 \\ 0 & 1.5 < t < 3.0 \\ 1.0 & 3.0 \le t < 4.0 \\ 0 & 4.0 \le t \end{cases}$$

respectively.

The externally applied axial load $\sigma(t, x)$ can be related to the base acceleration $a_0(t)$ as follows. Recalling that τ denotes the internal tension in the beam, equilibrium in the x direction yields $D_x\tau = \rho a_0$, or

$$\tau(t, x) = -\rho(\ell - x)\, a_0(t) + \tau(t, \ell). \tag{4.2}$$

For the tip body we have

$$-\tau(t, \ell) = m a_0(t) - mc \sin\delta\, D_t^2 D_x u(t, \ell). \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\tau(t, x) = -\rho(\ell - x)a_0(t) - ma_0(t) + mc \sin\delta \cdot D_t^2 D_x u(t, \ell),$$

from which we immediately conclude that $\sigma(t, x) = -a_0(t)(\rho(\ell - x) + m)$. Taking $EI = 1.0$, $\rho = 2.0$, $m = 4.0$, $J = 0.4$, $c = 0.2$, $\delta = \pi/3$ and $\ell = 1.0$, displacement data at positions $x_j = 0.75, 0.87, 1.0$, at times $t_j = 4.0, 4.5, 5.0$ was generated using the first 3 natural modes for the unforced, fixed base beam/tip body system. The start-up values were chosen as $EI_0 = 0.7$ and $\rho_0 = 2.5$. The results are given in Table 4.3.

Table 4.3.

N	$\overline{EI}^N$	$\overline{\rho}^N$	$\overline{J}^N$
2	1.0003	2.0635	0.35×10^{-3}
3	1.0014	2.0668	0.22×10^{-3}
4	1.0007	2.0404	0.26×10^{-3}
5	.9994	2.0274	0.35×10^{-3}
True Values:	1.0	2.0	

Once again the eractic convergence is most probably attributable to using an approximate solution to generate observations.

We note that strictly speaking, our theory requires that $a_0 \varepsilon C^1$. However, as is evidenced by the example above the scheme appears to perform satisfactorily on problems involving a_0 which are discontinuous.

Although it was not considered in the present effort, it is possible to include terms which model viscous damping in the partial differential equations and boundary conditions. Furthermore, it is possible to extend our approximation theory and corresponding convergence results to include the ability to estimate parameters associated with the damping effects (see [1], [7], and [11]).

It is also possible to develop an approximation theory which is similar in spirit to the one presented here for the identification of models based upon the use of the higher-order Timoshenko theory to model the vibration of the beam. The Timoshenko theory includes effects due to shear deformation and rotatory inertia (see [2], [6], [7], and [19]).

Finally we note that while the scheme described in Section 3 is cubic spline based, because of the choice of state variables, it in fact relies upon quintics to represent displacement. We are currently developing a method using a somewhat more direct approach than the abstract operator formulation employed here which does use cubic splines to represent displacement. Starting with a weak/variational form of the underlying hybrid system, a cubic spline based Galerkin approach is used to construct a sequence of approximating identification problems wherein the constraints are given by a finite dimensional linear second order ordinary differential equation in the approximate displacement. Based upon preliminary results this scheme promises to be computationally more efficient than and to perform as well as the one which was presented above. Moreover, it will permit the identification of spatially varying EI and ρ under for less stringent smoothness and continuity hypotheses than the ones given in assumption A2. This work will be discussed in a forthcoming paper.

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References

[1] Banks, H.T. and J.M. Crowley, Parameter Estimation for Distributed Systems Arising in Elasticity, to appear in Proceedings of the NCKU/AAS Symposium on Engineering Sciences and Mechanics, National Cheng Kung University, Taiwan, 1982.

[2] Banks, H.T. and J.M. Crowley, Parameter Estimation in Timoshenko Beam Models, Icase Report No. 82-22, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1982.

[3] Banks, H.T. and J.M. Crowley, Parameter Estimation in Continuum Models, NASA CR-172132, ICASE Report No. 83-13, Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, VA, 1983.

[4] Belleni-Morante, A., Applied Semigroups and Evolution Equations, Clarendon Press, Oxford, 1979.

[5] Burns, J.A. and E.M. Cliff, An Approximation Technique for the Control and Identification of Hybrid Systems, Proceedings of the Third VPI&SU/AIAA Symposium, on the Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA, June 15-17, 1981.

[6] Clough, R.W. and J. Penzien, Dynamics of Structures, McGraw Hill, New York, 1975.

[7] Crowley, J.M., Numerical Methods of Parameter Identification for Problems Arising in Elasticity, Ph.D. Thesis, Brown University, May 1982.

[8] International Mathematical and Statistical Libraries, Edition 9, Houston, 1983.

[9] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1976.

[10] Krein, S.G., Linear Differential Equations in Banach Space, American Mathematical Society, Providence, 1971.

[11] Kunisch, K. and E. Graif, Parameter Estimation for the Euler-Bernoulli-Beam, ICASE Report No. 84-22, NASA Langley Research Center, Hampton, VA 23666, June 1984.

[12] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[13] Prenter, P. M., Splines and Variational Methods, John Wiley and Sons, New York, 1975.

[14] Richtmyer, R. O. and K. W. Morton, Difference Methods for Initial Value Problems, John Wiley and Sons, New York, 1957.

[15] Schultz, M. H., Spline Analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[16] Storch, J. and S. Gates, Planar Dynamics of a Uniform Beam with Rigid-Bodies Affixed to the Ends, Report CSDL-R-1619, Charles Stark Draper Laboratory, Cambridge, 1983.

[17] Storch, J. and S. Gates, Transverse Vibration and Buckling of a Cantilevered Beam with Tip Body Under Axial Acceleration, to appear in J. Sound Vibration.

[18] Storch, J. and S. Gates, Transverse Vibration and Buckling of a Cantilevered Beam with Tip Body Under Constant Axial Base Acceleration, Report CSDL-R-1675, Charles Stark Draper Laboratory, Cambridge, 1983.

[19] Warburton, G. B., The Dynamical Behavior of Structures, 2nd Editon, Pergamon press, Oxford, 1976.

[20] Yosida, K., Functional Analysis, Springer-Verlag, New York, 1966.

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16. Abstract

A cubic spline based Galerkin-like method is developed for the identification of a class of hybrid systems which describe the transverse vibration of flexible beams with attached tip bodies. The identification problem is formulated as a least squares fit to data subject to the system dynamics given by a coupled system of ordinary and partial differential equations recast as an abstract evolution equation (AEE) in an appropriate infinite dimensional Hilbertspace. Projecting the AEE into spline-based subspaces leads naturally to a sequence of approximating finite dimensional identification problems. The solutions to these problems are shown to exist, are relatively easily computed, and are shown to, in some sense, converge to solutions to the original identification problem. Numerical results for a variety of examples are discussed.

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