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Eli Turkel

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Hampton, Virginia 23665

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ACCELERATION TO A STEADY STATE FOR THE EULER EQUATIONS

Eli Turkel
Tel-Aviv University
and
Institute for Computer Applications in Science and Engineering

Abstract

A multi-stage Runge-Kutta method is analyzed for solving the Euler equations exterior to an airfoil. Highly subsonic, transonic and supersonic flows are evaluated. Various techniques for accelerating the convergence to a steady state are introduced and analyzed.

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Introduction

In this study we shall discuss ways of accelerating the approach to a steady state for the Euler equations. We first consider the time-dependent equations. However, since we are only interested in the steady state we shall feel free to alter the equations in any way that does not affect the steady state. Since the Euler equations constitute a nonlinear system of equations, it usually can not be proven that the modified system even approaches a steady state. Instead, most of the analysis will refer to linearized versions of the equations. Assuming that the steady state is unique once the modified equations reach a steady state, it must be the steady state of the original equations. We shall concentrate on the inviscid equations, though most of the techniques are also applicable to the Navier-Stokes equations.

We assumed that a body-fitted curvilinear grid has been constructed from some package. All that we require is the (x,y) coordinates of each zone. The finite difference equations will be derived using a finite volume approach. Though the cells can be of any shape in such an approach we shall assume that all cells are quadrilaterals. Near the trailing edge some zones may degenerate and the finite volume approach still holds. The use of mapping formulas rather than finite volumes yields essentially the same finite difference formulas. The finite volume approach naturally associates the value of the dependent variables with zonal averages. These averages are associated with values at the cell center. Hence, all boundaries are placed along cell faces. With a mapping formulation the natural implementation would be to place the variables at grid nodes. Furthermore, using a finite volume approach, the Jacobian of the transformation is identified with the area of a cell. This leads to a slightly different formulation for the Jacobian than

would be natural for a finite difference approach. The finite volume formulation leads to a straightforward generalization for axisymmetric coordinates.

For convenience of notation we shall only consider two-dimensional flows. The Euler equations can then be written as

$$w_t + f_x + g_y = 0, \tag{1.1}$$

$$\begin{aligned} w &= (\rho,\ \rho u,\ \rho v, E)^T, \\ f &= (\rho u,\ \rho u^2 + p,\ \rho uv,\ \rho uh)^T, \\ g &= (\rho v,\ \rho uv,\ \rho v^2 + p,\ \rho vh)^T, \\ h &= \frac{E + p}{\rho}. \end{aligned} \tag{1.2}$$

Integrating (1.1) inside a cell and using the divergence theorem we get

$$\frac{\partial}{\partial t} \iint_D w dV + \int_{\partial D} (f dy - g dx) = 0, \tag{1.3}$$

where we have chosen D to be a quadrilateral. Let two adjacent sides of the quadrilateral be given by ξ = constant and η = constant. The component of velocity perpendicular to the curve ξ = constant, denoted by q, is given by

$$q = \frac{d\xi}{dt} = \xi_x u + \xi_y v = \frac{y_\eta u - x_\eta v}{J}, \quad J = x_\xi y_\eta - x_\eta y_\xi, \tag{1.4a}$$

while the velocity component along the curve ξ = constant, denoted q_{11}, is given by

$$q_{11} = \frac{y_\eta v + x_\eta u}{J} . \tag{1.4b}$$

Similarly, letting r be the velocity component perpendicular to η = constant and r_{11} parallel to η = constant, we have

$$r = \eta_x u + \eta_y v = (-y_\xi u + x_\xi v)/J, \tag{1.4c}$$

$$r_{11} = (y_\xi v + x_\xi u)/J, \tag{1.4d}$$

r is proportional to q_{11}, only if the grid is orthogonal, i.e., $x_\xi y_\eta + y_\xi x_\eta = 0$.

Letting w_A denote the cell average

$$w_A = \frac{\iint w dv}{\iint dv} , \tag{1.5}$$

we replace (1.3) with

$$\frac{d}{dt} (w_A \cdot V) + \sum_{i=1}^{4} \int_{S_i} (f dy - g dx) = 0, \tag{1.6}$$

where S_i are the sides of the quadrilateral and V is its area. We evaluate the line integrals using the midpoint rule. Normalizing the mesh so that the length of each quadrilateral is one, we replace (1.6) by the approximation

$$\frac{d}{dt}(w_A \cdot V) = -\sum_{\xi=\text{const}} (fy_\eta - gx_\eta) + \sum_{\eta=\text{const}} (fy_\xi - gx_\xi). \tag{1.7}$$

With second-order accuracy we can find w at the cell face by averaging w from the cell centers. This averaging is done in FLO52ST without accounting for the difference in volumes of neighboring cells. Given w at the cell face, one can then calculate the fluxes. It is preferable to average w rather than average the fluxes to help couple the even and odd points. For efficiency the pressure is also averaged rather than computed from w. The error in this procedure is of the same order as the error in the scheme. Several numerical checks have confirmed that this pressure averaging does not introduce any additional errors.

In the following sections we discuss ways to integrate (1.8) in time and to accelerate the convergence to a steady state. To measure this acceleration we need a way of deciding when a numerical steady state has been reached. In [8] the measure used was $\Delta\rho/\Delta t$. However, once acceleration techniques are used, it is important to measure the true residual, i.e., the right-hand-side of (1.8). For exterior problems the mesh is finer near the body and is coarser in the far field. As such the maximum residual usually occurs in the far field where the volumes are large. This is true even though the true flux is small in the far field even relative to the zonal areas. An alternative is to normalize the true residual by the cell area which tends to emphasize the zones near the airfoil.

2. Runge-Kutta Methods

In 1960 Lax and Wendroff [11] introduced a one-step method which was second-order in space and time. Richtmyer [15] suggested a two-step method which is linearly equivalent to the Lax-Wendroff scheme. The advantage of the two-step scheme is that only flux evaluations are necessary rather than matrix multiplications.

A disadvantage of these two-step schemes is that the two steps are different, complicating the coding. A more important disadvantage is that the Courant number is one, even though two steps are used. The leapfrog method requires only one step and has the same stability requirement. Hence, the leapfrog scheme is twice as efficient as a two-step Lax-Wendroff method. However, the leapfrog scheme is nondissipative and hence not useful for shocked flows. Nevertheless, we are interested in multistage schemes which are more efficient than the standard two-stage schemes. Graves [7] used a three-stage Runge-Kutta scheme for the Navier-Stokes equations. Van der Houwen [21], [22] analyzed multistage schemes for both hyperbolic and parabolic problems. A fourth-order Runge-Kutta scheme for the Euler equations was popularized in [8]. For simplicity we only consider two-space dimensions although the extension to higher dimensions is straightforward. Let

$$w_t = f_x + g_y. \tag{2.1}$$

An N stage Runge-Kutta scheme is given by

$$w^{(k)} = w^{(0)} + \alpha_k \, \Delta t\left(D_x \, f^{(k-1)} + D_y \, g^{(k-1)}\right), \tag{2.2}$$

with $w^{(0)} = w^n$, $w^{(n)} = w^{n+1}$, $\alpha_N = 1$. We note that this method is, in general, only first-order accurate in time even when the amplification matrix agrees with that of a higher-order method. In many cases one can achieve second-order accuracy in time. When one is only interested in the steady state solution, this is not a drawback. For time-dependent problems one may wish to have higher accuracy. One can then replace (2.2) with a true higher order Runge-Kutta method. The advantage of (2.2) is that only two levels of storage are required, i.e., $w^{(0)}$ and $w^{(k)}$ at each stage.

A fundamental question is how to choose the parameters α_k, $k=1,\cdots,N-1$. To simplify the question, we only consider the linear problem with constant coefficients. We then Fourier transform (2.2) and consider the following amplification matrix

$$G = 1 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_N z^n, \tag{2.3}$$

z is the Fourier transform of $\Delta t(D_x w + D_y w)$. Hence, if we use central differencing in x and y, then $z = i \cdot \frac{\Delta t}{\Delta x} \cdot \lambda(A \sin\theta + B \sin\phi)$ where $\lambda(A)$ denotes an eigenvalue of A. θ, ϕ are the Fourier variables, $\Delta x = \Delta y$, $A = \frac{\partial f}{\partial w}$, $B = \frac{\partial g}{\partial w}$. The parameters β_i are given by

$$\begin{aligned} \beta_1 &= 1, \\ \beta_2 &= \alpha_{N-1}, \\ \beta_k &= \beta_{k-1}\, \alpha_{N-k+1}, \qquad k = 3,\cdots,N. \end{aligned} \tag{2.4}$$

One possibility is to choose the β_i so that Δt is as large as possible. If the iteration is close, in some sense, to being time accurate, this implies that we advance in time as fast as possible. This requires that $|z|$ be as large as possible while requiring that $G^* G \leq 1$. In general, this Δt is only required for some frequencies which depend on the scheme and the matrices A and B. Using central differences the values of z lie along the imaginary axis while for an upwinded scheme, z will lie on some curve in the complex plane.

Using a central difference scheme the maximum time step is given by

$$\frac{\Delta t}{\Delta x} \leq \frac{K}{\max\limits_{\theta,\phi} \rho(A \sin\theta + B \sin\phi)}, \tag{2.5}$$

where $\rho(A)$ is the spectral radius of A and K is the maximum that z can achieve in (2.3). Vichnevetsky [25] (see also [16]) has shown that when z is purely imaginary that

$$|z| \leq N-1 \quad \text{for an } N \text{ stage scheme.} \tag{2.6}$$

This maximum can be achieved and the formula is given by [16], [21]

$$G = P_N(z) = i^{N-1}\, T_{N-1}\left(\frac{-iz}{N-1}\right) + \frac{i^N}{2}\left[T_N\left(\frac{-iz}{N-1}\right) - T_{N-2}\left(\frac{-iz}{N-1}\right)\right]. \tag{2.7}$$

For N odd, this formula coincides with that given by Van der Houwen [21] and the formula is second-order accurate. For N even, the formula is only first-order accurate.

In practice, an artificial viscosity is needed to stabilize the scheme (see section 4). This can be modelled, in one dimension, by

$$w_t = w_x - \varepsilon w_{xxxx}. \tag{2.8}$$

Using central differences, z now corresponds to $z = 1 \sin\theta - 16\,\varepsilon \sin^4 \frac{\theta}{2}$. $|z|$ now lies in the left half plane. Plotting the stability region of (2.3), (2.7) we see that the stability region contains the imaginary axis up to $N-1$ and also includes portions of the right-half plane. The stability region also includes part of the negative real axis up to z of about two to three (depending on N). Hence, as long as ε is sufficiently small the previous results are valid. At $\theta = \pi$, $z = -16\varepsilon$ and so the stability is governed only by the artificial viscosity. A similar analysis holds when the Navier-Stokes terms are added to the differential equation.

In the previous analysis we assumed that the artificial viscosity is reevaluated at each stage. In practice the computation of the artificial viscosity is expensive and so the same artificial viscosity is used for each stage within a cycle. Hence, (2.8) is approximated by

$$w^{(k)} = w^{(0)} + \alpha_k\, \Delta t\left[D_0\, w^{(k-1)} - \varepsilon D_+^2\, D_-^2\, w^{(0)}\right]. \tag{2.9}$$

Let z be the Fourier transform of $\Delta t \cdot D_0\, w$ and let y be the Fourier transform of Δt (artificial viscosity). Then the amplification factor is

$$G = 1 - (z+y)\left[\beta_1 + \beta_2\, z + \cdots + \beta_{N-1}\, z^{N-2}\right] + \beta_N\, z^N + y. \tag{2.10}$$

Therefore, the previous analysis does not hold since G depends on both z and y and not on one complex number. In various computational trials the optimal parameters (2.7) worked when the artificial viscosity was reevaluated at each stage. However, for N greater than four, the optimal parameters were not stable when the viscosity was frozen. It is not clear how much of these details are dependent on the exact formulation of the artificial viscosity.

When the artificial viscosity is reevaluated at each stage, then the use of a N stage formula used twice is equivalent to some $2N$ stage formula. Hence, it is always more optimal to use a higher stage formula. However, when the artificial viscosity is frozen at each stage, this no longer need be true. With the second-order central difference scheme we have seen, (2.6), that the Courant number for an N stage scheme has a maximum of $N-1$. Thus, the efficiency per stage is $(N-1)/N$. Hence, for N large, we approach the efficiency of the leapfrog method. At $N = 10$, we already have 90% efficiency and there is not much benefit in using a higher stage method. However, since this scheme requires the reevaluation of the artificial viscosity at each stage, these higher stage schemes are not efficient. An alternative is to evaluate the viscosity M times within an N stage scheme. The relationship between M and N to achieve maximum stability is not known. Furthermore, in many cases, robustness, i.e., including sections of the left-half of the complex plane, is more important than maximal time steps.

Until now we have assumed that the stability condition is mainly governed by z near the imaginary axis, i.e., central differences with a small artificial viscosity. Equations with variable coefficients or nonlinearities require us to consider perturbations off the imaginary axis even without the

presence of viscous effects, see [15]. For parabolic equations the appropriate z are along the negative real axis. The optimal parameters for this case were first considered in [17]. This is given by

$$G = T_n\left(1 + z/N^2\right), \qquad 0 \leq z \leq -2N^2 \tag{2.11}$$

for an N stage scheme. However, this formula is not stable for large negative z if z is slightly off the real axis. These perturbations can be caused by variable coefficients, lower order terms or boundary conditions. One type of perturbation is considered in [22]. Another perturbation of (2.11) is

$$G = (1 - \varepsilon)T_N(y) + \varepsilon\, y\, T_N(y)/N^2, \quad \text{with} \quad y = 1 + z/N^2. \tag{2.12}$$

The stability range of (2.12) is slightly smaller than but contains a large region off the real axis when $\varepsilon > 0$. In both cases the stability region for an N stage scheme is proportional to N^2. Hence, the more stages that are used the more efficient the method is. This is in contradistinction to the hyperbolic case where the efficiency per stage quickly approaches its upper bound which is given by the leapfrog scheme. Hence, when the number of stages for a parabolic problem is large, we can approach the time steps of an implicit scheme while retaining the advantages of an explicit scheme.

Using an upwinded scheme for a hyperbolic equation also results in eigenvalues that are not near the imaginary axis except for the longest of wave lengths. When one considers a pure one-sided scheme together with boundary conditions then the matrix is upper (or lower) diagonal and so all the eigenvalues lie along the negative real axis. Some experiments with this case are presented in [20].

In our analysis we have assumed that one approaches a steady state fastest by using the largest possible time step. This is intuitively obvious if one is using a time-like iteration procedure. However, when the iteration process is not time-accurate, the criterion for the optimal α_i need not be connected with the time step. This is well known for A.D.I. type schemes. In section 6 we shall show that when one uses residual smoothing, that the best strategy is not to choose the largest possible time step, even in one-space dimension. Recently, Jameson [9] has developed a multigrid code using the Runge-Kutta scheme as the smoothing operator. For this method one suspects that one would choose the α_i so as to damp the high frequencies. Hence, one choice for the parameters, in one dimension, is to choose the α_i so that

$$\min_{\alpha_i} \max_{\theta_0 < |\theta| < \pi} |G^* G(\theta)|, \tag{2.13}$$

for some θ_0, Δt and with the additional constraint that $|G^* G| \leq 1$, $-\pi \leq \theta \leq \pi$. Since there is not a well-developed theory for the multigrid method for hyperbolic equations, it is not clear that (2.13) yields the optimal parameters. Computations indicate that one wishes some combination of large time steps together with good dampling properties at high frequencies for the multigrid procedure to be efficient.

3. Time Step in Generalized Coordinates

We consider the two-dimensional equation

$$w_t + Aw_x + Bw_y = 0, \tag{3.1}$$

where A and B are constant matrices which represent the gradient of the fluxes appearing in (2.1). We shall only consider Runge-Kutta methods in time and second-order central differences in space. The effect of the artificial viscosity on the time step is ignored (see previous section for a more complete discussion). From (2.5) we see that for any multistep Runge-Kutta method that the stability criterion is of the form

$$\Delta t \cdot d \leq K, \tag{3.2}$$

where d is the maximum (over θ and ϕ) spectral radius of $D = A \sin\theta + B \sin\phi$. The constant K depends on the parameters of the Runge-Kutta scheme.

The Euler equations in general coordinates (ξ,η) are

$$(JW)_t + f_\xi + g_\eta = 0, \tag{3.3}$$

where

$$w = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} \qquad f = \begin{bmatrix} \rho q \\ \rho q u + y_\eta\, p \\ \rho q v - x_\eta\, p \\ q(E+p) \end{bmatrix} \qquad g = \begin{bmatrix} \rho r \\ \rho r u - y_\xi\, p \\ \rho r v + x_\xi\, p \\ r(E+p) \end{bmatrix} \tag{3.4}$$

$$J = x_\xi y_\eta - x_\eta y_\xi, \tag{3.5}$$

$$q = y_\eta u - x_\eta v, \qquad r = x_\xi v - y_\xi u. \tag{3.6}$$

The Jacobian J is equal to the volume of the cell that appears in the finite volume approach (see section 1). Let

$$w = q \sin\theta + r \sin\phi, \tag{3.7}$$

$$a = y_\eta \sin\theta - y_\xi \sin\phi, \qquad b = x_\xi \sin\phi - x_\eta \sin\theta, \tag{3.8}$$

$$h = \frac{E + p}{\rho}, \tag{3.9}$$

$$S^2 = \frac{(\gamma - 1)}{2}(u^2 + v^2), \tag{3.10}$$

then

$$D = \begin{bmatrix} 0 & a & b & 0 \\ aS^2 - uw & w - (\gamma-2)au & bu - (\gamma-1)av & (\gamma-1)a \\ bS^2 - vw & av - (\gamma-1)bu & w - (\gamma-2)bv & (\gamma-1)b \\ -w(h-S^2) & ah - (\gamma-1)wu & bh - (\gamma-1)wv & \gamma w \end{bmatrix} \tag{3.11}$$

Let

$$T = \begin{bmatrix} \frac{S^2}{\rho c} & \frac{-(\gamma-1)u}{\rho c} & \frac{-(\gamma-1)v}{\rho c} & \frac{\gamma-1}{\rho c} \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ S^2 - c^2 & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \end{bmatrix} \tag{3.12}$$

Then

$$D_0 = TDT^{-1} = \begin{bmatrix} w & ac & bc & 0 \\ ac & w & 0 & 0 \\ bc & 0 & w & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \tag{3.13}$$

where a,b are given by (3.8) and w by (3.7). Hence,

$$d = |w| + \sqrt{a^2 + b^2}\, c. \tag{3.14}$$

Letting $\Delta\xi = \Delta\eta = 1$, then the time restriction is of the form

$$\Delta t \leq \frac{K \cdot J}{|w| + \sqrt{a^2+b^2}\,c} \leq \frac{K \cdot J}{|q| + |r| + \sqrt{x_\xi^2+y_\xi^2+x_\eta^2+y_\eta^2 + 2|x_\xi x_\eta + y_\xi y_\eta|}\,c}, \tag{3.15}$$

where r and q are given by (3.6). In the code FLO52ST the stability criterion is implemently slightly differently as

$$\Delta t \leq \frac{K \cdot J}{|q| + |r| + \left(\sqrt{x_\xi^2 + y_\xi^2} + \sqrt{x_\eta^2 + y_\eta^2}\right)c} . \tag{3.16}$$

For most exterior problems a highly stretched mesh is used. In these cases the time step is governed by the area of the smallest cell which is much smaller than the area of the cells in the far field. To avoid this difficulty we use a different time step in each zone. This destroys the time accuracy of the solution but accelerates the convergence to the steady state. This local time step was first used in [13].

To see the effect of using a local time step we consider the radially symmetric wave equation

$$u_{tt} = \frac{1}{r}(ru_r)_r . \tag{3.17}$$

Discretizing the time variable we have

$$\lim_{\Delta t \to 0} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t)^2} = \frac{1}{r}(ru_r)_r . \tag{3.18}$$

Let Δt be the local time step $(\Delta t)_j$ and rewrite (3.18) as

$$\lim_{\Delta t \to 0} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{(\Delta t_{min})^2} = \frac{c^2(r)}{r}(ru_r)_r , \qquad c(r) = \frac{(\Delta t)_j}{(\Delta t)_{min}} . \tag{3.19}$$

Therefore, using a local time step is equivalent to introducing an artificial wave speed that increases as one goes to the far field. Hence, the further a wave goes towards infinity, the faster it goes. As an example, we consider a mesh that increases exponentially with r. Then $(\Delta t)_j$ is proportional to r and so $c(r) = r$. Then (3.19) becomes

$$u_{tt} = r(ru_r)_r. \tag{3.20}$$

Let $s = \log(r)$, (3.20) becomes

$$u_{tt} = u_{ss}. \tag{3.21}$$

If we only allow outgoing waves, then the solution to (3.21) is

$$u = f(s-t) = f(\log(r) - t). \tag{3.22}$$

Hence, if we begin with a wave of compact support at $r = 1$, $t = 0$, then at time t_0, the wave is centered at $r = \exp(t_0)$, i.e., the wave has moved exponentially fast.

4. Artificial Viscosity

The Runge-Kutta method with central differencing in space has two difficulties. The first difficulty is that the highest frequency is not damped, i.e., for a linear problem the neighboring points decouple. This odd-even decoupling prevents the possibility of driving the residual to machine

zero. With the nonlinear equations the variables are averaged at the cell faces before forming the fluxes. This nonlinearity couples all the neighbors together. Nevertheless this coupling is weak and convergence to a steady state can be slow. In order to accelerate the convergence a fourth-order linear viscosity is added to each equation.

A second difficulty with central differences is that it does not enforce an entropy decrease across shocks. As such the central difference may converge to the wrong solution. We attempt to enforce the entropy condition by introducing an artificial viscosity. The fourth-order viscosity does not help near shocks. In fact there are theoretical indications that the fourth-order viscosity can be a destabilizing factor [14]. These observations are confirmed by numerical experiments. Hence the fourth-order viscosity is turned off in the neighborhood of shocks. In order to prevent oscillations at the shock an additional viscosity is added which behaves as a second derivative. This is in the spirit of the Navier-Stokes equations and seems to yield shocks without any overshoots. This viscosity is a nonlinear viscosity so that the formal order of accuracy is not affected by the addition of the artificial viscosity.

We wish the artificial viscosity to accomplish two contradictory purposes. We want these terms to accelerate the convergence to a steady state. This implies that we should choose the viscosity as large as possible without violating any stability restrictions. On the other hand, we wish the viscosity to be as small as feasible so as not to affect the accuracy of the solution. Formally, the viscosity is of higher order than the truncation error. However, for a finite mesh, increasing the viscosity will decrease the accuracy of the steady state.

In section 7 we will introduce an upwinded scheme that does not require the use of an artificial viscosity. The use of an artificial viscosity has both advantages and disadvantages. The basic disadvantage is its artificiality. An artificial viscosity is not aesthetically appealing. Furthermore, there are invariably constants which must be adjusted for each case. This makes the code less robust. An upwind scheme has a built-in viscosity which should automatically adjust itself to each situation. The advantage of an artificial viscosity is its flexibility. We can use the freedom of arbitrary constants or functions to tailor the code to an individual problem. This requires more work on the part of the user but it can be beneficial. The upwind schemes are more automatic. However, in cases where one does not want any viscosity, it is difficult to turn off since it is built into the algorithm.

In order to introduce the artificial viscosity into the Runge-Kutta scheme we modify (2.2) and get

$$w^{(k)} = w^{(0)} + \alpha_k \, \Delta t\left(Lw^{(k-1)} + V_2 - V_4\right), \tag{4.1}$$

where Lw represents the approximation to the Euler equations. V_4 is given by

$$V_4 = \frac{C_4}{128}\left[D_{+\xi}\, D_{-\xi}\left(\frac{J}{\Delta t}\, D_{+\xi}\, D_{-\xi}\, w\right) + D_{+\eta}\, D_{-\eta}\left(\frac{J}{\Delta t}\, D_{+\eta}\, D_{-\eta}\, w\right)\right], \tag{4.2}$$

and V_2 will be presented later. Δt is the local time step while J is the area of the zone. C_4 is an arbitrary constant (usually about 0.2) while D_+ and D_- represent forward and backward differences respectively. ξ and η

are the curvilinear coordinates and we have assumed $\Delta\xi = \Delta\eta = 1$. As one approaches the boundary, V_4 can no longer be generated by (4.2). One alternative is to extrapolate the variables to an artificial position across the boundary and then to use (4.2). This is equivalent to using a one-sided approximation to (4.2). The viscosity added by V_4 is a completely dissipative mechanism. If we look at the differential level (4.2), (with $V_2 = 0$) leads to an approximation of

$$w_t = Lw - (Kw_{xx})_{xx}, \qquad K = \frac{c_4 J}{\Delta t}. \tag{4.3}$$

Multiplying (4.3) by w and integrating over all space we get

$$\frac{1}{2}\frac{d}{dt}\int |w|^2 \, dx = \int (w, Lw)dx - \int w(Kw_{xx})_{xx} \, dx. \tag{4.4}$$

Integrating the last integral by parts twice and ignoring boundaries we get

$$\frac{1}{2}\frac{d}{dt}\int w^2 \, dx = \int (w, Lw)dx - \int K(w_{xx})^2 \, dx. \tag{4.5}$$

Hence, as long as K is positive the viscosity terms decrease the total energy. An alternative to (4.2) is to use

$$V_4 = \frac{c_4}{128}\left[D_{+\xi}\left(\frac{J}{\Delta t} D_{+\xi} D^2_{-\xi} w\right) + D_{+\eta}\left(\frac{J}{\Delta t} D_{+\eta} D^2_{-\eta} w\right)\right]. \tag{4.6}$$

This version contains both dissipative and dispersive characteristics. The same technique can be used on the finite difference level using summation-by-parts. One can then also include boundary terms to see their effect.

Eriksson and Rizzi [4] choose boundary conditions so as to maximize the dissipation of the artificial dissipation. We shall see later that this may introduce errors into the steady state approximation.

The form of V_2 is given by

$$V_2 = \frac{c_2}{4}\left[D_{+\xi}\ \theta^x\ D_{-\xi}\ w + D_{+\eta}\ \theta^y\ D_{-\eta}\ w\right]. \tag{4.7}$$

θ is a switch which measures the gradients of the flow. In smooth regions θ should be of the order of Δ^2 while near shocks θ should be of order unity. Hence, this viscosity is of fourth-order in smooth regions and does not affect the order of the scheme. θ^x is given by

$$\theta^x_{j+1/2,k} = \max\left(\overline{\theta}_{j,k}\ \overline{\theta}_{j+1,k}\right). \tag{4.8}$$

Two possibilities for $\overline{\theta}$ are

$$\overline{\theta}_{j,k} = \kappa\ \frac{|p_{j+1,k} - 2p_{j,k} + p_{j-1,k}|}{|p_{j+1,k} + 2p_{j,k} + p_{j-1,k}|}, \qquad \kappa \sim 1.0, \tag{4.9a}$$

or

$$\overline{\theta}_{j,k} = \kappa\ \frac{|p_{j+1,k} - 2p_{j,k} + p_{j-1,k}|^2}{|p_{j+1,k} - p_{j,k}|^2 + |p_{j,k} - p_{j-1,k}|^2 + \varepsilon}, \qquad \kappa \sim 0.05. \tag{4.9b}$$

As with the fourth-order viscosity, (4.7) cannot be used directly at the points next to a boundary. Salas (see [1]) suggests extrapolating the variables to a virtual point outside the domain and then using (4.7). Eriksson and Rizzi [4] again implement the boundary terms so as to increase

the dissipative effect. However, computations indicate that the viscosity V_2 introduces errors near the boundaries. This error consists of two parts. Near the leading edge false entropy is generated which then forms an entropy layer along the body. At the same time a pressure jump occurs at the trailing edge which violates the Kutta condition. This problem is especially noticeable at high angles of attack. FLO52ST was used to find the solution about a NACA 0012 at 10° angle of attack with a free stream Mach number of 0.3. The solution is completely subsonic and so the Euler and potential solutions should agree. With the standard code a noticeable pressure jump is generated at the trailing edge. Since the flow is subsonic one can set $V_2 = 0$. Without the second-order viscosity the pressure is continuous at the trailing edge. Furthermore, comparisons of the lift between the potential and Euler codes show that the Euler code underpredicts the lift by about 9% on a coarse 64×16 O mesh On a finer mesh the lift is improved but is still much worse than the corresponding prediction of the potential code. Removing the V_2 component of the viscosity improves the lift prediction though it is still 5% too low on the 64×16 mesh. Careful checks show that the pressure jump at the trailing edge is due to the tangential component of the V_2 viscosity. Hence, this difficulty is not caused by the difficulty in evaluating (4.7) near the boundary. The cause is that the switch (4.9) becomes large near the leading and trailing edge. Hence, these regions are treated as if there were a shock and a loss of entropy is created. This entropy layer extends over the length of the airfoil and is different on the upper and lower surfaces. Hence, the stagnation pressure on the upper and lower surfaces are different leading to a pressure jump at the trailing edge. If the tangential component of the V_2 viscosity is set equal to zero then the pressure jump disappears.

However, there is no improvement in the lift prediction. Thus the lift is more sensitive to the normal component of the viscosity.

In [3] it is suggested that the viscosity V_2 be multiplied by a linear factor which is zero near the body and one in the far field. This improves the accuracy in many cases, however, it is ad hoc. An alternative is to multiply the viscosity by M^4. In stationary flow, shocks only occur in supersonic regions and so the viscosity is not changed near shocks. However, there is a stagnation point at the trailing edge and so the viscosity is turned off near the trailing edge.

5. Enthalpy Damping

With a second-order system in time one can add artificial terms that depend on the first-time derivative. Though such terms destroy the time accuracy they do not affect the steady state solution. These terms can be chosen so as to speed up the convergence to a steady state. Such applications have been used for SOR or the full potential equation. However, for a first-order system one cannot, in general, add on lower order terms since they are not zero in the steady state. For the Euler equations it is known that the total specific enthalpy is constant along each streamline in the steady state. If all the streamlines originate from a constant reservoir then the enthalpy will be constant in the entire region even in the presence of shocks. This constant enthalpy is known a priori from the inflow boundary condition. Therefore, one can add artificial terms to each equation that depend on the deviation of the local enthalpy from the steady state enthalpy. Such a forcing term is zero in the steady state and we will show that it can

accelerate the approach to the steady state. For simplicity of presentation we shall assume a one-dimensional isentropic fluid. We therefore consider the following modified equations.

$$\rho_t + u\rho_x + \rho u_x = \phi \tag{5.1a}$$

$$u_t + uu_x + \frac{c^2}{\rho}\rho_x = 0 \tag{5.1b}$$

$$\phi = -\alpha\rho(h - h_0), \tag{5.1c}$$

$$h = \frac{E + p}{\rho} = \frac{c^2}{\gamma - 1} + \frac{u^2}{2}, \tag{5.1d}$$

differentiating these equations, one obtains

$$\begin{aligned}\rho_{tt} + 2u\rho_{xt} + (u^2 - c^2)\rho_{xx} + u_t\,\rho_x + \rho_t\,u_x + 2uu_x\,\rho_x \\ - \rho u_x^2 - \rho\left(\frac{c^2}{\rho}\right)_x \rho_x = \phi_t + u\phi_x,\end{aligned} \tag{5.2}$$

also

$$\phi_t + u\phi_x = -\alpha[(h - h_0 + c^2)\rho_t + u(h - h_0)\rho_x]. \tag{5.3}$$

We ignore all terms involving the product of derivatives and then freeze the coefficients. We then have

$$\rho_{tt} + 2u\rho_{xt} - (c^2 - u^2)\rho_{xx} + \alpha(h - h_0 + c^2)\rho_t - \alpha u(h - h_0)\rho_x = 0. \tag{5.4}$$

This is a convective wave equation in which α multiplies the first time derivative of ρ. This is similar to the acceleration procedure used for the full potential equation. To reduce this to a standard wave equation we introduce new independent variables

$$\eta = x, \qquad \tau = \frac{Mx}{c\sqrt{1 - M^2}} + \sqrt{1 - M^2}\, t, \qquad M = \frac{u}{c}, \tag{5.5}$$

where we assume that $M \leq 1$. With this change, (5.4) becomes

$$\rho_{\tau\tau} - (c^2 - u^2)\rho_{\eta\eta} + \frac{\alpha}{\sqrt{1 - M^2}}\left[(1 - 2M^2)(h - h_0) + (1 - M^2)c^2\right]\rho_\tau - \alpha u(h - h_0)\rho_\eta = 0 \tag{5.6}$$

or

$$\rho_{\tau\tau} - d^2\, \rho_{\eta\eta} + K\rho_\tau + L\rho_\eta = 0. \tag{5.7}$$

Using a finite difference scheme we choose K proportional to $d/\Delta\eta$ so as to reach a steady state rapidly. In terms of the original variables this implies

$$\alpha\Delta t \quad \text{proportional to} \quad \frac{1 - M}{(1 - 2M^2)(h - h_0) + c^2(1 - M^2)}. \tag{5.8}$$

Note: Since α depends on $1/\Delta t$ the enthalpy damping is not a low order term and so it affects the stability limit.

As noted above this procedure is valid only for subsonic flow. For the potential equation it is well known that adding a ϕ_t term is not advisable in supersonic regions. Using enthalpy damping for the Euler equations it has been found experimentally that using a small amount (relatively) of enthalpy damping in the supersonic regions can accelerate convergence, especially for problems which are mainly supersonic.

In the code FLO52ST $\alpha = a_1(1 - M^2) + \alpha_2$, α_1, α_2 constant. In the above derivation we used the primitive variables. For the conservative variables, (5.1) is replaced by

$$\rho_t + L_1 = -\alpha\rho(h - h_0)$$

$$(\rho u)_t + L_2 = -\alpha\rho u(h - h_0) \tag{5.9}$$

$$(\rho v)_t + L_3 = -\alpha\rho v(h - h_0),$$

where L_i represent the standard Euler terms. The finite difference approximation to (5.9) is

$$\rho^{n+1} - \rho^n = -\Delta t L_1 - \alpha\Delta t\rho^{n+1}(h^n - h_0) \tag{5.10}$$

or

$$\rho^{n+1} = \frac{\rho^n - \Delta t L_1}{1 + \alpha\Delta(h^n - h_0)} . \tag{5.11}$$

We stress then in these formulas, α is always positive independent of the sign of $(h - h_0)$. Thus, the right-hand-side of (5.9) is not the equivalent of a forcing function that gives rise to an exponential decay in an ordinary differential equation. Replacing $(h - h_0)$ by $|h - h_0|$ in the definition of ϕ destroys the enthalpy damping and can lead to divergence.

For the energy equation we append to (5.1)

$$S_t + L_4 = 0. \tag{5.12}$$

Deriving the equation for $E - h_0 \rho$ one gets a forcing term that depends on $(h - h_0)^2$ which can lead to difficulties. In [8] this was fixed in a heuristic manner. An alternative is to replace (5.12) by

$$h_t + L_5 = -\beta(h - h_0) \tag{5.13}$$

where β may depend on the Mach number. Let $\hat{E} = E - h_0 \rho$ then combining (5.13) with (5.9) yields

$$\hat{E}_t = -(h - h_0)\left(\alpha\hat{E} + \frac{\beta}{\gamma}\rho\right) = -\left[\alpha(h - h_0) + \frac{\beta}{\gamma}\right]\hat{E} + \frac{\beta}{\gamma} p. \tag{5.14}$$

We note that when β is large we force the enthalpy to be equal to its steady state value. Hence (5.14) is a generalization of the isoenergetic systems.

Computations show that the enthalpy damping is also useful in removing temporal oscillations. Using the enthalpy damping, with a Mach number dependence, yields a more monontone convergence to a steady state than in the absence of damping. This property is especially useful if one uses an acceleration procedure on top of the Runge-Kutta scheme.

6. Residual Smoothing

Residual smoothing was first introduced by Lerat [12] for use with the Lax-Wendroff scheme. Jameson [9] later introduced a similar technique in conjunction with the Runge-Kutta schemes. We shall later compare the effects on both these methods. However, we first consider a two-step Runge-Kutta method. This scheme is given by

$$u^{(1)} = u^n + \alpha\Delta t\ Qu^{(n)}$$
$$u^{(2)} = u^n + \Delta t\ Qu^{(1)} \tag{6.1}$$

where Q denotes all the spatial derivatives. When an artificial viscosity is used it can be used either in both steps or else frozen at the same value for both stages. The residual smoothing is then given by

$$\prod_j \left(1 - \frac{\beta_j}{4}\,\delta_{xx}^{(j)}\right)\left(u^{n+1} - u^n\right) = u^{(2)} - u^n, \tag{6.2}$$

where the product is over all space dimensions. One can also replace the operator on the left-hand-side of (6.2) with a full multidimensional elliptic operator. Since this operator only involves constant coefficients on a rectangular region (in computational space), it can be inverted by a fast solver. However, we shall see that one should not use large time-steps in conjunction with (6.2) even in one-space dimension. Since the difficulties are not concerned with splitting errors there is not much of an advantage to consider multidimensional operators that are not in split form. Numerical experiments indicate that a multidimensional Laplacian is more effective in accelerating the flow to a steady state than (6.2) but not sufficiently fast to warrant the additional cost of inverting the multidimensional matrix even with a fast solver.

We now consider the constant coefficient problem in one-space dimension. Let Q be the second-order central difference and ignore the artificial viscosity. Taking the Fourier transform of (6.2) for $u_t = u_x$, we get

$$\left(1 + \beta \sin^2 \frac{\theta}{2}\right)(G-1) = i\lambda \sin \theta - \alpha\lambda^2 \sin^2 \theta$$
$$\lambda = \Delta t/\Delta x \tag{6.3}$$

or

$$G = 1 + \frac{i\lambda \sin \theta - \alpha\lambda^2 \sin^2 \theta}{1 + \beta \sin^2 \frac{\theta}{2}} . \tag{6.4}$$

The original two-step Runge-Kutta scheme $(\beta = 0)$ is stable when

$$\lambda \leq \frac{\sqrt{2\alpha-1}}{\alpha} . \tag{6.5}$$

For general β, the scheme is stable if we choose

$$\alpha \geq 1/2$$
$$\beta > 2\alpha\lambda\left(\lambda - \frac{\sqrt{2\alpha-1}}{\alpha}\right) . \tag{6.6}$$

Alternatively, if we replace β by $\sigma\lambda^2$, and $\alpha \geq 1/2$, $\sigma \geq 2\alpha$, then the scheme is unconditionally stable.

In three-space dimensions, (6.4) is replaced by

$$\left(1 + \sigma\lambda^2 \sin^2 \frac{\theta}{2}\right)\left(1 + \sigma\lambda^2 \sin^2 \frac{\phi}{2}\right)\left(1 + \sigma\lambda^2 \sin^2 \frac{\xi}{2}\right)(G-1) = i\lambda K - \alpha\lambda^2 K^2, \tag{6.7}$$

with $K = \sin \theta + \sin \theta + \sin \xi$. For stability we require that $G^* G \leq 1$. This occurs if and only if

$$-2\alpha\left(1 + \sigma\lambda^2 \sin^2 \frac{\theta}{2}\right)\left(1 + \sigma\lambda^2 \sin^2 \frac{\phi}{2}\right)\left(1 + \sigma\lambda^2 \sin^2 \frac{\xi}{2}\right) + \alpha^2 \lambda^2 K^2 + 1 \leq 0. \tag{6.8}$$

A sufficient condition for stability is that

$$-2\alpha\left[1 + \sigma\lambda^2\left(\sin^2 \frac{\theta}{2} + \sin^2 \frac{\phi}{2} + \sin^2 \frac{\xi}{2}\right)\right] + \alpha^2 \lambda^2 K^2 + 1 \leq 0. \tag{6.9}$$

Hence, the three-dimensional scheme is unconditionally stable if

$$\sigma \geq \max_{\rho,\phi,\xi} \frac{\alpha[\sin \theta + \sin \phi + \sin \xi]^2}{2\left(\sin^2 \frac{\theta}{2} + \sin^2 \frac{\phi}{2} + \sin^2 \frac{\xi}{2}\right)} . \tag{6.10}$$

Therefore, we have shown that a sufficient condition for the three-dimensional scheme (6.2) to be unconditionally stable is

$$\alpha \geq 1/2 , \qquad \sigma \geq 6\alpha. \tag{6.11}$$

Hence, the three-dimensional version of (6.2) is unconditionally stable. Moreover, since the Runge-Kutta scheme gives a steady state which is independent of Δt and (6.2) is in delta form we conclude that the steady state solution to (6.2) is also independent of Δt. We point out that the version of residual smoothing proposed by Lerat does not have a steady state independent of Δt. This is because the Thommen scheme used by Lerat has a Δt dependent steady state. Since the time steps used are not very large relative to the explicit time step this dependence on Δt may not be noticeable to graphical accuracy. We also note that for the one-dimensional Lax-Wendroff two-step methods (Richtmyer, MacCormack, etc.) and for the two-

dimensional Burstein scheme that the solution after the intermediate step is independent of Δt.

We wish to stress that even though (6.2) is unconditionally stable, choosing a large time step is not the best strategy. Even in one-space dimension choosing a very large time step severly retards the convergence to a steady state. This is not true, in one-space dimension, for the Lax-Wendroff scheme, if β depends on the matrix of the differential equation (see [12]). Hence, the use of residual smoothing with the Runge-Kutta scheme is fundamentally different than the backward Euler method. For the residual splitting method the inefficiency of large time steps has nothing to do with splitting errors. Let $\beta = \sigma\lambda^2$, (6.4) becomes

$$G(\theta) = 1 + \frac{i\lambda \sin\theta - \alpha^2 \lambda^2 \sin^2\theta}{1 + \sigma\lambda^2 \sin^2\frac{\theta}{2}}, \qquad \begin{matrix} \alpha \geq 1/2 \\ \sigma \geq 2\alpha \end{matrix} . \tag{6.12}$$

For large λ, (6.12) becomes

$$G(\theta) \simeq -\frac{\alpha^2}{\sigma} \frac{\sin^2\theta}{\sin^2\theta/2} . \tag{6.13}$$

We thus see that without artificial viscosity that the highest frequency, $\theta = \pi$ is not damped.

If we add an artificial fourth-order dissipation with coefficient ν at each stage, then (6.12) is replaced by

$$G(\theta) = 1 + \frac{\lambda\left(i \sin\theta - \nu \sin^4\frac{\theta}{2}\right) + \lambda^2\left(i \sin\theta - \nu \sin^4\frac{\theta}{2}\right)^2}{1 + \sigma\lambda^2 \sin^2\frac{\theta}{2}} . \tag{6.14}$$

As λ gets larger, we get

$$G(\pi) \sim 1 + \frac{\nu^2}{\sigma} > 1, \tag{6.15}$$

and so the scheme is not stable for large λ. We next consider the case that the same artificial viscosity is used at both stages. In that case (6.12) is replaced by

$$G(\theta) = 1 + \frac{i\lambda \sin\theta\left[1 + \alpha\lambda\left(i \sin\theta - \nu \sin^4 \frac{\theta}{2}\right)\right] - \lambda\nu \sin^4 \frac{\theta}{2}}{1 + \sigma\lambda^2 \sin^2 \frac{\theta}{2}} . \tag{6.16}$$

As λ approaches infinity, the coefficient of the artificial viscosity goes to zero relative to the denominator and so we recover (6.13) in the limit. Hence, when we freeze the artificial viscosity, the scheme remains unconditionally stable. However, when λ is large, the highest frequency is damped only a small amount proportional to $1/\lambda$. Therefore, if we wish to minimize G, we do not wish to choose λ large. An alternative is to choose a finite λ so that $\int_0^\pi G^2(\theta)d\theta$ is minimized.

We also note that if one adds the viscous terms from the Navier-Stokes equations, then a similar analysis holds. Hence, if one wants the scheme to be unconditionally stable then the Navier-Stokes terms should be frozen and one evaluated once per cycle. If the Navier-Stokes terms are reevaluated at each stage then the residual smoothing should be applied at every stage and not after the second stage.

The above analysis was for a two-stage Runge-Kutta scheme. For a multi-stage scheme one should apply the residual smoothing after every even stage. If the total number of stages is odd then one should do an extra residual smoothing after the cycle is completed. One can also show that the residual

smoothing will not stabilize a one-stage Runge-Kutta with central differences for a hyperbolic problem but the method is unconditionally stable for a one-stage method for parabolic problems. In fact it gives the backward Euler methods. Hence, for the Navier-Stokes equations one should use residual smoothing after every stage.

7. Highly Subsonic Flow

It is well known that for very subsonic flow that the flow can be considered as incompressible. The use of the compressible fluid equations is considered as inefficient, since the fluid velocity is much smaller than the speed of sound. The use of an explicit scheme requires Δt to be bounded by $1/c$. However, the physical properties change over time scales of order $1/u$ which is much larger. Similar arguments hold for viscous flows with a high Reynolds number. Hence, it is usually agreed that explicit schemes are inefficient for highly subsonic flows. We shall now show that if one is only interested in the steady state then a minor change to the code can greatly increase the efficiency of the explicit method. Even if one is using an implicit method the following changes should increase the efficiency of the scheme since all waves have similar speeds.

The Euler equations are expressed as

$$w_t + f_x + g_y = 0, \tag{7.1}$$

where (x,y) represent general curvilinear coordinates (see (3.3)). Since we are only interested in the steady state we replace (7.1) by the system

$$M^{-1} w_t + f_x + g_y = 0. \tag{7.2}$$

The requirements on M are that the matrix be nonsingular and that the original initial boundary value problem still be well-posed. It is straightforward to solve (7.2) with an explicit scheme. With an implicit method only the diagonal portion of the matrix to be inverted is changed. Though the code solves (7.2) we shall only analyze the constant coefficient problem

$$M^{-1} w_t + Aw_x + Bw_y = 0, \tag{7.3}$$

where the matrices M, A, B are constant and $A = \frac{\partial f}{\partial w}$, $B = \frac{\partial g}{\partial w}$. Let $w^{(0)} = Tw$, $A_0 = T\,A\,T^{-1}$, $B_0 = T\,B\,T^{-1}$, $M_0^{-1} = T\,M^{-1}\,T^{-1}$, where T is given by (3.12). Then (7.3) can be converted to

$$M_0^{-1} w_t^{(0)} + A_0 w_x^{(0)} + B_0 w_y^{(0)} = 0 \tag{7.4}$$

with

$$A_0 = \begin{bmatrix} q & c & 0 & 0 \\ c & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \quad B_0 = \begin{bmatrix} r & 0 & c & 0 \\ 0 & r & 0 & 0 \\ c & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \tag{7.5}$$

where q and r, defined in (3.6), are the contravariant velocity components.

If $M_0 = I$, then we have not changed the eigenvalues or the stability condition of (7.1). We now consider the case that $u^2 + v^2 << c^2$. We wish to choose M_0^{-1} so that the eigenvalues of $M_0 A_0$ and $M_0 B_0$ are independent of c. In addition, we wish M_0 to be positive definite. This will imply that (7.3) is a symmetric hyperbolic system and so is well-posed. One choice is

$$M_0^{-1} = \begin{bmatrix} \frac{c^2}{z^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{7.6}$$

where $z^2 = \max(\varepsilon, u^2 + v^2)$. ε is introduced so that the matrix M_0^{-1} is not singular at stagnation points. In particular, $\varepsilon = .01c$ seems to give reasonable results. Transforming back to the curvilinear coordinates we define

$$Q = \begin{bmatrix} s^2 & -u & -v & 1 & s^2 = \frac{u^2 + v^2}{2} \\ us^2 & -u^2 & -uv & u & \\ vs^2 & -uv & -v^2 & v & h = \frac{c^2}{\gamma-1} + s^2 \\ hs^2 & -uh & -vh & h & \end{bmatrix} \tag{7.7}$$

Then

$$M^{-1} = I + dQ, \qquad M = I + eQ, \tag{7.8}$$

where

$$d = \frac{\gamma-1}{c^2}\left(\frac{c^2}{z^2} - 1\right), \qquad e = \frac{\gamma-1}{c^2}\left(\frac{z^2}{c^2} - 1\right). \tag{7.9}$$

We note that given the first row of Q, the following rows are derived by multiplying the first row by u,v,h, respectively. Hence, the product of Q times a vector requires only six multiplications. For use in a Runge-Kutta scheme we evaluate the fluxes in (7.2) as usual including the artificial viscosity. The vector of the changes in the variables Δw is then multiplied by the matrix M where the elements of Q are evaluated at the previous stage. The variables at the next stage can then be evaluated.

Let $\overline{M}^2 = \frac{z^2}{c^2}$, then the largest eigenvalue of $D = A \sin\theta + B \sin\phi$ is given by

$$\lambda = \frac{|w|(1 + \overline{M}^2) + \sqrt{w^2(1 - \overline{M}^2) + 4(a^2 + b^2)z^2}}{2}, \tag{7.10}$$

where w, a, b are given in (3.8), (3.9). We see that at a stagnation point $\overline{M} = O(\varepsilon)$ and $\lambda = O(\varepsilon)$. For $\overline{M} = 1$, $\lambda = |w| + \sqrt{a^2 + b^2}\, c$. Hence, at low Mach numbers the largest eigenvalues (and hence the time step) is independent of the sound speed c. At transonic sound speeds the largest eigenvalue is comparable to the case with $M = I$. Comparing with (3.16) we see that the preconditioned form (7.2) allows the use of a larger time step for all subsonic flows. Applications to the incompressible Navier-Stokes equations are presented in [19] together with extensions to supersonic flow.

8. Upwind Schemes

At each stage of the Runge-Kutta scheme we have replaced the flux derivatives by central differences. This necessitated the use of an artificial viscosity. This stabilized the scheme in both smooth regions and

provided an entropy condition in shocked regions. To avoid this artificial aspect one can replace the derivatives by appropriate upwind differences.

Presently this is implemented using the flux splitting described by Van Leer [24] for the Euler equations. At each zone face ξ,η = constant, we calculate

$$w^{-}_{i+1/2,j} = w_i + 1/2\,\delta_i\, w \tag{8.1a}$$

$$w^{+}_{i-1/2,j} = w_i - 1/2\,\delta_i\, w. \tag{8.1b}$$

δ_i is constructed from the forward and backward differences using a switch. This switch prevents overshoots when the variables change dramatically over a zone width. The flux is then split into plus and minus contributions depending on the sign of the eigenvalues. These fluxes are calculated in a rotated system using the velocity components parallel and perpendicular to each coordinate direction. The result fluxes are then rotated back to yield the Cartesian fluxes. We then combine the plus and minus contributions from neighboring cells to obtain the flux at the face of the cell. This is done independently in each direction. Given the fluxes at all four faces of the cell we update the variables to the next stage. The coefficient of the Runge-Kutta scheme given by (2.7) were appropriate for central differences. Optimal values of the parameters for one-sided schemes are not known. Further details of the scheme are presented in [20].

Because of the extra logic involved in an upwind scheme it is more costly per stage than a central difference scheme. Though a central difference scheme requires an artificial viscosity it can be calculated once and then reused at each stage of the Runge-Kutta scheme. Since the viscosity of an

upwind scheme is built in, it is more difficult to perform some operations once and then reuse them during the k stage scheme. A four-stage scheme using upwind differences requires about three times as much computer time as the corresponding four-stage central difference scheme. In addition, the enthalpy damping technique of section 5 can not be used. This occurs since the enthalpy is not a constant in the steady state when using flux splitting. The stability limit for the one-sided scheme is only about two-thirds of that allowed for the central difference scheme. Hence, the present version of the upwind scheme is about five times slower than the central difference scheme to reach a steady state with a given tolerance. The chief advantage of the upwind scheme is its robustness. There is no need to choose constants for the artificial viscosity. Preliminary test indicate that the upwind scheme works for a larger range of inflow Mach numbers than the central difference scheme.

9. Boundary Conditions

In addition to advancing the scheme in the interior, it is necessary to supply boundary conditions. At the airfoil the normal component of the velocity is zero. Using the finite volume approach, it is only necessary to know the pressure on the airfoil. This pressure is found by the normal momentum equation. It is also necessary to give boundary conditions in the far field. When the flow is subsonic at infinity one should specify three conditions at inflow and one condition at outflow.

In one-space dimension, in order to decrease the energy as fast as possible and reach a steady state rapidly, one should specify characteristic conditions. Diagonalizing the wave equation one gets

$$u_t = u_x, \qquad v_t = v_x, \qquad 0 \leq x \leq 1$$
$$u(1,t) = \alpha v(1,t), \qquad v(0,t) = \beta u(0,t) \tag{9.1}$$

u,v are the characteristic variables. Specifying characteristic boundary conditions is equivalent to $\alpha = \beta = 0$. From (9.1) it follows that

$$\frac{d}{dt}\int_0^1 (u^2 + v^2)dx = (\alpha^2 - 1)v^2(1,t) + (\beta^2 - 1)u^2(0,t). \tag{9.2}$$

Hence, choosing $\alpha = \beta = 0$ minimizes the right-hand-side.

Therefore at inflow, one should specify three characteristic variables. However, we require that the enthalpy be constant over the entire field in the steady state. To achieve this it is necessry to specify enthalpy at inflow. This condition replaces one of the characteristic boundary conditions. For nonlinear problems it is more appropriate to specify the Riemann variables rather than characteristic variables. Hence, at inflow we specify

$$p/\rho^\gamma \tag{9.3a}$$

$$v \tag{9.3b}$$

$$h = \frac{E + p}{\rho}. \tag{9.3c}$$

u is the component of velocity perpendicular to the boundary while v is the component parallel to the boundary. The fourth boundary condition is found by extrapolation from the interior. For stability it is preferable to extrapolate characteristic variables [5]. Hence, at the inflow we extrapolate

$$u - \frac{2c}{\gamma-1} \, . \tag{9.3d}$$

At the outflow boundary we reverse the procedure and specify

$$u - \frac{2c}{\gamma-1} \, , \tag{9.4a}$$

and extrapolate

$$p/\rho^2 \tag{9.4b}$$

$$v \tag{9.4c}$$

$$h \tag{9.4d}$$

An alternative to (9.4) is to use nonreflecting boundary conditions [2].

Numerical experiments indicate that the far field boundary exerts a much greater influence on the drag and lift coefficients than does the boundary condition at the airfoil.

10. Conclusions

We have discussed many of the components of the code FLO52ST. This code gives a rapid solution to the Euler equations in both two- and three-space dimensions for a variety of geometries and a range of Mach numbers.

With central differences it is easy to increase the accuracy of the differences to fourth-order or spectral accuracy. Using an O mesh all variables are periodic around the airfoil. Therefore, high-order differences do not encounter any boundary difficulties. Using spectral methods a Fourier scheme would be appropriate. With a C mesh all boundaries are in the far field. Hence, one can simply reduce the order of accuracy near the outer

boundaries where the flow is smooth enough for second-order accuracy to be adequate. The approximation normal to the boundary requires more care with regard to boundary conditions at the airfoil surface. Hence, it is reasonable to consider second-order differences normal to the airfoil while using higher-order methods parallel to the body. In this case one must be careful in approximating the metric derivatives so that the constant flow in the far field remains a solution to the finite difference equations. Work is also in progress on extending the code to the Navier-Stokes equations.

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16. Abstract

A multi-stage Runge-Kutta method is analyzed for solving the Euler equations exterior to an airfoil. Highly subsonic, transonic and supersonic flows are evaluated. Various techniques for accelerating the convergence to a steady state are introduced and analyzed.

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