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PART II: THE LINEAR QUADRATIC OPTIMAL
CONTROL PROBLEM

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LEGENDRE-TAU APPROXIMATION FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

PART II: THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

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Abstract

The numerical scheme based on the Legendre-tau approximation is proposed to approximate the feedback solution to the linear quadratic optimal control problem for hereditary differential systems. The convergence property is established using Trotter ideas. The method yields very good approximations at low orders and provides an approximation technique for computing closed-loop eigenvalues of the feedback system. A comparison with existing methods (based on "averaging" and "spline" approximations) is made.

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INTRODUCTION

This paper is the continuation of the study [9] on the use of Legendre-tau approximation for functional differential equations (FDE) and concerns the problem of constructing feedback solutions to linear quadratic regulator problems for hereditary systems. This problem has received a rather extensive study and we refer to [14], [2] and [4] for the summary of the earlier contributions. Our approach is based upon the pioneering work of Banks - Burns [2] who clarified the idea of approximating FDE by systems of finite dimensional ordinary differential equations and applied it to optimal control problems; i.e., the convergence of a particular numerical scheme (so-called 'averaging' approximation) is established, using the Trotter-Kato theorem of linear semigroups. Recently, Gibson [8] has developed the approximation theory for the Riccati equations associated with a hereditary system and applied it to the averaging approximation scheme.

The purposes of this paper are: (i) to apply the basic idea developed in [9] to the linear quadratic regulator problem, (ii) to prove convergence of numerical approximations of the feedback control laws and, (iii) to demonstrate the feasibility of our numerical schemes.

For the multiple point delay case, the solution to the algebraic Riccati equation (ARE) has jump discontinuities as shown in [8]. With this consideration, an extended version of the scheme described in [9] is developed for such a case in Section 3. As pointed out in [9] and will be discussed in Sections 3 and 5, the tau approximation differs from the standard Galerkin approximation and because of this, the theory developed in [8] needs to be modified to prove convergence of approximate solutions to ARE.

An important question that arose in [8] was concerned with the preservation of exponential stability under approximation (i,e., conjecture 7.1 in [8]). At this time, we have not been able to answer this question for the Legendre-tau approximation. However, some of the numerical computations for several examples indicate that the conjecture holds for the tau approximation. Moreover, the related property (i.e., the uniform boundedness of approximate solutions to ARE) is proved for certain special cases in Lemma 5.4. The other discussion contained in [8] is to argue the strong convergence of approximate solutions to ARE. Under the same condition as given in Lemma 5.4, one can prove it for the tau approximation. However, instead of arguing this, we state a rather interesting result in Theorem 5.1. It says that if a sequence of approximate solutions to ARE converges weakly to the solution to ARE, then the closed loop system which results from the approximate feedback control law is exponentially stable for sufficiently large orders of approximation.

As will be discussed in Section 6, the tau method may offer considerable improvements over other methods (e.g., those discussed in [4], [8]) and it gives a good approximation to the closed loop eigenvalue.

The following is a brief summary of the contents of this paper. In Section 2 we review the equivalence results between FDE and abstract Cauchy problems on the product space $\mathbb{R}^n \times L_2$ and results on the regulator problem for hereditary differential systems. In Section 3 we introduce the numerical scheme based on the Legendre-tau approximation for the multiple point delay case and the basic convergence of approximate semigroups using the Trotter-Kato theorem. In Section 4 we show how one can use the numerical scheme described in Section 3 to obtain the feedback solutions. In Section 5 we state the basic convergence property of approximate solutions to ARE.

Finally, in Section 6 we present numerical results and compare these results for those obtained by other methods [4], [8]).

Throughout this paper the following notation will be used. $r > 0$ stands for the largest delay time appearing in the FDE. The Hilbert space of $\mathbb{R}^n$-valued square integrable functions on the interval $[a,b]$ is denoted by $L_2([a,b];\mathbb{R}^n)$. When the underlying space and interval can be understood from the context, we will abbreviate the notation and simply write L_2, $L_2^{loc}([0,\infty);\mathbb{R}^n)$, or L_2^{loc}, is the space of $\mathbb{R}^n$-valued locally square integrable functions on the semi-infinite interval $[0,\infty)$. H^k is the Sobolev space of $\mathbb{R}^n$-valued functions f on a compact interval with $f^{(k-1)}$ absolutely continuous and $f^{(k)} \in L_2$. We denote by Z the product space $\mathbb{R}^n \times L_2([-r,0];\mathbb{R}^n)$. Given an element $z \in Z$, $\eta \in \mathbb{R}^n$ and $\phi \in L_2$ denote the two coordinates of $z : z = (\eta,\phi)$. The bracket $\langle\cdot,\cdot\rangle_H$ stands for the inner product in the Hilbert space H, and the subscript for the underlying Hilbert space will be omitted when understood from the context. $\|\cdot\|$ denotes the norm for elements of a Banach space and for operators between Banach space, while $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$.

If X and Y are Banach spaces, then the space of bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$. $\mathcal{D}(A)$ denotes the domain of a linear operator A. χ_I denotes the characteristic function of the interval I. Fianlly, for any function ϕ of independent variable θ, we shall use $\dot{\phi}$ or $\frac{\partial}{\partial\theta}\phi$ to denote the derivative of ϕ with respect to θ.

2. RICCATI EQUATIONS

In this section, we state the type of problems to be considered and recall some results on the linear quadratic regulator problem for hereditary differential systems.

Given $(\eta,\phi) \in Z$ and $u \in L_2^{loc}([0,\infty), \mathbb{R}^n)$, we consider the initial value problem

$$\frac{d}{dt} x(t) = \int_{-r}^{0} d\mu(\theta)\, x(t+\theta) + Bu(t), \tag{2.1}$$

$$x(0) = \eta, \quad x(\theta) = \phi(\theta), \quad \theta \in [-r,0),$$

where μ is a matrix-valued function of bounded variation on $[-r,0]$ with the form

$$\mu(\theta) = \sum_{i=0}^{\ell} A_i\, \chi_{(-\theta_i,0]}(\theta) + \int_{-r}^{\theta} A(s)ds \tag{2.2}$$

with $0 = \theta_0 < \theta_1 < \cdots < \theta_\ell = r$. A_i and $A(\cdot)$ are $m\times n$ matrices, the elements of the latter being square integrable on $[-r,0]$. Alternatively, for $t > 0$

$$\int_{-r}^{0} d\mu(\theta)x(t+\theta) = \sum_{i=0}^{\ell} A_i\, x(t-\theta_i) + \int_{-r}^{0} A(\theta)x(t+\theta)d\theta.$$

It is well known [2], [5], [6] that for $(\eta,\phi) \in Z$ and $u \in L_2^{loc}$, (2.1) admits a unique solution $x \in L_2([-r,T];\mathbb{R}^n) \cap H^1([0,T];\mathbb{R}^n)$ for any $T > 0$, and that (2.1) can be formulated as an evolution equation on Z

$$\frac{d}{dt} z(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \qquad t > 0 \tag{2.3}$$

where $z(t) = \big(x(t), x(t+\cdot)\big) \in Z$, $t > 0$ and $u = (Bu,0) \in Z$ for

$u \in \mathbb{R}^m$. The infinitesimal generator A is defined by

$$\mathcal{D}(A) = \{(\eta,\phi) \in Z \mid \eta = \phi(0) \text{ and } \dot{\phi} \in L_2\} \tag{2.4}$$

and for $(\phi(0),\phi) \in \mathcal{D}(A)$

$$A(\phi(0),\phi) = \left(\int_{-r}^{0} d\mu(\theta)\phi(\theta),\dot{\phi}\right). \tag{2.5}$$

The C_0-semigroup generated by A on Z will be denoted by $\{S(t)\ t \geq 0\}$.

Consider the optimal control problem on a finite interval $[0,1]$: for given initial $(\eta,\phi) \in Z$,

$$\text{minimize } J(u;[0,T]) = \int_0^T \left(|Cx(t)|^2 + |u(t)|^2\right)dt + |Rx(T)|^2, \tag{2.6}$$

over $u \in L_2([0,T];\mathbb{R}^m)$ subject to (2.1). Here C and R are $p\times n$ matrices. Within the framework of (2.3), (2.6) can be written as

$$J(u) = \int_0^T \left(|\mathcal{C}\, z(t)|^2 + |u(t)|^2\right)dt + |\mathcal{R}z(T)|^2$$

where $\mathcal{C}(\eta,\phi) = C\eta$ and $\mathcal{R}(\eta,\phi) = R\eta$ for $(\eta,\phi) \in Z$. It then follows from [1], [7] that the optimal solution u^0 to (2.6) is given by

$$u^0(t) = -\mathcal{B}^* \Pi(t)z^0(t), \qquad t \geq 0 \tag{2.7}$$

where $\Pi(\cdot)$ is the unique solution, within class of non-negative (definite) self-adjoint operators for which $\langle\Pi(t)z,z\rangle$ is absolutely continuous on

[0,T], of the Riccati equation

$$\frac{d}{dt}\langle \Pi(t)z,z\rangle = -2\langle Az,\Pi(t)z\rangle + \langle B^* \Pi(t)z, B^* \Pi(t)z\rangle - \langle Cz,Cz\rangle$$

$$\text{for all} \quad z \in \mathcal{D}(A) \qquad (2.8)$$

$$\Pi(T) = R^*R ,$$

and $z^0(t)$ satisfies the evolution equation

$$\frac{d}{dt} z^0(t) = \left(A - B^* B\Pi(t)\right) z^0(t), \quad t > 0 \qquad (2.9)$$

$$z^0(0) = (\eta,\phi).$$

Now we consider the optimal control problem on the infinite interval. For given initial data $z = (\eta,\phi)$, minimize the cost functional

$$J(u,z) = \int_0^\infty \left(|Cz(t)|^2 + |u(t)|^2\right)dt, \qquad (2.10)$$

subject to (2.3).

<u>Definition 2.1.</u>

(i) (A,B) is stabilizable if there exists a bounded operator K such that $A - BK$ generates a uniformly exponentially stable semigroup.

(ii) (C,A) is detectable if (A^*, C^*) is stabilizable.

Remark 2.2. For hereditary differential systems, the condition (ii) is equivalent to

$$z \in \mathcal{D}(A), \quad Az = \lambda z \quad \text{and} \quad Cz = 0$$

for $\lambda \in \mathbb{C}^+$ imply that $z \equiv 0$. Moreover, (ii) holds if

$$\operatorname{rank}\left[\Delta(\lambda)^T, C^T\right] = n \qquad \text{for all } \lambda \in \mathbb{C}^+$$

where $\Delta(\lambda) = \lambda I - \int_{-r}^{0} d\mu(\theta)e^{\lambda\theta}$.

An operator $\Pi \in \mathcal{L}(Z)$ is a solution of the algebraic Riccati equation (ARE) if

$$2\langle Az,\Pi z\rangle - \langle B^* \Pi z, B^* \Pi z\rangle + \langle Cz,Cz\rangle = 0$$

$$\text{for all } z \in \mathcal{D}(A). \qquad \text{(ARE)}$$

The next theorem follows from [7], [17].

Theorem 2.3.

(i) If (A, B) is stabilizable, then (ARE) has a self-adjoint, non-negative solution.

(ii) If (C,A) is detectable, then (ARE) has at most, one self-adjoint, non-negative solution. Moreover, if Π denotes the said solution, then $A - BB^* \Pi$ generates a uniformly exponentially stable semigroup.

(iii) <u>If</u> $(\mathcal{A},\mathcal{B})$ <u>is stabilizable and</u> $(\mathcal{C},\mathcal{A})$ <u>is detectable, then (ARE) has a unique self-adjoint, non-negative solution and the optimal control to (2.10) is given by</u>

$$u^0(t) = -\mathcal{B}^* \Pi z^0(t), \tag{2.11}$$

<u>where</u> $z^0(t)$ <u>is the mild solution to</u>

$$\frac{d}{dt} z^0(t) = (\mathcal{A} - \mathcal{B}\mathcal{B}^* \Pi) z^0(t)$$

$$z^0(0) = z.$$

In what follows, we assume that condition (iii) in Theorem 2.3 holds and recall some of the important results due to Gibson [8].

Theorem 2.4. <u>If</u> Π <u>is the self-adjoint, non-negative solution to (ARE), then</u>

$$\Pi Z \subset \mathcal{D}(\mathcal{A}^*).$$

Note that $\mathcal{D}(\mathcal{A}^*)$ consists of elements $(y,\psi) \in Z$ for which $z(\theta) = \psi(\theta) - \sum_{i=1}^{\ell} A_i^T \chi_{(-\theta_i,0]}\, y$ is absolutely continuous on $[-r,0]$ with $z(-r) = 0$, [16]. If we write Π as a matrix of operators on $Z = \mathbb{R}^n \times L_2$;

$$\Pi = \begin{bmatrix} \Pi^{00} & \Pi^{01} \\ \Pi^{10} & \Pi^{11} \end{bmatrix}, \tag{2.12}$$

where Π^{00} is a non-negative, symmetric n×n matrix, $\Pi^{10}(\cdot)$ is a square integrable matrix function on $[-r,0]$, $\Pi^{01} = \Pi^{10*}$ and

$$\Pi^{01}\,\phi = \int_{-r}^{0} \Pi^{10}(\theta)^T\,\phi(\theta)d\theta, \qquad \phi \in L_2,$$

and Π^{11} is a non-negative, self-adjoint operator on L_2, then from (2.11) the optimal control u^0 may be written as

$$u^0(t) = -B^T\Big(\Pi^{00}\,x(t) + \int_{-r}^{0} \Pi^{10}(\theta)^T\,x(t+\theta)d\theta\Big).$$

From Theorem 2.4 we have

Theorem 2.5. $\Pi^{10}(\cdot)$ <u>is piecewise absolutely continuous on</u> $[-r,0]$ <u>with the jump conditions at</u> $-\theta_i$, $1 < i < \ell-1$

$$\Pi^{10}\big((-\theta_i)^+\big) - \Pi^{10}\big((-\theta_i)^-\big) = A_i^T\,\Pi^{00}.$$

<u>Also</u>,

$$\Pi^{10}(-r) = A_\ell^T\,\Pi^{00}.$$

Let us define an operator on Z×Z

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}$$

with $\mathcal{D}(H) = \mathcal{D}(A) \times \mathcal{D}(A^*)$. Then

Theorem 2.6. H <u>is closed and densely defined and has compact resolvent. For a complex number</u> λ with $\mathrm{Re}\,\lambda < 0$,

$$\lambda \in \sigma(A - \mathcal{B}\mathcal{B}^*\Pi) \quad \underline{\text{if}} \quad \lambda \in \sigma(H).$$

<u>The algebraic and geometric multiplicities of</u> λ <u>as an eigenvalue of</u> $A - \mathcal{B}\mathcal{B}^*\Pi$ <u>are finite and are identical to the respective multiplicities of</u> λ <u>as an eigenvalue of</u> H. <u>Moreover</u>, λ <u>is an eigenvalue of</u> H <u>if</u> $\det \hat{\Delta}(\lambda) = 0$ <u>where</u>

$$\hat{\Delta}(\lambda) = \lambda I - \begin{bmatrix} \int_{-r}^{0} d\mu(\theta)\, e^{\lambda\theta} & -BB^T \\ -C^T C & \int_{-r}^{0} d\mu(\theta)^T\, e^{\overline{\lambda}\theta} \end{bmatrix}.$$

3. LEGENDRE-TAU APPROXIMATIONS

As pointed out in Section 2, for the multiple point delay case, $\Pi^{10}(\cdot)$ has jump discontinuities. If we were to try to approximate the solution to $\Pi^{10}(\cdot)$ using a series of polynomials on $[-r,0]$, we would observe the so-called Gibbs phenomena. To avoid this difficulty, we proceed as follows. For simplicity of exposition we deal with the system of the form;

$$\frac{d}{dt}\, x(t) = A_0\, x(t) + A_1\, x\big(t + (-\theta_1)^+\big) + A_2\, x(t - \theta_2)$$

$$+ \int_{-r}^{0} A(\theta)\, x(t + \theta)d\theta + Bu(t), \qquad (3.1)$$

with $-r = -\theta_2 < -\theta_1 < 0$.

Alternatively, if $z(t,\theta) = x(t + \theta)$, then

$$\frac{\partial}{\partial t}\, z(t,\theta) = \frac{\partial}{\partial t}\, z(t,\theta), \qquad -r < \theta < 0 \qquad (3.1a)$$

$$\frac{d}{dt}\, z(t,0) = A_0\, z(t,0) + A_1\, z(t,-\theta_1) + A_2\, z(t,-r) + \int_{-r}^{0} A(\theta)\, z(t,\theta) + Bu(t). \qquad (3.1b)$$

The approximate solution $z^N(t,\theta)$ is assumed to be represented as

$$z^N(t,\theta) = \sum_{k=0}^{N} a_k\, p_k^{(1)}(\theta)\chi_{(-\theta_1,0]}(\theta) + \sum_{k=0}^{N} b_k\, p_k^{(2)}(\theta)\chi_{[-r,-\theta_1]}(\theta), \qquad (3.2)$$

where

$$p_k^{(1)}(\theta) = p_k\big((2\theta + \theta_1)/\theta_1\big),$$

$$p_k^{(2)}(\theta) = p_k\big((2\theta + \theta_1) + \theta_2 - \theta_1)/(\theta_2 - \theta_1)\big),$$

for $0 < k < N$ and $\{p_k\}_{k\geq 0}$ are the Legendre polynomials on $[-1,1]$. Note that $(\partial/\partial\theta)z^N$ is given by the following as an element in H^{-1}:

$$\frac{\partial}{\partial\theta} z^N(t,\theta) = \sum_{k=0}^{N} (2/\Delta_1)a_k(t)\ \dot{p}_k^{(1)}(\theta)\chi_{(-\theta_1,0]}(\theta)$$

$$+ \sum_{k=0}^{N} (2/\Delta_2)b_k(t)\ \dot{p}_k^{(2)}(\theta)\chi_{[-r,-\theta_1]}$$

$$+ \Big(\sum_{k=0}^{N} a_k(t)\ p_k^{(1)}(\theta) - \sum_{k=0}^{N} b_k(t)\ p_k^{(2)}(\theta)\Big)\ \delta(\theta - \theta_1),$$

where $\Delta_1 = \theta_1$, $\Delta_2 = \theta_2 - \theta_1$, and $\delta(\cdot)$ is the delta function. The underlying ideas of the tau method for approximating (3.1) are: (i) equating (3.1a) in the sense that

$$\langle \frac{\partial}{\partial t} z^N(t,\theta) - \frac{\partial}{\partial\theta} z^N(t,\theta),\ f\rangle_{L_2} = 0 \tag{3.3}$$

for all

$$f \in \{f \in L_2 \mid f = \sum_{k=0}^{N-1} \alpha_k\ p_k^{(1)}\ \chi_{(-\theta_1,0]} + \sum_{k=0}^{N} \beta_k\ p_k^{(2)}\ \chi_{[-r,-\theta_1]}, \alpha_k, \beta_k \in \mathbb{R}^n\},$$

and (ii) imposing (3.1b) on the approximate solution $z^N(t,\theta)$. From (i) we obtain (2N+1) equations;

$$\begin{cases} \frac{d}{dt} a_k(t) = (2/\Delta_1)\ (Sa)_k, & 0 \le k \le N-1 \\ \frac{d}{dt} b_k(t) = (2/\Delta_2)\ (Sb)_k + \sum_{i=0}^{N} \big((-1)^i\ a_i - b_i\big)(2k+1)/\Delta_2, & 0 \le k \le N\ . \end{cases} \tag{3.4}$$

where S is the matrix representation of the derivative $\partial/\partial\theta$ (i.e., if the vector a is associated with a series of Legendre polynomials whose coefficients are the components of a, then the components of Sa give the

Legendre coefficients of the derived series), and is given by

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & \cdots & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 & \cdots & 5 & 0 \\ \vdots & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2N-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2N-1 \end{bmatrix} \otimes I \quad (3.5)$$

for N even. Here I is the $n \times n$ identity matrix and $\otimes$ denotes Kronecker product. From (ii) we obtain an equation for a_N, i.e.,

$$\frac{d}{dt}\left(\sum_{k=0}^{N} a_k(t)\right) = \int_{-r}^{0} d\mu(\theta) z^N(t,\theta) + Bu(t)$$

or

$$\frac{d}{dt} a_N(t) = -\sum_{k=0}^{N-1} \frac{d}{dt} a_k(t) + \int_{-r}^{0} d\mu(\theta)\, z^N(t,\theta) + Bu(t). \quad (3.6)$$

Hence, from (3.4) and (3.6) we obtain a system of ordinary differential equations for $\mathrm{col}(b_0,\cdots,b_N,a_0,\cdots,a_N)$:

$$\frac{d}{dt}\begin{bmatrix} \beta^N \\ \alpha^N \end{bmatrix} = A^N \begin{bmatrix} \beta^N \\ \alpha^N \end{bmatrix} + B^N u(t) \quad (3.7)$$

$$B^N = (e_{2N+2} \otimes I)B,$$

where $\alpha^N = \mathrm{col}(a_0,a_1,\cdots,a_N)$, $\beta^N = \mathrm{col}(b_0,b_1,\cdots,b_N)$ and

$e_{2N+1} = \mathrm{col}(0,0,\cdots,1) \in \mathbb{R}^{2N+2}$. If we define the matrices $J^{(1)}$, $J^{(2)}$ and $\tilde{S}$ by

$$J^{(1)} = [\overbrace{u,\ u,\ \cdots\ ,u}^{N+1}] \otimes I$$

$$J^{(2)} = [\overbrace{u,\ -u,\ \cdots,\ u,\ (-1)^N u}^{N+1}] \otimes I$$

with $u = \mathrm{col}(1,3,\cdots,2N+1)$ and,

$$\tilde{S} = \begin{bmatrix} S \\ (0\ \ -1,\ -3,\ \cdots\ ,\ -\frac{N(N+1)}{2}) \otimes I \end{bmatrix},$$

then A^N is given by $A^N = A_0^N + A_\mu^N$ where,

$$A_0^N = \begin{bmatrix} \frac{1}{\Delta_2}(2\tilde{S} - J^{(1)}) & \frac{1}{\Delta_2} J^{(2)} \\ \emptyset & \frac{1}{\Delta_1}\tilde{S} \end{bmatrix}$$

and

$$A_\mu^N = \begin{bmatrix} \emptyset & \emptyset \\ \emptyset & \emptyset \\ F_0\ \cdots\ F_N & D_0\ \cdots\ D_N \end{bmatrix}$$

with

$$D_k = \left(A_0 + (-1)^k A_1\right) + \int_{-\theta_1}^{0} A(\theta)\, p_k^{(1)}(\theta)d\theta$$

and

$$F_k = (-1)^k A_2 + \int_{-r}^{-\theta_1} A(\theta)\, p_k^{(2)}(\theta)d\theta, \qquad \text{for} \quad 0 < k < N.$$

Note that in the case when $\theta_1 = r$, the approximation scheme described above for (3.1) is exactly the same as that given in [9]. Let us introduce the orthogonal projection Q^N on Z. For any $z = (\eta,\phi)$, Q^N is defined by

$$Q^N z = \left(\eta, \sum_{k=0}^{N-1} a_k\, p_k^{(1)}(\theta)\chi_{(-\theta_1,0]} + \sum_{k=0}^{N} b_k\, p_k^{(2)}(\theta)\chi_{[-r,-\theta_1]}\right),$$

where

$$b_k = \frac{2k+1}{\Delta_2} \int_{-r}^{-\theta_1} \phi(\theta)p_k^{(2)}(\theta)d\theta, \qquad 0 < k < N, \tag{3.8}$$

$$a_k = \frac{2k+1}{\Delta_1} \int_{-\theta_1}^{0} \phi(\theta)p_k^{(1)}(\theta)d\theta, \qquad 0 < k < N-1, \tag{3.9}$$

and, we define the projection operator L^N on Z by

$$L^N z = Q^N z + a_N\left(0,\ p_k^{(1)}\ \chi_{(-\theta_1,0]}\right)$$

$$a^N = \eta - \sum_{k=0}^{N-1} a_k\, p_k^{(1)}(0) = \eta - \sum_{k=0}^{N-1} a_k. \tag{3.10}$$

Immediately, one can obtain the following lemma.

Lemma 3.1 <u>If</u> $(\eta,\phi) = L^N z$, <u>then</u> $\phi(0) = \eta$. <u>For</u> $N \geq 1$,

$$L^N Q^N = L^N \quad \text{and} \quad Q^N L^N = Q^N. \tag{3.11}$$

<u>Moreover, if</u>

$$L^N z = \sum_{k=0}^{N} a_k\left(p_k^{(1)}(0),\ p_k^{(1)} \chi_{(-\theta_1,0]}\right) + \sum_{k=0}^{N} b_k\left(0,\ p_k^{(2)} \chi_{[-r,-\theta_1]}\right)$$

<u>and</u>

$$Q^N z = \sum_{k=0}^{N-1} a_k\left(0,\ p_k^{(1)} \chi_{(-\theta_1,0]}\right) + \eta(1, \textcircled{|}) + \sum_{k=0}^{N} b_k\left(0,\ p_k^{(2)} \chi_{[-r,\theta_1]}\right),$$

<u>then we have</u>

$$\Omega^N\left(b_0,\cdots,b_N,\ a_0,\cdots,a_{N-1},\ a_N\right)^T = \left(b_0,\cdots,b_N,a_0,\cdots,a_{N-1},\eta\right)^T$$

<u>where</u>

$$\Omega^N = \left[\begin{array}{c|c} I & \textcircled{|} \\ \hline \textcircled{|} & \begin{array}{cc} I & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 1\ 1 \cdots & 1 \end{array} \end{array}\right].$$

As shown in [9], the tau method can be interpreted as follows. Let

$$z^N(t) = \left(z^N(t,0),\ z^N(t,\cdot)\right) \in Z \tag{3.12}$$

where $z^N(t,\cdot)$ is given by (3.2). Then $z^N(t)$, $t > 0$ satisfies

$$\frac{d}{dt} z^N(t) = L^N \mathcal{A} z^N + L^N \mathcal{B} u(t) \tag{3.13}$$

$$z^N(0) = L^N z.$$

From (3.11), the approximating solution

$$\tilde{z}^N(t) = Q^N z^N(t), \quad t > 0 \tag{3.14}$$

satisfies

$$\frac{d}{dt} \tilde{z}^N(t) = \mathcal{A}^N \tilde{z}^N(t) + \mathcal{B} u(t) \tag{3.15}$$

$$\tilde{z}^N(0) = Q^N z.$$

where

$$\mathcal{A}^N = Q^N \mathcal{A} L^N.$$

The following lemma concerns the question of stability of the tau approximation for (3.1). Following an idea in [2], we define the norm $\| \ \|_g$ on Z by

$$\|z\|_g = |\eta|^2 + \int_{-r}^{0} |\phi|^2 g(\theta) d\theta \qquad \text{for} \quad z = (\eta,\phi) \in Z,$$

where g is the piecewise constant function on $[-r,0]$ defined by

$$g(\theta) = \begin{cases} 1, & \theta \in [-r,-\theta_1) \\ 2, & \theta \in (-\theta_1,0] \end{cases} \tag{3.16}$$

Lemma 3.2. Let $\{S^N(t),\ t \geq 0\}$ be the semigroup on Z^N generated by $A^N = Q^N A L^N$. Then there exists a positive constant ω such that

$$\|S^N(t)\|_g \leq e^{\omega t}, \qquad t \geq 0.$$

Proof: Let us consider the inner product $\langle\cdot,\cdot\rangle_g$ on Z:

$$\langle(\eta^1,\phi^1),(\eta^2,\phi^2)\rangle_g = \langle\eta^1,\eta^2\rangle_{\mathbb{R}^n} + \int_{-r}^{0} \langle\phi^1,\phi^2\rangle_{\mathbb{R}^n}\, g(\theta)d\theta. \tag{3.17}$$

It suffices to show that $A^N - \omega I$ is dissipative on Z with the norm $\|\cdot\|_g$; i.e., for all $z \in Z$,

$$\langle A^N z,\ z\rangle_g \leq \omega\|z\|_g^2.$$

Let

$$(\eta,\phi) = L^N z = \Big(n,\ \sum_{k=0}^{N} a_k\, p_k^{(1)}\, \chi_{(-\theta_1,0]} + \sum_{k=0}^{N} b_k\, p_k^{(2)}\, \chi_{[-r,-\theta_1]}\Big),$$

where a_k, b_k are given by (3.8) - (3.10) and let $(\eta,\tilde{\phi}) = Q^N z$. Since p_N is orthogonal to all polynomials of degree at most $N-1$, it follows from

(3.4), (3.6), and (3.7) that

$$\langle A^N z, z\rangle_g = \langle \int_{-r}^{0} d\mu(\theta)\phi(\theta), \eta\rangle + \int_{-r}^{-\theta_1} \langle \dot{\phi}(\theta),\phi(\theta)\rangle d\theta$$

$$+ 2\int_{-\theta_1}^{0} \langle \dot{\phi}(\theta),\phi(\theta)\rangle d\theta + \langle \sum_{i=0}^{N} \left((-1)^i a_i - b_i\right) \sum_{k=0}^{N} b_k\rangle. \quad (3.18)$$

Note that

$$\phi(0) = \eta,\ \phi\left((-\theta_1)^-\right) = \sum_{k=0}^{N} b_k \quad \text{and} \quad \phi\left((-\theta_1)^+\right) = \sum_{k=0}^{N} (-1)^k a_k.$$

Then the right-hand side of (3.18) becomes

$$= \langle \int_{-r}^{0} d\mu(\theta)\phi(\theta),\phi(0)\rangle + \frac{1}{2}\left(|\phi((-\theta_1)^-)|^2 - |\phi(-r)|^2\right) + |\phi(0)|^2 - |\phi((-\theta_1)^+)|^2$$

$$+ \langle \phi\left((-\theta_1)^+\right),\phi\left((-\theta_1)^-\right)\rangle - |\phi\left((-\theta_1)^-\right)|^2$$

$$= \langle A_0\, \phi(0) + A_1\, \phi\left((-\theta_1)^+\right) + A_2\, \phi(-r) + \int_{-r}^{0} A(\theta)\phi(\theta)d\theta, \phi(0)\rangle$$

$$- \frac{1}{2}\,|\phi(-r)|^2 - \frac{1}{2}\,|\phi\left((-\theta_1)^+\right) - \phi\left((-\theta_1)^-\right)|^2 - \frac{1}{2}\,|\phi\left((-\theta_1)^+\right)|^2 + |\phi(0)|^2$$

$$\leq \left(1 + |A_0| + \frac{1}{2}\,|A_1^T A_1| + \frac{1}{2}|A_2^T A_2|\right)\,|\phi(0)|^2$$

$$+ \left(\int_{-r}^{0} |A(\theta)|^2 \, d\theta\right)^{1/2} \left(\int_{-r}^{0} |\tilde{\phi}(\theta)|^2 \, d\theta\right)^{1/2} |\phi(0)|$$

$$\leq \omega \| Q^N z \|_g^2 \leq \omega \| z \|_g^2 ,$$

where

$$\omega = 1 + |A_0| + \frac{1}{2} |A_1^T A_1| + \frac{1}{2} |A_2^T A_2| + \frac{1}{2} \| A(\cdot) \|_{L_2},$$

and we used the relation: $2\langle x,y\rangle \leq |x|^2 + |y|^2$ for $x,y \in \mathbb{R}^n$, and the fact the Q^N is symmetric w.r.t. $\langle \cdot,\cdot \rangle_g$ - inner product. (Q.E.D)

To establish convergence for the tau approximation, we will use the Trotter-Kato theorem (see Theorem 4.6 in [10]).

Theorem 3.3. Let $S(t)$ and $S^N(t)$, $N \geq 1$ be C_0-semigroups acting on a Banach space X with infinitesimal generators A and A^N respectively. Assume that the following conditions are satisfied:

(i) (stability). There exists a constant ω such that

$$\| S(t) \|_X \leq e^{\omega t} \quad \text{and} \quad \| S^N(t) \|_X \leq e^{\omega t}, \qquad t \geq 0.$$

(ii) (consistency). There exists a subset ∞ contained in $\mathcal{D}(A) \cap \bigcap_{N=1}^{\infty} \mathcal{D}(A^N)$ which together with $(\lambda I - A)$ for some $\lambda > 0$ is dense in X and such that $A^N \phi - A\phi$ for all $\phi \in \mathcal{D}$ as $N \to \infty$. Then for all $\phi \in X$,

$$\| S^N(t)\phi - S(t)\phi \| \to 0,$$

uniformly on bounded t-intervals.

In our discussions X is the Hilbert space Z equipped with the inner product (3.17). We will prove the consistency of the tau approximation in Section 5 (see Lemma 5.2).

Remark: Although we will not pursue the details here, one can prove that the adjoint semi-groups $S_N^*(t)$ also converge strongly to $S^*(t)$ uniformly on bounded t-intervals.

4. AN APPROXIMATION SCHEME FOR THE RICCATI EQUATION

In this section, we discuss an approximation scheme for the regulator problem (2.10) based upon the Legendre-tau approximation.

Let us consider the Nth approximate problem to (2.10)

$$\text{Minimize} \quad J^N(u,\tilde{z}) = \int_0^\infty \left(| C \hat{z}^N(t)|^2 + |u(t)|^2 \right. dt, \tag{4.1}$$

subject to (3.15):

$$\frac{d}{dt}\tilde{z}^N(t) = A^N \tilde{z}^N(t) + B^N u(t),$$

$$z^N(0) = \tilde{z} = Q^N z.$$

It follows from Theorem 2.3 that if (A^N, B) is stabilizable and (C, A^N) is detectable, then there exists a unique solution Π^N to $(\mathrm{ARE})^N$:

$$(A^N)^* \Pi^N + \Pi^N A^N - \Pi^N BB^* \Pi^N + C^*C = \bigcirc, \qquad (\mathrm{ARE})^N$$

and the optimal solution to (4.1) is given by

$$u^N(t) = -B^* \Pi^N \tilde{z}^N(t) \tag{4.2}$$

where $\tilde{z}^N(t)$, $t > 0$ satisfies

$$\frac{d}{dt}\tilde{z}^N(t) = (A^N - BB^* \Pi^N)\tilde{z}^N(t)$$

$$z^N(0) = \tilde{z}.$$

In terms of the Legendre coordinate system,

$$\tilde{z}^N(t) = \sum_{k=0}^{N-1} a_k(t)\left(0,\ p_k^{(1)} \chi_{(-\theta_1,0]}\right)$$

$$+ \eta(t)(1, \bigcirc) + \sum_{k=0}^{N} b_k(t)\left(0,\ p_k^{(2)} \chi_{[-r,-\theta_1]}\right).$$

It then follows from Lemma 3.1 and (3.7) that

$$\tilde{\xi}^N(t) = (b_0,\cdots,b_N,\ a_0,\cdots,a_{N-1},\ \eta)^T, \qquad t > 0$$

satisfies

$$\frac{d}{dt}\,\tilde{\xi}^N(t) = \tilde{A}^N\,\tilde{\xi}^N(t) + B^N\,u(t), \tag{4.3}$$

$$\tilde{\xi}^N(0) = \tilde{\xi},$$

where $\tilde{A}^N = \Omega^N\,A^N(\Omega^N)^{-1}$ and $\tilde{\xi}$ is the vector representation of $Q^N\,z$ in terms of Legendre coordinates. A^N, B^N Ω^N are given in Section 3. Thus we can write (4.1) as

$$\text{Minimize}\quad J^N(u,\tilde{\xi}) = \int_0^\infty \left(|\tilde{C}^N\,\tilde{\xi}^N(t)|^2 + |u(t)|^2\right)dt \tag{4.4}$$

subject to (4.3), where $\tilde{C}^N = C^N(\Omega^N)^{-1}$ with

$$C^N = \Big(\bigoplus_{n\times(N+1)} |C|\ C|\cdots|C\Big)\ \varepsilon\ \mathbb{R}^{p\times 2n(N+1)}.$$

Hence the optimal solution u^N to (4.1) can be also given by

$$u^N(t) = -(B^N)^T\,\Sigma^N\,\tilde{\xi}^N(t),$$

where Σ^N satisfies the matrix Riccati equation

$$(\tilde{A}^N)^T\,\Sigma^N + \Sigma^N\,\tilde{A}^N - \Sigma^N\,B^N(B^N)^T\,\Sigma^N + (\tilde{C}^N)^T\,\tilde{C}^N = \bigoplus. \tag{4.5}$$

If (A^N,B) is stabilizable and (C,A^N) is detectable, then (4.5) has a unique, symmetric, non-negative definite solution and it can be computed effectively by Potter's method (e.g., [13], [11]) which involves the eigenvalue-eigenvector decomposition of the matrix

$$H^N = \begin{bmatrix} \tilde{A}^N & -B^N(B^N)^T \\ -(\tilde{C}^N)^T\,\tilde{C}^N & -(\tilde{A}^N)^T \end{bmatrix} . \tag{4.6}$$

Let us define the matrix $\hat{\Sigma}^N$ by

$$\hat{\Sigma}^N = \Lambda^N\,\Sigma^N\,\Lambda^N$$

where

$$\Lambda^N = \mathrm{diag}\Big(\frac{\Delta_2}{1},\cdots,\frac{\Delta_2}{2k+1},\cdots,\frac{\Delta_2}{2N+1},\frac{\Delta_1}{1},\cdots,\frac{\Delta_1}{2k+1},\cdots,\frac{\Delta_1}{2N-1},\ 1\Big),$$

and define the matrices $\sigma_{i,j}$, $0 < i,j < 2N+1$ by

$$\sigma_{ij} = (e_{i+1} \otimes I)^T\,\hat{\Sigma}^N\,(e_{j+1} \otimes I), \tag{4.7}$$

where e_i is the ith unit vector in $\mathbb{R}^{2N+2}$; i.e., $e_i = (\underbrace{0,\cdots,0}_{i-1},1,0\cdots0)^T$.

Lemma 4.1. <u>Suppose</u> (A^N,B) <u>is stabilizable and</u> (C,A^N) <u>is detectable.</u> <u>Then for</u> $z = (\eta,\phi) \in Z$, $\Pi^N z = (y,\psi)$ with

$$y = \sigma\eta + \int_{-\theta_1}^{0} \sum_{k=0}^{N-1} (\sigma_{\bullet,k+N+1})\, P_k^{(1)}(\theta)\phi(\theta)d\theta + \int_{-r}^{-\theta_1} \sum_{k=0}^{M} (\sigma_{\bullet,k}) P_k^{(2)}(\theta)\phi(\theta)d\theta,$$

and

$$\psi(\theta) = \sum_{i=0}^{N-1} [(\sigma_{i+N+1,\bullet})\eta + \int_{-\theta_1}^{0} \sum_{k=0}^{N-1} (\sigma_{i+N+1,k+N+1}) P_k^{(1)}(\theta)\phi(\theta)d\theta$$

$$+ \int_{-r}^{-\theta_1} \sum_{k=0}^{N} (\sigma_{i+N+1,k}) P_k^{(2)}(\theta)\phi(\theta)d\theta] P_i^{(1)}(\theta)\chi_{(-\theta_1,0]}$$

$$+ \sum_{i=0}^{N} [(\sigma_{i,\bullet})\eta + \int_{-\theta_1}^{0} \sum_{k=0}^{N-1} (\sigma_{i,k+N+1}) P_k^{(1)}(\theta)\phi(\theta)d\theta$$

$$+ \int_{-r}^{-\theta_1} \sum_{k=0}^{N} (\sigma_{i,k})\, P_k^{(2)}(\theta)\phi(\theta)d\theta] P_i^{(2)}(\theta)\chi_{[-r,-\theta_1]},$$

where the symbol $(\bullet)$ stands for $2N+1$, $\sigma = \sigma_{2N+1,2N+1}$.

Proof: It is known [17] that

$$\langle \Pi^N z,z\rangle_Z = \langle \Sigma^N \tilde{\xi}, \tilde{\xi}\rangle_{\mathbb{R}^{2n(N+1)}} = \min J^N(u)$$

for all $z \in Z$, where $\tilde{\xi}$ is the vector representation of $Q^N z$. Since Σ^N and Π^N are symmetric,

$$\langle \Pi^N z^1,z^2\rangle_Z = \langle \Sigma^N \tilde{\xi}^1,\tilde{\xi}^2\rangle_{\mathbb{R}^{2n(N+1)}} \tag{4.8}$$

for all $z^i = (\eta^i, \phi^i) \in Z$, $i = 1,2$, where $\tilde{\xi}^i$ is the vector representation of $Q^N z^i$ for $i = 1,2$. Note that

$$(\Lambda^N)^{-1} \tilde{\xi}^i = (\underline{\beta}^i, \underline{\alpha}^i, \gamma^i)^T, \quad i = 1,2,$$

where for $i = 1,2$

$$\beta_k^i = \int_{-r}^{-\theta_1} \phi^i(\theta)\, P_k^{(2)}(\theta) d\theta$$

$$\alpha_k^i = \int_{-\theta_1}^{0} \phi^i(\theta)\, P_k^{(1)}(\theta) d\theta \tag{4.9}$$

$$\gamma^i = \eta^i .$$

Then

$$\langle \Sigma^N \tilde{\xi}^1, \tilde{\xi}^2 \rangle = (\underline{\beta}^1, \underline{\alpha}^1, \gamma^1)\, \hat{\Sigma}^N (\underline{\beta}^2, \underline{\alpha}^2, \gamma^2)^T.$$

Now, if $\Pi^N(\eta^1, \phi^1) = (y, \psi)$, then

$$\langle \Pi^N(\eta^1,\phi^1), (\eta^2,\phi^2) \rangle_2 = \langle \eta^2, y \rangle + \int_{-\theta_1}^{0} \langle \phi^2(\theta), \psi(\theta) \rangle d\theta + \int_{-r}^{-\theta} \langle \phi^2(\theta),$$

Equating (4.8), we obtain

$$y = \sigma\eta^1 + \sum_{k=0}^{N-1} (\sigma_{\cdot,k+N+1}) \alpha_k^1 + \sum_{k=0}^{N} (\sigma_{\cdot,k}) \beta_k^1$$

and

$$\psi(\theta) = \sum_{i=0}^{N-1} [(\sigma_{i+N+1,\cdot})\eta^1 + \sum_{i=0}^{N-1} (\sigma_{i+N+1,k+N+1})\ \alpha_k^1$$

$$+ \sum_{k=0}^{N} (\sigma_{i+N+1,k})\beta_k^1]P_i^{(1)}\ \chi_{(-\theta_1,0]}$$

$$+ \sum_{i=0}^{N} [(\sigma_{i,\cdot})\eta^1 + \sum_{k=0}^{N-1} (\sigma_{i,k+N+1})\alpha_k^1$$

$$+ \sum_{k=0}^{N} (\sigma_{i,k})\beta_k^1]\ P_i^{(2)}\ \chi_{[-r,-\theta_1]},$$

which completes the proof along with (4.9). (Q.E.D.)

Corollary 4.2. <u>The optimal solution</u> u^N <u>to (4.1) can be written in the operator form:</u>

$$u^N(t) = -\mathcal{K}^N\ \tilde{z}^N(t).$$

$\mathcal{K}^N \in \mathcal{L}(Z, \mathbb{R}^m)$ <u>is given by</u>

$$\mathcal{K}^N z = B^T(\Pi_N^{00}\ \eta + \int_{-r}^{0} \Pi_N^{10}(\theta)^T\ \phi(\theta)d\theta), \quad \text{for} \quad z = (\eta,\phi) \in Z,$$

<u>where</u>

$$\Pi_N^{00} = \sigma,$$

<u>and</u>

$$\Pi_N^{10}(\theta) = \sum_{i=0}^{N-1} (\sigma_{i+N+1,\cdot})P_i^{(1)}(\theta)\ \chi_{(-\theta_1,0]}$$

$$+ \sum_{i=0}^{N} (\sigma_{i,\cdot})P_i^{(2)}(\theta)\ \chi_{[-r,-\theta_1]}, \qquad -r < 0 < 0,$$

Proof: Since $\mathcal{B}^*(\eta,\phi) = B^T \eta$, the corollary follows from Lemma 4.1 and (4.2). (Q. E. D.)

Remark: For single point delay case, we are able to prove that if $(\mathcal{A},\mathcal{B})$ is stabilizable ($(\mathcal{C},\mathcal{A})$ is detectable), then for sufficiently large N $(\mathcal{A}^N,\mathcal{B})$ is stabilizable ($(\mathcal{C},\mathcal{A}^N)$ is detectable), which will be discussed in the forthcoming paper. The proof is based upon the characterization of detectability in Remark 2.2.

5. CONVERGENCE PROOF

In this section we discuss the convergence property of Π^N. It is easy to show that for $k > 2$

$$\mathcal{D}^k = \mathcal{D}(\mathcal{A}^k) \subseteq \{(\phi(0),\phi) \in Z \mid \dot{\phi}(0) = \int_{-r}^{0} d\mu(\theta)\phi(\theta) \text{ and } \phi \in H^k\}, \tag{5.1}$$

and is dense in Z. Let us introduce the graph norm on $\mathcal{D}^k$;

$$\|z\|_{\mathcal{D}^k} = \sum_{i=0}^{k} \|\mathcal{A}^i z\|_Z^2 \qquad \text{for } z \in \mathcal{D}^k.$$

Note that $\|\phi\|_{H^k} \leq \|z\|_{\mathcal{D}^k}$ for all $z = (\phi(0),\phi) \in \mathcal{D}^k$.

Theorem 5.1: If $\{\Pi^N\}$ is uniformly bounded on Z and $(\mathcal{C},\mathcal{A})$ is detectable, then

(i) Π^N converges weakly to Π which is the unique solution to ARE,

(ii) there exists an integer N_0 such that if $N > N_0$, then

$$\|S^N(t)\| < Me^{-\omega t}$$

for some positive constants M and ω, where $\{S^N(t), t > 0\}$ is the semigroup on Z generated by $\mathcal{A} - \mathcal{B}(\mathcal{B}^N)^* \Pi^N$.

Proof of (i): Since $\{\Pi^N\}$ is uniformly bounded on Z, by Theorem 6.5 in [8], there exists a subsequence $\{\Pi^{N_j}\}$ which converges weakly to some non-negative, self-adjoint operator Π. If $(\mathcal{C},\mathcal{A})$ is detectable, then from Theorem 2.3, ARE has at most, one non-negative, self-adjoint solution. Hence, we only need to show Π satisfies ARE. Without loss of generality we can assume that Π^N converges weakly to Π. Note that for $N > 1$, Π^N satisfies $(ARE)^N$. Since $\dim(R^m) < \infty$ $\mathcal{B}^* \Pi^N$ converges strongly $\mathcal{B}^* \Pi$. It now follows from Lemma 5.2 that

$$2\langle \mathcal{A}z,\Pi z\rangle - \langle \mathcal{B}^* \Pi z, \mathcal{B}^* \Pi z\rangle + \langle \mathcal{C}z,\mathcal{C}z\rangle = 0, \qquad (5.2)$$
$$\text{for all } z \in \mathcal{D}^2.$$

Since $\mathcal{D}^2 = \mathcal{D}(\mathcal{A}^2)$ is dense in $\mathcal{D}(\mathcal{A})$, a simple limit argument shows that (5.2) holds for all $z \in \mathcal{D}(\mathcal{A})$; i.e., Π is a solution to ARE.

Proof of (ii): First of all, we note that $\mathcal{A} = \mathcal{A} - \mathcal{B}\mathcal{B}^* \Pi$ generates a uniformly exponentially stable semigroup $\{\tilde{S}(t), t > 0\}$ on Z; i.e., there exist positive constants $\tilde{M}$ and $\tilde{\omega}$ such that

$$\|\tilde{S}(t)\| \leq \tilde{M}e^{-\tilde{\omega}t}, \qquad t \geq 0. \tag{5.3}$$

For $z \in \mathcal{D}(A)$

$$(A - \mathcal{B}\mathcal{B}^* \Pi^N)z = \tilde{A}z - - \mathcal{B}(\mathcal{B}^* \Pi - \mathcal{B}^* \Pi^N)z.$$

Thus,

$$S^N(t) = \tilde{S}(t)z + \int_0^t \tilde{S}(t-s)\, \mathcal{B}(\mathcal{B}^* \Pi - \mathcal{B}^* \Pi^N)\, S^N(s)z ds, \qquad \text{for all } z \in Z. \tag{5.4}$$

For $t \geq r$, we may write (5.4) as

$$S^N(t)z = \tilde{S}(t-r)\tilde{z} + \int_r^t \tilde{S}(t-s)\, \mathcal{B}\, (\mathcal{B}^* \Pi - \mathcal{B}^* \Pi^N)\, S^N(s)z ds, \tag{5.5}$$

with

$$\tilde{z} = \tilde{S}(r)z + \int_0^r \tilde{S}(r-s)\, \mathcal{B}(\mathcal{B}^* \Pi - \mathcal{B}^* \Pi^N)S^N(s)z ds. \tag{5.6}$$

From [5] we have that $\tilde{S}(r)z \in \mathcal{D}(\tilde{A})$ for all z and $\|\tilde{S}(r)z\| \leq \gamma_1 \|z\|_Z$ for some positive constant γ_1. If

$$z(t) = \big(x(t), x(t+\cdot)\big) = \int_0^t \tilde{S}(t-s)\, \mathcal{B}u(s)ds, \qquad t \geq 0,$$

then $x(t) \in H^1([-r,T]; \mathbb{R}^n)$ for any $T \geq 0$ and satisfies

$$\begin{aligned} \frac{d}{dt} x(t) &= \int_{-r}^0 d\mu(\theta)x(t+\theta) - BB^T \Pi^{00} x(t) \\ &\qquad - BB^T \int_{-r}^0 \Pi^{10}(\theta)^T x(t+\theta)d\theta + Bu(t) \\ &\equiv \int_{-r}^0 d\tilde{\mu}(\theta)\, x(t+\theta) + Bu(t). \end{aligned}$$

Hence for $u \in L_2^{loc}([0,\infty); \mathbb{R}^n)$, $z(t) \in \mathcal{D}(\tilde{A})$ and

$$\tilde{A} \int_0^t \tilde{S}(t-s) \quad u(s)ds = \left(\int_{-r}^0 d\tilde{\mu}(\theta)\ x(t+\theta),\ \psi(t+\cdot)\right), \tag{5.7}$$

where

$$\psi(t) = \int_{-r}^0 d\tilde{\mu}(\theta)\ x(t+\theta) + Bu(t), \quad t > 0,$$

and

$$\psi(t) = 0 \quad \text{for} \quad t < 0.$$

Here we note that

$$\left\| \int_{-r}^0 d\tilde{\mu}\, x(\cdot+\theta) \right\|_{L_2([a,b],\, \mathbb{R}^n)} \leq \gamma_2 \|x\|_{L_2([a-r,b],\, \mathbb{R}^n)} \tag{5.8}$$

for $b > a > 0$, where $\gamma_2 = \int_{-r}^0 |d\tilde{\mu}|$. Since $\{\Pi^N\}$ is uniformly bounded, it now follows from (5.6) and (5.7) that $\tilde{z} \in \mathcal{D}(\tilde{A})$ for all $z \in Z$ and

$$\|\tilde{z}\|_{\mathcal{D}(\tilde{A})} \leq \gamma_3 \|z\|_Z, \tag{5.9}$$

for some positive constant γ_3. From (5.5) and (5.7), $S^N(t)\ z \in \mathcal{D}(\tilde{A})$, $t > r$ for $z \in Z$ and

$$\tilde{A}S^N(t)z = \tilde{S}(t-r)\tilde{A}\tilde{z} + \tilde{\ } \int_r^t \tilde{S}(t-s)\mathcal{B}F^N\left(\tilde{A}S^N(x)z\right)ds, \tag{5.10}$$

where $F^N : Z \to \mathbb{R}^m$ is given by

$$F^N = (\mathcal{B}^*\ \Pi - \mathcal{B}^*\ \Pi^N)\tilde{A}^{-1}.$$

Since $0 \notin P\sigma(\tilde{A})$, $(\tilde{A})^{-1}$ exists and moreover, it is compact [17]. Note that

$$(F^N)^* = (\tilde{A}^*)^{-1}(\Pi B - \Pi^N \mathcal{B}) \in \mathcal{L}(\mathbb{R}^m, Z).$$

Since $\Pi^N \mathcal{B}$ converges weakly to $\Pi\mathcal{B}$ as $N \to \infty$ and $(\tilde{A})^{-1}$ is compact, $(F^N)^*$ converges strongly to zero. Hence, the finite dimensionality of $\mathbb{R}^m$ implies

$$\|(F^N)^*\| = \|F^N\| \to 0 \quad \text{as} \quad N \to \infty,$$

i.e., for any $\varepsilon > 0$ there exists an integer $N_0(\varepsilon)$ such that $\|F^N\| < \varepsilon$ for $N > N_0$.

For $z \in Z$, let us define the Z-valued function $\beta^N(t)$, $t > r$ by

$$\beta^N(t) = \tilde{A}\, S^N(t)z.$$

Then from (5.7) and (5.10)

$$\beta^N(t) = \tilde{S}(t-r)\tilde{A}\tilde{z} + \Big(\int_{-r}^{0} d\tilde{\mu}(\theta)x(t+\theta),\ \psi(t+\cdot)\Big),$$

where for $t > 0$

$$\psi(t) = \int_{-r}^{0} d\tilde{\mu}(\theta)x(t+\theta) + \mathcal{B}F^N\, \beta^N(t)$$

and

$$\big(x(t), x(t+\cdot)\big) = \int_{r}^{t} \tilde{S}(t-s)\mathcal{B}F^N\, \beta^N(s)ds, \qquad t > r \tag{5.11}$$

with $x(t) = 0$, $t < r$. Now from (5.8), for $T > r$

$$\left(\int_r^T \|\beta^N(t)\|^2 dt\right)^{1/2} < \|\tilde{A}\tilde{z}\|\left(\int_r^T \|\tilde{S}(t-r)\|^2\, dt\right)^{1/2} + \gamma_2\left(\int_0^T |x(t)|^2\, dt\right)^{1/2}$$

$$+ \gamma_2 \left(\int_r^T \int_{t-2r}^t |x(s)|^2\, ds\, dt\right)^{1/2}$$

$$+ |B|\, \|F^N\| \left(\int_r^T \int_{t-r}^t \|\beta^N(s)\|^2\, ds\, dt\right)^{1/2},$$

and from (5.3) and (5.11),

$$< \left(\tilde{M}/(2\tilde{\omega})^{1/2}\right) \|\tilde{A}\tilde{z}\|$$

$$+ |B|\, \|F^N\| \left(\frac{\tilde{M}}{\tilde{\omega}} \gamma_2\left(1 + (2r)^{1/2}\right) + (r)^{1/2}\right)\left(\int_r^T \|\beta^N(s)\|^2\, ds\right)^{1/2}$$

where we used Fubini's theorem and Young's inequality. Thus, from (5.9)

$$\int_r^T \|\beta^N(t)\|_N^2\, dt < \left(\frac{\tilde{M}^2}{\tilde{\omega}}\right)\gamma_3^2\, \|z\|_Z^2 + 2\|F^N\|^2\, \gamma^2 \int_r^T \|\beta^N(s)\|^2\, ds,$$

where

$$\gamma = |B|\left(\frac{\tilde{M}}{\tilde{\omega}} \gamma_2\left(1 + (2r)^{1/2}\right) + (r)^{1/2}\right).$$

If we choose ε such that $2\varepsilon^2 \gamma^2 < 1/2$, then it follows that for $T > r$

$$\int_r^T \|\beta^N(t)\|_N^2\, dt < 2\left(\frac{\tilde{M}^2}{\tilde{\omega}}\right)\gamma_3^2\, \|z\|_Z^2.$$

Note that $S^N(t)z = \tilde{A}^{-1}\, \beta^N(t)$, $t > r$ and $\tilde{A}^{-1} \in L(Z)$. Hence, for $T > r$

$$\int_r^T \|S^N(t)z\|^2\, dt \leq 2\left(\frac{\tilde{M}^2}{\tilde{\omega}}\right)\gamma_3^2\, \|\tilde{A}^{-1}\|^2\, \|z\|_Z^2 .$$

It now follows from Lemma 7.4 in [8] that there exists positive constant M and ω such that

$$\|S^N(t)\| \leq Me^{-\omega t}, \quad t \geq 0 \quad \text{for} \quad N \geq N_0(\varepsilon). \qquad \text{(Q.E.D.)}$$

Lemma 5.2. $\|(A^N - A)z\| \to 0$ <u>as</u> $N \to \infty$

<u>for all</u> $z \in \mathcal{D}^k$, $k \geq 2$.

To prove this lemma, we need the following technical lemma.

Lemma 5.3. <u>Let us define the projection operator</u> P^N <u>of</u> $L_2[-1,1]$ by

$$P^N f = \sum_{k=0}^{N} f_k P_k$$

$$f_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)\, P_k(x)dx.$$

<u>Then for any positive integer</u> m, <u>there exists a constant</u> K <u>such that</u>

$$|P^N f(\pm 1) - f(\pm 1)| \leq KN^{-m+1/2}\, \|f\|_{H^m}$$

<u>and</u>

$$\left|\frac{d}{dx}(P^N f)(\pm 1) - \frac{d}{dx} f\right| \leq KN^{-m+5/2}\, \|f\|_{H^m}.$$

<u>Proof:</u> Note that for $k > 1$, P_k satisfies

$$LP_k + k(k+1)P_k = 0,$$

where is the differential operator:

$$(Lf)(x) = \frac{d}{dt}\left((1 - x^2)\frac{d}{dx} f\right).$$

Thus for $k > 1$ and $f \in H^1$

$$f_k = -\frac{2k+1}{2k(k+1)}\int_{-1}^{1} f(x)\, LP_k\, dx = \frac{2k+1}{2k(k+1)}\int_{-1}^{1} (1 - x^2)\frac{d}{dx} P_k \frac{d}{dx} f dx.$$

Using the relation

$$(1 - x^2)\frac{d}{dx} P_k = \frac{k(k+1)}{2k+1}\left(P_{k+1} - P_{k-1}\right), \tag{5.12}$$

we obtain

$$f_k = \frac{1}{2}\int_{-1}^{1}\left(P_{k+1} - P_{k-1}\right)\frac{d}{dx} f dx.$$

It then follows that

$$\left(P^N f\right)(\pm 1) = \sum_{k=0}^{N} (\pm 1)^k a^k = a_0 + \left(\frac{1}{2}\sum_{k:\text{even}} \pm \frac{1}{2}\sum_{k:\text{odd}}\right)\int_{-1}^{1}\left(P_{k+1} - P_{k-1}\right)\dot{f} dx.$$

If N is even, then

$$(P^N f)(\pm 1) = a_0 - \frac{1}{2}\int_{-1}^{1} (P_1 \pm P_0)\dot{f}dx + \frac{1}{2}\int_{-1}^{1} (P_{N+1} \pm P_N)\dot{f}dx$$

$$= \frac{1}{2}\int_{-1}^{1} fdx - \frac{1}{2}\int_{-1}^{1} (x \pm 1)\dot{f}dx + \frac{1}{2}\int_{-1}^{1} (P_{N+1} \pm P_N)\dot{f}dx$$

$$= f(\pm 1) + \frac{1}{2}\int_{-1}^{1} (P_{N+1} \pm P_N)\dot{f}dx. \tag{5.13}$$

Similarly, for N odd,

$$(P^N f)(\pm 1)\ f(\pm 1) + \frac{1}{2}\int_{-1}^{1} (P_N \pm P_{N+1})\dot{f}dx. \tag{5.14}$$

If $m = 2k+1$, $k \geq 0$, then

$$\int_{-1}^{1} P_N \dot{f}dx = \left(-\frac{1}{N(N+1)}\right)^k \int_{-1}^{1} (L^k P_N)\ \dot{f}dx = \left(-\frac{1}{N(N+1)}\right)^k \int_{-1}^{1} P_N(L^k\dot{f})dx.$$

And, if $m = 2k+2$, $k \geq 0$, then

$$\int_{-1}^{1} P_N \dot{f}dx = \left(-\frac{1}{N(N+1)}\right)^{k+1} \int_{-1}^{1} (1-x^2)\frac{d}{dx} P \frac{d}{dx}(L^k \dot{f})dx$$

and using (5.12)

$$= \left(-\frac{1}{N(N+1)}\right)^k \frac{1}{2N+1}\int_{-1}^{1} (P_{N+1} - P_{N-1})\frac{d}{dx}(L^k \dot{f})dx.$$

Since L^k is a differentiable operator of order $2k$ with polynomials, coefficients on $[-1,1]$, there exists a constant c_k for $k \geq 0$ such that

$$\|L^k \dot{f}\| \leq c_k \|f\|_{H^{2k+1}}$$

and

$$\|\frac{d}{dx}(L^k \dot{f})\| \leq c_k \|f\|_{H^{2k+2}}.$$

Now, the first inequality of the lemma follows from (5.13) and (5.14).

To prove the second inequality, we note that

$$\frac{d}{dx}(P^N f)(\pm 1) = \sum_{k=0}^{N} (\mp 1)^k \frac{k(k+1)}{2} a_k = \frac{1}{2} \sum_{k=0}^{N} (\mp 1)^k \int_{-1}^{1} (Lf)P_k \, dx.$$

Then the same arguments as above enable us to obtain the second inequality.

(Q.E.D.)

<u>Proof of Lemma 4.2:</u> From the definition (3.7) of L^N

$$z^N = L^N z = (\phi^N(0), \phi^N)$$

$$\phi^N = \phi^{(1)} \chi_{(-\theta_1,0]} + \phi^{(2)} \chi_{[-r,-\theta_1]},$$

where

$$\phi^{(1)} = \sum_{k=0}^{N} a_k P_k^{(1)} \qquad \text{on } (-\theta_1,0],$$

$$\phi^{(2)} = \sum_{k=0}^{N} b_k P_k^{(2)} \qquad \text{on } [-r,-\theta_1]$$

and $\{a_k\}$ and $\{b_k\}$ are given by (3.8) - (3.10). It then follows from (3.4) and (3.6) that

$$A^N z = (\eta^N, \psi^N)$$

with

$$\eta^N = \int_{-r}^{0} d\mu(\theta)\phi^N$$

$$\psi^N = \dot{\phi}^{(1)} \chi_{(-\theta_1,0]} + \dot{\phi}^{(2)} \chi_{[-r_1,-\theta_1]}$$

$$+ \left(\phi^{(1)}(-\theta_1) - \phi^{(2)}(-\theta_1)\right) \sum_{k=0}^{N} \frac{2k+1}{\Delta_2} P_k^{(2)} \chi_{[-r,-\theta_1]}$$

Thus, for $z \in \mathcal{D}(A)$

$$\delta = \|(A^N - A)z\| \leq \|\dot{\phi}^{(1)} - \dot{\phi}\|_{L_2[-\theta_1,0]} + \|\dot{\phi}^{(2)} - \dot{\phi}\|_{L_2[-r,-\theta_1]}$$

$$+ |\phi^{(1)}(-\theta_1) - \phi^{(2)}(-\theta_1)|\left(\sum_{k=0}^{N} \frac{2k+1}{\Delta_2}\right)^{1/2}$$

$$+ \left|\int_{-r}^{0} d\mu(\theta)\left(\phi^N(\theta) - \phi(\theta)\right)\right|$$

$$= \delta_1 + \delta_2 + \delta_3 + \delta_4 . \tag{5.15}$$

Here, we note that

$$\phi^{(1)} = \tilde{\phi}^{(1)} + \left(\phi(0) - \tilde{\phi}^{(1)}(0)\right)P_N^{(1)} \quad \text{on} \quad [-\theta_1,0], \tag{5.16}$$

where

$$\tilde{\phi}^{(1)} = \sum_{k=0}^{N-1} a_k P_k^{(1)} \quad \text{on} \quad [-\theta_1,0].$$

For $z = (\phi(0),\phi) \in \mathcal{D}(A)$,

$$\int_{-r}^{0} d\mu(\theta)(\phi^N(\theta) - \phi(\theta))$$

$$= A_1(\phi^{(1)}(-\theta_1) - \phi(-\theta_1)) + A_2(\phi^{(2)}(-r) - \phi(-r))$$

$$+ \int_{-\theta_1}^{0} A(\theta)(\tilde{\phi}^{(1)} - \phi)d\theta + \int_{-r}^{-\theta_1} A(\theta)(\phi^{(2)} - \phi),$$

where from (5.16)

$$\phi^{(1)}(-\theta_1) = \tilde{\phi}^{(1)}(-\theta_1) + (\pm 1)^N (\tilde{\phi}^{(1)}(0) - \phi(0)).$$

It then follows that

$$\delta_4 < |A_1| \left(|\tilde{\phi}^{(1)}(-\theta_1)-\phi(-\theta_1)| + |\tilde{\phi}^{(1)}(0)-\phi(0)| \right) + |A_2| \; |\phi^{(2)}(-r)-\phi(-r)|$$

$$+ \|A(\cdot)\|_{L_2} \left(\|\tilde{\phi}^{(1)}-\phi\|_{L_2[-\theta_1,0]} + \|\phi^{(2)}-\phi\|_{L_2[-r,-\theta_1]} \right).$$

It now follows from Lemma 5.3 and Lemmas 3.1 - 3.2 in [9] that

$$|\delta_4| < K\left((2|A_1| + |A_2|)N^{-k + 1/2} + 2\|A(\cdot)\|_{L_2} N^{-k} \right) \|z\|_{\mathcal{D}^k}$$

$$< K_4 \, N^{-k+ 1/2} \, \|z\|_{\mathcal{D}^k} .$$

From Lemma 3.2 in [9]

$$\delta_2 < K_2 \, N^{-k+3/2} \, \|z\|_{\mathcal{D}^k}.$$

From (5.16)

$$\delta_1 < \|\dot{\tilde{\phi}}^{(1)} - \dot{\phi}\|_{L_2[-\theta_1,0]} + \sqrt{N(N+1)}\,|(\tilde{\phi}^{(1)}(0) - \phi(0)|$$

where we used the fact that

$$\int_{-1}^{1} |\dot{P}_N(\theta)|^2 \, d\theta = N(N+1).$$

It then follows from Lemma 5.3 and Lemma 3.2 in [9] that

$$\delta_1 < K_1 \, N^{-k+3/2} \, \|z\|_{\mathcal{D}^k}.$$

Since

$$\begin{aligned}
&|\phi^{(1)}(-\theta_1) - \phi^{(2)}(-\theta_1)| \\
&\qquad < |\phi^{(1)}(-\theta_1) - \phi(-\theta_1)| + |\phi^{(2)}(-\theta_1) - \phi(-\theta_1)| \\
&\qquad < |\tilde{\phi}^{(1)}(-\theta_1) - \phi(-\theta_1)| + |\tilde{\phi}^{(1)}(0) - \phi(0)| + |\phi^{(2)}(-\theta_1) - \phi(-\theta_1)|,
\end{aligned}$$

it follows from Lemma 5.3 that

$$\delta_3 < K_3 \, N^{-k+3/2} \, \|z\|_{\mathcal{D}^k}.$$

Hence from (5.15)

$$\|(A^N - A)\, z\|_Z \leq \tilde{K}\, N^{-k+\,3/2}\, \|z\|_{\mathcal{D}^k}, \qquad k \geq 2,$$

where $\tilde{K}$ is independent of N. (Q.E.D.)

The next lemma concerns the uniform boundedness of $\{\Pi^N\}$ in Theorem 5.1 (i).

<u>Lemma 5.4.</u> <u>Consider the system with the form</u>

$$\frac{d}{dt}\, x(t) = \sum_{i=0}^{\ell} A_i\, x(t - \theta_i) + Bu(t). \tag{5.17}$$

<u>If the pair</u> (A_0,B) <u>is controllable and the range of</u> B <u>contains the range of</u> A_i, $1 \leq i \leq \ell$, <u>then</u> $\{\Pi^N\}$ <u>is a uniformly bounded sequence on</u> Z.

<u>Proof</u>: For simplicity of exposition we consider the case, $\ell = 2$. The approximate solution $z^N(t) = (z^N(t,0),\, z^N(t,\cdot)) \in Z$ of initial value problem (5.17) satisfies

$$\begin{cases} \dfrac{d}{dt}\, z^N(t,0) = A_0\, z^N(t,0) + A_1\, z^N\big(t,(-\theta_1)^+\big) + A_2\, z^N(t,-r) + Bu(t), \\[2ex] \dfrac{\partial}{\partial t}\, z^N(t,\theta) = \dfrac{\partial}{\partial \theta}\, z^N(t,\theta) \qquad\qquad -r \leq \theta \leq 0 \end{cases} \tag{5.18}$$

where the second equation holds in the sense of (3.3). Since (A_0,B) is controllable, then there exists an $m\times n$ matrix K such that the matrix

(A_0-BK) has distinct negative real eigenvalues λ_i, $1 \leq i \leq n$ with $\max_{1\leq i\leq n} \lambda_i \leq -3/2$. Since the eigenvalues of (A_0-BK) are distinct, there exists a nonsingular matrix P such that

$$P^{-1}(A_0 - BK)P = \Lambda = \mathrm{diag}(\lambda_1,\cdots,\lambda_n).$$

Let us consider the feedback control low to (4.1):

$$\tilde{u}(t) = -Kz^N(t,0) - (B^T B)^{-1} B^T \left(A_1 z^N\left(t,(-\theta_1)^+\right) + A_2 z^N(t_1, -r)\right). \tag{5.19}$$

Then (5.18) has the closed loop equation;

$$\begin{cases} \frac{d}{dt} z^N(t,0) = (A_0 - BK)z^N(t,0), \\ \\ \frac{\partial}{\partial t} z^N(t,\theta) = \frac{\partial}{\partial \theta} z^N(t,\theta). \end{cases} \tag{5.20}$$

If $\hat{z}^N(t) = \left(P^{-1} z^N(t,0),\ P^{-1} z^N(t,\cdot)\right)$, $t > 0$, then $\hat{z}^n(\cdot)$ satisfies

$$\frac{d}{dt} \hat{z}^N(t,0) = \Lambda \hat{z}^N(t,0)$$

$$\frac{\partial}{\partial t} \hat{z}^N = \frac{\partial}{\partial \theta} \hat{z}^N.$$

By using the same arguments given in the proof of Lemma 3.1, we obtain

$$\frac{1}{2}\frac{d}{dt}\|Q_N\,\hat{z}^N(t)\|_g^2 < \langle\Lambda\hat{z}(t,0),\ \hat{z}^N(t,0)\rangle + |\hat{z}^N(t,0)|^2$$

$$-\frac{1}{2}\,|\hat{z}^N(t,(-\theta_1)^+)^2 - \frac{1}{2}|\hat{z}^N(t,-r)|^2$$

$$< -\frac{1}{2}\left(|\hat{z}^N(t,0)|^2 + |\hat{z}^N(t_1(-\theta_1)^+|^2 + |\hat{z}^N(t,-r)|^2\right),$$

where we used the fact that $\Lambda < -\frac{3}{2}I$. Integration of this with respect to t yields

$$\|Q^N\,\hat{z}^N(t)\|_g^2 - \|Q^N\,\hat{z}^N(0)\|_g^2$$

$$< -\int_0^t\left(|\hat{z}^N(s,0)|^2 + |\hat{z}^N(s,(-\theta_1)^+)|^2 + |\hat{z}^N(s,-r)|^2\right)ds,$$

for all $t > 0$. Thus, for all $t > 0$

$$\int_0^t |\hat{z}^N(s,0)|^2\,ds,\quad \int_0^t |\hat{z}^N(s,(-\theta_1)^+)|^2\,ds,$$

and

$$\int_0^t |\hat{z}^N(s,-r)|^2\,ds < \|Q^N\,\hat{z}^N(0)\|_g^2 < \|\hat{z}\|_g^2 < \sigma_{max}(P^{-T}\,P^{-1})\,\|z\|_g^2\,.$$

Since $\|z\|_g^2 < 2\|z\|_Z^2$, it now follows from (5.9) and (5.20)

$$\langle\Pi^N\,z,z\rangle < J^N(\tilde{u},z) = \int_0^\infty\left(|Cz^N(t,0)|^2 + |\tilde{u}(t)|^2\right)dt < \beta\,\|z\|_2^2$$

for some positive constant β. Since Π^N is nonnegative, self-adjoint, for $N > 1$, $\Pi^N < \beta I$. (Q.E.D.)

6. NUMERICAL EXAMPLES AND CONCLUSIONS

In this section, we discuss some numerical examples which demonstrate the feasibility of the Legendre-tau method for approximating the optimal feedback solution. We only consider examples of optimal control on the infinite interval. We solved the Riccati equation (4.5) for the matrix Σ^N using Potter's method. All computations were performed using MATLAB developed by Cleve Moler [12] which provides easy access to matrix software developed by LINPACK and EISPACK projects.

The Nth feedback control is given by

$$u^N(t) = -B^T\left(\Pi_N^{00}\, x(t) + \int_{-r}^{0} \Pi_N^{10}\,(\theta)^T\, x(t+\theta)d\theta\right), \tag{6.1}$$

where Π_N^{00} and Π_N^{10} are given in terms of the coefficients of Σ^N in Corollary 4.2. The strong convergence of Π^N to Π implies $\Pi_N^{00} \to \Pi^{00}$ and $\Pi_N^{10} \to \Pi^{10}$ in $L_2([-r,0];\ \mathbb{R}^{n\times n})$. We also discuss below how closely Π_N^{00} and Π_N^{10} approximate the conditions described in Theorem 2.5 and how closely the eigenvalues of the Nth Hamiltonian matrix H^N in (4.6) approximate the closed-loop eigenvalues of $A - BB^*\,\Pi$.

<u>Example 6.1.</u> (Gibson [8], Example 8.1)

Consider the scalar differential equation

$$\frac{d}{dt}\, x(t) = x(t) + x(t-1) + u(t), \tag{6.2}$$

and the performance index of (2.6) is

$$J(u,(\eta,\phi)) = \int_0^\infty (x^2(t) + u^2(t))dt. \tag{6.3}$$

For each N, Π_N^{00} is a scalar and $\Pi_N^{10}(\cdot)^T \varepsilon L_2([-1,0];R)$ and $B \equiv 1$ in (6.1). Table I shows the numerical results for Π_N^{00} and the expansion coefficients of Π_N^{10}, i.e.,

$$\Pi_N^{10}(\theta) = \sum_{k=0}^{N-1} a_k^N P_k(2\theta + 1), \qquad -1 < \theta < 0,$$

and how closely we have approximated the boundary condition (2.14).

Table I

	N	2	4	6	8
	Π_N^{00}	2.8139	2.8094	2.8094	2.8094
	k=0	1.4222	1.4267	1.4267	1.4267
	1	-1.0844	-1.0438	-1.0438	-1.0438
	2		0.2919	0.2919	0.2919
	3		-0.0424	-0.0420	-0.0420
$\{a_k^N\}$	4			0.0046	0.0046
	5			-0.0004	-0.0004
	6				2.3×10^{-5}
	7				1.2×10^{-6}
	$\lvert\Pi_N^{00} - \Pi_N^{10}(-1)\rvert$	0.3074	0.0046	2.3×10^{-5}	5.3×10^{-8}

For comparison, the following are obtained using the average (AVE) scheme [8] and the linear spline (SPL) scheme [4].

$$\Pi_{74}^{00}(\text{AVE}) = 2.8130$$

$$\Pi_{32}^{00}(\text{SPL}) = 2.8091.$$

Note that both schemes have not fully converged yet. However, for the Legendre-tau method, the result for $N = 4$ appears to give a fairly good approximation of the optimal feedback; e.g.,

$$|\Pi_4^{00} - \Pi_8^{00}| = 4.4 \times 10^{-7},$$

$$\|\Pi_4^{10} - \Pi_8^{10}\|_{L_2[-1,0]} = 1.5 \times 10^{-3}.$$

Table II compares $\Pi_{74}^{10}(\theta)(\text{AVE})$ and $\Pi_4^{10}(\theta)(\text{L-T})$ where L-T denotes the Legendre-tau approximation.

Table II

$\|\Pi_{74}^{10}(\theta)\ (\mathrm{AVE}) - \Pi_{4}^{10}(\theta)\ (\mathrm{L-T})\|_{L_2} = 1.9 \times 10^{-2}$

θ	$\Pi_{64}^{10}(\theta)$ (AVE)	$\Pi_{4}^{10}(\theta)$ (L-T)
0.0	0.6435	0.6323
-0.1	0.7273	0.7225
-0.2	0.8258	0.8273
-0.3	0.9607	0.9519
-0.4	1.1023	1.1013
-0.5	1.2694	1.2807
-0.6	1.4965	1.4951
-0.7	1.7315	1.7497
-0.8	2.0480	2.0494
-0.9	2.3748	2.3994
-1.0	2.7541	2.8048

The oscillatory behavior exhibited by the spline approximation to Π^{10} [4] has not been observed for the Legendre-tau approximation.

Table III shows the eigenvalues λ_i^N of H^N which give the relatively small equation error $|\det \hat{\Delta}(\lambda_i^N)|$ where $\hat{\Delta}(\lambda)$ is given by (2.5); i.e., in this example

$$\hat{\Delta}(\lambda) = \left(\lambda - 1 - e^{-\lambda}\right)(\lambda + 1 + e^{\lambda}) - 1.$$

In the table, the numbers inside ( ) stand for the corresponding equation errors $|\det \hat{\Delta}(\lambda)|$ to the eigenvalues λ_i^N.

Table III

	N	2	4
$\{\lambda_i^N\}$	i=1	-1.4032 (.019)	-1.4011 (1.9×10^{-6})
	2		-1.6351 ± 4.1627 (.38)

	N	6	8
$\{\lambda_i^N\}$	i=1	-1.4011 (3.2×10^{-11})	-1.4011 (2.4×10^{-15})
	i=2	-1.6343 ± 4.1827 (8.2×10^{-4})	-1.8343 ± 4.1827 (4.5×10^{-7})
	i=3		-2.4284 ± 10.6698 (2.3)

N	16
i=1	-1.4011 (1.2×10^{-14})
i=2	-1.6344 ± 4.1827 (6.7×10^{-14})
i=3	-2.4256 ± 10.6890 (2.1×10^{-10})
i=4	-3.1695 ± 23.3811 (4.5×10^{-4})

Example 6.2. (Gibson [8], Example 8.3)

We consider the problem of minimizing

$$J(u) = \int_0^\infty \left(y^2(t) + \dot{y}^2(t) + u^2(t)\right)dt, \tag{6.4}$$

subject to the harmonic oscillator with delayed restoring force and delayed damping given by

$$\frac{d^2}{dt^2} y(t) + \frac{d}{dt} y(t-1) + y(t-1) = u(t). \tag{6.5}$$

If we define $x(t) \in \mathbb{R}^2$ by

$$x(t) = \left(y(t), \frac{d}{dt} y(t)\right)^T,$$

then (6.4) and (6.5) are equivalent to

$$J\left(u;(\eta,\phi)\right) = \int_0^\infty \left(|x(t)|^2 + u^2(t)\right)dt$$

and

$$\frac{d}{dt} x(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

respectively.

The optimal control in feedback form is

$$u(t) = -\Pi^{00}_{,21}\, x_1(t) - \Pi^{00}_{,22}\, x_2(t)$$

$$- \int_{-1}^{0} \left(\Pi^{10}_{,12}(\theta)\, x_1(t+\theta) + \Pi^{10}_{,22}(\theta)\, x_2(t+\theta)\right)d\theta, \tag{6.6}$$

where $\Pi^{00}_{,i,j}$ and $\Pi^{10}_{,i,j}(\theta)$ are the (i,j)-elements of the matrix Π^{00} and $\Pi^{10}(\theta)$, respectively. The Nth feedback control law is

$$u^N(t) = -\Pi^{00}_{N,21}\, x_1(t) - \Pi^{00}_{,22}\, x_2(t)$$

$$- \int_{-1}^{0} \left(\Pi^{10}_{N,12}(\theta)\, x_1(t+\theta) + \Pi^{10}_{N,22}(\theta)\, x_2(t+\theta)\right) d\theta. \tag{6.7}$$

Note that if we define $\xi(t) \in \mathbb{R}^2$ by

$$\xi(t) = \left(y(t),\ \frac{d}{dt}\, y(t) + y(t)\right)^T,$$

then (6.4) and (6.5) are equivalent to

$$J\left(u;\ (\tilde{\eta},\tilde{\phi})\right) = \int_0^\infty \left(\xi(t)^T\, Q\, \xi(t) + u^2(t)\right) dt, \tag{6.8}$$

with

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

and

$$\frac{d}{dt}\, \xi(t) = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \xi(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \xi(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{6.9}$$

respectively. Here, the initial conditions

$$\tilde{\eta} = (0,0)^T \quad \text{and} \quad \xi_2(\theta) = 0, \qquad -1 < \theta < 0, \tag{6.10}$$

yield $\xi(t) \equiv 0$, $t > 0$, regardless of the initial function $\xi_1(\theta)$, $-1 < \theta < 0$. Hence, for the initial conditions in (6.10) and any initial history $\xi_1(\cdot)$, the optimal control is $u(t) = 0$, $t > 0$. Therefore, the optimal control $u(t)$ must have the form

$$u(t) = -\tilde{\Pi}^{00}_{,21}\, \xi_1(t) - \tilde{\Pi}^{00}_{,22}\, \xi_2(t) - \int_{-1}^{0} \tilde{\Pi}^{10}_{,22}(\theta)\, \xi_1(t+\theta)d\theta. \qquad (6.11)$$

where $\tilde{\Pi}$ corresponds to the minimization problem to (6.8) and (6.9).

Note that $\xi_2(t) = x_1(t) + x_2(t)$, $t > 1$. Hence, it follows from (6.6) and (6.11) that

$$\Pi^{10}_{,12} = \Pi^{10}_{,22}.$$

Similarly,

$$\Pi^{10}_{N,12} = \Pi^{10}_{N,22}, \quad N > 1.$$

Numerically, we have the results in Table IV.

Table IV

$$\Pi_2^{00} = \begin{bmatrix} 2.1407 & 1.2988 \\ 1.2988 & 1.8611 \end{bmatrix} \qquad \Pi_4^{00} = \begin{bmatrix} 2.1387 & 1.2963 \\ 1.2963 & 1.8579 \end{bmatrix}$$

$$\Pi_6^{00} = \begin{bmatrix} 2.1387 & 1.2963 \\ 1.2963 & 1.8579 \end{bmatrix} \qquad \Pi_8^{00} = \begin{bmatrix} 2.1387 & 1.2963 \\ 1.2963 & 1.8579 \end{bmatrix}$$

	N	2	4	6	8
	k=0	-0.8846	-0.8821	-0.8821	-01.8821
	1	0.8971	0.8969	0.8969	0.8969
	2		-0.0835	-0.0835	-0.0835
	3		-0.0031	-0.0030	-0.0030
$\{a_k^N\}$	4		0.0014	0.0014	
	5			-0.0001	-0.0001
	6				2.4×10^{-6}
	7				2.4×10^{-7}
$\lvert\Pi_N^{10}(\theta) - A_1^T \Pi_N^{00}\rvert$		0.2182	0.0024	1.3×10^{-5}	3.5×10^{-8}

In the Table IV, $\{a_k^N\}_{k=0}^{N-1}$ are the expansion coefficients of $\Pi_{N,12}^{10}(\theta)$; i.e.,

$$\Pi_{N,12}^{10}(\theta) = \sum_{k=0}^{N-1} a_k^N P_k(2\theta + 1), \qquad -1 < \theta < 0.$$

Note that

$$\lvert\Pi_4^{00} - \Pi_8^{00}\rvert = 6.3 \times 10^{-7},$$

$$\|\Pi^{10}_{4,12} - \Pi^{10}_{8,12}\|_{L_2} = 4.8 \times 10^{-4}.$$

Again, one can see that the result for $N = 4$ gives a fairly good approximation. For comparison, the following are obtained by AVE and SPLINE schemes:

$$\Pi^{00}_{22} = \begin{bmatrix} 2.1034 & 1.2574 \\ 1.2574 & 1.8123 \end{bmatrix} \qquad \text{(AVE)}$$

$$\Pi^{00}_{16} = \begin{bmatrix} 2.1389 & 1.2963 \\ 1.2963 & 1.8576 \end{bmatrix} . \qquad \text{(SPL)}$$

Table V compares $\Pi^{10}_{22,1,2}(\theta)(\text{AVE})$, ([8], p. 137) and $\Pi^{10}_{4,1,2}(\theta)(\text{L-T})$.

Table V

$$\|\Pi^{10}_{22,1,2}(\theta)(\text{AVE}) - \Pi^{10}_{4,1,2}(\theta)(\text{L-T})\|_{L_2} = 7.0 \times 10^{-2}$$

θ	$\Pi^{10}_{22,1,2}(\theta)(\text{AVE})$	$\Pi^{10}_{4,1,2}(\theta)(\text{L-T})$
0.0	-0.1152	-0.0719
-0.1	-0.2247	-0.2033
-0.2	-0.3449	-0.3462
-0.3	-0.4750	-0.5003
-0.4	-0.6147	-0.6652
-0.5	-0.7631	-0.8404
-0.6	-1.0013	-1.0257
-0.7	-1.1698	-1.2206
-0.8	-1.3455	-1.4247
-0.9	-1.5278	-1.6378
-1.0	-1.7160	-1.8593

In this example, the closed-loop eigenvalues of $A - BB^* \Pi$ are roots of the characteristic equation $\det \hat{\Delta}(\lambda) = 0$, where

$$\hat{\Delta}(\lambda) = \begin{bmatrix} \lambda I - A_0 - e^{-\lambda} A_1 & -BB^T \\ -I & \lambda I + A_0 + e^{\lambda} A_1 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Table VI lists the eigenvalues λ_i^N of H^N which lie in the left half plane of C and give the relatively small equation error $|\det \hat{\Delta}(\lambda_i^N)|$.

Table VI

	N	2	4
$\{\lambda_i^N\}$	i=1	1.3983 (.0342)	-1.3893 (3.3×10^{-6})
	i=2	-0.7358 ± 1.2207i (.0277)	-0.7339 ± 1.2235i (3.4×10^{-6})

	N	6	8
$\{\lambda_i^N\}$	i=1	-1.3893 (5.4×10^{-11})	-1.3893 (3.9×10^{-14})
	i=2	-0.7339 ± 1.2235i (6.2×10^{-11})	-0.7339 ± 1.2235i (2.1×10^{-14})
	i=3	-2.0927 ± 7.4395i (78.1)	-2.0890 ± 7.4619 (.474)

Example 6.3 Here we deal with the equation which has multiple point delays

$$\frac{d}{dt}x(t) = x(t) + 2x(t-1) + x(t-2) + u(t), \tag{6.12}$$

with the cost functional

$$J(u,(\eta,\phi)) = \int_0^\infty \left(x^2(t) + u^2(t)\right)dt.$$

For each N, the Nth feedback control law is

$$u^N(t) = -\Pi_N^{00}\, x(t) - \int_{-2}^{0} \Pi_N^{10}(\theta)\, x(t+\theta)d\theta,$$

where Π_N^{00} is a scalar and $\Pi_N^{10}(\cdot) \in L_3([-2,0];\, R)$ is given by

$$\Pi_N^{10}(\theta) = \sum_{k=0}^{N} b_k^N\, P_k(2\theta + 3)\, \chi_{[-2,-1]}(\theta) + \sum_{k=0}^{N-1} a_k^N\, P_k(2\theta + 1)\, \chi_{(-1,0]}(\theta),$$

$$-2 < \theta < 0.$$

Table VII shows the numerical results for Π_N^{00} and $\Pi_N^{10}(\theta)$ and how closely we have approximated the jump condition (2.13) and the boundary condition (2.14)

Table VII

N	2	4	6	8
Π_N^{00}	3.2159	3.2074	3.2073	3.2074
b_0	1.5306	1.5246	1.5244	1.5243
b_1	-1.220	-1.2205	-1.2214	-1.2216
b_2	0.3295	0.3990	0.3972	0.3969
b_3		-0.0583	-0.0590	-0.0595
b_4		-0.0002	-0.0050	-0.0049
b_5			-0.0008	-0.0001
b_6			-0.0011	-0.0001
b_7				-0.0005
b_8				-0.0002
a_0	3.3767	3.3911	3.3914	3.3914
a_1	-2.8081	-2.6999	-2.7004	-2.7006
a_2		0.8479	0.8477	0.8478
a_3		-0.1119	-0.1092	-0.1094
a_4			0.0083	0.0080
a_5			-0.0018	-0.0009
a_6				-0.0002
a_7				0.0004
$\lvert\Pi_N^{10}(-2) - \Pi_N^{00}\rvert$	0.1352	0.0047	6.0×10^{-4}	6.2×10^{-5}
$\lvert\Pi_N^{10}((-1)^+) - \Pi_N^{10}((-1)^-) - 2\Pi_{00}^N\rvert$	0.8865	0.0090	2.0×10^{-4}	8.7×10^{-5}

We have the function values of $\Pi_N^{10}(\theta)$ in Table VIII.

Table VIII

θ	$\Pi_2^{10}(\theta)$	$\Pi_4^{10}(\theta)$	$\Pi_6^{10}(\theta)$	$\Pi_8^{10}(\theta)$
-2.0	3.0807	3.2026	3.2067	3.2074
-1.9	2.6586	2.6892	2.6879	2.6877
-1.8	2.2761	2.2518	2.2492	2.2498
-1.7	1.9331	1.8834	1.8830	1.8831
-1.6	1.6297	1.5769	1.5769	1.5784
-1.5	1.3658	1.3252	1.3280	1.3277
-1.4	1.1415	1.1213	1.1229	1.1233
-1.3	0.9567	0.9583	0.9574	0.9581
-1.2	0.8114	0.8292	0.8266	0.8265
-1.0	0.6396	0.6451	0.6443	0.6440
-1.0	6.1848	7.0508	7.0587	7.0587
-0.9	5.6232	5.9500	5.9477	5.9482
-0.8	5.0615	5.0047	5.0025	5.0031
-0.7	4.4999	4.2014	4.2026	4.2025
-0.6	3.9383	3.5267	3.5303	3.5298
-0.5	3.3767	2.9671	2.9706	2.9704
-0.4	2.8150	2.5094	2.5103	2.5106
-0.3	2.2534	2.1399	2.1375	2.1378
-0.2	1.6918	1.8454	1.8413	1.8411
-0.1	1.1301	1.6123	1.6111	1.6107
0.0	0.5685	1.4272	1.4360	1.4362

In this example, the closed loop characteristic equation is given by

$$\Delta(\lambda) = \left(\lambda - 1 - 2e^{-\lambda} - e^{-2\lambda}\right)\left(\lambda + 1 + 2e^{\lambda} + e^{2\lambda}\right) - 1 = 0.$$

Table IX shows the eigenvalues of H^N in the same manner as before.

Table IX

N	2	4
λ_1^N	-1.5217 (.2172)	-1.5174 (3.0×10^{-5})
λ_2^N	0.9524 ± 2.4826 i (2.090)	-0.9028 ± 2.5445 i (.0031)
λ_3^N		-0.6103 ± 5.0272 i (.8349)

N	6	8
λ_1^N	-1.5174 (6.9×10^{-10})	-1.5174 (6.0×10^{-14})
λ_2^N	-0.9029 ± 2.5445 i (7.0×10^{-7})	-0.9029 ± 2.5445 i (4.6×10^{-11})
λ_3^N	-0.5890 ± 5.0114 i (.0030)	-0.5889 ± 5.0114 i (2.7×10^{-6})
λ_4^N	-1.3588 ± 8.7500 i (10.18)	-1.3159 ± 8.7703 i (0.1018)
λ_5^N		-1.0595 ± 11.4781 i (4.108)

The numerical results presented here reveal that numerically one has strong convergence of Π^N for the Legendre-tau approximation. At this time, we hve not been able to prove the strong convergence of Π^N in the general case (except for certain special cases described in Lemma 5.4). It requires a careful study of the asymptotic behavior of the spectra of ^{N}. However, the efficiency of the numerical schemes is most important from the practical point of veiw. We observe, from the numerical results of this section, that the Legendre-tau method provides faster convergence and better approximation at low orders (i.e., small N) than the AVE and SPLINE schemes. In the above examples, the results corresponding to $N = 4$ give a fairly good approximation of the optimal feedback gain.

As further evidence of the usefulness of the Legendre-tau approximation, one can use it as an approximation technique for computing closed-loop eigenvalues of the feedback system. Note that eigenvalues close to the origin are approximated quite well at low orders on the above examples.

From these observations, we believe the Legendre-tau approximation scheme offers one of the favorable methods for construction of feedback gains. In future investigations, our efforts for constructing feedback gains for delay systems will be combined with the approach to finite-order compensentor design for distributed parameter systems [15], developed by J. M. Schumacher to develop a design procedure for the construction of compensentors for delay systems.

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Final Report

16. Abstract

The numerical scheme based on the Legendre-tau approximation is proposed to approximate the feedback solution to the linear quadratic optimal control problem for hereditary differential systems. The convergence property is established using Trotter ideas. The method yields very good approximations at low orders and provides an approximation technique for computing closed-loop eigenvalues of the feedback system. A comparison with existing methods (based on "averaging" and "spline" approximations) is made.

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