Aeroelastic Effects in Multi-Rotor Vehicles With Application to a Hybrid Heavy Lift System

Part I: Formulation of Equations of Motion

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GRANT NAG2-116
AUGUST 1984
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Prepared for
Ames Research Center
under Grant NAG2-116

NASA
National Aeronautics
and Space Administration
Scientific and Technical
Information Branch
1984
PREFACE

This report presents a set of governing coupled differential equations for a model representing a Hybrid Heavy Lift Airship (HHLA). These equations serve as the basis of a numerical study aimed at determining the aeroelastic stability and structural response characteristics of the HHLA. These results will be presented in a follow on report which will represent Part II of this study.

The research effort reported herein was carried out in the Mechanics and Structures Department at UCLA by Dr. C. Venkatesan and Professor P. Friedmann who served as the principal investigator.

The authors want to take this opportunity to express their gratitude to the grant monitor Dr. H. Miura for his numerous constructive comments and suggestions.
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NOMENCLATURE

\( A_T \) - Total cross-sectional area of the blade
\( a \) - Lift curve slope
\( a_{pk} \) - Acceleration of a point \( p \) on \( k \)th blade
\( b \) - Blade semichord
\( c_{do} \) - Drag coefficient for blade
\( C(k) \) - Theodorsen's lift deficiency function
\( D_\alpha \) - Drag force on envelope
\( D_k \) - Drag force per unit length on \( k \)th blade
\( EI_z \) - Bending stiffness of the supporting structure in \( X_s-Y_s \) plane
\( EI_y \) - Bending stiffness of the supporting structure in \( X_s-Z_s \) plane
\( e \) - Blade offset
\( \hat{e}_x, \hat{e}_y, \hat{e}_z \) - Unit vectors along \( X, Y, Z \) axes
\( EI_2 \) - Bending stiffness of blade in lead-lag
\( EI_3 \) - Bending stiffness of blade in flap
\( \xi_F, \xi_L, \xi_T \) - Damping coefficients
\( GJ \) - Torsional stiffness of the supporting structure/blade
\( h_1 \) - Distance between origin \( O_s \) and C.G. of the underslung weight
\( h_2 \) - Distance between hub center and C.G. of the fuselage
\( h_3 \) - Distance between origin \( O_s \) and center of buoyancy of envelope
\( h_4 \) - Distance between origin \( O_s \) and C.G. of the envelope
\( h_5 \) - Distance between C.G. of the supporting structure to the origin \( O_s \)
\( I_{F1}, I_{F2} \) - Moments of inertia of the fuselage \( F_1 \) and \( F_2 \)
\( I_{MB1}, I_{MB2} \) - Principal moments of inertia per unit length of the blade about cross-sectional axes
\( [I] \) - Inertia tensor
$K_{PB}, K_{BC}$ - Root spring stiffness in flap and lag, representing blade stiffness

$K_{PB}, K_{PH}$ - Stiffness of the root springs representing blade torsional stiffness and control link stiffness

$L_{1}, L_{2}$ - Lift due to rotor systems 1, 2

$L_{B}$ - Buoyant lift on envelope

$L_{C}$ - Circulatory flow lift

$L_{NC}$ - Noncirculatory flow lift

$L_{F1}, L_{F2}$ - Distance between the origin $O_{S}$ of the supporting structure and C.G.'s of fuselages $F_1$ and $F_2$

$m$ - Mass per unit length of the supporting structure/blade

$M_{F1}, M_{F2}$ - Fuselage masses of fuselages $F_1, F_2$

$M_{a}$ - Moment due to envelope

$M_{x}, M_{y}, M_{z}$ - Elastic moments in torsion, flap and lag

$N$ - Number of blades

$N_{N1}, N_{N2}, N_{N2}$ - Number of normal modes used in modeling the supporting structure

$O_{S}$ - Origin located at the center line of the supporting structure

$O_{H1}, O_{H2}$ - Hub Centers

$P_{F1}, P_{F2}$ - Thrust force

$P_{T1}, P_{T2}$ - Thrust force

$q_{ik}, q_{Ak}, q_{Dk}$ - Distributed blade inertia, aerodynamic and damping moments

$r_{pk}$ - Position of vector at a point $p$ on $k^{th}$ blade
t - Time

$u_k, v_k, w_k$ - $k^{th}$ blade deformation in axial, lead-lag and flap directions

$v_s, w_s/v_{sl}, w_{sl}$ - Elastic deflection of the supporting structure

$V$ - Free stream velocity

$V_F$ - Forward velocity of the vehicle

$W_{UN}$ - Underslung weight

$W$ - Weight

$X_A$ - Offset between the elastic center and the aerodynamic center in blade cross-section

$X_I$ - Offset between the elastic center and the mass center in blade cross-section

$X_T$ - Offset between the elastic center and the tension center in blade cross-section

$x_k$ - Coordinate along $k^{th}$ blade elastic axis

$y_{ok}, z_{ok}$ - Blade cross-sectional coordinate

$\alpha_R$ - Angle of forward tilt of the rotor plane

$\beta_p$ - Blade precone angle

$\beta_k$ - Flap angle for $k^{th}$ blade

$\zeta_k$ - Lead-lag angle for $k^{th}$ blade

$\phi_k$ - Torsional angle for $k^{th}$ blade

$\varepsilon$ - Basis for orders of magnitude comparison associated with typical elastic blade slopes

$\eta_{ok}, \xi_{ok}$ - Blade cross-sectional principal axis coordinate

$\eta_i$ - $i^{th}$ mode shape

$\theta_k, \theta_{ok}$ - $k^{th}$ blade collective pitch

$\theta_{lc}, \theta_{ls}$ - Cyclic pitch components

$\theta_{Bk}$ - Pretwist in blade

$\theta_{Gk}$ - Geometric pitch in $k^{th}$ blade
\( \theta_{zs} - \theta_{ys} - \theta_{xs} \) - Rigid body perturbational rotation in yaw-pitch-roll

\( \lambda_k \) - Inflow ratio at \( k^{th} \) blade

\( \nu_k \) - Induced velocity at the \( k^{th} \) blade cross-section

\( \rho \) - Density of the material of the blade

\( \rho_A \) - Density of air

\( \phi_{ik} \) - Inflow angle

\( \phi_s, \phi_{sl} \) - Elastic twist in supporting frame

\( \mu \) - Advance ratio

\( \psi_k \) - Azimuth angle of \( k^{th} \) blade

\( \omega_k \) - Angular velocity of \( k^{th} \) blade

\( \omega_1 \) - Natural frequency of the supporting structure in \( i^{th} \) mode of vibration

\( \Omega \) - Rotor r.p.m.

**Subscripts**

EN - Envelope

F1,F2 - Fuselages \( F_1,F_2 \)/forces acting at fuselage C.G. \( O_{F1},O_{F2} \)

H - Hub center

H1,H2 - Forces acting at hub center \( O_{H1},O_{H2} \) of rotor systems \( R_1,R_2 \)

R - Rotor, quantities refer to R system

S,s - Supporting structure, s system

S1,s1 - s1 system

T - Thrust force

UN - Underslung weight

W - Gravity loads

x,y,z - x,y,z components

( )_x - \( \frac{d}{dx} \)

1,2,3,4,5 - Quantities refer to the corresponding coordinate system
SUMMARY

This report presents a set of governing coupled differential equations for a model representing a Hybrid Heavy Lift Airship (HHLA). The model consists of a bouyant envelope, multiple rotor systems, an underslung weight and thrusters, all attached to a flexible supporting structure. The dynamic equations are written for the individual blade with hub motions, for the rigid body motions of the whole model and also for the flexible modes of the supporting structure. The purpose of these equations is to serve as the basis of a numerical study aimed at determining the aeroelastic stability and structural response characteristics of the HHLA.
1. INTRODUCTION

Hybrid Heavy Lift Airship (HHLA) or Hybrid Heavy Lift Helicopter (HHLH) is useful for providing heavy lift capability whose potential applications are for logging, construction, coast guard surveillance and military heavy lift capability. These vehicles combine a buoyant envelope lift with lift and control forces generated by a multiple rotor system. A rough sketch of a HHLA configuration is shown in Fig. 1. Such a configuration is different from the conventional rotorcraft which have been considered in the past. It is well known that the aeroelastic and structural dynamic response problems are crucial for the safe design of a successful rotorcraft. Therefore it is essential to consider the basic aeroelastic and dynamic behavior of HHLA type vehicles so that the potential aeroelastic instability modes and structural dynamic features can be simulated and identified in the design process.

It has been established that rotary-wing aerelasticity is inherently nonlinear [Ref. 1]. Aeroelastic studies performed in both industry and research organization are indicative of this aspect. Thus the correct treatment of a wide class of problems in this field requires a consistent development of a mathematical model which includes geometrically nonlinear effects, due to the inclusion of finite slopes in the inertia, structural and aerodynamic operators. It is also well known [Ref. 2] that the unsteady aerodynamic environment in rotorcraft is complicated. Accurate mathematical models, including the unsteady wake effects are rarely incorporated in aeroelastic analyses. In HHLA type vehicles these difficulties will be further compounded by interference buoyant lift. Therefore, it is clear that study of some basic aeroelastic effects in HHLA type vehicles is important for the effective design of such vehicles.

Some of the typical problems that might be encountered by the unique configuration represented by HHLA type vehicles are described below.

(a) **Isolated Blade Instabilities**: These instabilities are of the flap-lag, flap-pitch or coupled flap-lag-torsion type and can occur both in hover and forward flight. Even if some existing rotor systems which are expected to be free of these isolated blade type instabilities, a wake excited flap-pitch or coupled flap-lag-torsion flutter can occur at low thrust and low inflow...
[Refs. 2, 3 and 4]. This situation could potentially be of interest for HHLA type vehicles when the rotors are lightly loaded and the buoyance ratio \( \beta \) is large [buoyance ratio \( \beta = \text{buoyant lift/vehicle gross weight} \)]. Furthermore, it is reasonable to concentrate primarily on the hover case for HHLA type vehicles because the forward speed of HHLA type vehicles will be low (i.e. \( \mu < 0.20 \)). It was shown [Ref. 5] that forward flight is frequently stabilizing.

(b) **Coupled Rotor/Support System Instabilities:** A rotor mounted on a moving or flexible support system can have additional instabilities when compared to an isolated blade. On the ground a mechanical instability can occur known as ground resonance [Refs. 2 and 6] and this instability is known to be sensitive to the flexibility and damping of the landing gear system. In flight, the coupled rotor/support system can experience an aeromechanical instability usually denoted as air-resonance [Refs. 2 and 6]. All these instabilities could be encountered in a HHLA type vehicle, because furthermore, the buoyancy effect and the flexibility of the supporting structure could modify these instabilities in an unexpected manner.

(c) **Vibration Problems:** The vibration levels in helicopters have two peaks, when plotted as a function of advance ratio [Ref. 6]. One peak occurs at relatively low advance ratios and the second at high advance ratios. Since, the advance ratio for the HHLA is low, this type of vehicle could experience considerable vibration levels. Thus, it is necessary to estimate the vibration levels and the resulting dynamic stresses to determine the fatigue life of the structure.

To gain a fundamental understanding of aeroelastic effects which could be encountered on HHLA type vehicles due to their unique features (such as buoyancy, multiple rotors, flexible supporting structure and underslung load), a study of an idealized, simple model, representative of a typical HHLA vehicle, shown in Figure 2, was selected. This report presents a detailed derivation of the equations of equilibrium which cover the dynamics of this system.
2. AEROELASTIC MODEL OF AN HHLA

2.1 Introduction and Assumption

To study the basic aeroelastic problems which could be encountered in an HHLA type configuration, a typical configuration shown schematically in Fig. 2 will be considered. The essential features of the configuration are:

(a) A flexible supporting structure with bending stiffness $E_{Iy}(x)$ in the $X_S-Z_S$ plane, bending stiffness $E_{Iz}(x)$ in the $Y_S-X_S$ plane ($Y_S$ coordinate is normal to the figure), a torsional rigidity $GJ(x)$ and a mass distribution $m(x)$.

(b) Two rotor systems capable of providing lift, each having an arbitrary number of blades $N$, are attached rigidly to the ends of the flexible structure. The distance between the center line of the structure to the hub center for the rotor systems is $h_2$.

(c) Two masses $M_{F1}$, $M_{F2}$ having inertias $I_{F1}$ and $I_{F2}$ respectively are attached to the ends of the flexible structure. These masses and inertias represent the helicopters. The distance between the origin $O_s$ fixed in the supporting structure to the C.G.'s of the fuselages $F_1$ and $F_2$ are $l_{F1}$ and $l_{F2}$ respectively. The C.G. of the supporting structure is at a distance $h_5$ from the origin $O_S$. Furthermore it is assumed that the C.G.'s of the supporting structure and fuselages are on the X-axis.

(d) A weight $W_{UN}$ is attached to the structure. Its C.G. is at a distance $h_1$ from the origin $O_s$. This weight can move freely or it can be locked in a fixed position with respect to the flexible structure.

(e) An envelope, providing the buoyant lift $L_B$ and drag $D_B$ acting at its center of pressure, is attached to the structure. The center of pressure is at a distance $h_3$ from the origin $O_s$. The C.G. of the envelope is at a distance $h_4$ from the origin $O_s$.

(f) Concentrated axial loads $P_{T1}$, $P_{T2}$ simulate thrusters.

Using this model, the dynamic equations of motion for the combined system consisting of two rotors, flexible structure, buoyant envelope and load $W_{UN}$ are derived. The derivation requires four ingredients: blade equations with support motions, equations for the flexible structure connecting the rotors, equations representing the forces and moments introduced by the envelope and finally a representation of the dynamics of the load $W_{UN}$.

Certain assumptions are introduced before writing the dynamic equations for this system, these are given below:
(1) The rotor blades are assumed to be rigid with equivalent root springs representing the flexibility of the blade.
(2) The rotor blades are attached to the hub with an offset \( e \) from the axis of rotation (Hub center).
(3) The blade feathering axis is preconed by an angle \( \beta \). The blade has no torque offset, sweep or droop.
(4) The feathering axis coincides with the elastic axis of the blade.
(5) The blade cross-section is symmetric and has four distinct points: elastic center, mass center, aerodynamic center and tension center (Fig. 3).
(6) The structural damping in the blade is assumed to be of the viscous type.
(7) The rotor shaft is rigid.
(8) The rotor speed is constant.
(9) The rotor consists of three or more blades.
(10) Two-dimensional quasi-steady aerodynamics is used to obtain the aerodynamic loads. There is no reverse flow and stall. The compressibility effect is neglected.
(11) The C.G. of the fuselages are on the center line of the supporting structure such that the individual C.G.'s lie on a straight line.
(12) The underslung mass is rigidly attached to the structure.
(13) The elastic deformations of the supporting structure are at least one order of magnitude lower than that of the blade deformation.
(14) Flexible structure is modeled by using free-free beam modes with arbitrary mass and stiffness distribution.
(15) Aerodynamic forces and moments due to the envelope are modeled by using the model provided in Ref. 7. Quasisteady aerodynamic theory is used for blade aerodynamics, and aerodynamic interference between the rotor and the envelope is neglected.

Based on these assumptions, the dynamic equations of motion for the model are derived, using force and moment equilibrium conditions at the connecting points as was done in Ref. 8.

2.2 Ordering Scheme
When deriving equations of motion for such a multi-rotor system, a large number of higher order terms has to be considered. Previous research has clearly indicated that many higher order terms can be neglected systematically by using
an ordering scheme [Refs. 1,8]. Warmbrodt and Friedmann [Ref. 8] and Levin have, in their derivation of coupled rotor/fuselage equations, assigned in a judicious manner, appropriate orders of magnitude for various terms encountered in the coupled rotor/fuselage equations. The ordering scheme employed in this study follows this approach. By assuming fuselage rotations of order ε many additional terms will appear in the coupled rotor/fuselage equations. Such an ordering scheme was recently used in Ref. 9. In the earlier derivations, the fuselage was assumed to have only rigid body degrees of freedom and the orders of magnitude of the corresponding perturbed quantities are \(0(\varepsilon^{3/2})\). In the present case, the fuselage/supporting structure is being considered flexible and orders of magnitude are assigned also to the deformation of the supporting structure. An order of magnitude of \(0(\varepsilon^2)\) is assigned to the elastic deformations of the structure so that this effect appears in the hub motion while at the same time the number of terms in the equations remains manageable.

The basis of the ordering scheme is a small dimensionless parameter \(\varepsilon\) which represents typical blade slopes due to elastic deflections. It is known that for helicopter blades \(\varepsilon\) is in the range

\[0.1 \leq \varepsilon \leq 0.2\]

The ordering scheme is based on the assumption that

\[1 + 0(\varepsilon^2) \simeq 1\]

i.e. terms of the order of \(0(\varepsilon^2)\) are neglected in comparison with unity. The orders of magnitude for the various parameters governing this problem are given below.

\[
\begin{aligned}
\cos \psi_k, \sin \psi_k, & \frac{x_k}{R}, \frac{\rho A b R}{m}, \frac{h_1}{R}, \frac{h_2}{R}, \frac{h_3}{R}, \frac{h_4}{R}, \frac{h_5}{R}, \frac{\rho_{F1}}{R}, \frac{\rho_{F2}}{R} = 0(1) \\
\frac{1}{\Omega} \frac{\partial}{\partial \tau} ( ) = \frac{3}{\partial \psi} ( ) = 0(1) \\
\frac{R}{\partial x_k} ( ) = \frac{3}{\partial x_k} = 0(1) \\
\theta_{0k}, \theta_{Gk} = 0(\varepsilon^k)
\end{aligned}
\]

*J. Levin, "Formulation of Helicopter Air Resonance Problem in Hover with Active Controls", M.S. Thesis, Mechanics and Structures Department, University of California, Los Angeles, September 1981.
\[ \beta_k, \phi_k, \psi_k, \frac{e}{R}, \frac{b}{R}, \beta_p, \lambda_k, \theta_{Gk}, \ddot{\theta}_{Gk} = O(\varepsilon) \]

\[ \frac{C_{do}}{a}, \frac{R_{xs}}{R}, \frac{R_{ys}}{R}, \frac{R_{zs}}{R}, \theta_{xs}, \theta_{ys}, \theta_{zs} = O(\varepsilon^{3/2}) \]

\[ \frac{u_k}{R}, \frac{x_A}{R}, \frac{x_I}{R}, \frac{u_s}{R}, \frac{v_s}{R} = O(\varepsilon^2) \]

\[ \frac{x_T}{R} = O(\varepsilon^3) \]

\[ v_{s,x}, u_{s,x}, \phi_s = O(\varepsilon^{3/2}) \]

\[ \frac{I_{MB3}}{mR^2} = O(\varepsilon^3) \]

\[ \frac{I_{MB2}}{mR^2} = O(\varepsilon^7) \]
3. COORDINATE SYSTEMS

In the derivation of equations of motion of the HHLA model, various reference coordinate systems are used. The transformation relation between quantities referred in the various inertial, noninertial coordinate systems to be established before deriving the equations of motion. The relation between two orthogonal coordinate systems with axes $X_i$, $Y_i$, $Z_i$ and $X_j$, $Y_j$, $Z_j$ with $\vec{e}_{xi}$, $\vec{e}_{yi}$, $\vec{e}_{zi}$ and $\vec{e}_{xj}$, $\vec{e}_{yj}$, $\vec{e}_{zj}$ as unit vectors along the respective axes is

$$
\begin{bmatrix}
\vec{e}_{xi} \\
\vec{e}_{yi} \\
\vec{e}_{zi}
\end{bmatrix} = [T_{ij}]
\begin{bmatrix}
\vec{e}_{xj} \\
\vec{e}_{yj} \\
\vec{e}_{zj}
\end{bmatrix}
$$

(3.1)

where $[T_{ij}]$, the transformation matrix, can be found using the Euler angles required to rotate the $j$-system so as to make it parallel to $i$-system.

The $S$-system (Fig. 2) is an inertial system whose origin $O_s$ is fixed at the center line of the supporting structure in the unperturbed state with the $Z_s$ axis vertically upwards passing through the center of gravity of the envelope and $X_s$ is directed aft.

The $Sl$-system is a noninertial coordinate system whose origin is also fixed at the same point $O_s$ of the supporting structure. This is a body fixed coordinate system which moves along with the body during perturbational motion. The $S$-system and $Sl$-system coincide with each other in the unperturbed state of the model.

The $R$-system is another inertial system fixed at the center $O_H$ of the unperturbed hub. The directions of the axes of this system are parallel to that of the $S$-system. It should be noted that in the development of rotor blade equations, only one general rotor system with hub motions is considered. Consequently one set of rotor coordinate systems will be defined. These definitions are valid for all the rotor systems in the model. The only difference that will occur are the different hub motions due to the relative positions of the hubs with respect to the origin of the $S$-system. This is accounted for in the derivation by deriving a general expression for the motion of hub center $O_H$ due to the rigid body translation and rotation and due to the elastic deformations of the supporting structure.
The 1-system is a body fixed system with its origin fixed at the center of the hub \( O_H \) (Fig. 4). Prior to perturbational motion the 1-system coincides with the R-system. It is assumed that the 1-system and SI-system are parallel systems, because as pointed out earlier that the elastic deformation slopes of the supporting structure are of order \( O(\epsilon^{7/2}) \). So any small rotational motion given to the hub fixed 1-system, due to the elastic deformation of the supporting structure, is assumed negligible.

The perturbational translational motion at the hub center \( O_H \) due to the rigid body motion and the elastic deformation of the structure is written as

\[
\hat{R}_H = R_x \hat{x}_R + R_y \hat{y}_R + R_z \hat{z}_R
\]  

(3.2)

and if \( \theta_z, \theta_y, \theta_x \) represent the yaw-pitch-roll rotations of the structure then the transformation matrix \([T_{1R}]\) can be written as

\[
[T_{1R}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_x & \sin \theta_x \\
0 & -\sin \theta_x & \cos \theta_x
\end{bmatrix} \begin{bmatrix}
\cos \theta_y & 0 & -\sin \theta_y \\
0 & 1 & 0 \\
\sin \theta_y & 0 & \cos \theta_y
\end{bmatrix} \begin{bmatrix}
\cos \theta_z & \sin \theta_z & 0 \\
-\sin \theta_z & \cos \theta_z & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Since \( \theta_x, \theta_y, \theta_z \) are of order \( O(\epsilon^{3/2}) \), the sines and cosines can be replaced with \( \sin \theta = \theta \) and \( \cos \theta = 1 \). Thus

\[
[T_{1R}] = \begin{bmatrix}
1 & \theta & -\theta \\
\theta & -\theta & 1 \\
\theta & \theta & -\theta
\end{bmatrix}
\]

(3.3)

Rotating 2k-system is a blade fixed coordinate system which rotates with the \( k \)th blade. This 2k-system is rotated from the 1-system by the azimuth angle, \( \psi_k \), of the \( k \)th blade (Fig. 4) about \( Z_1 \) axis. The transformation matrix is

\[
[T_{21}] = \begin{bmatrix}
\cos \psi_k & \sin \psi_k & 0 \\
-\sin \psi_k & \cos \psi_k & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(3.4)
Rotating the 2k-system by an angle $-\beta_p$ (precone angle) about $Y_{2k}$-axis and translating the origin to the $k^{th}$ blade bearing by a distance $e_x e_x2k$, gives the 3k-system (Fig. 5). The $X_{3k}$-axis is along the elastic axis of the undeformed $k^{th}$ blade. Since $\beta_p$ is of the order $\theta(\varepsilon)$, $\sin \beta_p \approx \beta_p$ and $\cos \beta_p \approx 1$. The rotation matrix is

$$
[T_{32}] = \begin{bmatrix}
1 & 0 & \beta_p \\
0 & 1 & 0 \\
-\beta_p & 0 & 1
\end{bmatrix}
$$

(3.5)

The 4k-system (Fig. 6) is fixed in the cross-section of the $k^{th}$ blade. Translating 3k system an amount $x_k \hat{e}_{x3k}$ gives the 4k system at the cross-section $x_k$ of the $k^{th}$ blade prior to elastic deformation. During elastic deformation of the $k^{th}$ blade, i.e., flap, lag and torsion, the 4k system translates and rotates with the cross-section. The origin of the 4k system after the deformation is given as

$$(x_k + \xi_k) \hat{e}_{x3k} + \eta_k \hat{e}_{y3k} + \omega_k \hat{e}_{z3k}
$$

(3.6)

The rotation of the 4k system is obtained by Euler angles $-\beta_k, \zeta_k, \phi_k$. These angles represent the flap-lag-torsional rotation of the $k^{th}$ blade at location $x_k$. The sequence of rotation is flap-lag-torsion. The transformation matrix $[T_{43}]$ is

$$
[T_{43}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi_k \sin \phi_k & \sin \phi_k \\
\sin \phi_k & \cos \phi_k & 0
\end{bmatrix}
\begin{bmatrix}
\cos \zeta_k & \sin \zeta_k & 0 \\
-\sin \zeta_k & \cos \zeta_k & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \beta_k & 0 & \sin \beta_k \\
0 & 1 & 0 \\
-\sin \beta_k & 0 & \cos \beta_k
\end{bmatrix}
$$

(3.7)

Since the angles $\phi_k, \zeta_k, \beta_k \approx \theta(\varepsilon)$, the transformation matrix can be simplified by assuming $\sin \theta \approx \theta$ and $\cos \theta \approx 1$, and be written as

$$
[T_{43}] = \begin{bmatrix}
1 & \zeta_k & \beta_k \\
-\zeta_k \phi_k - \beta_k \zeta_k & 1 & \phi_k - \beta_k \zeta_k \\
-\beta_k - \phi_k \zeta_k & -\phi_k & 1
\end{bmatrix}
$$

(3.7)
In our model, we have considered the blade as a rigid blade with root springs. So, the relation between the translation and rotation is

\[ v_k = x_k \xi_k \quad \text{and} \quad w_k = -(-x_k \partial_k) = x_k \phi_k \]  

(3.8)

To facilitate the description of the blade element aerodynamics, the 5k system (Fig. 7) is defined by removing the torsional twisting of the blade from the 4k system which gives the rotation matrix

\[
[T_{54}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(-\phi_k) & \sin(-\phi_k) \\
0 & -\sin(-\phi_k) & \cos(-\phi_k)
\end{bmatrix}
\]  

(3.9)

when \( \phi_k \) is small, it can be written as

\[
[T_{54}] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -\phi_k \\
0 & \phi_k & 1
\end{bmatrix}
\]  

(3.10)

Summary of Coordinate Systems

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Coordinate System</th>
<th>Unit Vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>Inertial system. Origin ( O_S ) fixed at the center line of the undisturbed supporting structure with ( Z_S ) axis passing through the C.G. of the envelope</td>
<td>( \hat{e}<em>{xs}, \hat{e}</em>{ys}, \hat{e}_{zs} )</td>
</tr>
<tr>
<td>S1</td>
<td>Noninertial body fixed. Origin at the point ( O_{S1} ) on the supporting structure</td>
<td>( \hat{e}<em>{xs1}, \hat{e}</em>{ys1}, \hat{e}_{zs1} )</td>
</tr>
<tr>
<td>R</td>
<td>Inertial. Fixed at the undeformed hub center ( O_H ). ( S ) and ( R ) are parallel systems</td>
<td>( \hat{e}<em>{rx}, \hat{e}</em>{ry}, \hat{e}_{rz} )</td>
</tr>
<tr>
<td>l</td>
<td>Noninertial body fixed. Origin at the center of rotor hub ( O_{H_l} ). ( S1 ) and ( l ) are parallel systems</td>
<td>( \hat{e}<em>{xl}, \hat{e}</em>{yl}, \hat{e}_{zl} )</td>
</tr>
<tr>
<td>k</td>
<td>Description</td>
<td>Origin and Axes</td>
</tr>
<tr>
<td>----</td>
<td>-----------------------------------------------------------------------------</td>
<td>-----------------</td>
</tr>
<tr>
<td>2k</td>
<td>Rotates with k\textsuperscript{th} blade. Origin at the center of the rotor hub $O_H$.</td>
<td>$\hat{e}<em>{x2k}$, $\hat{e}</em>{y2k}$, $\hat{e}_{z2k}$</td>
</tr>
<tr>
<td>3k</td>
<td>Rotates with k\textsuperscript{th} blade. Origin at the k\textsuperscript{th} blade pitch bearing. Preconed. $x_{3k}$ axis coincident with blade elastic axis in undeformed position</td>
<td>$\hat{e}<em>{x3k}$, $\hat{e}</em>{y3k}$, $\hat{e}_{z3k}$</td>
</tr>
<tr>
<td>4k</td>
<td>Rotates with k\textsuperscript{th} blade. Origin at the elastic axis of the deformed blade cross-section at a distance $x_k$</td>
<td>$\hat{e}<em>{x4k}$, $\hat{e}</em>{y4k}$, $\hat{e}_{z4k}$</td>
</tr>
<tr>
<td>5k</td>
<td>Rotates with k\textsuperscript{th} blade. Origin at the elastic axis of the deformed blade cross-section at a distance $x_k$. Torsional rotation of the blade not included</td>
<td>$\hat{e}<em>{x5k}$, $\hat{e}</em>{y5k}$, $\hat{e}_{z5k}$</td>
</tr>
</tbody>
</table>
4. MOTION OF THE VEHICLE

The unconstrained vehicle has six rigid body degrees of freedom and also has elastic deformation of the supporting structure. Before presenting the equations of motion for the blade and the structure, it is necessary to establish certain kinematical relations between the vehicle motion and the hub displacement, because the blade inertia and aerodynamic loads are affected by the hub displacement. In this section, the position vector of the origin of the hub centers $O_{H1}$ and $O_{H2}$ and the rotation vector at the rotor systems, due to the perturbations in rigid body translation and rotation, and elastic deformation of the supporting structure, is derived. Subsequently these expressions are used in writing the blade loads.

4.1 Kinematical Relations

The sequence of perturbational motion of the vehicle model [Fig. 8], assumed to take place, consists first of rigid body translation of the origin $O_s$ of the supporting structure, i.e. origin of the $S$-system, then rigid body rotation in the sequence yaw-pitch-roll and finally in the perturbed position, the elastic deformation of the structure. Referring to Fig. 8, the perturbational translational motion occurs along $X_s$, $Y_s$, and $Z_s$ coordinate axes. After the rigid body rotation, the body fixed axes system is referred as $S_1$-system. The elastic deformations occur in the $S_1$-system [Fig. 9].

During rigid body perturbational translation, the origin $O_s$ is moved through a distance,

$$\hat{r}_{0s} = R_{xs} \hat{x}_s + R_{ys} \hat{y}_s + R_{zs} \hat{z}_s$$ (4.1)

Then the model is rotated about $Z_s$ axis through an angle $\theta_{zs}$ representing yaw motion, followed by a rotation $\theta_{ys}$ about the yawed $Y_s$ axis representing pitch. To represent roll, a rotation $\theta_{xs}$ is introduced about the yawed-pitched $X_s$ axis. The new position of the body axis system is $S_1$. The transformation of unit vectors from $S_1$-system to $S$-system is given by
Figure 9 refers to the perturbed state of the model after rigid body motion. It is assumed that the elastic deformations of the structure occur in this state. The elastic deformations are bending in $X_{s1}, Y_{s1}$ plane, bending in $X_{s1}, Z_{s1}$ plane and torsion about $X_{s1}$ axis. The deformations are represented by:

1. $v_{s1}$ along $Y_{s1}$
2. $w_{s1}$ along $Z_{s1}$
3. $\phi_{s1}$ along $X_{s1}$

Position vector of the C.G. of the fuselage $F_2$ (is point $O_{F2}$) after the deformation is

$$\overrightarrow{OF_2} = \ell_{F2} \hat{e}_{xs1} + v_{s1} | \ell_{F2} \hat{e}_{ys1} + w_{s1} | \ell_{F2} \hat{e}_{zs1} \quad (4.3)$$

where the symbol $|_{\ell_{F2}}$ refers to the value of the appropriate displacement at location $\ell_{F2}$. The position vector of the origin of hub in rotor system 2 (i.e. point $O_{H2}$) after the elastic deformation is

$$\overrightarrow{OH_2} = (\ell_{F2} - h_2 w_{s1}, x | \ell_{F2}) \hat{e}_{xs1} + (v_{s1} | \ell_{F2} - h_2 \phi_{s1} | \ell_{F2}) \hat{e}_{ys1} + (h_2 + w_{s1} | \ell_{F2}) \hat{e}_{zs1} \quad (4.4)$$

Due to these deformations, there is also a rotation. The rotation along axes at $O_{F2}$ and $O_{H2}$ are given by

$$\overrightarrow{\phi_{OF2}} = \overrightarrow{\phi_{OH2}} = \phi_{s1} | \ell_{F2} \hat{e}_{xs1} - v_{s1}, x | \ell_{F2} \hat{e}_{ys1} + v_{s1}, x | \ell_{F2} \hat{e}_{zs1} \quad (4.5)$$
Using equations 4.1, 4.2 and 4.4, the position vector of $O_{H2}$ after the perturbational rigid body motions and the elastic deformation, can be written as (in the inertial system $S$)

$$\mathbf{R}_{OH2} = [R_{xs} + (\ell_{F2} - h_2 \omega_s l_{F2})] + (v_{s1}|_{\ell_{F2}} - h_2 \phi_s l_{F2}) (-\dot{\theta} + \dot{\theta} \theta \dot{\theta}) \\
+ (h_2 + w_s l_{F2}) (\dot{\theta} + \dot{\theta} \theta \dot{\theta}) + \hat{\theta}_{xs}$$

$$+ [R_{ys} + (\ell_{F2} - h_2 \omega_s l_{F2})] \theta_{zs} + (v_{s1}|_{\ell_{F2}} - h_2 \phi_s l_{F2}) \\
+ (h_2 + w_s l_{F2}) (-\dot{\theta} + \dot{\theta} \theta \dot{\theta}) + \hat{\theta}_{ys}$$

$$+ [R_{zs} + (\ell_{F2} - h_2 \omega_s l_{F2}) (-\dot{\theta} + \dot{\theta} \theta \dot{\theta}) + (v_{s1}|_{\ell_{F2}} - h_2 \phi_s l_{F2}) \theta_{xs} \\
+ (h_2 + w_s l_{F2})] \hat{\theta}_{zs} \tag{4.6}$$

This relation can be more compactly written as

$$\mathbf{R}_{OH2} = R_x \hat{\theta}_{xs} + R_y \hat{\theta}_{ys} + R_z \hat{\theta}_{zs} \tag{4.7}$$

The perturbational displacement at the hub center $O_{H1}$ of the rotor system 1 can be obtained by replacing $\ell_{F2}$ in Eq. 4.6 by $\ell_{F1}$.

The velocity at $O_{H2}$ due these perturbational motions is

$$\dot{\mathbf{R}}_{OH2} = \Omega [\dot{R}_{xs} - h_2 \dot{\omega}_s l_{F2} + h_2 (\dot{\theta} + \dot{\theta} \theta \dot{\theta} + \dot{\theta} \theta \dot{\theta})] \hat{\theta}_{xs} \\
+ \Omega [\dot{R}_{ys} + \ell_{F2} \dot{\theta} \theta + v_{s1}|_{\ell_{F2}} - h_2 \phi_s l_{F2} \\
+ (h_2 + w_s l_{F2}) (\theta + \theta \theta \theta + \theta \theta \theta)] \hat{\theta}_{ys}$$

$$+ \Omega [\dot{R}_{zs} - \ell_{F2} \dot{\theta} \theta + \dot{\omega}_s l_{F2}] \hat{\theta}_{zs} \tag{4.8}$$
where (') indicates derivative with respect to the nondimensional parameter \( \psi \), (\( \psi = \Omega t \)). The acceleration at \( O_{H2} \) due to these perturbational motion is

\[
\ddot{R}_{OH2} = \Omega^2 [R_{xs} - h_2 \omega_{sl,x}|k_{F2} + h_2(\theta_{ysz} + \theta_{ysz} + 2\theta_{ysz} + \theta_{ysz}) \hat{e}_{xs} \\
+ \Omega^2 [R_{xs} + k_{F2} \theta_{xs} + \omega_{sl} |l_{F2} - h_2 \phi_{sl} |l_{F2} \\
+ h_2 (-\theta_{xs} + \theta_{ysz} + 2\theta_{ysz} + \theta_{ysz})] \hat{e}_{ys} \\
+ \Omega^2 [R_{zs} - k_{F2} \theta_{ys} + \omega_{sl} |l_{F2} ]] \hat{e}_{zs}
\]

Equations 4.8 and 4.9 can be more compactly written as

\[
\ddot{\dot{R}}_{OH2} = (R_x \dot{\theta}_{xs} + R_y \dot{\theta}_{ys} + R_z \dot{\theta}_{zs}) \Omega
\]

\[
\ddot{R}_{OH2} = (R_x \dot{\theta}_{xs} + R_y \dot{\theta}_{ys} + R_z \dot{\theta}_{zs}) \Omega^2
\]

The perturbational velocity and acceleration at the hub center \( O_{H1} \) can be obtained by replacing \( k_{F2} \) in Equations 4.8 and 4.9 by \( -l_{F1} \).

Rigid body angular velocities are \( \dot{\theta}_{zs} \Omega \) about \( Z_s \) axis and \( \dot{\theta}_{ys} \Omega \) about yawed \( Y_s \) axis and \( \dot{\theta}_{xs} \Omega \) about yawed-pitched \( X_s \) axis. The angular velocity of the model due to rigid body rotation alone is

\[
\ddot{\omega}_{rigid} = \Omega(\dot{\theta}_{xs} - \dot{\theta}_{ysz}) \hat{e}_{xs} + \Omega(\dot{\theta}_{ysz} + \dot{\theta}_{ysz}) \hat{e}_{ys} \\
+ \Omega(\dot{\theta}_{zs} - \dot{\theta}_{xsz}) \hat{e}_{zs}
\]

Angular velocity at \( O_{F2} \) and \( O_{H2} \) due to elastic deformation is (from equation 4.5)

\[
\ddot{\omega}_{E} = \Omega(\dot{\phi}_{sl}|g_{F2} \hat{e}_{xs} - \dot{w}_{sl,x}|l_{F2} \hat{e}_{ys} + \dot{v}_{sl,x}|l_{F2} \hat{e}_{zs})
\]

Combining equations (4.12) and (4.13), using equation (4.2), the angular velocity at the rotor hub due to the elastic deformation and due to the rigid
body rotation can be obtained. The angular velocity given in the s-system is

\[ \omega_{OH2} = \Omega (\dot{\theta}_{xs} - \dot{\theta}_{ys} \theta_{zs}) \hat{e}_{xs} + \Omega (\dot{\theta}_{ys} + \dot{\theta}_{xs} \theta_{zs}) \hat{e}_{ys} \]

\[ + \Omega (\dot{\theta}_{zs} - \dot{\theta}_{xs} \theta_{ys}) \hat{e}_{zs} \]

\[ + \Omega [\phi_{s1}|\ell_{P2} - \dot{\omega}_{s1,x}|\ell_{P2} (-\theta_{zs} + \theta_{ys} \theta_{xs}) + \dot{\nu}_{s1,x}|\ell_{P2} (\theta_{ys} + \theta_{zs} \theta_{xs})] \hat{e}_{xs} \]

\[ + \Omega [\phi_{s1}|\ell_{P2} \theta_{zs} - \dot{\omega}_{s1,x}|\ell_{P2} + \dot{\nu}_{s1,x}|\ell_{P2} (-\theta_{xs} + \theta_{ys} \theta_{zs})] \hat{e}_{ys} \]

\[ + \Omega [-\phi_{s1}|\ell_{P2} \theta_{ys} - \dot{\omega}_{s1,x}|\ell_{P2} \theta_{xs} + \dot{\nu}_{s1,x}|\ell_{P2}] \hat{e}_{zs} \]

which can be simplified to

\[ \omega_{OH2} = \Omega (\dot{\theta}_{xs} - \dot{\theta}_{ys} \theta_{zs} + \phi_{s1}|\ell_{P2}) \hat{e}_{xs} \]

\[ + (\dot{\theta}_{ys} + \dot{\theta}_{xs} \theta_{zs} - \dot{\omega}_{s1,x}|\ell_{P2}) \hat{e}_{ys} \]

\[ + (\dot{\theta}_{zs} - \dot{\theta}_{xs} \theta_{ys} + \dot{\nu}_{s1,x}|\ell_{P2}) \hat{e}_{zs} \]  

(4.14)

The Angular acceleration is

\[ \ddot{\omega}_{OH2} = \Omega^2 [(\ddot{\theta}_{xs} - \ddot{\theta}_{ys} \theta_{zs} - \ddot{\dot{\theta}}_{ys} \theta_{zs} + \ddot{\phi}_{s1}|\ell_{P2} \hat{e}_{xs} \]

\[ + (\ddot{\theta}_{ys} + \ddot{\theta}_{xs} \theta_{zs} + \ddot{\dot{\theta}}_{xs} \theta_{zs} - \ddot{\dot{\omega}}_{s1,x}|\ell_{P2}) \hat{e}_{ys} \]

\[ + (\ddot{\theta}_{zs} - \ddot{\theta}_{xs} \theta_{ys} - \ddot{\dot{\theta}}_{xs} \theta_{ys} + \ddot{\dot{\nu}}_{s1,x}|\ell_{P2}) \hat{e}_{zs} \]  

(4.15)

The angular velocity and acceleration at \( \theta_{H1} \) is obtained by replacing \( \ell_{P2} \) by \( -\ell_{F1} \) in equation (4.14) and (4.15).

Assuming the rotations due to the elastic deformation of the supporting structure to be small compared to those due of the rigid body rotation, causes the angular velocity at the hub to be a result of rigid body rotation alone.

Thus this expression becomes
\[ \ddot{\omega}_{\text{OH2}} = \Omega \{ (\dot{\theta}_{xs} - \dot{\theta}_{ys} \dot{\theta}_{zs}) \dot{e}_{xs} + (\dot{\theta}_{ys} + \dot{\theta}_{xs} \dot{\theta}_{zs}) \dot{e}_{ys} \]
\[ + (\ddot{\theta}_{zs} - \dot{\theta}_{xs} \dot{\theta}_{ys}) \dot{e}_{zs} \} \] (4.16)

and
\[ \dddot{\omega}_{\text{OH2}} = \Omega^2 \{ (\dddot{\theta}_{xs} - \dddot{\theta}_{ys} \dddot{\theta}_{zs} - \dddot{\theta}_{ys} \dddot{\theta}_{zs} ) \dot{e}_{xs} \]
\[ + (\dddot{\theta}_{ys} + \dddot{\theta}_{xs} \dddot{\theta}_{zs} + \dddot{\theta}_{xs} \dddot{\theta}_{zs} ) \dot{e}_{ys} \]
\[ + (\dddot{\theta}_{zs} - \dddot{\theta}_{xs} \dddot{\theta}_{ys} - \dddot{\theta}_{xs} \dddot{\theta}_{ys} ) \dot{e}_{zs} \} \] (4.17)

In the equations which follow the subscript \( s \) on the rotations will be deleted, since they are in essence rigid body rotations.

Equations (4.8), (4.9), (4.16) and (4.17) will be used for deriving the inertia and aerodynamic loads on the rotor blades.
5. EQUATIONS OF MOTION FOR THE ROTOR

When modelling the behavior of rotors the individual blades are considered first, the equations of motion of the individual blade is derived by writing the equation of dynamic equilibrium under the action of aerodynamic, inertia and structural loads. These equations are derived for the general \( k \)th blade. Subsequently these equations are coupled with the fuselage motion to provide the complete set of dynamical equations of motion for HHLA vehicle model [Fig. 2].

5.1 Blade Cross-Sectional Parameters

In the derivation of the equations of motion of the blade, certain cross-sectional parameters described below, are required. The 4k-system is the cross-sectional coordinate system. The location of any point in the cross-section from the origin of the 4k-system is

\[
y_{ok} \hat{e}_{y4k} + z_{ok} \hat{e}_{z4k}
\]

The principal axes of the symmetric blade cross-section are rotated from the 4k-system by the geometric pitch angle \( \theta_{Gk} \) [Fig. 6].

The geometrical pitch angle is defined as

\[
\theta_{Gk} = \theta_{Bk}(x) + \theta_{ok} + \theta_{lck} \cos \phi_{k} + \theta_{lsk} \sin \phi_{k}
\]  

(5.1)

where \( \theta_{Bk}(x) \) is the pretwist

\( \theta_{ok} \) is collective pitch

\( \theta_{lck} \) and \( \theta_{lsk} \) are cyclic pitch components

If \( \eta_{ok} \) and \( \xi_{ok} \) are the principal axes coordinates, the transformation for \( y_{ok} \) and \( z_{ok} \) is given by

\[
\begin{align*}
\begin{bmatrix}
y_{ok} \\
z_{ok}
\end{bmatrix} &=
\begin{bmatrix}
\cos \theta_{Gk} & -\sin \theta_{Gk} \\
\sin \theta_{Gk} & \cos \theta_{Gk}
\end{bmatrix}
\begin{bmatrix}
\eta_{ok} \\
\xi_{ok}
\end{bmatrix}
\end{align*}
\]  

(5.2)
Differentiating equation (5.2) with respect to $\psi$ yields

\[
\begin{pmatrix}
\dot{y}_{ok} \\
\dot{z}_{ok}
\end{pmatrix} = \delta_{Gk} \begin{pmatrix}
-z_{ok} \\
y_{ok}
\end{pmatrix}
\]  
(5.3)

and

\[
\begin{pmatrix}
\ddot{y}_{ok} \\
\ddot{z}_{ok}
\end{pmatrix} = \theta_{Gk} \begin{pmatrix}
-z_{ok} \\
y_{ok}
\end{pmatrix} - \delta_{Gk}^2 \begin{pmatrix}
0 \\
y_{ok}
\end{pmatrix}
\]  
(5.4)

Also required are expressions which define quantities involved in performing integration over the blade cross sectional area. Defining:

\[
\int_{A_T} \rho dA = m; \int_{A_T} \rho y_{ok} dA = m x_I; \int_{A_T} \rho z_{ok} dA = 0
\]

\[
\int_{A_T} \rho y_{ok}^2 dA = I_{MB3}; \int_{A_T} \rho z_{ok}^2 dA = I_{MB2}; \int_{A_T} \rho y_{ok} z_{ok} dA = 0
\]

from these, it follows that

\[
\int_{A_T} \rho y_{ok} dA = m x_I \cos \theta_{Gk}; \int_{A_T} \rho z_{ok} dA = m x_I \sin \theta_{Gk}
\]

\[
\int_{A_T} \rho y_{ok}^2 dA = I_{MB3} \cos^2 \theta_{Gk} + I_{MB2} \sin^2 \theta_{Gk}
\]

\[
\int_{A_T} \rho z_{ok}^2 dA = I_{MB3} \sin^2 \theta_{Gk} + I_{MB2} \cos^2 \theta_{Gk}
\]

(5.5)

\[
\int_{A_T} \rho z_{ok} z_{ok} dA = (I_{MB3} - I_{MB2}) \sin \theta_{Gk} \cos \theta_{Gk}
\]

In these integrals, $\rho$ is the density of the material and $A_T$ is the total cross-sectional area of the blade.
5.2 Equations of Motion for the Individual Blade

Dynamic equations of equilibrium for a blade are obtained using the Newtonian approach. The equations are obtained by combining the structural operator with the inertial, aerodynamic and structural damping loads. Since, the rigid, offset hinged, spring restrained model of the blade is used in this study, the various distributed loads are integrated over the blade length and then combined together to give the equation of motion. The various distributed loads obtained first, are described in the following sections.

5.2.1 Distributed Inertia Loads on the Blade

The distributed inertia loads on the \( k \)th blade are obtained by first determining the acceleration at a general point 'P' on the blade. The loads per unit volume are found from D'Alembert's principle and they are integrated over the cross-section to give the distributed blade loads per unit length of the blade.

Acceleration at a point in the blade.

The absolute acceleration at a point 'P', viewed from a translating and rotating coordinate system with respect to an inertial frame, is given by

\[
\ddot{a}_{pk} = \ddot{R}_o + \dot{R}_{pk} + 2\omega_k \times \dot{r}_{pk} + \omega_k \times \ddot{r}_{pk} + \omega_k \times (\omega_k \times r_{pk})
\]  

(5.6)

where \( \ddot{R}_o \) is the position vector of the origin of the moving coordinate system with respect to the inertial system. 

\( R_{pk} \) is the position vector of the point 'P' in the \( k \)th blade from the origin of the moving reference system and \( \omega_k \) is the angular velocity of the moving coordinate system.

In the present rotor blade analysis, the inertial system is the \( R \)-system whose origin is fixed at the undeformed hub location and the the moving reference frame is the \( 2k \)-system whose origin moves and rotates with the blade.

The position vector of a point 'P' on the \( k \)th blade is

\[
\mathbf{r}_{pk} = e \hat{x}_{2k} + (x_k u_k) \hat{x}_{3k} + v_k \hat{y}_{3k} + w_k \hat{z}_{3k} + y_{ok} \hat{a}_{4k} + z_{ok} \hat{a}_{24k}
\]

(5.7)

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Transforming all the unit vectors to the 2k-system

\[ \tilde{\mathbf{r}}_{p2k} = \hat{e}_{x2k} [(e + x_k + u_k) - \beta_p^x v_k + y_{ok} (-\zeta_k - \phi_k (\beta_p + \beta_k) + \beta_p \beta_k \zeta_k)] \\
+ \hat{e}_{y2k} [v_k + y_{ok} + z_{ok} (-\phi_k)] \\
+ \hat{e}_{z2k} [\beta_p (x_k + u_k) + w_k + y_{ok} (\phi_k - \zeta_k (\beta_p + \beta_k) - \beta_p \beta_k \phi_k)] \\
+ z_{ok} (1 - \beta_p \beta_k + \beta_p \phi_k \zeta_k)] \]  

(5.8)

Applying the ordering scheme and substituting \( \omega_k = x_k \beta_k \) and \( v_k = x_k \gamma_k \) (based on rigid blade approximation), equation (5.8) becomes

\[ \tilde{\mathbf{r}}_{p2k} = \hat{e}_{x2k} [(e + x_k + y_{ok} (-\zeta_k - \phi_k (\beta_p + \beta_k)) + z_{ok} (-\beta_p - \beta_k + \phi_k \zeta_k)] \\
+ \hat{e}_{y2k} [x_k \gamma_k + y_{ok} + z_{ok} (-\phi_k)] \\
+ \hat{e}_{z2k} [\beta_k x_k + x_k \beta_p + y_{ok} (\phi_k - \zeta_k (\beta_p + \beta_k)) + z_{ok}] \]

(5.9)

Taking the first and second derivative of \( \tilde{\mathbf{r}}_{p2k} \) and applying the ordering scheme yields

\[ \tilde{\mathbf{r}}_{p2k} = \Omega \hat{e}_{x2k} [-\beta_p^x \dot{\gamma}_k x_k + \dot{u}_k + y_{ok} < -\zeta_k - \phi_k (\beta_p + \beta_k) - \beta_p \beta_k G_k - \beta_p \phi_k G_k > \\
+ z_{ok} < -\beta_p \dot{\gamma}_k + \phi_k \zeta_k + \phi_k \gamma_k > ] \\
+ \Omega \hat{e}_{y2k} [x_k \ddot{r}_k + y_{ok} < -\phi_k \dot{G}_k > + z_{ok} < -\phi_k - \dot{G}_k > ] \\
+ \Omega \hat{e}_{z2k} [x_k \ddot{\gamma}_k + y_{ok} \dot{\phi}_k - \zeta_k (\beta_p + \beta_k) - \beta_p \beta_k \gamma_k + \dot{G}_k > \\
+ z_{ok} < -\beta_p \phi_k \zeta_k + \beta_p \phi_k \gamma_k + \beta_p \phi_k \zeta_k - \phi_k \dot{G}_k > ] \\
+ \dot{\gamma}_k (\beta_p + \beta_k)] \]

(5.10)
\[
\ddot{r}_{2k} = \Omega^2 \dot{a}_{x2k} [\ddot{u}_k - \beta_p \dot{\beta}_k x_k + y_{ok} - \ddot{\phi}_k (\beta_p + \beta_k) - 2\ddot{\phi}_k \dot{\beta}_k - \dot{\phi}_k \beta_k \\
- \ddot{\theta} G_k (\beta_k + \beta_p) - 2\ddot{\theta} G_k \dot{\beta}_k > \\
+ z_{ok} < -\beta_k + \phi_k \dot{\zeta}_k + 2\dot{\phi}_k \dot{\zeta}_k + \phi_k \ddot{\zeta}_k + 2\dot{\theta} G_k \ddot{\phi}_k + \dot{\theta} G_k \dot{\phi}_k > \\
+ \Omega^2 \dot{a}_{y2k} [\dot{\zeta}_k x_k + y_{ok} < -\ddot{\theta} G_k - 2\ddot{\phi} G_k \dot{\phi}_k - \ddot{\phi} G_k \dot{\phi}_k > + z_{ok} < -\ddot{\theta} G_k - \dot{\phi}_k > ] \\
+ \Omega^2 \dot{a}_{z2k} [\beta_p u_k + \beta_k x_k + y_{ok} < \dot{\phi}_k - \dot{\phi}_k (\beta_p + \beta_k) - 2\ddot{\phi}_k \dot{\beta}_k - \dot{\phi}_k \beta_k + \ddot{\phi}_k > \\
+ z_{ok} < -\beta_p \ddot{\phi}_k + \beta_p \dot{\phi}_k \zeta_k + 2\beta_p \dot{\phi}_k \ddot{\phi}_k + \beta_p \dot{\phi}_k \dot{\phi}_k > \\
- 2\dot{\theta} G_k \ddot{\phi}_k + 2\dot{\theta} G_k \dot{\phi}_k \beta_k + 2\dot{\theta} G_k \dot{\phi}_k \beta_k \\
- \ddot{\phi} G_k \dot{\phi}_k + \ddot{\phi} G_k \dot{\phi}_k \beta_k (\beta_p + \beta_k) - \dot{\phi}_k > ]
\] (5.11)

The angular velocity of the k\textsuperscript{th} blade is

\[
\bar{\omega}_k = \bar{\omega}_{0H} + \Omega \dot{a}_{z1}
\]

where \(\bar{\omega}_{0H}\) is the angular velocity at the hub center \(O_H\) due to the fuselage motion and it is given by equation (4.16). The angular velocity of the k\textsuperscript{th} blade in 2k system is (using equation (4.16) and noting that the R and S systems are parallel and inertial systems)

\[
\bar{\omega}_{2k} = \Omega \dot{a}_{x2k} [\cos \psi_k < \dot{\phi}_k - \theta_k \dot{\phi}_k > + \sin \psi_k < \dot{\psi}_k > + \theta_k \dot{\psi}_k > ] \\
+ \Omega \dot{a}_{y2k} [\cos \psi_k < \dot{\phi}_k + \theta_k \dot{\phi}_k > + \sin \psi_k < -\dot{\psi}_k > + \theta_k \dot{\psi}_k > ] \\
+ \Omega \dot{a}_{z2k} [1 + \dot{\theta}_z]
\]

which can be written as

\[
\bar{\omega}_{2k} = \Omega [\omega_x \dot{a}_{x2k} + \omega_y \dot{a}_{y2k} + (1 + \omega_z) \dot{a}_{z2k}]
\] (5.12)
The angular acceleration $\dot{\omega}_{2k}$ is

$$
\ddot{\omega}_{2k} = \Omega^2 \dot{e}_{x2k} [\cos\psi \left< \ddot{\theta}_x - \dot{\theta}_y \dot{\theta}_z - \dot{\theta}_z \dot{\theta}_x + \ddot{\theta}_y + \dot{\theta}_x \dot{\theta}_z \right> + \\
\sin\psi \left< \ddot{\theta}_y + \dot{\theta}_x \dot{\theta}_z + \dot{\theta}_z \dot{\theta}_x - \dot{\theta}_x \dot{\theta}_y \right> ]
$$

$$
\dot{\omega}_{2k} = \Omega^2 \dot{e}_{y2k} \left[ \cos\psi \left< \ddot{\theta}_y + \dot{\theta}_x \dot{\theta}_z + \dot{\theta}_z \dot{\theta}_x - \dot{\theta}_x \dot{\theta}_y \right> + \\
\sin\psi \left< \ddot{\theta}_z - \dot{\theta}_x \dot{\theta}_y + \dot{\theta}_y \dot{\theta}_x \right> \right]
$$

$$
\dot{\omega}_{2k} = \Omega^2 \dot{e}_{z2k} \left[ \ddot{\theta}_z - \dot{\theta}_x \dot{\theta}_y + \dot{\theta}_y \dot{\theta}_x \right]
$$

which can be written as

$$
\ddot{\omega}_{2k} = \Omega^2 \left[ \dot{\omega}_{x2k} \dot{e}_{x2k} + \dot{\omega}_{y2k} \dot{e}_{y2k} + \dot{\omega}_{z2k} \dot{e}_{z2k} \right]
$$

(5.13)

In terms of dimensionless derivative in time, $(\psi)$, $(\dot{\psi})$ is replaced by $\Omega^2 (\psi)$ in equation (5.6).

The acceleration is

$$
\dddot{p}_{2k} = \Omega^2 \left[ \dddot{R}_o + \dddot{p}_{2k} + 2\dddot{\omega}_{2k} \times \dddot{r}_{p2k} + \dddot{\omega}_{2k} \times \dddot{r}_{p2k} + \dddot{\omega}_{2k} \times (\dddot{\omega}_{2k} \times \dddot{r}_{p2k}) \right]
$$

(5.14)

In equation (5.14), all the quantities except the first term is relative to the 2k-system. The first term, i.e. the acceleration of the origin of the 2k-system is in the inertial system, as given by in equation (4.11). This contribution can be transformed into components parallel to 2k-system, as indicated below.

$$
\dddot{R}_o = \Omega^2 \dot{e}_{x2k} \left[ \cos\psi \left< \dddot{R}_x + \dddot{R}_y \dot{R}_z - \dddot{R}_z \dot{R}_y \right> + \\
\sin\psi \left< \dddot{R}_y - \dddot{R}_x \dot{R}_z + \dddot{R}_z \dot{R}_x \right> \right]
$$

$$
\dddot{R}_o = \Omega^2 \dot{e}_{y2k} \left[ \cos\psi \left< \dddot{R}_y - \dddot{R}_x \dot{R}_z + \dddot{R}_z \dot{R}_x \right> + \\
\sin\psi \left< \dddot{R}_x - \dddot{R}_y \dot{R}_z + \dddot{R}_z \dot{R}_y \right> \right]
$$

$$
\dddot{R}_o = \Omega^2 \dot{e}_{z2k} \left[ \dddot{R}_z - \dddot{R}_x \dot{R}_y + \dddot{R}_y \dot{R}_x \right]
$$

(5.15)
Equation (5.15) has been obtained after applying the ordering scheme. The various terms in equations (5.12) and (5.15) contain the rigid body motion and the elastic deformation of the supporting structure. The various other terms in equation (5.14) are given below. These expressions are also obtained after applying the ordering scheme.

The Coriolis Term

\[ \begin{align*}
2\omega_{2k} \times \dot{r}_{p2k} &= 2\Omega^2 \delta_{x2k} \left[ \omega \dot{x}_k \beta_k - \dot{x}_k \tau_k - \omega \dot{z}_k \right] \\
&+ y_{ok} \left( \phi_k - \tau_k \left( \beta_p + \beta_k \right) - \tau_k \dot{\theta}_k + \dot{\theta}_Gk \right) \\
&+ z_{ok} \left( -\omega \tau_k \left( \phi_k + \theta_Gk \right) \right) \\
&+ 2\Omega^2 \delta_{y2k} \left[ \omega \dot{u}_k - \dot{x}_k \dot{u}_k + \omega \dot{z}_k \dot{x}_k - \omega \dot{x}_k \dot{z}_k \right] \\
&+ y_{ok} \left( \tau_k - \phi_k \left( \beta_p + \beta_k \right) - \phi_k \dot{\theta}_k - \beta_k \dot{\theta}_Gk - \beta_p \dot{\theta}_Gk - \omega \tau_k \right) \\
&+ \omega_x (\phi_k + \dot{\theta}_Gk) \\
&+ z_{ok} \left( -\dot{\beta}_k + \phi_k \tau_k + \dot{\phi}_k \tau_k + \dot{\tau}_k \dot{\theta}_Gk - \omega \dot{\beta}_k \right) \\
&+ 2\Omega^2 \delta_{z2k} \left[ \omega \dot{x}_k \tau_k - \omega \dot{x}_k \tau_k + \omega \beta \beta \dot{x}_k \right] \\
&+ y_{ok} \left( -\omega \phi_k \dot{\theta}_Gk - \omega \dot{x}_k \dot{\theta}_Gk - \omega \dot{y}_k \dot{\theta}_Gk \right) \\
&+ z_{ok} \left( -\omega \dot{x}_k + \omega \dot{z}_k - \omega \dot{y}_k \right) \left( \dot{\beta}_k + \phi_k \tau_k + \dot{\phi}_k \tau_k + \dot{\tau}_k \dot{\theta}_Gk \right)
\end{align*} \]

(5.16)
The Angular Acceleration Term

\[ \ddot{\omega}_{2k} \times \bar{r}_{p2k} = \Omega^2 \hat{e}_{x2k} \left[ \dot{\omega}_y \beta_p x_k - \dot{\omega}_z \tau_k + \dot{\omega}_x \tau_k \right] 
+ y_{ok} \left( \omega_{y'k} - \omega_z \right) 
+ z_{ok} \left( \omega_y - \omega_z (-\phi_k) \right) \]

\[ + \Omega^2 \hat{e}_{y2k} \left[ \dot{\omega}_z e + \dot{\omega}_z x_k - \dot{\omega}_z \beta_k + \dot{\omega}_x \beta_k \right] 
+ y_{ok} \left( \omega_z (-\tau_k - \phi_k (\beta_p + \beta_k)) - \right. 
\left. \omega_x (\phi_k - \tau_k (\beta_p + \beta_k)) \right) 
+ z_{ok} \left( \omega_z (-\beta_p - \beta_k) - \omega_x \right) \]

\[ + \Omega^2 \hat{e}_{z2k} \left[ \dot{\omega}_x x_k \tau_k - \dot{\omega}_x y - \dot{\omega}_x \right] 
+ y_{ok} \left( \omega_x - \omega_y (-\tau_k) \right) 
+ z_{ok} \left( \omega_x (-\phi_k) - \omega_y (-\beta_k - \beta_p + \phi_k \tau_k) \right) \] 

(5.17)
The Centripetal Acceleration Term

\[ \ddot{\omega}_{2k} \times (\ddot{\omega}_{2k} \times \ddot{r}_{p2k}) = \Omega^2 \hat{e}_{2k} \{ - (x_k + x_k \omega + e + x_k \omega_z) + y_{ok} < (\beta_k - \beta_p + \phi_k \tau_k - \omega_z \tau_k) - \omega \phi_k > \\
+ z_{ok} < (-\beta_k - \beta_p + \phi_k \tau_k - \omega \beta_k - \omega \beta_p) + \omega_x > \\
+ \Omega^2 \hat{e}_{2k} \{ - x_k \tau_k - \omega_x \tau_k \}
\]

\[ + y_{ok} < (1 + 2 \omega_z) > \\
+ y_{ok} < \beta_k + \phi_k \omega_k z + \omega_y + \phi_k \omega_k z > ]
\]

\[ + \Omega^2 \hat{e}_{2k} \{ \omega_x x_k + \omega_x e + \omega_x x_k + \omega_y x_k \tau_k > \\
+ y_{ok} < \omega_y + \omega_x \omega_z - \omega_x \tau_k > \\
+ z_{ok} < \omega_x (-\beta_k - \beta_p + \phi_k \tau_k - \omega \beta_k - \omega \beta_p) - \omega_x^2 \\
- \omega_y (\omega_y + \phi_k + \phi_k \omega_z) > ] \]  

(5.18)
Combining the various terms, i.e. equations (5.11), (5.15), (5.16), (5.17), (5.18) the three components of the acceleration can be written as

\[ a_{p,2k} - \Omega^2 \left( -x_k - 2x_k\omega + e - 2x_k\omega_k + \cos\psi_k R_x + \sin\psi_k R_y \right) + y_{ok} [\xi_k + \phi_k (\beta_0 + \beta_k) + 2\omega z_k + \omega \phi_k + \dot{\omega} \phi_k - \dot{\omega}] \\
+ 2\omega y_k + 2\omega \beta_{Gk} + 2\phi \beta_{Gk} - \dot{\xi}_k \\
+ \phi_k (\beta_0 + \beta_k) - 2\phi \beta_{Gk} - \phi_k \beta_{Gk} - 2\phi \beta_{Gk} - \phi_k \beta_{Gk} \right] \\
+ z_{ok} [\beta_k + \beta_k - \phi_k z_k + 2\omega \beta_k + 2\omega \beta_k + \omega + \dot{\omega} + \phi_k \omega \dot{z}] \\
+ 2\phi \beta_k + 2\phi \beta_{Gk} + 2\omega \phi_k (\beta_k + \beta_{Gk}) - \phi_k \beta_k + \phi_k \beta_{Gk} \right] \\
+ \phi_k \beta_k + 2\phi \beta_{Gk} + \phi_k \beta_{Gk} \}

or in a more compact form

\[ a_{p,2k} = \Omega^2 \left[ a_{p,2k}^{(c)} + y_{ok} a_{p,2k}^{(y)} + z_{ok} a_{p,2k}^{(z)} \right] \]  \hspace{1cm} (5.19)

The three terms correspond respectively to 1) the constant part of \( x \)-component of the acceleration over the cross-section of the blade at a distance \( x_k \), 2) the part dependent on \( y_{ok} \) and 3) the term dependent on \( z_{ok} \) respectively. This separation facilitates integration of those terms over the cross-section of the blade to yield inertia forces. It also helps in identifying the orders of magnitude of various terms.

The \( y \)-component of the acceleration is

\[ a_{p,y2k} = \Omega^2 \left( -x_k \xi_k - \omega z_k \xi_k + \omega \beta_{y_k} x_k + \omega \beta_{y_k} x_k - \omega z_k x_k + \omega x_k \xi_k + \right. \\
+ \omega x_k - \omega x_k \beta_{y_k} x_k - \omega x_k \beta_{y_k} x_k + \xi_k x_k + \\
\left. \cos\psi_k R_y + \sin\psi_k R_x \right) + 2\omega_k - 2\beta_{p_k} x_k - 2\omega x_k \beta_k \]
The z-component of the acceleration is

\[
a_{p_{z2k}} = \Omega^2 \left[ a_{\mathbf{p}_{z2k}}^{(c)} + y_{\mathbf{ok}} a_{\mathbf{p}_{y2k}}^{(y)} + z_{\mathbf{ok}} a_{\mathbf{p}_{y2k}}^{(z)} \right]
\]

(5.20)

Transforming these acceleration components from 2k to 3k systems yields
Before evaluating the distributed inertia forces and moments, it is worth noting down the relative orders of magnitude of the leading terms in various acceleration components. They are

\[ a_{p_{x2k}}^{(c)} = 0(1) \quad a_{p_{x2k}}^{(y)} = 0(\epsilon) \quad a_{p_{x2k}}^{(z)} = 0(\epsilon) \]

\[ a_{p_{y2k}}^{(c)} = 0(\epsilon) \quad a_{p_{y2k}}^{(y)} = 0(1) \quad a_{p_{y2k}}^{(z)} = 0(\epsilon) \ (5.23) \]

\[ a_{p_{z2k}}^{(c)} = 0(\epsilon) \quad a_{p_{z2k}}^{(y)} = 0(\epsilon) \quad a_{p_{z2k}}^{(z)} = 0(\epsilon^2) \]

This information is useful in neglecting higher order terms even before evaluating the integrals to obtain inertia forces and moments.

**Distributed Inertia Forces**

The distributed inertia force per unit length is obtained from the D'Alembert's principle, for the k\textsuperscript{th} blade.
\[ \ddot{p}_{I_k} = \int_{A_k} - \rho \ddot{p}_k \, dA \]  

(5.24)

where \( \rho \) is the material density of the blade.

Substituting the various components of the acceleration and using the cross-sectional parameters defined earlier, the distributed inertia forces are obtained, in the blade fixed 3k system. The acceleration components in the 3k system (from eq. (5.22)) are

\[
a_{px3k} = \Omega^2 \{ -x_k \ddot{r}_k - e - 2x_k \dot{z} \dot{\theta}_k + \ddot{R}_x \cos \psi_k + \ddot{R}_y \sin \psi_k
\]

\[
+ y_{ok} \left[ \ddot{r}_k - \dot{z}_k + \dot{\phi}_k (\beta_p + \beta_k) - \dot{\phi}_k \beta_k + 2\theta \ddot{z}_k - \ddot{\theta}_z + 2\phi_k \dot{\theta}_k \right]
\]

\[
- 2\phi_k \ddot{\phi}_k - 2\theta \dot{\theta}_k \beta_k - \ddot{\theta}_k \beta_k
\]

\[
+ \cos \psi_k < \phi_k \dot{\theta}_y + 2\phi_k \dot{\theta}_y + 2\theta \dot{\theta}_k \dot{\theta}_k + \beta_p \dot{\theta}_x + 2\theta \dot{\theta}_k
\]

\[
+ \sin \psi_k < -\dot{\phi}_k \dot{\theta}_x - 2\phi_k \dot{\theta}_x - 2\theta \dot{\theta}_k \dot{\theta}_k + \beta_p \dot{\theta}_y + 2\theta \dot{\theta}_k
\]

\[
+ z_{ok} \left[ -\dot{r}_k + \dot{z}_k \right]
\]

\[
+ \ddot{\theta}_k \dot{r}_k + \ddot{\theta}_k \dot{z}_k + \dot{\phi}_k \dot{\theta}_x + \ddot{\phi}_k \dot{\theta}_x
\]

\[
+ \ddot{\phi}_k \dot{\theta}_x
\]

\[
(5.25)
\]

\[
a_{py3k} = \Omega^2 \{ -x_k \ddot{r}_k + x_k \ddot{\theta}_k + 2u_k - 2\beta_p x_k - 26 z x_k \dot{r}_k + \ddot{\theta}_z (x_k + e)
\]

\[
+ \cos \psi_k < \theta_x (\beta_p + \beta_k) x_k + \dot{R}_y - 2x_k \beta_k \dot{\theta}_x
\]

\[
+ \sin \psi_k < -\ddot{\theta}_y (\beta_p + \beta_k) x_k - \ddot{R}_x - 2x_k \beta_k \dot{\theta}_y
\]

\[
y_{ok} \cdot \left[ -1 - 2\theta \dot{z}_k - 2\dot{r}_k \right]
\]

\[
+ z_{ok} \left[ \dot{r}_k - \dot{z}_k \right]
\]

\[
+ \ddot{\theta}_k \dot{r}_k - \ddot{\theta}_k \dot{z}_k + \ddot{\phi}_k \dot{\theta}_x - \ddot{\phi}_k \dot{\theta}_x
\]

\[
+ 2\dot{r}_k \beta_k - 2\theta_k \beta_k - \ddot{\theta}_k \beta_k - \ddot{\theta}_k \beta_k
\]

\[
\text{sin} \psi_k \} \]  

(5.26)
\[ a_{p_{3k}} = \Omega^2 \left( x_k \dot{\beta}_k + \dot{R}_z + \beta_p (x_k + e + 2x_k \dot{\zeta}_k + 2x_k \dot{\theta}_z) \right) + \cos \psi_k < (x_k + e) (-\dot{\theta}_y + 2\dot{\theta}_x) + x_k \ddot{\zeta}_k (\dot{\theta}_y + 2\dot{\theta}_x) + 2x_k \dddot{\theta}_k \dot{\zeta}_k - \beta_p R_x \right) \\
+ \sin \psi_k < (x_k + e) (\dot{\theta}_y + 2\dot{\theta}_x) + x_k \ddot{\zeta}_k (\dot{\theta}_y - 2\dot{\theta}_x) + 2x_k \dddot{\theta}_k \dot{\zeta}_k - \beta_p R_x \right) \\
+ y_{ok} [\phi_k - \dot{\zeta}_k \beta_k - 2r_k \ddot{\beta}_k - \ddot{\zeta}_k \dot{\beta}_k + \ddot{\beta}_k - \beta_p r_k + \beta_p \dot{\beta}_k \\
+ \cos \psi_k < \ddot{\phi}_x + 2\dot{\phi}_y + \dddot{\zeta}_k (\ddot{\phi}_y - 2\dddot{\phi}_x) + 2\ddot{\zeta}_k \dot{\phi}_y > \\
+ \sin \psi_k < \ddot{\phi}_y - 2\dot{\phi}_x - \dddot{\zeta}_k (\ddot{\phi}_x + 2\dddot{\phi}_y) - 2\ddot{\zeta}_k \dot{\phi}_x ] \\
+ z_{ok} [-\beta_p \dot{\beta}_k - \beta_p^2 \dot{\beta}_k - \beta_p \phi_k \dot{r}_k - 2\beta_p \beta_k (\dot{\beta}_k + \dot{r}_k) - \beta_p \dot{\beta}_k - 2\beta_p \dot{r}_k \\
- 2\beta_p \ddot{r}_k (\dot{\phi}_k + \dot{\theta}_k) - 2\beta_p \dot{r}_k \dot{\phi}_x \\
- 2\beta_p \dot{\beta}_k \dot{r}_k + \beta_p \dot{\beta}_k + \beta_p \ddot{r}_k - \ddot{\beta}_p \dot{r}_k \dot{\phi}_x + \dddot{\beta}_p \dot{r}_k \dot{\phi}_x \\
- \dddot{\beta}_p \dot{r}_k \dot{\phi}_x - \dddot{\beta}_p \dot{r}_k \dot{\phi}_x \\
+ \cos \psi_k < - 2\dot{\phi}_x (\dot{\phi}_k + \theta_{Gk}) - 2\dot{\phi}_y (\dot{\beta}_k + \dot{\beta}_k + \dot{r}_k + \dot{\theta}_k) \\
+ \phi_k (\ddot{\beta}_k + \dot{\beta}_k) + 2\dot{\phi}_x (\ddot{\beta}_k + \dot{\beta}_k + \dot{r}_k + \dot{\theta}_k) \\
+ \sin \psi_k < - 2\dot{\phi}_x (\dot{\phi}_k + \dot{\theta}_{Gk}) + 2\dot{\phi}_y (\ddot{\beta}_k + \dot{\beta}_k + \dot{r}_k + \dot{\theta}_k) > \\
(5.27) \\
Substituting equations (5.25 through 5.27) in equation (5.24) the distributed inertia loads are evaluated. The components of these loads are \\
\[ p_{I_{3k}} = m\Omega^2 [x_k + e + 2x_k \dot{r}_k + 2x_k \dot{\theta}_z - \ddot{R}_x \cos \psi_k - \ddot{R}_y \sin \psi_k ] \]
(5.28)
\[ P_{Iy3k} = m \Omega^2 \left[ x_k \zeta_k + 2x_{k} \zeta_k z - (x_k + e) \theta_z - x_k \zeta_k - 2u_k + 2x_{k} \beta_k \theta + x_1 \cos \theta G_k \right] \]
\[ + \cos \psi_k < x_k \beta \theta x + x_k \beta \theta x + 2x_{k} \beta \theta y - \ddot{R}_y > \]
\[ + \sin \psi_k < x_k \beta \theta y + 2x_{k} \beta \theta y + 2x_{k} \beta \theta y + \ddot{R}_y > \]  
(5.29)

\[ P_{Iz3k} = m \Omega^2 \left[ -x_k \dot{\beta}_k - \ddot{R}_z - \beta_k < x_k + e + 2x_k \theta_k + 2x_k \zeta_k > \right] \]
\[ + \cos \psi_k < \beta R \dot{x} - 2x_k \zeta_k x + (x_k + e)(\ddot{\theta}_y - 2\dot{\theta}_y) - x_k \zeta_k (\ddot{\theta}_x + 2\dot{\theta}_x) > \]
\[ + \sin \psi_k < \beta R \dot{y} - 2x_k \zeta_k y - (x_k + e)(\ddot{\theta}_x + 2\dot{\theta}_y) - x_k \zeta_k (\ddot{\theta}_y - 2\dot{\theta}_x) > \]  
(5.30)

**Distributed Inertia Moment**

The distributed moment per unit length for the \( k \)th blade is also obtained by D'Alembert principle by taking the integral of the vector product given below.

\[ \vec{q}_{Ik} = \iint_{A_T} \left[ - \rho(y_{ok} \hat{e}_{y4k} + z_{ok} \hat{e}_{z4k}) \times \vec{a}_{pk} \right] dA \]  
(5.31)

The moments are also evaluated in \( 3k \) system, and are given in component form below.

\[ q_{Ix3k} = \iint_{A_T} - \rho[(y_{ok} - z_{ok} \phi_k) a_{pz3k} - (z_{ok} - y_{ok} \beta_{k} \zeta_k + y_{ok} \phi_k) a_{py3k}] dA \]

\[ q_{Iy3k} = \iint_{A_T} - \rho[(z_{ok} - y_{ok} \beta_{k} \zeta_k + y_{ok} \phi_k) a_{px3k} - (-y_{ok} \zeta_k - y_{ok} \phi_{k} \beta_k - z_{ok} \beta_k + z_{ok} \phi_k \zeta_k) a_{pz3k}] dA \]

\[ q_{Iz3k} = \iint_{A_T} - \rho[(-y_{ok} \zeta_k - y_{ok} \phi_{k} \beta_k - z_{ok} \beta_k + z_{ok} \phi_k \zeta_k) a_{py3k} - (y_{ok} - z_{ok} \phi_k) a_{px3k}] dA \]
Substituting for the acceleration components from equations (5.25) through (5.27) and making use of the integrals given in equation (5.5) the components of the distributed inertia moments (or torques per unit span) are evaluated. These expressions are given below.

\[ q_{I_{x3k}} = \Omega^2 \left\{ m x I \cos \theta_{Gk} \left[ -x_k \dot{\alpha}_k - \frac{R_k}{2} - \frac{\beta}{p} (x_k + e + 2x_k \dot{\alpha}_k + 2x_k \ddot{\alpha}_k) \right. \right. \]

\[ \left. \left. - x_k \ddot{\alpha}_k \phi_k + x_k \dot{\alpha}_k \ddot{\alpha}_k + x_k \phi_k \dddot{\alpha}_k \right. \right. \]

\[ + \cos \psi_k \left( x_k + e \right) \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) - x_k \ddot{\alpha}_k \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) \]

\[ - 2x_k \ddot{\alpha}_k \frac{\beta}{p} + 2x_k \ddot{\alpha}_k \frac{\beta}{p} \phi \frac{R_k}{x_k} \]

\[ + \sin \psi_k \left( x_k + e \right) \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) - x_k 

\[ \left. \ddot{\alpha}_k \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) \right. \right. \]

\[ \left. \left. - 2x_k \ddot{\alpha}_k \phi_k + \frac{R_k}{x_k} \phi_k \right] \right. \]

\[ + m x I \sin \theta_{Gk} \left[ -x_k \ddot{x}_k + x_k \dddot{x}_k + 2 \frac{\beta}{p} \beta \frac{x_k}{k} - 2 \theta \frac{\beta}{p} \frac{x_k}{k} \right. \]

\[ \left. \left. - 2 \theta \frac{\beta}{p} \frac{x_k}{k} \ddot{x}_k \right) \right. \]

\[ + \theta \ddot{x}_k \left( x_k + e \right) \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) + \ddot{x}_k \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) \]

\[ + \frac{\phi_k}{x_k} \left( \dddot{x}_k + \frac{2 \theta}{y} \right) \right. \right. \]

\[ + \theta \ddot{x}_k \left( x_k + e \right) \left( \dot{\alpha}_k + \frac{2 \theta}{y} \right) - 2 \theta \frac{\beta}{p} \frac{x_k}{k} \ddot{x}_k \]

\[ + \frac{\phi_k}{x_k} \left( \dddot{x}_k + \frac{2 \theta}{y} \right) \right. \right. \]

\[ + \frac{\phi_k}{x_k} \left( \dddot{x}_k + \frac{2 \theta}{y} \right) \right. \]

\[ + (L_{MB2} \sin^2 \theta_{Gk} + L_{MB3} \cos \theta_{Gk} \sin^2 \theta_{Gk}) \left[ - \frac{\phi_k}{x_k} + \dddot{\alpha}_k \beta_k + \dddot{\alpha}_k \beta_k \right. \]

\[ - \left. \phi_k \beta_k \left( \dot{\alpha}_k - \frac{2 \theta}{y} \right) \right. \]

\[ + \frac{\phi_k}{x_k} \left( -1 + 2 \theta \right) \beta_k \theta_k \]
\begin{align*}
+ \cos \psi_k &\left< \frac{\ddot{y}}{x} + \left( \ddot{\theta}_y + \zeta_k \left( \ddot{\theta}_x - \ddot{\theta}_z \right) + 2 \ddot{\zeta}_k \dot{\theta}_y \right) \right> \\
+ \sin \psi_k &\left< \frac{\ddot{y}}{x} - \left( \ddot{\theta}_y - \ddot{\theta}_x \right) - \zeta_k \left( \ddot{\theta}_x + \ddot{\theta}_z \right) - 2 \ddot{\zeta}_k \dot{\theta}_x \right> \\
+ (I_{MB3} \sin^2 \theta_{Gk} + I_{MB2} \cos^2 \theta_{Gk}) [\dot{\phi}_k - \dot{\phi}_k + 2 \ddot{\phi}_k \theta_k - \dot{\theta}_z (\beta + \theta_k) - 2 \ddot{\theta}_k] \\
&\quad + 2 \ddot{\phi}_k \zeta_k + 2 \ddot{\phi}_k \zeta_k + 2 \ddot{\theta}_k \theta_{Gk} \\
- 2 \ddot{\theta}_z \beta_k - \dot{\theta}_z \cos \psi_k - \dot{\theta}_y \sin \psi_k \\
+ (I_{MB3} - I_{MB2}) \sin \theta_{Gk} \cos \theta_{Gk} [\theta_{Gk} - \ddot{\theta}_z - 2 \ddot{\zeta}_k] \right> \\
\end{align*}

\begin{align*}
q_{I_{y3k}} = \Omega^2 \{ m x_i \cos \theta_{Gk} [\dot{\phi}_k (x_k + e + 2 x_k \dot{\zeta}_k + 2 x_k \dot{\theta}_k) - \beta_k \dot{x}_k \dot{y}_k \\
- \dot{\zeta}_k (x_k \zeta_k + R_z + \beta x_k) + \cos \psi_k < - \dot{\phi}_k \dot{R}_z - \dot{\zeta}_k (x_k + e) (- \ddot{\theta}_y + 2 \ddot{\phi}_k) \right> \\
+ \sin \psi_k < - \dot{\phi}_k \ddot{R}_y - \dot{\zeta}_k (x_k + e) (\ddot{\theta}_x + 2 \ddot{\phi}_k) \right> \\
+ m x_i \sin \theta_{Gk} [x_k + e + 2 x_k \dot{\zeta}_k + 2 x_k \dot{\theta}_k - \ddot{R}_x \cos \psi_k - \ddot{R}_y \sin \psi_k] \\
+ (I_{MB3} \cos \theta_{Gk} + I_{MB2} \sin \theta_{Gk}) [\ddot{\phi}_k \zeta_k - \ddot{\phi}_k \zeta_k - \ddot{\phi}_k \zeta_k + \ddot{\phi}_k \zeta_k + \ddot{\zeta}_k \ddot{\theta}_z \\
+ (\beta + \theta_k) (\ddot{\zeta}_k - \dot{\zeta}_k + \ddot{\phi}_k \theta_k + \ddot{\phi}_k \theta_k - 2 \ddot{\theta}_k \zeta_k - \ddot{\theta}_k \zeta_k) \\
- \ddot{\theta}_z \beta_k (\beta + \theta_k) - 2 \ddot{\theta}_z \beta_k - \ddot{\theta}_z \beta_k \\
+ 2 \dot{\phi}_k \beta_k \theta_{Gk} + 2 \ddot{\phi}_k \beta_k \theta_{Gk} + \ddot{\phi}_k \beta_k \theta_{Gk} \\
- \ddot{\phi}_k \beta_k \theta_{Gk} (\theta_x + 2 \ddot{\theta}_y) \right> \\
\end{align*}
\begin{align*}
- \phi_k \left( \phi_k \overline{\theta}_y + 2 \phi_k \overline{\theta}_y + 2 \overline{\theta}_x \overline{\theta}_y + \beta \overline{\theta}_p \overline{\theta}_x + 2 \beta \overline{\theta}_p \overline{\theta}_x \right) \\
+ \sin \psi_k \left( - \zeta_k \left( \overline{\theta}_y - 2 \overline{\theta}_x - \zeta_k \left( \overline{\theta}_x + 2 \overline{\theta}_y \right) \right) - 2 \zeta_k \overline{\theta}_x \\
- \phi_k \beta_k \left( \overline{\theta}_y - 2 \overline{\theta}_x \right)
\end{align*}

\begin{align*}
\left[ \right] + (I_{MB3} \sin^2 \theta_k + I_{MB2} \cos^2 \theta_k) \left[ \beta_k - \beta_k - \beta \overline{\theta}_p \zeta_k - 2 \zeta \left( \overline{\theta}_k + \overline{\theta}_p \right) - \phi_k \overline{\theta}_x \right] \\
- 2 \phi_k - 2 \overline{\theta}_k (1 + \zeta_k) - 2 \overline{\theta}_x \left( \phi_k + \theta_k \right) \\
- \phi_k \zeta_k - 2 \phi_k \zeta_k - \phi_k \zeta_k - \theta \overline{\theta}_k \\
- \theta \overline{\theta}_y \cos \psi_k + \theta \overline{\theta}_x \sin \psi_k
\end{align*}

\begin{align*}
+ (I_{MB3} - I_{MB2}) \sin \theta_k \cos \theta_k \left[ - \zeta_k + \zeta_k - 2 \overline{\theta}_k \left( \overline{\theta}_k + \overline{\theta}_p \right) - 2 \zeta \overline{\theta}_k + \theta \overline{\theta}_x \right] \\
- 4 \phi_k \beta \overline{\theta}_k + 2 \phi_k \beta \overline{\theta}_k + \phi_k \beta \overline{\theta}_k + 2 \overline{\theta}_k \beta \overline{\theta}_k
\end{align*}

\begin{align*}
+ \phi_k \beta \overline{\theta}_k - 2 \phi_k \beta \overline{\theta}_k \\
+ \cos \psi_k \left( -2 \phi_k \overline{\theta}_y - 2 \phi_k \overline{\theta}_y - 2 \overline{\theta}_x \overline{\theta}_y - (\beta_k + \beta_k) \left( \overline{\theta}_y + 2 \overline{\theta}_x \right) \right) \\
+ \sin \psi_k \left( 2 \phi_k \overline{\theta}_x + 2 \phi_k \overline{\theta}_x + 2 \overline{\theta}_x \overline{\theta}_x - (\beta_k + \beta_k) \left( \overline{\theta}_y + 2 \overline{\theta}_x \right) \right)
\end{align*}

\[ q_{1k3k} = \Omega (\sin \theta_k \sin \theta_k + \Gamma \cos \theta_k + \Gamma \sin \psi_k) \\
+ \sin \theta_k \sin \theta_k [\phi_k \left( x_k + e + 2 \overline{x}_k \zeta_k + 2 \overline{x}_k \overline{\theta}_x \right) \\
+ \beta_k \left( -x_k \zeta_k + x_k \zeta_k + \theta \overline{x}_k \right) \\
+ \beta_k \left( \overline{x}_k \zeta_k + \overline{x}_k \zeta_k + \theta \overline{x}_k \right) \\
+ \overline{x}_k \zeta_k + \overline{x}_k \zeta_k]
\]
\[
+ \cos \psi_k < -\phi_k \dot{R}_x + \beta_k \dot{R}_y > \\
+ \sin \psi_k < -\phi_k \dot{R}_y - \beta_k \dot{R}_x > \\
+ (\Gamma_{MB3} \cos^2 \theta_{Gk} + \Gamma_{MB2} \sin^2 \theta_{Gk}) \left[ - \phi_k \dot{\tau}_k + \beta_k \dot{R}_k - \phi_k \dot{\beta}_k - \theta_k \dot{z} + 2 \phi_k \dot{\theta}_{Gk} \\
- 2 \phi_k \beta_k \tau_k - \phi_k \dot{\beta}_k - 2 \theta_{Gk} \dot{\beta}_k - \dot{\theta}_{Gk} \beta_k - 2 \dot{\tau}_k \dot{\beta}_k \\
+ \cos \psi_k \phi_k \dot{\theta}_y + 2 \phi_k \dot{\theta}_y + 2 \theta_{Gk} \dot{\theta}_y + \beta_k \dot{\theta}_x + 2 \beta_k \dot{\theta}_y > \\
+ \sin \psi_k < -\phi_k \dot{\theta}_x - 2 \phi_k \dot{\theta}_x - 2 \theta_{Gk} \dot{\theta}_x + \beta_k \dot{\theta}_y - 2 \beta_k \dot{\theta}_y > \\
+ (\Gamma_{MB3} \sin^2 \theta_{Gk} + \Gamma_{MB2} \cos^2 \theta_{Gk}) \left[ - \phi_k \dot{\tau}_k + \beta_k \dot{R}_k - \phi_k \dot{\beta}_k + 2 \phi_k \dot{\theta}_z \left( \beta_k + \beta_p \right) \\
+ \phi_k \dot{\theta}_z + 2 \phi_k \dot{\theta}_z + 2 \theta_{Gk} \dot{\theta}_z + (\beta_k + \beta_p) \theta_k + \phi_k \dot{\tau}_k \\
+ 2 \phi_k \dot{\tau}_k + \phi_k \dot{\beta}_k + 2 \theta_{Gk} \dot{\tau}_k + \theta_{Gk} \dot{\beta}_k > \\
+ \beta_k \phi_k \dot{\tau}_k - \phi_k \dot{\theta}_z - (\beta_k + \beta_p) \theta_k - 2 \beta_k \dot{\theta}_z \dot{\beta}_k \dot{\theta}_z - 2 \beta_k \dot{\theta}_z \dot{\beta}_k > \\
- \phi_k \dot{\tau}_k \left( \phi_k \dot{\beta}_k - \beta_k \dot{\theta}_k \right) - \phi_k \dot{\theta}_x - 2 \beta_k \dot{\theta}_x > \\
+ \cos \psi_k < -\phi_k \dot{\theta}_y - \beta_k \dot{\theta}_x + \phi_k \dot{\tau}_k \dot{\theta}_x > \\
+ \sin \psi_k < \phi_k \dot{\theta}_x - \beta_k \dot{\theta}_y + \phi_k \dot{\tau}_k \dot{\theta}_y > \\
+ (\Gamma_{MB3} - \Gamma_{MB2}) \cos \theta_{Gk} \sin \theta_{Gk} \left[ - \beta_k \dot{R}_p + 2 \theta_{z} \dot{R}_p + 2 \phi_k \dot{\theta}_z + 2 \phi_k \dot{\tau}_k + 2 \theta_{Gk} \dot{\tau}_k \\
+ 2 \theta_{z} (\phi_k + \theta_{Gk}) + 2 \phi_k \dot{\tau}_k + 2 \theta_{Gk} \dot{\tau}_k > \\
- 2 \beta_k \dot{R}_k - 2 \tau_k \dot{\beta}_k > \\
+ \cos \psi_k \theta_k \dot{\tau}_k \dot{\beta}_k \dot{\theta}_k > \\
+ \sin \psi_k < -\theta_k \dot{\tau}_k \dot{\beta}_k \dot{\theta}_k > \right] \right] \\
\tag{5.34}
\]
5.2.2 Distributed Aerodynamic Loads

Greenberg [Ref. 10] has derived expressions for unsteady lift and moment on a two-dimensional airfoil executing harmonic motion in a pulsating stream of incompressible fluid. This derivation is an extension of Theodorsen's unsteady aerodynamic theory [Ref. 11]. The lift and moment expressions consist of two contributions. The first contribution is due to circulatory flow and the second one is due to noncirculatory flow. Greenberg has assumed that the circulatory lift and noncirculatory lift are acting in the same direction, i.e. normal to the resultant flow. However, other researchers using this theory have introduced their interpretation. For example Hodges and Ormiston [Ref. 12] assumed that the circulatory lift acts normal to the resultant flow and the noncirculatory lift acts normal to the blade chord. An examination of the alternative mathematical expressions for the unsteady lift indicates that assuming the noncirculatory part of the lift to be perpendicular to the blade chord is somewhat more convenient. In this study it is assumed that the circulatory lift acts normal to the resultant flow and the noncirculatory lift acts normal to the blade chord, for mathematical convenience.

The lift and moment expression as given by Greenberg are

\[
L_C = 2\pi \rho A b \left[ V_o \alpha + \sigma V_o \alpha c(k_v) e^{i\omega t} + \right.
\]

\[
\left[ b \left( \frac{1}{2} - a \right) \beta + V_o \beta \right] c(k_B) + \dot{h} c(k_h)
\]

\[
+ \sigma V_o \beta c(k_{v+\beta}) e^{i\omega t} \right] \quad (5.35)
\]

\[
L_{NC} = \pi \rho A b^2 \left[ \dot{h} + V_B + V(\alpha + \beta) - ab\beta \right] \quad (5.36)
\]

where \( L_C \) is the circulatory lift

\( L_{NC} \) is noncirculatory lift

Referring to Fig. 10

- \( h \) is the vertical displacement of the axis of rotation (positive downward)
- \( \alpha \) is the constant part of the angle of attack
- \( \beta \) is the time varying part of the angle of attack
- \( V_o \) is the constant part of the stream velocity
\( \sigma V_o e^{i\omega t} \) is the varying part of the stream velocity
\[ V = V_o \left( 1 + \sigma e^{i\omega t} \right) \]

\( ba \) is the position of the torsion axis (axis of rotation of the airfoil) measured from the center of the airfoil section.

The total moment due to both circulatory and noncirculatory parts is
\[ M = \pi \rho_A b^2 \left[ \frac{h + V}{2} \left( \alpha + \beta \right) - \frac{h}{2} \left( \frac{1}{2} - a \right) \beta - b^2 \left( \frac{1}{8} + a^2 \beta \right) \right] + 2\pi \rho_A b^2 \left( a + \frac{1}{2} \right) \left[ v_o \alpha + \sigma V_o \alpha c(k_v) e^{i\omega t} \right] + \left[ b \left( \frac{1}{2} - a \right) \beta + v_o \beta \right] c(k_B) + \frac{h}{2} c(k_H) + \sigma V_o \beta c(k_v + \beta) e^{i\omega t} \]

where \( M \) is the pitching moment about the axis of rotation (positive nose up).

For low frequency oscillations of the rotor blades, the reduced frequency, \( k \), is low and one can introduce the assumption that the Theodorsen's lift deficiency function \( c(k) \) is unity. This is equivalent to the quasisteady assumption. Furthermore from Figs. 3 and 10 one has
\[ ba = -b + x_A + \frac{b}{2} = x_A - \frac{b}{2} \]

Substituting for 'ba' from equation (5.38) and replacing \( c(k) \) by unity, the lift and moment equations become
\[ L = 2\pi \rho_A b h \left[ \frac{1}{2} + V (\alpha + \beta) + (b - x_A) \beta \right] \]
\[ L_{NC} = \pi \rho_A b^2 \left[ \frac{1}{2} + V (\alpha + \beta) - (x_A - \frac{b}{2}) \beta \right] \]
\[ M = \pi \rho_A b^2 \left\{ (x_A - \frac{b}{2}) \left[ \frac{1}{2} + V (\alpha + \beta) + \frac{1}{2} \beta - \frac{1}{2} \beta \right] V \frac{b}{2} \beta - b^2 \beta \right\} + 2\pi \rho_A b \left[ \frac{h}{2} + V (\alpha + \beta) + (b - x_A) \beta \right] \]

Equations (5.39) – (5.41) can be rewritten in a modified form by replacing the quantity \( 2\pi \) by the incompressible lift curve slope 'a', also replacing \( \alpha + \beta \) by \( \tilde{\alpha} \), which represents the total effective angle of attack, where \( \alpha \) represents the constant part thus \( \tilde{\alpha} = \beta \); \( \tilde{\beta} = \beta \) and
\[ L_C = \rho_A \, a \, b \, V \{ h + V\ddot{a} + (b - x_A) \dddot{a} \} \] (5.42)

\[ L_{NC} = \frac{1}{2} \rho_A \, a \, b^2 \{ \dddot{h} + 2 \ddot{V_a} + \dddot{V_a} - (x_A - \frac{b}{2}) \dddot{a} \} \] (5.43)

\[ M = \frac{1}{2} \rho_A \, a \, b^2 \{ (x_A - \frac{b}{2}) [ \dddot{h} + 2 \ddot{V_a} + \dddot{V_a} - (x_A - \frac{b}{2}) \dddot{a}] - \frac{V_b}{2} - \frac{b^2}{8} \dddot{a} \}
+ a \rho_a \, b \, V \, x_A \{ h + V\ddot{a} + (b - x_A) \dddot{a} \} \] (5.44)

Next the various velocity components, relative to the oscillating rotor blade have to be identified. Let \( V_{Ak} \) be the free stream velocity and \( \vec{V}_{Ek} \) be the velocity at any point on the elastic axis of the \( k \)th blade due to its oscillation, the net air flow velocity for the \( k \)th blade is then
\[ \vec{V}_k = \vec{V}_{Ak} - \vec{V}_{Ek} \] (5.45)

For a rotor blade in forward flight with constant velocity, the free stream velocity is
\[ \vec{V}_{Ak} = V_F \cos \alpha_R \hat{e}_{Rx} - (V_F \sin \alpha_R + \nu_k) \hat{e}_{Rz} \] (5.46)

where \( V_F \) is the forward velocity of the model
\( \alpha_R \) is the angle of forward tilt of the rotor plane
\( \nu_k \) is the induced velocity

Equation (5.46) can be written in terms of nondimensional quantities \( \mu \) and \( \lambda_k \) where \( \mu \) is the advance ratio = \( V_F \cos \alpha_R / OR \) and \( \lambda_k \) is the inflow ratio = \( (V_F \sin \alpha_R + \nu_k) / OR \). Hence
\[ \vec{V}_{Ak} = OR(\mu \hat{e}_{Rx} - \lambda_k \hat{e}_{Rz}) \] (5.47)

This velocity can be written in terms of components along the \( 2k \)-system
\[ \vec{V}_{A2k} = OR \{ [\cos \psi_k \mu + \lambda_k \theta_y] + \sin \psi_k \mu (\theta_y \hat{e}_x - \theta_z) - \lambda_k \hat{e}_x \} \hat{e}_{x2k} \]

\[ + [\cos \psi_k \mu (\theta_y \hat{e}_x - \theta_z) - \lambda_k \theta_y] + \sin \psi_k (- \mu - \lambda_k \theta_y) \} \hat{e}_{y2k} \]
The velocity at any point on the elastic axis due to blade deformation is

\[ \vec{v}_{E\text{k}} = \vec{R}_{OH} + \vec{r}_{pk} + \vec{\omega}_k \times \vec{r}_{pk} \]  

(5.49)

where \( \vec{R}_{OH} \) is the velocity of the hub center

\( \vec{r}_{pk} \) is the velocity of the point 'p' on the elastic axis of the blade as seen in the rotating reference frame

\( \vec{\omega}_k \) is the angular velocity of the rotating reference frame

and

\[ \vec{r}_{pk} = e \hat{e}_{x2k} + (x_k + u_k) \hat{e}_{x3k} + w_k \hat{e}_{z3k} + v_k \hat{e}_{y3k} \]  

(5.50)

The various terms in equation (5.49), in 2k system, are

\[ \vec{r}_{pk} = \Omega(\beta \beta_k x_k + u_k) \hat{e}_{x2k} + x_k \hat{e}_{y2k} + x_k \hat{e}_{z2k} \]  

(5.51)

(Based on rigid blade assumption, i.e., \( w_k = x_k \beta_k \) and \( v_k = x_k \gamma_k \)) and

\[ \vec{R}_{OH} = \Omega(\hat{e}_{x2k} [\cos\psi_k < R_x - R \theta y z + R \theta y > + \sin\psi_k < R_y - R \theta x z + R \theta x > ] + \hat{e}_{y2k} [\cos\psi_k < R_y - R \theta x z + R \theta x > + \sin\psi_k < -R_x - R \theta y z + R \theta y > ] + \hat{e}_{z2k} [R_z - R \theta y x + R \theta y ] \]  

(5.52)

and

\[ \vec{\omega}_k \times \vec{r}_{pk} = \Omega[\hat{e}_{x2k} [-x_k \hat{e}_x - x_k \hat{e}_y \theta + \cos\psi_k \hat{e}_y (\beta x_k + \beta_k x_k) > + \sin\psi_k < -\theta x (\beta x_k + \beta_k x_k) > ] + \hat{e}_{y2k} [x_k + e + x_k \hat{e}_z ] \]
Combining equations (5.51) - (5.53)

\[\hat{\mathbf{v}}_{EC2k} = \hat{\mathbf{e}}_{x2k} \{ -x_k \mathbf{e}_k - x_k \mathbf{e}_k \theta + u_k - \beta \mathbf{e}_p \}
+ \cos \psi_k < R - R \theta - R \theta + \theta (x_k \mathbf{e}_p + x_k \mathbf{e}_y) > 
+ \sin \psi_k < -x - R \theta - R \theta > 
+ \hat{\mathbf{e}}_{y2k} \{ -x_k \mathbf{e}_k - x_k \mathbf{e}_z + \cos \psi_k < -x - R \theta - R \theta > 
+ \sin \psi_k < -x - R \theta - R \theta > 
+ \hat{\mathbf{e}}_{z2k} \{ -x_k \mathbf{e}_k - x_k \mathbf{e}_z + \cos \psi_k < -x - R \theta - R \theta > 
+ \sin \psi_k < -x - R \theta - R \theta > 
\] (5.54)

Substituting equations (5.48) and (5.54) in equation (5.45) and applying the ordering scheme yields

\[\hat{\mathbf{v}}_{2k} = \Omega \left\{ \hat{\mathbf{e}}_{x2k} \{ x_k \mathbf{e}_k (1 + \theta^2) - u_k + \beta \mathbf{e}_p \mathbf{e}_k \} + \cos \psi_k < -x - R \theta > 
+ \sin \psi_k < -x - R \theta > 
+ \hat{\mathbf{e}}_{y2k} \{ -x_k \mathbf{e}_k - x_k \mathbf{e}_z + \cos \psi_k < -x - R \theta > 
+ \sin \psi_k < -x - R \theta > 
+ \hat{\mathbf{e}}_{z2k} \{ -x_k \mathbf{e}_k - x_k \mathbf{e}_z + \cos \psi_k < -x - R \theta > 
+ \sin \psi_k < -x - R \theta > 
\right\} \]
This velocity is again transformed into components along the 5k system where 5k system is defined as the one whose origin is fixed in the deformed blade elastic axis and rotated from 4k system by removing the elastic torsional rotation. In the 5k system, the velocity components are

\[
V_{x5k} = \Omega\{-\varepsilon_k - u_k - x_k \beta_k \beta_k + (\beta_k + \beta_p)(-\lambda_k R - R_z + \mu R \dot{\theta}_k)
- \sin \psi_k (x_k \zeta_k \dot{\theta}_x + (x_k + e) \dot{\theta}_x) \}
\]

\[
V_{y5k} = \Omega[-x_k - e - x_k \zeta_k - x_k \dot{\theta}_z + \cos \psi_k (\mu R - R_y)
+ \sin \psi_k (\mu R + R_x > )]
\]

\[
V_{z5k} = \Omega[-x_k \beta_k - \lambda_k R + \mu R \dot{\theta}_y - R_z - x_k \zeta_k (\beta_k + \beta_p)
- \cos \psi_k (x_k \zeta_k \dot{\theta}_y - (x_k + e) \dot{\theta}_y + (\beta_k + \beta_p)(\mu R - R_x))
- \sin \psi_k (x_k \zeta_k \dot{\theta}_y + (x_k + e) \dot{\theta}_x + (\beta_k + \beta_p)(\mu R \dot{\theta}_z - R_y) ]
\]

For the evaluation of the unsteady aerodynamic forces and moments, the various velocity terms in equations (5.42) - (5.44) have to be identified. Figure 11 shows that

\[
V = -V_{y5k}; \quad \hat{h} = V_{z5k} \text{ and } \hat{\alpha} = \theta_0 + \phi_k
\]

Substituting these in the lift and moment expressions, the loads per unit length on the kth blade becomes

\[
L_{Ck} = \rho A a b V_{y5k} [-V_{z5k} + V_{y5k} (\theta_0 + \phi_k) - (b - x_3 \dot{\theta}_3 + \phi_k) \Omega]
\]
The drag force $D_k$ is
\[ D_k = \rho_A a b \left( \frac{c_{d0}}{a} \right) \left[ V_{y5k}^2 + V_{z5k}^2 \right] \] (5.63)

The inflow angle $\phi_{ik}$ is
\[ \phi_{ik} = \tan^{-1} \left( \frac{V_{z5k}}{V_{y5k}} \right) \] (5.64)

According to the assumption made previously, the circulatory lift acts normal to the resultant flow and the noncirculatory lift acts normal to the blade chord. Resolving the lift and drag forces along the 5k system (Fig. 11), the force components per unit length are

\[ p_{A_y5k} = -\rho_A a b V_{y5k} \left[ -V_{z5k} + V_{y5k} \left( \theta_{Gk} + \phi_k \right) - \left( b - x_A \right) \left( \theta_{Gk} + \phi_k \right) \Omega \right] \sin \phi_{ik} \]
\[ - \frac{1}{2} \rho_A a b^2 \left[ V_{z5k} - V_{y5k} \left( \theta_{Gk} + \phi_k \right) \Omega - V_{y5k} \left( \theta_{Gk} + \phi_k \right) \right] \]
\[ - \left( x_A - \frac{b}{2} \right) \left( \dot{\theta}_{Gk} + \ddot{\phi}_k \right) \Omega^2 \right] \sin \left( \theta_{Gk} + \phi_k \right) \]
\[ - \rho_A a b \left( \frac{c_{d0}}{a} \right) \left[ V_{y5k}^2 + V_{z5k}^2 \right] \cos \phi_{ik} \] (5.65)

\[ p_{A_z5k} = \rho_A a b V_{y5k} \left[ -V_{z5k} + V_{y5k} \left( \theta_{Gk} + \phi_k \right) - \left( b - x_A \right) \left( \theta_{Gk} + \phi_k \right) \Omega \right] \cos \phi_{ik} \]
\[ + \frac{1}{2} \rho_A a b^2 \left[ V_{z5k} - V_{y5k} \left( \theta_{Gk} + \phi_k \right) \Omega - V_{y5k} \left( \theta_{Gk} + \phi_k \right) \right] \]
\[ - \left( x_A - \frac{b}{2} \right) \left( \dot{\theta}_{Gk} + \ddot{\phi}_k \right) \Omega^2 \right] \cos \left( \theta_{Gk} + \phi_k \right) \]
\[- \rho_A \, a \, b \left( \frac{c_{do}}{a} \right) \left[ v_{y5k}^2 + v_{z5k}^2 \right] \sin \phi_{ik} \]  

(5.66)

and the torsional moment, per unit length is

\[
q_{Ax5k} = \rho_A \, a \, b \left[ v_{y5k} \, \dot{x}_A - v_{z5k} \left( \theta_{Gk} + \phi_{ik} \right) \right] + \frac{1}{2} \, \rho_A \, a \, b^2 \left[ \left( x_A - \frac{b}{2} \right) \left( v_{z5k} - v_{y5k} \left( \theta_{Gk} + \phi_{ik} \right) \right) \right] \]

(5.67)

Assuming that angle $\phi_{ik}$ is small, the following approximations are made.

\[
\sin \phi_{ik} \approx \phi_{ik} = \frac{v_{z5k}}{v_{y5k}}
\]

and

\[
\cos \phi_{ik} \approx 1
\]

And also

\[
\sin(\theta_{Gk} + \phi_{ik}) \approx \sin \theta_{Gk} + \phi_{ik} \cos \theta_{Gk}
\]

\[
\cos(\theta_{Gk} + \phi_{ik}) \approx \cos \theta_{Gk} - \phi_{ik} \sin \theta_{Gk}
\]

Note that $v_{z5k}/v_{y5k}$ is $O(e)$, $\phi_{ik}$ is $O(e)$, $\theta_{Gk}$ is $O(e^2)$ and $\left( \frac{c_{do}}{a} \right)$ is $O(e^{3/2})$.

Using these approximations, the force components per unit length become

\[
\rho_{A} v_{y5k} = \rho_A \, a \, b \left[ v_{z5k} + v_{y5k} \left( \theta_{Gk} + \phi_{ik} \right) \right] + \left( b - x_A \right) \left( \theta_{Gk} + \phi_{ik} \right) \Omega
\]

\[
- \rho_A \, a \, b \left( \frac{c_{do}}{a} \right) v_{y5k}^2
\]

\[
- \frac{1}{2} \rho_A \, a \, b \left[ v_{z5k} - v_{y5k} \left( \theta_{Gk} + \phi_{ik} \right) \right] \Omega - v_{y5k} \left( \theta_{Gk} + \phi_{ik} \right) \]

\[
- \left( x_A - \frac{b}{2} \right) \left( \theta_{Gk} + \phi_{ik} \right) \Omega \sin \theta_{Gk} + \phi_{ik} \cos \theta_{Gk} \right] \]

(5.68)
\[ P_{Az5k} = \rho_A a b \left[ -v_{z5k} + v_{y5k} (\theta_G + \phi_k) - (b - x_A)(\theta_G + \phi_k) \Omega \right] \\
+ \frac{1}{2} \rho_A a b^2 \left[ v_{z5k} - v_{y5k} (\dot{\theta}_G + \dot{\phi}_k) \Omega - v_{y5k} (\theta_G + \phi_k) \right] \\
- (x_A - \frac{b}{2}) (\dot{\theta}_G + \dot{\phi}_k) \Omega^2 \right] \left[ \cos(\theta_G) - \phi_k \sin(\theta_G) \right] \]

(5.69)

Where the appropriate velocity components have to be substituted in these equations. These velocity components are given in equations (5.56)-(5.58). The aerodynamic forces per unit length of the blade, in component form are

\[ P_{Ay5k} = \rho_A a b \Omega^2 \left[ \frac{c_{do}}{a} \left[ \chi_k^2 \frac{1 + 2\alpha_k + 2\beta_k}{2\chi_k} + 2\chi_k \beta_k \right] \right. \\
- \beta_k \beta_k \left( \begin{array}{c} \beta_k \theta_G + \phi_k - \beta_k + \tau_k (\theta_G + \phi_k) \\ - \tau_k (\beta_p + \beta_k) + \theta_G \theta_G > \\
+ \tau_k (\beta_p + \beta_k) < \theta_G + \phi_k - \beta_k > \\
- \chi_k [ \beta_k < - \lambda_k R + e(\theta_G + \phi_k) + \mu R \theta_y - R_z > \\
+ \lambda_k R < \theta_G + \phi_k - \beta_k + \tau_k (\theta_G + \phi_k) - \tau_k (\beta_p + \beta_k) + \theta_G \theta_G > \\
+ (-\mu R \theta_y + R_z) < \theta_G + \phi_k - \beta_k + \tau_k \theta_G > \\
+ \tau_k (\beta_p + \beta_k) < - \lambda_k R + \beta_k b (\theta_G + \phi_k) > \\
- \lambda_k R \beta_k < - \lambda_k R + e(\theta_G + \phi_k) + \mu R \theta_y - R_z > \\
+ (\mu R \theta_y - R_z) < - \lambda_k R + e(\theta_G + \phi_k) + \mu R \theta_y - R_z > \\
- \lambda_k R b (\theta_G + \phi_k) \right] \\
\]

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- \cos \psi_k \left[ 2 \frac{c_{do}}{a} \left[ x_k \left( \mu R \zeta_k + \mu \theta z + R_y \right) \right]^{\bullet} \right.

- x_k^2 \left[ \beta_k \theta_y - \zeta_k \theta_x \theta_k \right. + \left. \theta_y \left( \theta_k + \phi_k - \beta_k \right) + \theta_k \theta_0 \theta_y \right]

- x_k \left[ \beta_k < \mu R (\beta_p + \beta_k) + \theta_k \left( -\mu R \zeta_k - \mu R \theta z - R_y \right) - \phi_k \mu R \zeta_k \right]^{\bullet}

- \theta_y < \lambda_k R - \mu R \theta y + R_z >

+ \zeta_k (\beta_p + \beta_k) < \mu R (\beta_p + \beta_k) >

+ \theta_k \left( \beta_p + \beta_k \right) < \mu R (\beta_p + \beta_k) >

- \lambda_k \theta \theta y - \zeta_k \left( \theta G_k + \phi_k \right) \mu R (\beta_p + \beta_k)

+ \theta_y < \text{e} \theta G_k + \mu R \theta y - R_z >

+ \left( \theta \theta G_k - \zeta_k (\beta_p + \beta_k) \right) < -\mu R (\beta_p + \beta_k) >

- \lambda_k R < \mu R (\beta_p + \beta_k) + \theta_k \left( -\mu R \zeta_k - \mu R \theta z - R_y \right) - \mu R \zeta_k \phi_k

+ \left( \mu R \theta y - R_z \right) < \mu R (\beta_p + \beta_k) - \mu R \zeta_k \theta G_k

- \lambda_k R < \mu R (\beta_p + \beta_k) >

+ \left[ e (\theta G_k + \phi_k) + \mu R \theta y - R_z \right] < \mu R (\beta_p + \beta_k) >

+ \mu R (\beta_p + \beta_k) \text{b} \left( \theta G_k + \phi_k \right)

- \sin \psi_k \left[ \frac{2c_{do}}{a} \left[ x_k < \mu R \left( 1 + \zeta_k \theta + \theta z - R_x \right) + \theta \mu R \right] \right.

- x_k^2 \left[ \beta_k \theta x - \theta_k \zeta_k \theta y - \theta x \left( \theta_k + \phi_k - \beta_k \right) - \theta_k \theta_k \theta x \right]
\[- x_k [ \dot{\theta}_k - \mu R (\theta_{G_k} + \phi_k) + \dot{\theta}_{G_k} R_x ]
\]
\[+ \frac{\dot{\theta}}{x} < \lambda_k R - \mu R \dot{\theta}_y + \dot{R}_z >
\]
\[+ \tau_k (\beta_k + \dot{\beta}_k) < - \mu R (\dot{\theta}_{G_k} + \phi_k) >
\]
\[\dot{\theta}_{G_k} < e_{\theta} + (\beta_k + \dot{\beta}_k) (-\mu R \dot{\theta}_z - R_y) >
\]
\[+ \lambda_k R \dot{\theta}_x - \dot{\theta}_x < e_{\theta} + \mu R \dot{\theta}_y + \dot{R}_z >
\]
\[\lambda_k R < - \mu R (\dot{\theta}_{G_k} + \phi_k) + \dot{\theta}_{G_k} R_x >
\]
\[+ (\mu R \dot{\theta}_y - R_z) < - \mu R (\dot{\theta}_{G_k} + \phi_k) >
\]
\[+ \cos \psi_k \sin \psi_k [ - \frac{c_d \alpha}{a} [ \mu R (\mu R \dot{x}_k + \mu R \dot{\theta}_z + R_y) ]
\]
\[- x_k^2 [2 \dot{\theta}_y \dot{\theta}_x]
\]
\[- x_k [ - \dot{\theta}_x < 2 \mu R (\beta_k + \dot{\beta}_k) - 2 \mu R \dot{\theta}_y \dot{\theta}_x >
\]
\[+ \dot{\theta}_y < - \mu R (\dot{\theta}_{G_k} + \phi_k) >
\]
\[+ \mu R \dot{\theta}_{G_k} e_{\theta} - \mu R (\beta_k + \dot{\beta}_k) < \mu R \dot{\theta}_{G_k} + \mu R \phi_k >
\]
\[+ 2 \mu R \dot{\theta}_{G_k} \dot{R}_x (\beta_k + \dot{\beta}_k)
\]
\[+ \cos^2 \psi_k [ x_k^2 \dot{\theta}_y^2 - x_k [2 \dot{\theta}_y (\dot{\beta}_k + \dot{\beta}_k) \mu R - \mu R \dot{\theta}_{G_k} \dot{\theta}_y ] + \mu R^2 (\beta_k + \dot{\beta}_k)^2
\]
\[+ \mu R (\beta_k + \dot{\beta}_k) < - \mu R \dot{\theta}_{G_k} (\dot{\theta}_{G_k} + \phi_k) + \dot{\theta}_{G_k} (-\dot{R}_y - \mu R \dot{\theta}_z)>
\]
\[+ \sin^2 \psi_k [ x_k^2 \dot{\theta}_x^2 + x_k [ - \mu R \dot{\theta}_{G_k} (\dot{\theta}_x + \dot{\theta}_k \dot{y}) - \mu R \dot{\phi}_k \dot{\theta}_x]
\]
\[- \mu R \dot{\theta}_{G_k} < e_{\theta} + (\beta_k + \dot{\beta}_k) (-\mu R \dot{\theta}_z - R_y)>
\]
\[- \frac{c_d \alpha}{a} < \mu R^2 - 2 \mu R \dot{R}_x > ] \}

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\[- \frac{1}{2} \rho_A a b^2 \omega^2 \left\{ - x_k \left[ (\ddot{R}_k - (\theta_{Gk} + \phi_k)) - \sin \theta_{Gk} + \phi_k \cos \theta_{Gk} - \sin \theta_{Gk} \right] \right. \\
+ \sin \theta_{Gk} \left( \mu R R \dot{y} - R_z \right) - \cos \psi_k \left[ - x_k (\ddot{\theta} - \dot{x}) \sin \theta_{Gk} \right. \\
+ \left. \left[ \mu R \ddot{\theta}_k - \mu R (\theta_{Gk} + \phi_k) \right] \right\} < \sin \theta_{Gk} + \phi_k \cos \theta_{Gk} > \\
- \mu R \theta_{Gk} \sin \theta_{Gk} \right\}

\sin \psi_k \left[ x_k (\ddot{\theta}_x + \ddot{\theta}_y) \sin \theta_{Gk} - \left[ \mu R (\beta_p + \beta_k) \right. \right. \\
+ \mu R (\ddot{\theta}_Gk + \phi_k) \left. \left. \right\} \right. < \sin \theta_{Gk} + \phi_k \cos \theta_{Gk} > + \mu R \theta_{Gk} \sin \theta_{Gk} \left. \right\}

(5.70)

\[ P_{A_{z5k}} = \rho_A a b \omega^2 \left\{ x_k^2 [ \theta_{Gk} + \phi_k - \beta_k + \dddot{\tau}_k (\theta_{Gk} + \phi_k) - \dddot{\tau}_k (\beta_p + \beta_k) + \dddot{\theta}_{Gk} \right. \\
+ \dddot{\phi}_k \right\} < \theta_{Gk} + \phi_k - \beta_k > + \dddot{\theta}_{Gk} \}

\sin \psi_k \left[ - x_k^2 \dddot{\theta}_y \right. \\
+ \left. x_k \left[ \mu R (\beta_p + \beta_k) + \theta_{Gk} (\dddot{\tau}_k - \dddot{R}_y - \mu R \dot{\theta}_z) - \mu R \dddot{\phi}_{Gk} \left. \right\} \right. \\
+ \dddot{\tau}_k < \mu R (\beta_p + \beta_k) > + \dddot{\theta}_{Gk} < - \mu R \dddot{\tau}_k > + \dddot{\phi}_k > + \mu R (\beta_p + \beta_k) + \lambda_k \mu R \dddot{\phi}_k \left. \right\}

\sin \psi_k \left[ x_k^2 \dddot{\theta}_x \right. \\
+ \left. x_k \left[ - \mu R (\theta_{Gk} + \phi_k) + \dddot{\phi}_k + \dddot{\tau}_k \theta_{Gk} - \mu R (\theta_{Gk} + \phi_k) \dddot{\tau}_k \right. \right. \\
- \mu R \theta_{Gk} - \mu R \theta_{Gk} + \phi_k - \beta_k > + \dddot{R}_x \theta_{Gk} \right. \\

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It can be seen from the expressions for the aerodynamic forces that the contribution due to \((\theta_{Gk} + \phi_k)\) associated with equation (5.40) is absent because of ordering scheme. Equation (5.71) can be written as

\begin{equation}
\begin{aligned}
\rho A_{z5k} &= \rho A_{z5k}^{(1)} + \rho A_{z5k}^{(2)} \cos \theta_{Gk} \\
\end{aligned}
\end{equation}

where \(\rho A_{z5k}^{(1)}\) represents the circulatory term and \(\rho A_{z5k}^{(2)}\) represents the non-circulatory term.

The aerodynamic moment per unit length can be written as
\[ q_{A_{x5k}} = x_{A}^{(1)} + (x_{A} - \frac{b}{2}) p_{A_{z5k}} - \frac{b^2}{8} \left( \dot{\theta}_{G_k} + \ddot{\phi}_k \right) \Omega^2 \frac{1}{2} \rho_{A} a b^2 \]

\[ + \frac{1}{2} \rho_{A} a b^2 b \left( \dot{\theta}_{G_k} + \dot{\phi}_k \right) \Omega^2 \left\{ -x_{k} - e - \cos \psi_k \mu R \zeta_k - \sin \psi_k \mu R \right\} \]

\[ = \rho_{A} a b \Omega^2 \left[ x_{k} \left( \theta_{G_k} + \phi_k - \beta_k + \zeta_k \right) \left( \theta_{G_k} + \phi_k \right) - \zeta_k \left( \beta_p + \beta_k \right) \right. \]

\[ + \theta_{G_k} + \zeta_k < \dot{\theta}_{G_k} + \dot{\phi}_k - \beta_k > + \theta_{z G_k} \right] \]

\[ + x_{k} \left[ - \lambda_k R + e \left( \theta_{G_k} + \phi_k \right) + \mu R \theta_y - R_z - \lambda_k R \zeta_k \right. \]

\[ + e < \dot{\theta}_{G_k} + \dot{\phi}_k - \beta_k > + \beta_k \left( \theta_{G_k} + \phi_k \right) - e \lambda_k R \]

\[ - \cos \psi_k \left[ - x_{k}^2 \dot{\theta}_{y} + x_{k} \left[ \mu R \left( \beta_p + \beta_k \right) \right. \right. \]

\[ + \theta_{G_k} \left( - \mu R \zeta_k - \mu R \theta_y - \mu R \theta_z \right) = \mu R \zeta_k \phi_k \]

\[ + \zeta_k < \mu R \left( \beta_p + \beta_k \right) + \theta_{G_k} \left( - \mu R \zeta_k - \mu R \theta_y - \mu R \theta_z \right. \]

\[ + \left( \phi_k - \beta_k \right) \left( - \mu R \zeta_k \right) \right] + e \mu R \left( \beta_p + \beta_k \right) + \lambda_k R \mu R \zeta_k \]

\[ - \sin \psi_k \left[ x_{k} \dot{\theta}_{x} + x_{k} \left[ - \mu R \left( \theta_{G_k} + \phi_k \right) \right. \right. \]

\[ + \mu R \left( \theta_{G_k} + \phi_k \right) - \mu R \left( \theta_{G_k} + \phi_k \right) \left. \right] \mu R \theta_z \theta_{G_k} \]

\[ - \mu R \left< \theta_{G_k} + \phi_k - \beta_k > + \mu R \theta_{G_k} \right. \]

\[ - \mu R < \zeta_k \left( \theta_{G_k} + \phi_k \right) - \zeta_k \left( \beta_p + \beta_k \right) + \theta_{G_k} \theta_{G_k} \right] \]

\[ + e \left[ - \mu R \left( \theta_{G_k} + \phi_k \right) + \lambda_k R \mu R \right. \]

\[ - \mu R < e \left( \theta_{G_k} + \phi_k \right) + \mu R \theta_y - R_z > \mu R \left( \theta_{G_k} + \phi_k \right) \]

(5.73)
These aerodynamic force and moment expressions are transformed into components in 3k system because the blade dynamic equations of motion are written in 3k system. The appropriate components are

\[ p_{A_{x3k}} = -\zeta_k p_{A_{y5k}} - \beta_k p_{A_{z5k}} \]  
(5.75)

\[ p_{A_{y3k}} = p_{A_{y5k}} \]  
(5.76)

\[ p_{A_{z3k}} = -\beta_k \zeta_k p_{A_{y5k}} \]  
(5.77)

\[ q_{A_{x3k}} = q_{A_{x5k}} \]  
(5.78)

\[ q_{A_{y3k}} = \zeta_k q_{A_{x5k}} \]  
(5.79)

\[ q_{A_{z3k}} = \beta_k q_{A_{x5k}} \]  
(5.80)
5.2.3 Distributed Structural Damping Loads

The structural damping incorporated in this analysis is of a viscous equivalent type. The damping forces per unit span of the blade for flap, lag and torsion are respectively given as

Flap \[ p_{Dz3k} = \Omega x_k \beta_k g_F \] \hspace{1cm} (5.81)

Lead-lag \[ p_{Dy3k} = -\Omega x_k \xi_k g_L \] \hspace{1cm} (5.82)

Torsion \[ q_{Dx3k} = -\Omega \phi_k g_T \] \hspace{1cm} (5.83)

No attempt is made to eliminate these terms by considering the ordering schemes since these terms serve the purpose of determining the effect of damping on stability.

5.2.4 Rotor Blade Equations

In this section, the individual blade equations of motion are presented. For the rigid, offset hinged, spring restrained blade model used in this study, the distributed inertia, aerodynamic and structural damping loads are integrated over the length of the blade and moment equilibrium at the spring restrained hinge is enforced. The loads due to inertia, aerodynamic and damping, integrated over the blade span, are given in the following sections. Finally the blade equations are obtained by enforcing moment equilibrium at the root of the blade.

Inertia Loads

The forces and moments at the blade root due to the inertia forces are

\[ \bar{P}_{I3k} = \int_{0}^{R-e} \bar{P}_{I3k} \, dx_k \] \hspace{1cm} (5.84)

\[ \bar{Q}_{I3k} = \int_{0}^{R-e} ( \bar{q}_{I3k} + \bar{r}_{p3k} \times \bar{P}_{I3k} ) \, dx_k \] \hspace{1cm} (5.85)

where \( \bar{P}_{I3k} \) are the distributed inertia force on the \( k^{th} \) blade and \( \bar{Q}_{I3k} \) is the distributed inertia moment about the elastic axis. These quantities are derived in previous sections of this report.

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Recall that the position vector of any point "p" on the deformed elastic axis of the blade is given by

\[
\vec{r}_{p3k} = (x_k + u_k) \hat{e}_{x3k} + x_k \zeta_k \hat{e}_{y3k} + x_k \beta_k \hat{e}_{z3k}
\]  

(5.86)

and \( u_k \) is the axial displacement primarily due to geometric shortening

\[
u_k = -\frac{1}{2} \int_0^{x_k} (\zeta_k^2 + \beta_k^2) \, dx_k
\]

\[
= -\frac{1}{2} x_k (\zeta_k^2 + \beta_k^2)
\]

(5.87)

and

\[
u_k = - x_k (\zeta_k \zeta_k + \beta_k \beta_k)
\]

(5.88)

It is assumed that the inflow is constant over the disk and the pretwist of the blade is zero. Hence in integrations over the blade span \( \lambda \) and \( \theta_{ck} \) remain constants. Mass per unit length of the blade is also assumed to be constant.

The components of the inertia forces at the root of the blade in 3k system are

\[
P_{I_{x3k}} = m\omega^2 \left[ \frac{(R-e)^2}{2} + (R-e) \omega + \frac{(R-e)^2}{2} \frac{\dot{\zeta}_k}{\zeta_k} + \frac{(R-e)^2}{2} \frac{\dot{\theta}_z}{\theta_z} + \frac{(R-e)^2}{2} \frac{\dot{\zeta}_k}{\zeta_k} \right]
\]

(5.89)

\[
P_{I_{y3k}} = m\omega^2 \left[ \frac{(R-e)^2}{2} \zeta_k^2 + 2\dot{\zeta}_k \dot{\theta}_z - \dot{\theta}_z + 2\dot{\theta}_k \dot{\theta}_z \right] - (R-e) \frac{\dot{\theta}_z}{\theta_z} + 2 \frac{(R-e)^2}{2} \left[ \zeta_k \zeta_k + \beta_k \beta_k \right] + (R-e) x_1 \cos \theta_{ck}
\]

\[
+ \frac{(R-e)^2}{2} \frac{\dot{\zeta}_k}{\zeta_k} + \beta_k \dot{\theta}_k + 2\beta_k \dot{\theta}_k > - (R-e) \frac{\dot{\zeta}_k}{\zeta_k} \right] \]

(5.90)
\[ p_{I_{z3k}} = m\omega^2 \left[ -\frac{(R-e)^2}{2} \right] + (R-e) \frac{\beta_k}{R_z} - \beta_p + \frac{(R-e)^2}{2} \left( 1 + 2\dot{\theta}_z + \dot{\zeta}_k \right) + (R-e) e \]

\[ + \cos \psi_k < (R-e) \left[ \beta_p \frac{\ddot{\theta}}{R_x} + e \left( \dot{\theta}_y - 2\dot{\theta}_x \right) \right] \]

\[ - \frac{(R-e)^2}{2} \left[ 2\ddot{z}_k - (\dot{\theta}_y - 2\dot{\theta}_x) + \zeta_k \left( \ddot{\theta}_x + 2\dot{\theta}_y \right) \right] \]

\[ + \sin \psi_k < (R-e) \left[ \beta_p \frac{\ddot{\theta}}{R_y} - e \left( \ddot{\theta}_x + 2\dot{\theta}_y \right) \right] \]

\[ - \frac{(R-e)^2}{2} \left[ 2\ddot{z}_k \ddot{\theta}_y + (\ddot{\theta}_x + 2\dot{\theta}_y) + \zeta_k \left( \ddot{\theta}_y - 2\dot{\theta}_x \right) \right] > \]

(5.91)

The components of the inertia moments, after applying the ordering scheme, are

\[ Q_{I_{x3k}} = \int_0^R \left( q_{I_{x3k}} + x_k r_k p_{I_{z3k}} - x_k \beta_k p_{I_{z3k}} \right) dx_k \]

\[ = \zeta_k \left\{ m\omega^2 \left[ -\frac{(R-e)^3}{3} \right] + \frac{(R-e)^2}{2} \frac{\beta_k}{R_z} - \beta_p + \frac{(R-e)^3}{3} \dot{\theta}_z + \frac{(R-e)^3}{3} \ddot{z}_k \right\} \]

\[ + \cos \psi_k < (R-e) \left[ \beta_p \frac{\ddot{\theta}}{R_y} - \frac{(R-e)^3}{3} \ddot{\theta}_x + \frac{(R-e)^3}{3} \ddot{\theta}_y \right] \]

\[ - \frac{(R-e)^2}{2} \left[ 2\ddot{z}_k \dot{\theta}_y + (\ddot{\theta}_x + 2\dot{\theta}_y) + \zeta_k \left( \ddot{\theta}_y - 2\dot{\theta}_x \right) \right] > \]

\[ - \beta_k \left\{ m\omega^2 \left[ \frac{(R-e)^3}{3} \right] + \frac{(R-e)^3}{3} \ddot{z}_k \ddot{\theta}_z - \frac{(R-e)^3}{3} \ddot{\theta}_z \right\} \]

\[ - \frac{(R-e)^3}{3} \ddot{z}_k + \frac{(R-e)^3}{3} \ddot{z}_k \beta_k \ddot{\theta}_x + \frac{(R-e)^3}{3} \beta_k \dot{\theta}_x \left( \ddot{\theta}_y - 2\dot{\theta}_x \right) \]

\[ + \cos \psi_k < \frac{(R-e)^3}{3} \beta_p \frac{\ddot{\theta}}{R_x} + \frac{(R-e)^3}{3} \beta_k \frac{\ddot{\theta}}{R_x} + \frac{(R-e)^3}{3} \beta_k \frac{\ddot{\theta}}{R_y} + \frac{(R-e)^3}{3} \begin{array}{c} \beta \beta_k \frac{\ddot{\theta}}{R} \end{array} \]
\[ + \sin \psi_k < \left( \frac{R-3}{3} \right)^3 \beta p y + \left( \frac{R-3}{3} \right)^3 \beta k y + \left( \frac{R-e}{3} \right)^3 2\beta_k y + \left( \frac{R-e}{2} \right) R > 1 \]

\[ + \Omega^2 \left\{ m x_i \cos \theta G_k \left[ - \left( \frac{R-e}{2} \right)^2 \beta_k - \left( \frac{R-e}{2} \right) \right. \right. \]

\[ - \beta_p < \left( \frac{R-e}{2} \right)^2 + \left( \frac{R-e}{2} \right) e + \left( \frac{R-e}{2} \right)^2 2z_k + \left( \frac{R-e}{2} \right) 2 \theta > \]

\[ - \left( \frac{R-e}{2} \right) z_k \phi_k + \left( \frac{R-e}{2} \right) \phi_k \left( \frac{R-e}{2} \right) \phi_k \left( \frac{R-e}{2} \right) \phi_k \]

\[ + \cos \psi_k < \left( \frac{R-e}{2} \right)^2 \phi_k x \cos \theta x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \]

\[ + \sin \psi_k < - \left( \frac{R-e}{2} \right)^2 \left( \theta y + 2 \theta x \right) - \left( \frac{R-e}{2} \right) e \left( \theta x + 2 \theta y \right) \]

\[ - \left( \frac{R-e}{2} \right) z_k \phi_k \left( \theta x + 2 \theta y \right) - \left( \frac{R-e}{2} \right) 2z_k \theta x \]

\[ + \left( \frac{R-e}{2} \right) \left( \beta p y + (R-e) \phi_k x \right) > \]

\[ + m x_i \sin \theta G_k \left[ - \left( \frac{R-e}{2} \right)^2 \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \right. \]

\[ - \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \]

\[ + \cos \psi_k < \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \]

\[ + \sin \psi_k < - \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \left( \frac{R-e}{2} \right) \phi_k x \]

\[ + \left( \frac{R-e}{2} \right) \left( \phi_k x + 2 \theta x \right) > \]

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\[ + (R-e) \{ I_{MB3} \cos^2 \theta_{Gk} + I_{MB2} \sin^2 \theta_{Gk} \} \left[ -\dot{\phi}_k - \dot{\zeta}_k \beta_k + 2\zeta_k \beta_k \right. \\
+ \zeta_k \beta_k - \theta_{Gk} + \beta_p (\zeta_k - \theta_z) \\
+ \phi_k (1 - 2\theta_z - 2\zeta_k) + \beta_k \tau_k \right. \\
+ \cos \psi_k < - \left( \dot{\theta}_x + 2\dot{\theta}_y + \zeta_k \left( \dot{\theta}_y - 2\dot{\theta}_x + 2\zeta_k \theta_y \right) \right) > \\
+ \sin \psi_k < - \left( \dot{\theta}_y - 2\dot{\theta}_x - \zeta_k \left( \dot{\theta}_x + 2\dot{\theta}_y - 2\zeta_k \theta_x \right) \right) > \\
+ (I_{MB3} \sin^2 \theta_{Gk} + I_{MB2} \cos^2 \theta_{Gk}) \left[ \dot{\phi}_k - \dot{\phi}_k + 2\dot{\phi}_k \theta \right. \\
- \dot{\theta}_z (\beta_p + \beta_k) - 2\dot{\theta}_k + 2\dot{\phi}_k \zeta_k \\
+ 2\dot{\phi}_k \zeta_k + 2\zeta_k \theta_{Gk} - 2\dot{\theta}_z \beta_k - \theta_{Gk} \\
- \dot{\theta}_x \cos \psi_k - \dot{\theta}_y \sin \psi_k \right. \\
\left. \right] + (I_{MB3} - I_{MB2}) \sin \theta_{Gk} \cos \theta_{Gk} \left[ -1 - 2\dot{\theta}_z - 2\tau_k \right] \] (5.92)

\[
Q_{I_{y3k}} = \int_0^{R-e} \left( q_{I_{y3k}} + x_k \beta_k p_{I_{x3k}} - (x_k + u_k) p_{I_{z3k}} \right) dx_k
\]

\[
= \beta_k \left\{ m_1^2 \left[ \frac{(R-e)^3}{3} + \frac{(R-e)^2}{2} e + \frac{(R-e)^3}{3} \right] \dot{\theta}_z + \frac{(R-e)^3}{3} \dot{\tau}_k \\
- \frac{(R-e)^2}{2} \dot{R_x} \cos \psi_k - \frac{(R-e)^2}{2} \dot{R_y} \sin \psi_k \right\} \\
- m_1^2 \left[ - \frac{(R-e)^3}{3} \dot{\beta}_k - \frac{(R-e)^2}{2} \dot{R_z} \\
- \dot{\beta}_p < \frac{(R-e)^3}{3} + \frac{(R-e)^2}{2} e + \frac{(R-e)^3}{3} \right] \dot{\theta}_z + \frac{(R-e)^3}{3} \dot{\tau}_k > \\
+ \cos \psi_k < \frac{(R-e)^2}{2} \dot{\beta}_p \dot{R_x} - \frac{(R-e)^3}{3} 2\dot{R_k} \theta_x + \frac{(R-e)^3}{3} \left( \dot{\theta}_y - 2\dot{\theta}_x \right) \right. \\
\left. \right]
\]
The order of magnitude of leading terms in these expressions for the loads are listed below for convenience

\[ P_{I\times3k} = 0(1) \quad ; \quad P_{I\times y3k} = 0(\varepsilon) \quad ; \quad P_{I\times z3k} = 0(\varepsilon) \]

and

\[ Q_{I\times3k} = 0(\varepsilon^2) \quad ; \quad Q_{I\times y3k} = 0(\varepsilon) \quad ; \quad Q_{I\times z3k} = 0(\varepsilon) \]

The loads at the root due to distributed aerodynamic loads are
\[ \overline{F}_{A3k} = \int_0^{R-3} \overline{p}_{A3k} \, dx_k \]

and

\[ \overline{G}_{A3k} = \int_0^{R-e} (\overline{q}_{A3k} + \overline{r}_{p3k} \times \overline{p}_{A3k}) \, dx_k \]

\[ \overline{P}_{A \times 3k} = -\rho_A \alpha n^2 \beta_k \left( \frac{(R-e)^3}{3} \right) \left[ \theta_{Gk} + \phi_k - \beta_k + \xi_{k} (\theta_{Gk} + \phi_k) - \zeta_{k} (\beta_p + \beta_k) \right. \]

\[ \left. + \theta_{z Gk} + \xi_{k} (\theta_{Gk} + \beta_k) + \theta_{z Gk} \right] \]

\[ + \frac{(R-e)^2}{2} \left[ -\lambda_k R + e(\theta_{Gk} + \phi_k) + \mu R \theta_y - R_z - \lambda_k \zeta_{k} \right. \]

\[ \left. + e(\theta_{Gk} + \phi_k - \beta_k) + b(\theta_{Gk} + \phi_k) - (R-e) e \lambda_k R \right] \]

\[ - \cos \psi_k \left[ - \frac{(R-e)^3}{3} \theta_x + \frac{(R-e)^2}{2} \left[ \mu R (\beta_p + \beta_k) + \theta_{Gk} (\mu \zeta_{k} - \mu \theta_{Gk}) \right. \right. \]

\[ \left. \left. - \mu \zeta_{k} (\beta_p + \beta_k) \right] \right. \]

\[ + \theta_{Gk} (\mu \zeta_{k} - \mu \theta_{Gk}) \right. \]

\[ \left. + \left( R-e \right) \left[ e \mu R (\beta_p + \beta_k) + \lambda_k \mu \zeta_{k} \right] \right] \]

\[ - \sin \psi_k \left[ - \frac{(R-e)^3}{3} \theta_x + \frac{(R-e)^2}{2} \left[ -\mu R (\theta_{Gk} + \phi_k) + R \mu R - \mu R (\theta_{Gk} + \phi_k) \right. \right. \]

\[ \left. \left. - \mu R \theta_{Gk} - \mu R (\theta_{Gk} + \phi_k - \beta_k) + R \mu \theta_{Gk} \right] \right. \]

\[ \left. - \mu R \zeta_{k} (\theta_{Gk} + \phi_k) + \zeta_{k} (\beta_p + \beta_k) + \theta_{z Gk} \right] \]

\[ + (R-e) \left[ -e \mu R (\theta_{Gk} + \phi_k) + \lambda_k \mu R \right. \]

\[ \left. - \mu R < e (\theta_{Gk} + \phi_k) + \mu R \theta_y - \mu R \theta_z \right] \]

\[ - \mu R b (\theta_{Gk} + \phi_k) \right. \]

\[ + \cos \psi_k \sin \psi_k \left[ - \frac{(R-e)^2}{2} \mu R \theta_y \right. \]
\[ + (R-e) \left[ - \mu R \left( \beta_p + \beta_k \right) + 2 \theta_{Gk} \left( -2 \mu R \zeta_k - R - \mu R \theta_x \right) \right] \]

\[ - 2 \mu R \zeta_k \phi_k \right] \]

\[ + \cos^2 \psi_k \left[ - (R-e) \mu R \zeta_k^2 \left( \beta_p + \beta_k \right) \right] \]

\[ + \sin^2 \psi_k \left[ - \frac{(R-e)^2}{2} \mu R \theta_x - (R-e) \mu R \left( \theta_{Gk} + \phi_k \right) \right] \}

\[ - \frac{1}{2} \rho_a b^2 \Omega^2 \beta_k \left[ - \frac{(R-e)^2}{2} \theta_k - (\theta_{Gk} + \phi_k) \right] \]

\[ - \cos \psi_k \left[ (R-e) \left( \mu R \beta_k - \mu R \left( \theta_{Gk} + \phi_k \right) \right) \right] \]

\[ - \sin \psi_k \left[ (R-e) - \mu R \left( \beta_p + \beta_k \right) - \mu R \left( \theta_{Gk} + \phi_k \right) \right] \cos \theta_{Gk} \]

\[ - \rho_a b \Omega^2 \zeta_k \left[ - \frac{c \dot{d} \sigma}{a} \left[ \frac{(R-e)^3}{3} \right] \right] \]

\[ - \frac{(R-e)^3}{3} \left[ \beta_k < \theta_{Gk} + \phi_k - \beta_k \right] \]

\[ - \frac{(R-e)^2}{2} \left[ \beta_k \left( -\lambda_k R \right) + \lambda_k R \left( \theta_{Gk} + \phi_k - \beta_k \right) \right] \]

\[ + (-\mu R \theta_x + R) \theta_{Gk} \]

\[ + (R-e) \left[ - \lambda_k R < - \lambda_k R \right] \]

\[ - \cos \psi_k \left[ - \frac{(R-e)}{3} \right] \theta_{Gk} \]

\[ - \frac{(R-e)^2}{2} \left[ \beta_k \mu R \left( \beta_p + \beta_k \right) + (-\phi_k + \beta_k) \mu R \left( \beta_p + \beta_k \right) \right] \]

\[ + (R-e) \left[ - \lambda_k \mu R \left( \beta_p + \beta_k \right) - \lambda_k \mu R \left( \beta_p + \beta_k \right) \right] \]

\[ - \sin \psi_k \left[ - \frac{2 c \dot{d} \sigma}{a} \frac{(R-e)^2}{2} \mu R \right] \]

\[ - \frac{(R-e)^3}{3} \left[ - \theta_x \theta_{Gk} \right] - \frac{(R-e)^2}{2} \left[ \beta_k < - \mu R \left( \theta_{Gk} + \phi_k \right) \right] \]

\[ 60 \]
\begin{align*}
+ (R-e) \left[ - \lambda_k R \left( -\mu R \left( \theta_{Gk} + \phi_k \right) - \mu R \theta_{Gk} \left( \mu R \theta_y - \tilde{R}_z \right) \right) \right] \\
+ \cos \psi_k \sin \psi_k \left[ - \frac{(R-e)^2}{2} \theta_y \left( - \mu R \theta_{Gk} \right) \right] \\
+ (R-e) \left[ - \mu R \left( \beta_p + \beta_k \right) < \mu R \theta_{Gk} + \mu R \phi_k > \right] \\
+ \cos^2 \psi_k \left[ (R-e) \mu R^2 \left( \beta_p + \beta_k \right)^2 \right] \\
+ \sin^2 \psi_k \left[ \frac{(R-e)^2}{2} \left[ - \mu R \theta_{Gk} \theta_x \right] \right] \\
- \frac{c \rho_o a}{a} \left( R-e \right) \left[ \mu R \right]^2 ] \right] \\
+ \frac{1}{2} \rho_o a b^2 \Omega^2 \tau_k \{ (R-e) \left[ - \mu R \theta_{Gk} \sin \theta_{Gk} \right] \left( - \cos \psi_k \right) \} \\
\end{align*}

\( P_{A_y 3k} = \rho_o a b^2 \left\{ - \frac{c \rho_o a}{a} \left[ \frac{(R-e)^3}{3} \left( 1 + 2 \tau_k \theta_{Gk} + 2 \tau_k \theta_{Gk} \right) + \frac{2(R-e)^2}{2} e \right] \\
- \frac{(R-e)^3}{3} \left[ \beta_p + \beta_k \right] \left( \theta_{Gk} + \phi_k \right) - \beta_k + \tau_k \left( \theta_{Gk} + \phi_k \right) \right. \\
- \tau_k \left( \beta_p + \beta_k \right) + \theta_k \theta_{Gk} \right\} \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) - \beta_k \right\} \\
- \frac{(R-e)^2}{2} \left[ \beta_k \left[ -\lambda_k R + e \left( \theta_{Gk} + \phi_k \right) + \mu R \theta_y - \tilde{R}_z \right] \\
+ \lambda_k R \left( \theta_{Gk} + \phi_k \right) - \beta_k + \tau_k \left( \theta_{Gk} + \phi_k \right) - \tau_k \left( \beta_p + \beta_k \right) \right. \\
+ \beta_k \left( \theta_{Gk} + \phi_k \right) \\
+ \tau_k \left( \beta_p + \beta_k \right) \left. \left( \theta_{Gk} + \phi_k \right) \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
+ \tau_k \left( \beta_p + \beta_k \right) \left( \theta_{Gk} + \phi_k \right) \left. \right] \right. \\
\end{align*}
\[ + (R-e) \left[ - \lambda_k R < - \lambda_k R + e (\theta_{G_k} + \phi_k) + \mu R \dot{\theta} - R_z \right] \]
\[ + (\mu R \dot{\theta} - R_z) < - \lambda_k R + e \theta_{G_k} + \mu R \dot{\theta} - R_z \]
\[ - \lambda_k R \dot{b} (\theta_{G_k} + \phi_k) \]
\[ - \cos \psi_k \left[ \frac{2 \epsilon d \alpha}{a} \left[ \frac{(R-e)^2}{2} (\mu R \tau_k + \mu R \theta - R_y) \right] \right] \]
\[ - \frac{(R-e)^3}{3} \left[ - \beta_k \dot{\theta} - \tau_k \theta \dot{G}_k + \theta_k (\theta_{G_k} - \dot{b}_k) + \tau_k \theta \dot{G}_k \right] \]
\[ - \frac{(R-e)^2}{2} \left[ \beta_k < \mu R (\beta + \beta_k) + \theta_k (- \mu R \tau_k - \mu R \theta z - R_y) - \phi_k \mu R \tau_k \right] \]
\[ - \theta_k < \lambda_k R - \mu R \dot{\theta} + R_z \]
\[ + \tau_k (\beta + \beta_k) < \mu R (\beta + \beta_k) \]
\[ - \theta_k < - e \dot{G}_k + (\beta + \beta_k) (\mu R - R_x) \]
\[ + (\beta + \beta_k) < \mu R (\beta + \beta_k) \]
\[ - \lambda_k R \dot{\theta} - \tau_k (\theta_{G_k} + \phi_k) \mu R (\beta + \beta_k) \]
\[ + \theta_k < e \dot{G}_k + \mu R \dot{\theta} - R_z \]
\[ + e (\theta_{G_k} + \phi_k) < \mu R (\beta + \beta_k) \]
\[ + \mu R (\beta + \beta_k) b (\dot{G}_k + \phi_k) \]
\[-\sin\psi_k \left[ \frac{2c_0}{a} \left( \frac{(R-e)^2}{2} < \mu R \left( 1 + \zeta_k + \theta_z \right) - R_x \right) + e \mu R (R-e) \right] \]

\[-\frac{(R-e)^3}{3} \left[ \theta_k x - \theta G_k \zeta_k \theta_y - \theta x \left( \theta G_k + \Phi_k - \beta_k \right) - \theta G_k \zeta_k \theta_x \right] \]

\[-\frac{(R-e)^2}{2} \left[ \beta_k < -\mu R \left( \theta G_k + \Phi_k \right) + \theta G_k R_x \right. \]

\[+ \theta x < \lambda_k R - \mu R \theta y + R_z \]

\[+ \zeta_k \left( \beta_p + \beta_k \right) < -\mu R \left( \theta G_k + \Phi_k \right) \]

\[-\theta G_k < e \theta x + \left( \beta_p + \beta_k \right) \left( -\mu R \theta z - R_y \right) \]

\[+ \lambda_k \theta x < \theta y + e \theta G_k + \mu R \theta y - R_z \right] \]

\[+ (R-e) \left[ -\lambda_k R < -\mu R \left( \theta G_k + \Phi_k \right) + \theta G_k R_x \right. \]

\[+ \left( \mu R \theta y - R_z \right) < -\mu R \left( \theta G_k + \Phi_k \right) \] \]

\[+ \cos \psi_k \sin \psi_k \left[ -\frac{2c_0}{a} \left( \mu R \theta_k + \mu R \theta z + R_y \right) \right. \]

\[+ \frac{(R-e)^3}{3} \left( 2 \theta \theta_x \right) \]

\[-\frac{(R-e)^2}{2} \left[ -\theta x < 2 \mu R \left( \beta_p + \beta_k \right) - 2 \mu R \theta_k \theta_G_k \right. \]

\[+ \theta y < -\mu R \left( \theta G_k + \Phi_k \right) \] \]

\[+ (R-e) \left[ \mu R \theta_G_k e \theta y - \mu R \left( \beta_p + \beta_k \right) \left( \mu R \theta_G_k + \mu R \Phi_k \right) \right. \]

\[+ 2 \mu R \theta_G_k R_x \left( \beta_p + \beta_k \right) \] \]

\[+ \cos \psi_k \left[ \frac{(R-e)^3}{3} \theta \theta_y \right. \]

\[-\frac{(R-e)^2}{2} \left[ 2 \theta y \left( \beta_p + \beta_k \right) \mu R - \mu R \theta_k \theta G_k \theta y \right. \]

\[+ (R-e) \left[ \mu R^2 \left( \beta_p + \beta_k \right)^2 \right. \]
\[ + \mu R (\beta_p + \beta_k) < -\mu R e_k (\theta_{Gk} + \phi_k) + \theta_k (-R_y - \mu R \theta_z) > ] \\
+ \sin^2 \psi_k [ \frac{(R-e)^3}{3} \theta_x^2 ] \\
+ \frac{(R-e)^2}{2} [ -\mu R \theta_{Gk} ( \dot{\theta}_x + \zeta_k \dot{\theta}_y ) - \mu R \theta_k \dot{\theta}_x ] \\
+ (R-e) [ - \mu R \theta_{Gk} < e \theta_x + (\beta_p + \beta_k)(-\mu R \theta_z - \dot{R}_y) > \\
- \frac{c_d}{a} < \mu^2 R^2 - 2 \mu R R_x > ] \} \}

\[ - \frac{1}{2} \rho A a b^2 \Omega^2 \left\{ - \frac{(R-e)^2}{2} \left[ (\beta_k - (\theta_{Gk} + \phi_k)) < \sin \theta_{Gk} + \phi_k \cos \theta_{Gk} \right. \right. \\
- \sum_k \theta_{Gk} \sin \theta_{Gk} \left. \right] \\
+ (R-e) \sin \theta_{Gk} (\mu R \theta_x - \ddot{R}_z) \\
- \cos \psi_k \left[ - \frac{(R-e)^2}{2} (\dot{\theta}_y - \dot{\theta}_x) \sin \theta_{Gk} \right. \\
+ (R-e) \left[ (\mu R \theta_x - \mu R (\theta_{Gk} + \phi_k)) (\sin \theta_{Gk} + \phi_k \cos \theta_{Gk}) \right. \\
- \mu R \theta_{Gk} \sin \theta_{Gk} \left. \right] \\
- \sin \psi_k \left[ \frac{(R-e)^2}{2} (\dot{\theta}_x + \dot{\theta}_y) \sin \theta_{Gk} \right. \\
- (R-e) \left[ (\mu R (\beta_p + \beta_k) + \mu R (\theta_{Gk} + \phi_k)) \\
(\sin \theta_{Gk} + \phi_k \cos \theta_{Gk}) \right. \\
+ \mu R \theta_{Gk} \sin \theta_{Gk} \left. \right] \} \} \right. (5.96) \]
\[ p_{A_{23k}} = \rho \omega a b \Omega^2 \left\{ \frac{(R-e)^3}{3} \left[ \Theta_{Gk} + \phi_k - \beta_k + \zeta_k (\theta_{Gk} + \phi_k) - \zeta_k (\beta_p + \beta_k) \right. \right. \]
\[ + \left. \left. \Theta_z \theta_{Gk} + \zeta_k \theta_{Gk} + \phi_k - \beta_k \right] + \Theta_z \theta_{Gk} \right\} \]
\[ + \frac{(R-e)^2}{2} \left[ - \lambda_k R + \epsilon \left( \Theta_{Gk} + \phi_k \right) + \mu R \theta_y - R_z - \lambda_k R \zeta_k \right. \]
\[ + \epsilon \left( \Theta_{Gk} + \phi_k - \beta_k \right] + \epsilon \left( \Theta_{Gk} + \phi_k \right) \]
\[ - (R-e) \epsilon \lambda_k R \]
\[ - \cos\psi_k \left[ - \frac{(R-e)^3}{3} \Theta_y \right. \]
\[ + \frac{(R-e)^2}{2} \left[ \mu R \left( \beta_p + \beta_k \right) + \Theta_{Gk} \left( -\mu R \zeta_k - R_y - \mu R \theta_z \right) \right. \]
\[ - \mu R \zeta_k \phi_k + \zeta_k \mu R \left( \beta_p + \beta_k \right) \]
\[ + \Theta_{Gk} \left( - \mu R \zeta_k - R_y - \mu R \theta_z \right) - \mu R \zeta_k \left( \phi_k - \beta_k \right) \]
\[ + (R-e) \left[ \epsilon \mu R \left( \beta_p + \beta_k \right) + \lambda_k R \mu R \zeta_k \right] \]
\[ - \sin\psi_k \left[ \frac{(R-e)^3}{3} \Theta_x \right. \]
\[ + \frac{(R-e)^2}{2} \left[ - \mu R \left( \Theta_{Gk} + \phi_k \right) + \zeta_k \Theta_{Gk} - \mu R \left( \theta_{Gk} + \phi_k \right) \right. \]
\[ - \mu R \theta_{Gk} - \mu R \left( \Theta_{Gk} + \phi_k - \beta_k \right] + \zeta_k \Theta_{Gk} \right. \]
\[ + (R-e) \left[ - \epsilon \mu R \left( \Theta_{Gk} + \phi_k \right) + \lambda_k R \mu R \right. \]
\[ - \mu R \left( \Theta_{Gk} + \phi_k \right) + \mu R \theta_y - R_z \right. \]
\[ - \mu R \left( \Theta_{Gk} + \phi_k \right) \]
The aerodynamic moments at the blade root are

\[
\begin{align*}
Q_{A_{x3k}} &= \int_0^{R-e} \left( q_{A_{x3k}} + x_k \zeta_k v_{A_{y3k}} - x_k \beta_k v_{A_{y3k}} \right) \, dx_k \\
&= \rho_a \alpha b \zeta_k \left\{ \frac{(R-e)^4}{4} \left[ \theta_k + \phi_k - \beta_k + \zeta_k (\theta_k + \phi_k) - \tau_k (\beta_p + \beta_k) \right. \right. \\
&\quad \left. + \zeta_k \theta_k + \phi_k - \beta_k > + \zeta_k \theta_k \right] \\
&\quad + \frac{(R-e)^3}{3} \left[ - \lambda_k R + e (\theta_k + \phi_k) + \mu R \theta_y - \lambda_k R \zeta_k \right. \\
&\quad \left. + e (\theta_k + \phi_k - \beta_k) + b (\theta_k + \phi_k) \right] \\
&\quad + \frac{(R-e)^2}{2} e(-\lambda_k R) \\
&\quad - \cos\psi_k \left[ - \frac{(R-e)^4}{4} \theta_y \right. \\
&\quad + \frac{(R-e)^3}{3} \left[ \mu R (\beta_p + \beta_k) + \theta_k (\mu R \tau_k - \lambda_k R) - \mu R \theta_z \right] \right. \\
&\quad \left. \right) \\
\end{align*}
\]

(5.97)
\[-\mu \tau \dot{\phi}_k + \zeta_k \dot{\mu} R \left( \beta_p + \beta_k \right) \]
\[+ \theta G_k \left( -\mu \tau \dot{\phi}_k - R_y - \mu \dot{R} \theta_\zeta \right) \]
\[+ \left( \dot{\phi}_k - \beta_k \right) \left( -\mu \tau \dot{\phi}_k \right) \]
\[+ \frac{(R-e)^2}{2} \left[ \left( \dot{\mu} R \left( \beta_p + \beta_k \right) + \lambda_k R \mu \tau \dot{\phi}_k \right) \right] \]
\[- \sin \psi_k \left( \frac{(R-e)^4}{4} \right) \dot{\phi}_k \]
\[+ \frac{(R-e)^3}{3} \left[ -\mu R \left( \theta G_k + \phi_k \right) + \dot{R} \dot{\theta} G_k - \mu R \left( \theta G_k + \phi_k \right) \right] \]
\[- \mu \dot{R} \dot{\theta} G_k - \mu R \left( \theta G_k + \phi_k - \beta_k \right) + \dot{R} \dot{\theta} G_k \]
\[- \mu \ddot{R} < \dot{\theta} G_k \left( \theta G_k + \phi_k \right) = \dot{\theta} G_k \left( \beta_p + \beta_k \right) + \theta G_k \theta G_k > \]
\[+ \frac{(R-e)^2}{2} \left[ e < -\mu R \left( \theta G_k + \phi_k \right) > + \lambda_k R \mu R \right] \]
\[- \mu R < e \left( \theta G_k + \phi_k \right) + \mu R \theta G_k \right] \]
\[+ \cos \psi_k \sin \psi_k \left( \frac{(R-e)^3}{3} \right) \mu \dot{R} \theta \]
\[+ \frac{(R-e)^2}{2} \left[ -\mu R \left( \beta_p + \beta_k \right) + \theta G_k \left( -\mu \tau \dot{\phi}_k - R_y - \mu \dot{R} \theta_\zeta \right) \right] \]
\[- \mu \tau \phi_k > \]
\[+ \mu \tau \dot{\phi}_k < \mu R \left( \theta G_k + \phi_k \right) > \]
\[- \mu \dot{R} \theta G_k < -\mu \dot{R} \dot{\theta}_z - R_y > \right] \]
\[+ \cos^2 \psi \left[ -\frac{(R-e)^2}{2} \mu R \tau \dot{\phi}_k \left( \beta_p + \beta_k \right) \right] \]
\[+ \sin^2 \psi \left[ -\frac{(R-e)^3}{3} \mu R \theta \right] \]
\[ + \frac{(R-e)^2}{2} \left[ -\mu R - \mu R \left( \theta_{Gk} + \phi_k \right) + \dot{R}_x \theta_{Gk} > - \mu R \theta_{Gk} \dot{R}_x \right] \]

\[ + \frac{1}{2} \rho_A a b^2 \omega^2 \tau_k \left[ - \frac{(R-e)^3}{3} \dot{\beta}_k - \left( \theta_{Gk} + \phi_k \right) \right] \]

\[ - \cos \psi_k \left[ \frac{(R-e)^2}{2} \left[ \mu R \beta_k - \mu R \left( \theta_{Gk} + \phi_k \right) \right] > \right] \]

\[ - \sin \psi_k \left[ \frac{(R-e)^2}{2} \left[ - \mu R \left( \beta_p + \beta_k \right) - \mu R \left( \theta_{Gk} + \phi_k \right) \right] > \right] \cos \theta_{Gk} \]

\[ \rho_A a b \omega^2 \beta_k (1 + \zeta_k^2) \left\{ - \frac{c_{do}}{a} \left[ \frac{(R-e)^4}{4} \left[ 1 + 2 \zeta_k + \theta_{Gk} + \phi_k \right] \right] \right. \]

\[ \left. - \frac{(R-e)^4}{4} \left[ \beta_k < \theta_{Gk} + \phi_k - \beta_k + \zeta_k \left( \theta_{Gk} + \phi_k \right) \right] \right. \]

\[ - \zeta_k \left( \beta_p + \beta_k \right) + \theta_{Gk} \]

\[ + \zeta_k \left( \beta_p + \beta_k \right) \left[ \theta_{Gk} + \phi_k - \beta_k > \right] \]

\[ \left. - \frac{(R-e)^3}{3} \left[ \beta_k < - \lambda_k R + e \left( \theta_{Gk} + \phi_k \right) + \mu R \theta_{Gk} \right] > \right] \]

\[ + \lambda_k R \left( \theta_{Gk} + \phi_k - \beta_k + \zeta_k \left( \theta_{Gk} + \phi_k \right) - \zeta_k \left( \beta_p + \beta_k \right) \right] \]

\[ + \theta_{Gk} > \]

\[ + (-\mu R \theta_{Gk} + \phi_k) \left[ \theta_{Gk} + \phi_k - \beta_k + \zeta_k \theta_{Gk} > \right] \]

\[ + \zeta_k \left( \beta_p + \beta_k \right) \left[ - \lambda_k R \right] + \beta_k b \left( \theta_{Gk} + \phi_k \right) \]

\[ + \frac{(R-e)^2}{2} \left[ - \lambda_k - \lambda_k R + e \left( \theta_{Gk} + \phi_k \right) + \mu R \theta_{Gk} \right] > \]

\[ + (\mu R \theta_{Gk} + \phi_k) \left[ - \lambda_k R + e \theta_{Gk} + \mu R \theta_{Gk} \right] > \]

\[ + \mu R \theta_{Gk} \left[ \beta_k + \theta_{Gk} \right] \]

\[ - \cos \psi_k \left[ \frac{2c_{do}}{a} \left[ \frac{(R-e)^3}{3} \left[ \mu R \zeta_k + \mu R \theta_{Gk} + \theta_{Gk} \right] \right] \right. \]

\[ - \cos \psi_k \left[ \frac{2c_{do}}{a} \left[ \frac{(R-e)^3}{3} \left[ \mu R \zeta_k + \mu R \theta_{Gk} + \theta_{Gk} \right] \right] \right. \]

\[ 68 \]
\[-\frac{(R-e)^4}{4} \left[ -\dot{\beta}_k y - \zeta_k \dot{\theta}_k \dot{\theta}_Gk + \dot{\theta}_Gk + \dot{\phi}_k - \dot{\beta}_k \right] + \frac{(R-e)^3}{3} \left[ \dot{\beta}_k - \mu R (\beta_p + \beta_k) + \theta_Gk (-\mu \zeta_k^2 - \mu \theta_k R - \dot{R}_y) \right. \]

\[+\left(\lambda_k R - \mu R \theta_y + \dot{R}_z \right) (-\dot{\theta}_y) + \mu \zeta_k (\beta_p + \beta_k) - \mu R \theta_y \]

\[-\dot{\theta}_Gk \left( -\dot{e}_y + (\beta_p + \beta_k) (\mu R - \dot{R}_x) \right) \]

\[+ (-\dot{\phi}_k + \dot{\beta}_k) \mu R (\beta_p + \beta_k) - \lambda_k R \theta_y \]

\[-\zeta_k (\dot{\theta}_Gk + \dot{\phi}_k) \mu R (\beta_p + \beta_k) + \theta_y < e \dot{\theta}_Gk + \mu \theta_k \theta_Gk \]

\[+ \left( \dot{\theta}_Gk - \zeta_k (\beta_p + \beta_k) \right) < - \mu R (\beta_p + \beta_k) > \]

\[+ \frac{(R-e)^2}{2} \left[ -\lambda_k R (\mu R (\beta_p + \beta_k) + \theta_Gk (-\mu \zeta_k^2 - \mu \theta_k R - \dot{R}_y) - \mu \zeta_k \theta_Gk) \right. \]

\[+ \left( \mu R \theta_y - \dot{R}_z \right) < \mu R (\beta_p + \beta_k) - \mu \zeta_k \theta_Gk > \]

\[-\dot{\lambda}_k \mu R (\beta_p + \beta_k) \]

\[+ \left[ e (\theta_Gk + \dot{\phi}_k) + \mu R \theta_y - \dot{R}_z \right] \mu R (\beta_p + \beta_k) \]

\[+ \mu R (\beta_p + \beta_k) b (\theta_Gk + \dot{\phi}_k) \]

\[-\sin \psi_k \left[ \frac{2c_d}{a} \right] \left[ \frac{(R-e)^3}{3} < \mu R (1 + \dot{\gamma}_k + \dot{\theta}_z) - \dot{R}_x > + \frac{(R-e)^2}{2} \right. \]

\[-\mu R \right] \]

\[-\frac{(R-e)^4}{4} \left[ \dot{\beta}_k e \right. \theta_Gk \theta_y - \theta_k \dot{\theta}_Gk + \dot{\phi}_k + \dot{\beta}_k \left] + \dot{\phi}_k \right. \theta_Gk \theta_x \]

\[-\frac{(R-e)^3}{3} \left[ \dot{\beta}_k < - \mu R (\theta_Gk + \dot{\phi}_k) + \theta_Gk R \theta_x \right. \]

\[+ \dot{\theta}_x (\lambda_k R - \mu R \theta_y + \dot{R}_z) \left. + \dot{\gamma}_k (\beta_p + \beta_k) < - \mu R (\theta_Gk + \dot{\phi}_k) > \]

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\[- \theta_G < e_{x} + (\beta_p + \beta_k)(-\mu R \theta_x - R_y) + \lambda_k \theta_x \]
\[- \theta_x (e_{\theta_G} + \mu \theta_y - R_z) \]
\[- \frac{(R-e)^2}{2} \left[ - \lambda_k R < - \mu (\theta_{Gk} + \phi_k) + \theta_{Gk} R_x \right. \]
\[+ (\mu \theta_y - R_z) < - \mu R(\theta_{Gk} + \phi_k) \left. \right] \]
\[+ \cos \psi_k \sin \psi_k \left[ - \frac{2c_{do}}{a} \frac{(R-e)^2}{2} \left[ \mu R < \mu R_z + \mu R_z + R_y \right] \right. \]
\[- \frac{(R-e)^4}{4} \theta_k \theta \theta \]
\[- \frac{(R-e)^3}{3} \left[ - 2 \theta _x < \mu R(\beta_p + \beta_k) - \mu R \theta_k \theta _{Gk} \right. \]
\[+ \theta_y < - \mu R(\theta_{Gk} + \phi_k) \left. \right] \]
\[+ \frac{(R-e)^2}{2} \left[ \mu R \theta_k \theta _{Gk} \theta_x + \mu R \theta_k \theta _{Gk} \theta_y \right. \]
\[+ 2 \mu R \theta_k \theta_x (\beta_p + \beta_k) \left. \right] \]
\[+ \cos^2 \psi_k \left[ \frac{(R-e)^4}{4} \theta_k \theta \right. \]
\[- \frac{(R-e)^3}{3} \left[ 2 \theta _y (\beta_p + \beta_k) \mu R - \mu R \theta_k \theta_{Gk} \theta _y \right. \]
\[+ \frac{(R-e)^2}{2} \left[ \mu^2 R^2 (\beta_p + \beta_k)^2 \right. \]
\[+ \mu R(\beta_p + \beta_k) < - \mu R \theta_k (\theta_{Gk} + \phi_k) + \theta_{Gk} (-R_y - R \theta_z) \left. \right] \]
\[+ \sin^2 \psi_k \left[ \frac{(R-e)^4}{4} \right. \theta_x \theta \right. \]
\[+ \frac{(R-e)^3}{3} \left[ - \mu \theta_{Gk} (\theta_x + \zeta_k \theta_y) - \mu \theta_k \theta_x \right. \]
\[+ \frac{(R-e)^2}{2} \left[ - \mu \theta_{Gk} < e_{\theta_x} + (\beta_p + \beta_k)(-\mu \theta_z - R_y) \right. \]
\[- \frac{c_{do}}{a} < \mu^2 R^2 - 2 \mu R \theta_k \theta_x \left. \right] \]}
\[ + \frac{1}{2} \rho_A a b \Omega^2 x_A \{ \frac{(R-e)^3}{3} [ \theta_{Gk} + \phi_{Gk} - \beta_k + \zeta_k (\theta_{Gk} + \phi_{Gk}) - \zeta_k (\beta_p + \beta_k) \\
+ \theta_{Gk} + \zeta_k (\theta_{Gk} + \phi_{Gk}) - \beta_k + \zeta_k (\beta_p + \beta_k)] \\
+ \frac{(R-e)^2}{2} [- \frac{(R-e)^3}{3} \theta_{Gk} + \phi_{Gk} - \beta_k + \zeta_k (\theta_{Gk} + \phi_{Gk}) - \beta_k + \zeta_k (\beta_p + \beta_k)] \\
+ \frac{(R-e)^2}{2} (\theta_{Gk} + \phi_{Gk} - \beta_k + \zeta_k (\theta_{Gk} + \phi_{Gk}) - \beta_k + \zeta_k (\beta_p + \beta_k))] \\
\]
\[ + (\phi_k - \beta_k) \left( -\mu R^2 \theta_k \right) \]

\[ + (R-e) \left[ e \mu R \left( \beta_p + \beta_k \right) + \lambda_k R \mu R \theta \right] \]

\[ - \sin \psi_k \left[ \frac{(R-e)^3}{3} \right] \]

\[ + \frac{(R-e)^2}{2} \left[ -\mu R \left( \theta G_k + \phi_k \right) + R \theta G_k - \mu R \left( \theta G_k + \phi_k \right) \theta \right] \]

\[ - \mu R \theta G_k - \mu R \left( \theta G_k + \phi_k \right) \theta \theta G_k \]

\[ - \mu R \left( \theta G_k + \phi_k \right) - \theta \left( \theta G_k + \phi_k \right) \theta \theta G_k \]

\[ + (R-e) \left[ e \left( -\mu R \left( \theta G_k + \phi_k \right) \right) + \lambda_k R \mu R \right] \]

\[ - \mu R \left( \theta G_k + \phi_k \right) \theta \theta G_k \]

\[ + \cos \psi_k \sin \psi_k \left[ \frac{(R-e)^2}{2} \right] \]

\[ + \mu R \theta G_k \theta \theta G_k \]

\[ + (R-e) \left[ -\mu R \left( \beta_p + \beta_k \right) + \theta G_k \left( -\mu R \right) \left( -\mu R \right) \theta \right] \]

\[ + \mu R^2 \theta G_k \theta \theta G_k \]

\[ + \cos^2 \psi_k \left[ - \frac{(R-e)^2}{2} \right] \]

\[ + \sin^2 \psi_k \left[ - \frac{(R-e)^2}{2} \right] \]

\[ + (R-e) \left[ -\mu R \left( \theta G_k + \phi_k \right) + \theta \theta G_k \theta \theta G_k \right] \]
Neglecting higher order terms

\[ Q_{A_{y3k}} = \int_0^{R-e} \left( q_{A_{y3k}} + x_k \beta_k P_{A_{x3k}} - (x_k + u_k) P_{A_{z3k}} \right) \, dx_k \]

\[ = - \rho_a b^2 \Omega^2 \left( x_k - b \right) \left\{ \begin{array}{l}
(\text{R-e})^2 \frac{\Omega^2}{2} \left( \dot{\phi}_k - (\theta_{G_k} + \phi_{k}) \right) \\
- \cos \psi_k \left( \text{R-e} \right) \left[ \mu R \beta_k - \mu R (\theta_{G_k} + \phi_{k}) \right]
\end{array} \right\} \\
+ \frac{1}{2} \rho a b^2 \Omega^2 \left( x_k - b \right) \left\{ \begin{array}{l}
\frac{\Omega^2}{2} \frac{1}{2} \rho a b^2 \text{(R-e)}
\end{array} \right\}
\]
\[ + \frac{(R-e)^3}{3} \left[ \mu R (\beta_p + \beta_k) + \theta_{Gk} \left( -\mu R \zeta_k \cdot R_y - \mu R \theta_z \right) \right. \\
- \mu R \phi_k + \zeta_k \mu R (\beta_p + \beta_k) \\
+ \theta_{Gk} \left( -\mu R \zeta_k \cdot R_y - \mu R \theta_z \right) \\
- \mu R \zeta_k (\phi_k - \beta_k) \right] \\
+ \frac{(R-e)^2}{2} \left[ e \mu R (\beta_p + \beta_k) + \lambda_k \mu R \zeta_k \right] \\
- \sin \psi_k \left[ \frac{(R-e)}{4} \theta_x \right. \\
+ \frac{(R-e)^3}{3} \left[ -\mu R (\theta_{Gk} + \phi_k) + R_x \theta_{Gk} - \mu R (\theta_{Gk} + \phi_k) \right. \\
- \mu R \theta_z \theta_{Gk} - \mu R (\theta_{Gk} + \phi_k) - \beta_k \right. \\
+ \left. R_x \theta_{Gk} \right] \\
- \mu R \left( \theta_{Gk} + \phi_k \right) - \zeta_k (\beta_p + \beta_k) + \theta_{Gk} \right] \\
+ \frac{(R-e)^2}{2} \left[ e < - \mu R (\theta_{Gk} + \phi_k) > + \lambda_k \mu R \right. \\
- \mu R < e (\theta_{Gk} + \phi_k) + \mu R \theta_y - \mu R \theta_z > \\
- \mu R \theta_y \left( \theta_{Gk} + \phi_k \right) \right] \\
+ \cos \psi_k \left[ \frac{(R-e)^3}{3} \mu R \theta_y \right. \\
+ \frac{(R-e)^2}{2} \left[ -\mu R (\beta_p + \beta_k) + \theta_{Gk} \left( -\mu R \zeta_k \cdot R_y - \mu R \theta_z \right) \right. \\
- \mu R \zeta_k \phi_k \right. \\
+ \mu R \zeta_k \mu R (\theta_{Gk} + \phi_k) - \mu R \theta_{Gk} < - \mu R \theta_z - R_y > \right] \\
+ \cos^2 \psi_k \left[ - \frac{(R-e)^2}{2} \mu R \zeta_k (\beta_p + \beta_k) \right] \\
\]
\begin{align*}
+ \sin^2 \psi_k [ - \frac{(R-e)^3}{3} \mu R^2 x \\
+ \frac{(R-e)^2}{2} [- \mu R - \mu R (\theta_{Gk} + \phi_k) + \mu R \theta_{Gk} + \mu R \phi_k, \phi_k] \\
- \frac{1}{2} \rho_A \omega^2 \omega^2 [ - \frac{(R-e)^3}{3} \beta_k + (\theta_{Gk} + \phi_k) > \\
- \cos \psi_k < \frac{(R-e)^2}{2} [ \mu R \beta_k - \mu R (\theta_{Gk} + \phi_k) > \\
- \sin \psi_k < \frac{(R-e)^2}{2} [- \mu R (\beta_k + \beta_k) - \mu R (\theta_{Gk} + \phi_k)] \} \cos \theta_{Gk}
\end{align*}

(5.99)

\[ Q_{A_{23k}} = \int_0^{R-e} (q_{A_{23k}} + (x_k + u_k) p_{A_{23k}} - x_k \tau_k p_{A_{23k}}) \, dx_k \]

Neglecting higher order terms

\[ Q_{A_{23k}} = \int_0^{R-e} (x_k \beta_k p_{A_{23k}} - x_k \tau_k p_{A_{23k}}) \, dx_k \]

where

\[ p_{A_{23k}} = -\tau_k p_{A_{5k}} - \beta_k p_{A_{23k}} = -\tau_k p_{A_{23k}} - \beta_k p_{A_{23k}} \]

thus

\[ Q_{A_{23k}} = \int_0^{R-e} (x_k \beta_k p_{A_{23k}} + x_k \tau_k p_{A_{23k}}) \, dx_k \]

\[ = \rho_A a b \omega^2 \{ - \frac{c \omega}{a} \omega \omega \omega \cdot \omega [ \frac{(R-e)^4}{3} \beta_k + (\theta_{Gk} + \phi_k) - \tau_k (\beta_k + \beta_k) + \theta_{Gk} - \phi_k, \phi_k] \\
+ \tau_k (\beta_k + \beta_k) < \theta_{Gk} + \phi_k, \phi_k] \} \]

\[ - \frac{(R-e)^3}{3} \beta_k < - \lambda_k R + e (\theta_{Gk} + \phi_k) + \mu R \theta_y - \mu R \phi_z \]

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\[ \begin{align*}
&\quad + \lambda_k R < \theta_{Gk} + \phi_k - \beta_k + \zeta_k (\theta_{Gk} + \phi_k) - \zeta_k (\beta_k + \beta_k) - \theta_k \gamma_{Gk} > \\
&\quad + (\mu R \theta_y + R_z) < \theta_{Gk} + \phi_k - \beta_k + \zeta_k \theta_{Gk} > \\
&\quad + \zeta_k (\beta_k + \beta_k) < -\lambda_k R > + \beta_k b (\theta_{Gk} + \phi_k) \\
&\quad + \frac{(R-e)^2}{2} [- \lambda_k R < - \lambda_k R + e(\theta_{Gk} + \phi_k) + \mu R \theta_y - R_z > \\
&\quad + (\mu R \theta_y - R_z) < - \lambda_k R + e \theta_{Gk} + \mu R \theta_y - R_z > \\
&\quad \quad - \lambda_k R b (\theta_{Gk} + \phi_k)] \\
&\quad - \cos \psi_k \left[ \frac{2c d_o}{a} \left[ \frac{(R-e)^3}{3} \right] < \mu R \gamma_k + \mu R \theta_z + R_y > \right] \\
&\quad - \frac{(R-e)^4}{4} [- \beta_k \theta_y - \zeta_k \theta \theta_{Gk} + \theta y < \theta_{Gk} + \phi_k - \beta_k > + \zeta_k \theta \gamma_{Gk} > ] \\
&\quad - \frac{(R-e)^3}{3} [\beta_k \le \mu R (\beta_k + \beta_k) + \theta_{Gk} (-\mu R \gamma_k - \mu R \theta_z - R_y) - \phi_k \mu R \gamma_k >] \\
&\quad + (\lambda_k R - \mu R \theta_y + R_z) (-\theta_y) + \zeta_k (\beta_k + \beta_k) \mu R (\beta_k + \beta_k) \\
&\quad - \theta_{Gk} < - e \theta_y + (\beta_k + \beta_k) (\mu R - R_k) > \\
&\quad + (\zeta_k (\theta_{Gk} + \phi_k) \mu R (\beta_k + \beta_k) \\
&\quad - \theta \gamma_{Gk} (\theta_{Gk} + \phi_k) + \mu R \gamma_{Gk} > \\
&\quad + (e \theta_{Gk} + \mu R \gamma_{Gk} - R_z) \\
&\quad + (\theta \gamma_{Gk} - \zeta_k (\beta_k + \beta_k)) < - \mu R (\beta_k + \beta_k) > ] \\
&\quad + \frac{(R-e)^2}{2} [- \lambda_k R < \mu R (\beta_k + \beta_k) + \theta_{Gk} (-\mu R \gamma_k - \mu R \theta_z - R_y) \\
&\quad \quad - \mu R \gamma_k \phi_k > \right]
\end{align*} \]
\[ + (\mu R \theta_y - R_z) < \mu R (\beta_p + \beta_k) - \mu R \tau_k \theta_{Gk} > - \lambda_k \mu R (\beta_p + \beta_k) + (e (\theta_{Gk} + \phi_k) + \mu R \theta_y - R_z) \mu R (\beta_p + \beta_k) + \mu R (\beta_p + \beta_k) b (\theta_{Gk} + \phi_k) ] \\
- \sin \psi_k [ \frac{2c_{do}}{a} \left( \frac{(R-e)^3}{3} < \mu R (1 + \tau_k + \theta_z) - R_z > + \frac{(R-e)^2}{2} \right) e \mu R ] \\
- \frac{(R-e)^4}{4} [ \beta_k \theta_y - \theta_{Gk} \tau_k \theta_y - \theta_x < \theta_{Gk} + \phi_k - \beta_k > - \theta_{Gk} \tau_k \theta_x ] \\
- \frac{(R-e)^3}{3} [ \beta_k < - \mu R (\theta_{Gk} + \phi_k) + \theta_{Gk} \theta_x ] \\
+ \theta_x ( \lambda_k R - \mu R \theta_y + R_z ) + \tau_k (\beta_p + \beta_k) < - \mu R (\theta_{Gk} + \phi_k) > \\
- \theta_{Gk} < e \theta_x + (\beta_p + \beta_k)(-\mu R \theta_y - R_z ) > + \lambda_k \theta_y x \\
- \theta_x (e \theta_{Gk} + \mu R \theta_y - R_z ) ] \\
+ \frac{(R-e)^2}{2} [ - \lambda_k R < - \mu R (\theta_{Gk} + \phi_k) + \theta_{Gk} \theta_x > + (\mu R \theta_y - R_z ) < - \mu R (\theta_{Gk} + \phi_k) > ] ] \\
+ \cos \psi_k \sin \psi_k [ - \frac{2c_{do}}{a} \left( \frac{(R-e)^2}{2} \right) \mu R < \mu R \tau_k + \mu R \theta_y + R_z > ] \\
- \frac{(R-e)^4}{4} \left[ 2 \theta_y \theta_x \right. \\
- \frac{(R-e)^3}{3} [ - 2 \theta_x < \mu R (\beta_p + \beta_k) - \mu R \tau_k \theta_{Gk} > + \theta_y < - \mu R (\theta_{Gk} + \phi_k) > ] \\
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\[+ \frac{(R-e)^2}{2} \left[ \mu R \theta_{Gk} \varepsilon^2 y - \mu^2 R^2 (\beta_R + \beta_p) (\theta_R + \phi_{Gk}) \right.
- 2\mu R \theta_{Gk} \frac{R_x}{R_y} (\beta_R + \beta_p) \]
\[+ \cos^2 \psi_k \left\{ \frac{(R-e)^4}{4} \varepsilon^2 y - \frac{(R-e)^3}{3} \left[ 2\theta_R (\beta_R + \beta_p) \mu R + \mu R \varepsilon_{Gk} \theta_R \theta_{Gk} \right] \right. \]
\[+ \frac{(R-e)^2}{2} \left[ \mu^2 R^2 (\beta_R + \beta_p)^2 \right.
+ \mu R (\beta_R + \beta_p) < - \mu R \varepsilon_{Gk} (\theta_R + \phi_{Gk}) + \theta_{Gk} \right. \]
\[+ \sin^2 \psi_k \left\{ \frac{(R-e)^4}{4} \varepsilon^2 x + \frac{(R-e)^3}{3} \left[ -\mu R \theta_{Gk} (\theta_R + \theta_{Gk}) - \mu R \phi_{Gk} \phi_{Gk} \right] \right. \]
\[+ \frac{(R-e)^2}{2} \left[ -\mu R \theta_{Gk} < e^2 x + (\beta_R + \beta_p) (-\mu R \theta_R - R_y) \right. \]
\[- \frac{c a}{a} < \mu^2 R^2 \right. \]
\[- \frac{1}{2} \rho A a b^2 h^2 \left\{ - \frac{(R-e)^3}{3} \left[ < \beta_{Gk} - (\theta_R + \phi_{Gk}) \right. \right. \]
\[- (\sin \theta_{Gk} + \phi_{Gk} \cos \theta_{Gk}) \right. \]
\[- - \varepsilon_{Gk} \sin \theta_{Gk} \left. \right. \]
\[+ \frac{(R-e)^2}{2} \sin \theta_{Gk} \left. \right. \]
\[- \cos \psi_k \left[ - \frac{(R-e)^3}{3} \left( \theta_R - \theta_{Gk} \right) \sin \theta_{Gk} \right. \]
\[+ \frac{(R-e)^2}{2} \left[ (\mu R \theta_{Gk} - \mu R (\theta_{Gk} + \phi_{Gk})) (\sin \theta_{Gk} + \phi_{Gk} \cos \theta_{Gk}) \right. \]
\[- - \mu R \varepsilon_{Gk} \theta_{Gk} \sin \theta_{Gk} \]
\[- \sin \psi_k \left[ \frac{(R-e)^3}{3} \left( \theta_R + \theta_{Gk} \right) \sin \theta_{Gk} \right. \]
\[+ \frac{(R-e)^2}{2} \left[ - (\mu R (\beta_R + \beta_{Gk}) + \mu R (\theta_{Gk} + \phi_{Gk})) \right. \]
\[\left. \right. \]
\[- \left. \right. \right. \]
The orders of magnitude of the leading terms in the various aerodynamic loads are given for convenience

\[ P_{A_{x3k}} = 0(\varepsilon^{3/2}) \]
\[ P_{A_{y3k}} = 0(\varepsilon^{3/2}) \]
\[ P_{A_{z3k}} = 0(\varepsilon^{1/2}) \]
\[ Q_{A_{x3k}} = 0(\varepsilon^{3/2}) \]
\[ Q_{A_{y3k}} = 0(\varepsilon^{1/2}) \]
\[ Q_{A_{z3k}} = 0(\varepsilon^{3/2}) \]

**Damping Force**

Instead of assuming a distributed damping force representing the structural damping of the blade, one can assume a damping force proportional to the velocity and it is provided at the root of the blade. It can be written as
Flap: \[ Q_{Dy3k} = + \Omega \beta_k g_{SF} \] (5.101)

Lead-lag: \[ Q_{Dz3k} = - \Omega \zeta_k g_{SL} \] (5.102)

Torsion: \[ Q_{Dx3k} = - \Omega \phi_k g_{ST} \] (5.103)

These damping forces are proportional to the rate of change in angles in flap, lead-lag and torsion. Reason for this type of assumption is that in the present study the blade is modeled as a rigid blade with root springs.

**Equations of Motion**

Using the moment equilibrium condition at the blade root, the equations of motion for the kth blade can be written as follows.

**Flap:**

The flap equation is

\[
M_{B_k} + Q_{Iy3k} + Q_{Ay3k} + Q_{Dy3k} = 0
\] (5.104)

**Lead-lag:**

The lead-lag equation is

\[
M_{\zeta_k} + Q_{Iz3k} + Q_{Az3k} + Q_{Dz3k} = 0
\] (5.105)

**Torsion:**

The torsion equation is

\[
M_{\phi_k} + Q_{Ix3k} + Q_{Ax3k} + Q_{Dx3k} = 0
\] (5.106)

The elastic restoring moments due to the root springs are given in Appendix. The results are summarized below.

\[
M_{\beta_k} = \frac{(\beta_k - \phi_k \zeta_k)}{\Delta} \{ K_\beta + R_1 (K_{\zeta} - K_\beta) \sin^2 \theta_{Gk} \}
\]

\[
+ \frac{(\zeta_k + \phi_k \beta_k)}{\Delta} \frac{R_1 (K_{\zeta} - K_\beta) \sin \theta_{Gk} \cos \theta_{Gk}}
\]

\[
M_{\zeta_k} = -\frac{\beta_k + \phi_k \zeta_k}{\Delta} \{ K_{\zeta} - R_1 (K_{\zeta} - K_\beta) \sin^2 \theta_{Gk} \}
\]

\[
- \frac{(\beta_k - \phi_k \zeta_k)}{\Delta} \frac{R_1 (K_{\zeta} - K_\beta) \sin \theta_{Gk} \cos \theta_{Gk}}
\]
\[ M_{\phi_k} = -K_{\phi_1} (\phi_k - \zeta_k \beta_k) - K_{\phi_2} \phi_k \]

where
\[ K_\beta = \frac{K_\beta H K_{\beta_B}}{K_\beta H + K_{\beta_B}} ; \quad K_\zeta = \frac{K_\zeta H K_{\zeta_B}}{K_\zeta H + K_{\zeta_B}} \]
\[ R_1 = \frac{1/K_\beta_B - 1/K_\zeta_B}{(1/K_\beta_B + 1/K_\beta_H) - (1/K_\zeta_B + 1/K_\zeta_H)} \]
\[ \Delta' = 1 + R_1 (1-R_1) \sin^2 \theta_g (K_\zeta - K_\beta)^2 / K_\zeta K_\beta \]

For an articulated blade
\[ K_\beta = K_\zeta = K_{\phi_1} = 0 \quad \text{and} \quad K_{\phi_2} = \frac{K_{\phi_C} K_{\phi_B}}{K_{\phi_C} + K_{\phi_B}} \]

For a hingeless blade
\[ K_{\phi_2} = 0 \quad \text{and} \quad K_{\phi_1} = \frac{K_{\phi_C} K_{\phi_B}}{K_{\phi_C} + K_{\phi_B}} \]
6. EQUATIONS OF MOTION OF THE SUPPORTING STRUCTURE

6.1 General

The supporting structure, to which the envelope and the rotors of the HHLA vehicle modeled in this study (Fig. 2) are attached, is assumed to be flexible. The degrees of freedom associated with the supporting structure consist therefore of both rigid body and flexible degrees of freedom. The structure is idealized as a free-free beam and is represented by relatively few number of bending and torsional free vibration modes. The derivation of the equations of motion consists of two parts.

1) One part representing the rigid body degrees of freedom.
2) A second part representing the flexible modes of the structure.

In dealing with these two contributions to the total motion it is assumed that rigid body motions occur first. After establishing a perturbed state, consisting of rigid body translation and rotation, the elastic deformations are superposed. Thus the equations representing the rigid body degrees of freedom are written in S-system and the equations of motion representing the elastic structure are written in S1-system, see Figs. 8 and 9. To write the equations, various external loads have to be identified and properly transformed to the corresponding coordinate systems.

6.2 Loads

The various external loads acting on the model are illustrated in Fig. 12.

1. The rotor loads at hub center $O_{H1}$ and $O_{H2}$.
2. Aerodynamic loads on the envelope acting at the center of buoyancy $O_C$ and a static buoyant lift.
3. Thruster loads acting at $O_{F1}$ and $O_{F2}$.
4. Gravity loads acting at the respective center of mass of underslung weight, envelope weight, fuselages weight and the supporting structure weight. The respective center of mass locations are denoted by $O_{UN}$, $O_{EN}$, $O_{F1}$, $O_{F2}$, $O_S$.

6.2.1 Rotor Loads

The loads acting on the rotor blades are inertia and the aerodynamic loads. These distributed loads are transformed into forces and moments acting at the blade root. The moments are balanced by root springs and dampers. The loads
on the kth blade are derived in Chapter 5 and are given in 3k coordinate system. They are given in equations (5.89)-(5.100). Net combined loads acting at the root of the blade are

\[ \bar{p}_{3k} = \bar{p}_{13k} + \bar{p}_{A3k} \]  (6.1)

\[ \bar{q}_{3k} = \bar{q}_{13k} + \bar{q}_{A3k} + \bar{q}_{D3k} \]  (6.2)

These loads are transformed into loads acting parallel to 2k system, using the following transformation.

\[ \bar{p}_{2k3} = [T_{23}] \bar{p}_{3k} \]  (6.3)

\[ \bar{q}_{2k3} = [T_{23}] \bar{q}_{3k} \]  (6.4)

where \( \bar{p}_{2k3} \) and \( \bar{q}_{2k3} \) are forces and moments acting at the blade root (i.e. origin of the 3k system) whose components are given in 2k system. Transforming these loads into loads acting at the origin of the 2k system (i.e. hub center, either \( O_{H1} \) or \( O_{H2} \))

\[ \bar{p}_{2k} = \bar{p}_{2k3} \]  (6.5)

\[ \bar{q}_{2k} = \bar{q}_{2k3} + (e ^{x2k} \times \bar{p}_{2k3}) \]  (6.6)

\[ \bar{p}_{2k} = (P_{x3k} - \beta_{p} P_{z3k}) \hat{e}_{x2k} + P_{y3k} \hat{e}_{y2k} + (P_{z3k} + \beta_{p} P_{x3k}) \hat{e}_{z2k} \]  (6.7)

\[ \bar{q}_{2k} = (Q_{x3k} - \beta_{p} Q_{z3k}) \hat{e}_{x2k} + Q_{y3k} \hat{e}_{y2k} + (Q_{z3k} + \beta_{p} Q_{x3k}) \hat{e}_{z2k} \]  (6.8)

\[ + [-e (P_{z3k} + \beta_{p} P_{x3k}) \hat{e}_{y2k} + e P_{y3k} \hat{e}_{z2k}] \]

Again transforming these loads to 1 system

\[ \bar{p}_{1k} = [T_{12}] \bar{p}_{2k} \]  (6.9)

\[ \bar{q}_{1k} = [T_{12}] \bar{q}_{2k} \]  (6.10)
Expanding

\[ \tilde{P}_{1k} = [\cos \psi_k \begin{pmatrix} P_{x3k} \ - \beta_p P_{z3k} \ - \sin \psi_k P_{y3k} \end{pmatrix}] \hat{e}_x \]

\[ + [\sin \psi_k \begin{pmatrix} P_{x3k} \ - \beta_p P_{z3k} \ + \cos \psi_k P_{y3k} \end{pmatrix}] \hat{e}_y \]

\[ + [P_{z3k} + \beta_p P_{x3k}] \hat{e}_z \]

(6.11)

\[ \tilde{Q}_{1k} = \hat{e}_x \begin{pmatrix} \cos \psi_k \begin{pmatrix} Q_{x3k} \ - \beta_p Q_{z3k} \ - \sin \psi_k Q_{y3k} \end{pmatrix} \end{pmatrix} \]

\[ + \hat{e}_y \begin{pmatrix} \sin \psi_k \begin{pmatrix} Q_{x3k} \ - \beta_p Q_{z3k} \ + \cos \psi_k Q_{y3k} \end{pmatrix} \end{pmatrix} \]

\[ + \hat{e}_z \begin{pmatrix} Q_{z3k} + \beta_p Q_{x3k} + e P_{y3k} \end{pmatrix} \]

(6.12)

These loads are acting origin of the 1 system i.e. either at \( O_{H1} \) or \( O_{H2} \) depending on the rotor system. Total load due to all the blades in the two rotor systems are

**Rotor System 1**

\[ \tilde{P}_{1H1} = \sum_{k=1}^{N} \tilde{P}_{1k} \]

(6.13)

\[ \tilde{Q}_{1H1} = \sum_{k=1}^{N} \tilde{Q}_{1k} \]

(6.14)

**Rotor System 2**

\[ \tilde{P}_{1H2} = \sum_{k=1}^{N} \tilde{P}_{1k} \]

(6.15)

\[ \tilde{Q}_{1H2} = \sum_{k=1}^{N} \tilde{Q}_{1k} \]

(6.16)

The summation is over the number of blades in the individual rotor systems. As it was pointed out earlier that \( \tilde{P}_{1k} \) and \( \tilde{Q}_{1k} \) will be different for different blades and also for different rotor systems.

Transforming the loads acting at \( O_{H1} \) to the point \( O_{F1} \) (C.G. of fuselage F1) and that at \( O_{H2} \) to the point \( O_{F2} \) (C.G. of fuselage F2),

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\[
\bar{F}_{1F1} = \bar{F}_{1H1} \\
\bar{Q}_{1F1} = \bar{Q}_{1H1} + (h_2 \hat{e}_{zsl} \times \bar{F}_{1H1})
\]

\[
= \bar{Q}_{1H1} + [-h_2 P_{1H1} \hat{e}_{xsl} + h_2 P_{x1H1} \hat{e}_{ysl}] \\
\]

(6.18)

where \( P_{1H1} \) is the y-component of the vector \( \bar{F}_{1H1} \) and \( P_{x1H1} \) is the x-component of the vector \( \bar{F}_{1H1} \). It should be noted that according to our initial assumption the 1 system and the S1 system are assumed to be parallel. Similarly for the second rotor system

\[
\bar{F}_{1F2} = \bar{F}_{1H2} \\
\bar{Q}_{1F2} = \bar{Q}_{1H2} + (h_2 \hat{e}_{zsl} \times \bar{F}_{1H2})
\]

\[
= \bar{Q}_{1H2} + [-h_2 P_{1H2} \hat{e}_{xsl} + h_2 P_{x1H2} \hat{e}_{ysl}] \\
\]

(6.20)

The loads presented in equations (6.17)-(6.20) are acting at the points \( O_{F1} \) and \( O_{F2} \). The components of these loads are given in 1 system (S1 system), both the 1 system and S1 system are parallel systems. The components of these loads will be used in writing the equations of motion representing the elastic modes of the supporting structure. Writing these loads in terms of loads in the 3k system

\[
\bar{F}_{1F1} = \sum_{k=1}^{N} \{ \hat{e}_{x1} [\cos \psi_k < P_{x3k} - \beta_p P_{z3k}]^2 - \sin \psi_k < P_{y3k}] \\
+ \hat{e}_{y1} [\sin \psi_k < P_{x3k} - \beta_p P_{z3k}]^2 + \cos \psi_k < P_{y3k}] \\
+ \hat{e}_{z1} [P_{z3k} + \beta_p P_{x3k}] \}
\]

(6.21)

\[
\bar{Q}_{1F1} = \sum_{k=1}^{N} \{ \hat{e}_{x1} [\cos \psi_k < Q_{x3k} - \beta_q Q_{z3k}]^2 - \sin \psi_k < Q_{y3k}] \\
- e (P_{z3k} + \beta_p P_{x3k}) > \\
- h_2 (\sin \psi_k < P_{x3k} - \beta_p P_{z3k}]^2 + \cos \psi_k < P_{y3k}] \}
\]

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The expressions $\bar{P}_{I1}$ and $\bar{Q}_{I1}$ represent the forces and moments due to rotor systems $R_1$. These loads act at the point $O_{F1}$ on the supporting structure. Similarly, the loads due to the rotor system $R_2$ are $\bar{P}_{I2}$ and $\bar{Q}_{I2}$. These loads act at this point $O_{F2}$ on the supporting structure. Since the rotor loads are derived for a general rotor system, the expressions for these loads given in equations (6.21) and (6.22) are valid for both rotor systems $R_1$ and $R_2$. However, the components $P_{x3k}$, $P_{y3k}$, $P_{z3k}$, $Q_{x3k}$, $Q_{y3k}$, $Q_{z3k}$ could be different for the two rotor systems depending on the operating conditions and various other rotor parameters.

The rotor leads $\bar{P}_{I1}$, $\bar{Q}_{I1}$, $\bar{P}_{I2}$, $\bar{Q}_{I2}$ will be used in writing the equations of motion of the vehicle and the supporting structure. Hence, it is convenient to refer these loads in the body fixed $S_I$ coordinate system. From equations (6.21) and (6.22) it can be seen that these rotor loads are given as components along the hub fixed $I$ system. By definition, the hub fixed $I$ system and the body fixed $S_I$ system are parallel. Therefore, in subsequent parts of this report these rotor loads will be referred as $\bar{P}_{S_{I1}I}$ for $\bar{P}_{I1}$, $\bar{Q}_{S_{I1}I}$ for $\bar{Q}_{I1}$, $\bar{P}_{S_{I2}I}$ for $\bar{P}_{I2}$ and $\bar{Q}_{S_{I2}I}$ for $\bar{Q}_{I2}$. Again, these loads are transformed into components along the $S$ system acting at point $O_S$ on the structure. Finally, the individual rotor loads are added together to get the total rotor loads.

The total rotor loads at $O_S$ are (components along $S$ system)

$$\bar{P}_{SR} = [T_{S,S_I}] \bar{P}_{S_{I1}I} + [T_{S,S_I}] \bar{P}_{S_{I2}I}$$  (6.23)
\[
\tilde{\delta}_{SR} = [T_{S,S1} \tilde{\delta}_{S1F1} - \hat{\lambda}_{F1} \hat{e}_{xS1} \times \hat{F}_{S1F1}]
\]
\[
+ [T_{S,S1} \tilde{\delta}_{S1F2} + \hat{\lambda}_{F2} \hat{e}_{xS1} \times \hat{F}_{S1F2}] \quad (6.24)
\]

where \([T_{S,S1}]\) is the transformation matrix relating the body fixed noninertial S1 system to inertial S system.

\[
[T_{S,S1}] = \begin{bmatrix}
1 & \theta_x & \theta_y & \theta_z \\
\theta_x & 1 & \theta_{xy} & \theta_{xyz} \\
\theta_y & \theta_{xy} & 1 & \theta_{xyz} \\
\theta_z & \theta_{xyz} & \theta_{xyz} & 1 \\
\end{bmatrix}
\]

where \(\theta_x, \theta_y, \theta_z\) represent the Euler angles for rigid body rotation.

6.2.2 Aerodynamic Loads Due to the Envelope

The aerodynamic loads of the envelope are the buoyancy loads acting along the body axes. These are denoted respectively by \(\tilde{F}_{EN}\) and \(\tilde{Q}_{EN}\).

\[
\tilde{F}_{EN} = P_x \hat{e}_{xS1} + P_y \hat{e}_{yS1} + P_z \hat{e}_{zS1} + P^S \hat{e}_{zs}
\]
\[
= (P_x - \theta_y P_z) \hat{e}_{xS1} + (P_y + \theta_x P_z) \hat{e}_{yS1} + (P_z + P^S) \hat{e}_{zs} 
\]

\[
\tilde{Q}_{EN} = Q_x \hat{e}_{xS1} + Q_y \hat{e}_{yS1} + Q_z \hat{e}_{zS1} 
\]

These loads act at the center of buoyancy \(O_c\). The components of the dynamic loads are defined as [Ref. 7]

Forces \(P_x = C_x q V^{2/3}\), \(P_y = C_y q V^{2/3}\), \(P_z = C_z q V^{2/3}\) \(\quad (6.27)\)

Moments \(Q_x = C_n q V\), \(Q_y = C_m q V\), \(Q_z = C_n q V\) \(\quad (6.28)\)

Where \(V\) is the volume of the envelope
q is the dynamic pressure

\(P^s\) is the static lift on the envelope acting along \(Z_s\) axis

and \(c_x, y, z, c, m, n\) are coefficients

Transforming these loads to the point \(O_s\) on the supporting structure

\[
\bar{\mathbf{P}}_{s1eN} = \bar{\mathbf{P}}_{EN} \tag{6.29}
\]

\[
\bar{Q}_{s1eN} = \bar{Q}_{EN} + (h_3 \hat{e}_{zsl} \times \bar{\mathbf{P}}_{EN})
\]

\[
= \bar{Q}_{EN} + [-h_3 (P_y + \theta P^s_{x z}) \hat{e}_{xsl} + h_3 (P_x - \theta P^s_{y z}) \hat{e}_{ysl}] \tag{6.30}
\]

Expanding

\[
\bar{\mathbf{P}}_{s1eN} = (c_x q V^{2/3} - \theta P^s_{y z}) \hat{e}_{xsl} + (c_y q V^{2/3} + \theta P^s_{x z}) \hat{e}_{ysl}
\]

\[
+ (c_z q V^{2/3} + P^s_z) \hat{e}_{zsl} \tag{6.31}
\]

\[
\bar{Q}_{s1eN} = (c \hat{q} V - h_3 c_y q V^{2/3} - h_3 \theta P^s_{x z}) \hat{e}_{xsl}
\]

\[
+ (c_m q V + h_3 c_x q V^{2/3} - \theta P^s_{y z}) \hat{e}_{ysl}
\]

\[
+ c_n q V \hat{e}_{zsl} \tag{6.32}
\]

Writing these loads, along the \(S\) system, acting at \(O_s\)

\[
\bar{\mathbf{P}}_{SEN} = [T_{s}, S_1] \bar{\mathbf{P}}_{s1eN} \tag{6.33}
\]

\[
\bar{Q}_{SEN} = [T_{s}, S_1] \bar{Q}_{s1eN} \tag{6.34}
\]

6.2.3 Thruster Loads

Two thrusters simulated by thruster forces are assumed to act at the C.G. of the fuselages F1 and F2 i.e. \(O_{F1}\) and \(O_{F2}\). They are

\[
\bar{\mathbf{P}}_{TF1} = - \bar{P}_T \hat{e}_{xsl} \quad \text{and} \quad \bar{\mathbf{P}}_{TF2} = - \bar{P}_T \hat{e}_{xsl} \tag{6.35}
\]

Transforming these forces to the C.G. of the structure \(O_s\) and combining them

\[
\bar{\mathbf{P}}_{s1t} = - 2 \bar{P}_T \hat{e}_{xsl} \tag{6.36}
\]
in the S coordinate system

$$\mathbf{P}_{ST} = [T_{S,S1}] \mathbf{P}_{S1T} (6.37)$$

### 6.2.4 Gravity Loads

The gravity loads are due to the various masses which constitute the HHLA model. These loads act along \( \hat{e}_{z_s} \) direction, at the respective centers of mass. The various contributions are:

**Fuselage**

The gravity loads on the two fuselages act at \( O_{F1} \) and \( O_{F2} \). These are

$$\mathbf{P}_{WF1} = -W_{F1} \hat{e}_{z_s} (6.38)$$

$$\mathbf{P}_{WF2} = -W_{F2} \hat{e}_{z_s} (6.39)$$

Transforming these loads to the point \( O_S \) on the structure and adding them

$$\mathbf{P}_{SWF} = -(W_{F1} + W_{F2}) \hat{e}_{z_s} (6.40)$$

$$\mathbf{Q}_{SWF} = -l_{F1} \hat{e}_{x_{s1}} \times \mathbf{P}_{WF1} + l_{F2} \hat{e}_{x_{s1}} \times \mathbf{P}_{WF2}$$

where \( \mathbf{P}_{SWF} \) is the gravity force

\( \mathbf{Q}_{SWF} \) is the moment at \( O_S \) due to the gravity forces.

$$\mathbf{Q}_{SWF} = (-l_{F1} \hat{e}_{x_{s}} - \theta l_{F1} \hat{e}_{y_{s}} + \theta l_{F1} \hat{e}_{z_{s}}) \times (-W_{F1} \hat{e}_{z_{s}})$$

$$+ (l_{F2} \hat{e}_{x_{s}} + \theta l_{F2} \hat{e}_{y_{s}} - \theta l_{F2} \hat{e}_{z_{s}}) \times (-W_{F2} \hat{e}_{z_{s}})$$

$$= (\theta l_{F1} W_{F1} - \theta l_{F2} W_{F2}) \hat{e}_{x_{s}}$$

$$+ (-l_{F1} W_{F1} + l_{F2} W_{F2}) \hat{e}_{y_{s}} (6.41)$$

### Underslung Weight

It is assumed that the underslung mass is rigidly attached to the supporting structure. The gravity force on this mass acts at its C.G. \( O_{UN} \) at a distance \(-h_{1} \hat{e}_{z_{s1}}\) from \( O_S \).
The loads due to this mass at $O_S$ (in $S$ system) are

\[ \bar{P}_{SWUN} = -W_{UN} \hat{e}_{zs} \]  

(6.42)

\[ \bar{Q}_{SWUN} = -h_1 \hat{e}_{zsl} \times (-W_{UN} \hat{e}_{zs}) \]

\[ = (-h_1 (\theta_\theta \theta + \theta_\theta) \hat{e}_{xs} - h_1 (\theta_\theta \theta - \theta_\theta) \hat{e}_{ys} - h_1 \hat{e}_{zs}) \times (-W_{UN} \hat{e}_{zs}) \]

\[ = h_1 (\theta_\theta \theta - \theta_\theta) W_{UN} \hat{e}_{xs} - h_1 (\theta_\theta \theta + \theta_\theta) W_{UN} \hat{e}_{ys} \]  

(6.43)

**Envelope Weight**

The gravity load on the envelope acting at its C.G., $O_{EN}$ is $-W_{EN} \hat{e}_{zs}$. It is located at a distance $h_4 \hat{e}_{zsl}$ from $O_S$. Transforming this load to the point $O_S$ on the structure,

\[ \bar{P}_{SWEN} = -W_{EN} \hat{e}_{zs} \]  

(6.44)

\[ \bar{Q}_{SWEN} = h_4 \hat{e}_{zsl} \times (-W_{EN} \hat{e}_{zs}) \]

\[ = [h_4 (\theta_\theta \theta + \theta_\theta) \hat{e}_{xs} + h_4 (\theta_\theta \theta - \theta_\theta) \hat{e}_{ys} + h_4 \hat{e}_{zs}] \times (-W_{EN} \hat{e}_{zs}) \]

\[ = -h_4 (\theta_\theta \theta - \theta_\theta) W_{EN} \hat{e}_{xs} + h_4 (\theta_\theta \theta + \theta_\theta) W_{EN} \hat{e}_{ys} \]  

(6.45)

**Supporting Structure Weight**

The gravity force on the supporting structure is acting at its C.G., $O_S$, in $-\hat{e}_{zs}$ direction, it is given by

\[ \bar{P}_{SWS} = -W_S \hat{e}_{zs} \]  

(6.46)

\[ \bar{Q}_{SWS} = h_5 \hat{e}_{xsl} \times (-W_S \hat{e}_{zs}) \]

\[ = (h_5 \hat{e}_{xs} + \theta z h_5 \hat{e}_{ys} - \theta y h_5 \hat{e}_{zs}) \times (-W_S \hat{e}_{zs}) \]

\[ = (-\theta z h_5 W_S \hat{e}_{xs} + h_5 W_S \hat{e}_{ys}) \]  

(6.47)
6.3 Rigid Body Equations of Motion

Using the various loads derived in S coordinate system, the rigid body translational and rotational equations of motion can be written in the S system.

Let the rigid body perturbational translational motion of the point \( O \) be

\[
\dot{R}_S = R_{xs} \dot{e}_{xs} + R_{ys} \dot{e}_{ys} + R_{zs} \dot{e}_{zs}
\]  

(6.48)

Then the translational equations of motion become

\[
\frac{W}{g} \ddot{R}_S = \bar{F}_{SR} + \bar{F}_{SEN} + \bar{F}_{ST} + \bar{F}_{SWF} + \bar{F}_{SWUN} + \bar{F}_{SWEN} + \bar{F}_{SWS}
\]

(6.49)

where \( W = W_{EN} + W_{UN} + W_{F1} + W_{F2} + W_S \)

\( W_S \) is the supporting structure weight.

The equations of motion for the rotational degrees of freedom are

\[
\frac{d}{dt} [I \ddot{\omega}] = \bar{Q}_{SR} + \bar{Q}_{SEN} + \bar{Q}_{SWF} + \bar{Q}_{SWUN} + \bar{Q}_{SWEN} + \bar{Q}_{SWS}
\]

(6.50)

where \([I] = [T_{S1}, S]^T[I_S + I_{F1} + I_{F2} + I_{EN} + I_{UN}] [T_{S1}, S]\)

and \( \ddot{\omega} = \Omega (\dot{\theta}_x - \theta \dot{\theta}_x) \dot{e}_{xs} + (\dot{\theta}_y + \theta \dot{\theta}_y) \dot{e}_{ys} + (\dot{\theta}_z - \theta \dot{\theta}_z) \dot{e}_{zs} \)

The individual inertia tensors are given by

Structure:

\[
[I_S] = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
W_s \frac{g}{s} & 0 & h^2 \frac{g}{s}
\end{bmatrix}
\]

Fuselage:

\[
[I_{F1}] = \begin{bmatrix}
I_{xx} & 0 & -I_{xz} \\
0 & I_{yy} & 0 \\
-I_{xz} & 0 & I_{zz}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
W_{F1} \frac{g}{s} & 0 & \lambda_{F1}^2
\end{bmatrix}
\]

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\[
[I_{F2}] = \begin{bmatrix}
I_{xx} & 0 & -I_{xz} \\
0 & I_{yy} & 0 \\
-I_{xz} & 0 & I_{zz}
\end{bmatrix} + \frac{W_{F2}}{g} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & L_{F2}^2 \\
0 & 0 & 0
\end{bmatrix}
\]

The first matrix is defined about the body axes.

Envelope:
\[
[I_{EN}] = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix} + \frac{W_{EN}}{g} \begin{bmatrix}
h_4^2 & 0 & 0 \\
0 & h_4^2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Underslung weight:
\[
[I_{UN}] = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix} + \frac{W_{UN}}{g} \begin{bmatrix}
h_1^2 & 0 & 0 \\
0 & h_1^2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

6.4 Equations of Motion for Elastic Modes

The flexible supporting structure is assumed to be a free-free beam idealized by a small number of modes of vibration. The structure can have bending vibration in the two planes and a torsional vibration about its longitudinal axis. These elastic modes are assumed to occur in SI coordinate system, and therefore the equations of motion are written in SI system. Furthermore the structure is assumed to vibrate about its equilibrium position. Bending in \(x_{sl} - y_{sl}\) plane [Fig. 9]

The elastic deformation at any point on the structure due to the bending deformation is written as
\[
v_s(x_{sl}) = \sum_{i=1}^{NM} \eta_i(x_{sl}) \xi_i(t)
\]

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where \( \eta_i \) is the mode shape, \( \xi_i \) is the generalized coordinate or normal coordinate and \( NM \) is the number of modes.

The equations of motion written, in normal coordinate, are

\[
\{M\} \{\dot{\xi} \} + \{K\} \{\xi \} = \{Q\}
\]

(6.52)

Where \( \{Q\} \) represents the generalized forces

\( \{M\} \) is the generalized mass

\( \{K\} \) is the generalized stiffness

It is assumed that these vibration modes are small compared to the rigid body perturbational motion. So there is no coupling between the vibration modes. The generalized forces are (force at a location \( x \) modal displacement at the same location or moment \( x \) slope of the modal displacement curve)

\[
Q_i = P_{yS1FL} \eta_i(\xi_{F1}) + Q_{yS1FL} \eta_{i,x}(\xi_{F1}) + P_{yS1WF1} \eta_i(\xi_{F1}) + P_{yS1F2} \eta_i(\xi_{F2}) + Q_{yS1F2} \eta_{i,x}(\xi_{F2}) + P_{yS1WF2} \eta_i(\xi_{F2}) + P_{yS1EN} \eta_i(O_S) + Q_{yS1EN} \eta_{i,x}(O_S) + [P_{yS1WEN} + P_{yS1WUN}] \eta_i(O_S) + \int_{-\xi_{FL}}^{\xi_{FL}} P_{yS1WS} \eta_i(x) \, dx
\]

Terms with suffix 'S1W' refer to gravity loads referred in SI system. A typical element of the mass matrix and stiffness matrix are

\[
M_{ii} = \int_{-\xi_{FL}}^{\xi_{FL}} \eta_i \, m_s(x) \, dx + \frac{W_{F1}}{g} \eta_i^2(\xi_{F1}) + I_{zzF1} \eta_{i,x}^2(\xi_{F1}) + \frac{W_{F2}}{g} \eta_i^2(\xi_{F2}) + I_{zzF2} \eta_{i,x}^2(\xi_{F2}) + \frac{W_{EN}}{g} \eta_i^2(O_S) + I_{zzEN} \eta_{i,x}^2(O_S) + \frac{W_{UN}}{g} \eta_i^2(O_S) + I_{zzUN} \eta_{i,x}^2(O_S)
\]

and \( K_{ii} = \omega_i^2 \int_{-\xi_{FL}}^{\xi_{FL}} \eta_i \, m_s(x) \, dx \)

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where \( \omega_i \) is the \( i \)th mode natural frequency of the supporting structure
\( m_s \) is the mass of the structure/unit length

Bending \( x_{s1} - z_{s1} \) plane

The elastic deformation at any point is given by

\[
\omega_s(x_{s1}) = \sum_{i=N+1}^{NM1} \eta_i(x_{s1}) f_i(t)
\] (6.53)

The form equation of motion is similar to equation (6.46). The generalized forces are

\[
Q_1 = P_{z1F1} \eta_1(\ell_{F1}) + Q_{y1F1} \eta_{1,x}(\ell_{F1}) + Q_{y1WUN} \eta_{1,x}(0_s) \\
+ P_{z1F2} \eta_1(\ell_{F2}) + Q_{y1F2} \eta_{1,x}(\ell_{F2}) + Q_{y1WEN} \eta_{1,x}(0_s) \\
+ P_{z1EN} \eta_1(0_s) + Q_{y1EN} \eta_{1,x}(0_s)
\]

Typical elements of the mass and stiffness matrix are

\[
M_{11} = \int_{-\ell_{F2}}^{\ell_{F1}} \eta_1 m_s(x) \eta_1 dx + \frac{W_{F1}}{g} \eta_1^2(\ell_{F1}) + I_{yyF1} \eta_{1,x}^2(\ell_{F1}) \\
+ \frac{W_{F2}}{g} \eta_1^2(\ell_{F2}) + I_{yyF2} \eta_{1,x}^2(\ell_{F2}) \\
+ \frac{W_{EN}}{g} \eta_1^2(0_s) + I_{yyEN} \eta_{1,x}^2(0_s) \\
+ \frac{W_{EN}}{g} h_4 \eta_1^2(0_s) \\
+ \frac{W_{UN}}{g} \eta_1^2(0_s) + I_{yyUN} \eta_{1,x}^2(0_s) \\
+ \frac{W_{UN}}{g} h_4 \eta_{1,x}^2(0_s)
\]

and

\[
K_{11} = \omega_i^2 \int_{-\ell_{F2}}^{\ell_{F1}} \eta_1 m_s(x) \eta_1 dx
\]
Torsion

The torsional vibration of the supporting structure is about \( x_{S1} \) axis [Fig. 9]. The twist at any section due to the torsional deformation is given by the normal modes \( \eta_i(x) \) as

\[
\phi_s(x_{S1}) = \sum_{i=NM1+1}^{NM2} \eta_i(x_{S1}) \xi_i(t) \quad (6.54)
\]

The torsional vibration equation in normal mode is

\[
[\mathbf{I}] \ddot{\{\xi\}} + [\mathbf{K}] \{\xi\} = \{Q\} \quad (6.55)
\]

The generalized force \( Q_i \)'s are

\[
Q_i = Q_xS1F1 \eta_i(\ell_{F1}) + Q_xS1F2 \eta_i(\ell_{F2}) + Q_xS1EN \eta_i(O_S) \\
+ Q_xS1WEN \eta_i(O_S) + Q_xS1WUN \eta_i(O_S)
\]

Typical members of \([\mathbf{I}]\) and \([\mathbf{K}]\) matrices are

\[
I_{ii} = \int_{-\ell_{F1}}^{\ell_{F2}} \eta_i I_s(x) \eta_i \, dx + I_{xxF1} \eta_i^2(\ell_{F1}) + I_{xxF2} \eta_i^2(\ell_{F2}) \\
+ I_{xxEN} \eta_i^2(O_S) + I_{xxUN} \eta_i^2(O_S) \\
+ \frac{W_{EN}}{g} h_4 \eta_i^2(O_S) + \frac{W_{UN}}{g} h_1 \eta_i^2(O_S)
\]

and

\[
K_{ii} = \omega_i^2 \int_{-\ell_{F1}}^{\ell_{F2}} \eta_i I_s(x) \eta_i \, dx
\]

where \( I_s \) is the moment of inertia of the structure per unit length about its longitudinal axis.
7. CONCLUDING REMARKS

A complete set of dynamical equations of motion for a simple model of HHLA were derived in this report. These equations can be used to study the stability of HHLA and to obtain the various response quantities at different stations on the vehicle. For convenience, the equation numbers are summarized below with the physical degree of freedom which it represents.

Blade equations

- Flap Equation (5.104)
- Lead-Lag Equation (5.105)
- Torsion Equation (5.106)

Supporting Structure equations

- Rigid body translation Equation (6.49)
- Rigid body rotation Equation (6.50)
- Bending in $x_{s1}$-$y_{s1}$ plane Equation (6.52)
- Bending in $x_{s1}$-$z_{s1}$ plane Equation (6.52)
- Torsion about $x_{s1}$ axis Equation (6.55)

The coupled rotor/body equations of motion which have been derived in this study have considerable versatility and can be used to model a number of diverse rotary-wing configurations, which are listed below:

(a) Isolated rotor blade aeroelastic stability.
(b) Coupled rotor/fuselage dynamics for a single rotor.
(c) Response cyclic, collective and higher harmonic control inputs.
(d) Stability analysis of a tandem rotor system connected by a flexible structure.
(e) Dynamics, aeroelasticity, and aeroelastic response of a Hybrid Heavy Lift Airship.

Depending on the type of system which one intends to analyze the complete equations presented in this document have to be simplified to fit the specific application.

In a sequel to this report entitled "Aeroelastic Effects in Multirotor Vehicles, Part II: Method of Solution and Results Illustrating Coupled Rotor/Body Aeromechanical Stability", two separate coupled rotor/body problems are
solved with considerable detail. In the first case the equations are used to predict the aeromechanical stability problem of a single rotor helicopter in ground resonance, including the effect of the aerodynamic forces. For this case high quality experimental results are available, and the agreement between theory and test was found to be quite good. In the second case, the stability of a simplified model vehicle (Fig. 2) representing an HHLA type vehicle in hover is analyzed, and the basic aeroelastic characteristics of such a vehicle are obtained.

The various details of the solution such as: evaluation of the equilibrium position, stability equations in multiblade coordinates and appropriate methods of solution are given in the second report, which constitutes a sequel to the present report.
REFERENCES


Figure 1. Hybrid Heavy Lift Airship - Approximate Configuration
Figure 2. Sketch Showing the Main Ingredients of the Aeroelastic Model
Figure 3. Blade Cross-Section Configuration
Figure 4. Disturbed Rotor Hub and Rotor Blade Coordinate Systems
Figure 5. Rotor Blade Coordinate Systems
Figure 6. Undeformed and Deformed $k^{th}$ Blade Cross-Sections
Figure 7. Deformed $k^{th}$ Blade Cross-Section Coordinate Systems
Figure 8. Inertial S-Coordinate System (Coincident with Body Fixed S1-System in Unperturbed State)

Figure 9. Body Fixed S1-Coordinate System (Obtained After Rigid Body Perturbational Motion) Elastic Deformations of the Supporting Structure Occur in this S1-System
Figure 10. Geometry for Oscillating Airfoil in Pulsating Flow
Figure 11. Relative Flow Velocities

- $L_C$: Circulatory Lift Normal to the Resultant Flow
- $L_{NC}$: Noncirculatory Lift Normal to the Blade Chord

$V_R = \text{RESULTANT VELOCITY}$
Figure 12. HHLA Model
Appendix A: Equivalent Structural Model for Offset Hinged Spring Restrained Blades

A.1 General

In this Appendix the expressions for the elastic restoring moments acting at the hinge of a spring restrained blade are obtained. The root springs are representative of blade structural flexibility or they can represent flexures built into the blade, thus the model simulates the elastic properties of the configuration shown in Fig. A.1. The various expressions for the elastic restoring moments in flap, lead-lag and torsion, respectively, are derived. Subsequently these expressions are compared to similar equations obtained in previous studies. In this comparison both elastic hingeless blades as well as spring restrained equivalent models are considered. Furthermore it should be noted that the main advantage in using this simple model for an HHFA type vehicle consists of the capability of the model, to capture the essential behavior of both hingeless and articulated rotor configurations.

Ormiston and Hodges [Ref. 13] have derived one of the first models of this type, however Ref. 13 was restricted to the equivalent model of a hingeless spring restrained blade having only flap and lag degrees of freedom. The present model represents an extension of Ref. 13, to the case where both torsional blade flexibility as well as pitch link flexibility are incorporated in the blade model.

Peters [Ref. 14] has also derived a flap-lag model similar to Ormiston and Hodges. In both studies, Refs. 13 and 14, the flexibility of the hub has been also considered. A careful study of these two References reveals some discrepancies, which are associated primarily with the hub flexibility. The expressions for hub flexibility are also derived for our model since it was felt that these could be useful in some potential application. The final moment expressions obtained in this study are capable of representing the coupled flap-lag-torsional motion of both hingeless and articulated blades.

This Appendix is divided into three sections. In the first section the moment expressions, excluding hub flexibility, are derived for a spring restrained blade. The second section treats the special form of these equations for the case of an articulated blade. Finally the last section presents the moment equations for the blade including hub flexibility.
A.2 Moment Equations for a Hingeless Blade

The flexible elastic blade is modeled as a rigid blade with three orthogonal springs located at the root of the blade. These springs represent the flexibility of the blade in flap, lead-lag and torsion, respectively. In addition to these three springs, a torsional spring, in board of these three springs, is introduced to represent the control system or pitch link flexibility, see Fig. A1. It is assumed that the orientation of these springs does not change as the blade undergoes reorientation due to flap, lag and torsional motion. Consequently, the following question can be immediately raised. Since the torsion or twist of the blade is assumed to occur about the elastic axis of the blade which can have a different orientation as the blade flaps and lags, the spring representing the torsional stiffness of the blade should also change its orientation? The answer to this question is negative. Since the model is intended to represent a hingeless cantilevered blade for which the slope of the elastic axis at the root is always zero, irrespective of blade orientation. In our model, the torsional stiffness of the blade is represented by an equivalent torsional spring at the root. Hence, its orientation does not change with the blade motion. The spring stiffness are $K_\Omega$ for flaping, $K_{\tau_B}$ for lead-lag, $K_{\psi_B}$ for torsion and $K_{\psi_C}$ for the pitch link flexibility. From the physics of the problem it is clear that these springs are all torsional type springs.

In deriving these equations the sign convention is important. All counter clock-wise rotations and moments are taken as positive. The restoring moment in any torsional spring, due to a positive displacement (i.e., rotation), is clock-wise and hence negative.

Another important ingredient in this derivation is the coordinate system. Let $x, y, z$ be an orthogonal triad attached to the undeformed blade with zero pitch angle. The $X$-axis is along the elastic axis of the blade as shown in Fig. A2. The $X', Y', Z'$ system represents another orthogonal triad, rotated through an angle $\theta$ about $X$-axis in the counter clock-wise direction. The angle $\theta$ represents the collective pitch of the blade shown in Fig. A2. The root springs representing the blade flexibility are oriented as follows: $K_\Omega$ along $Y'$ axis, $K_{\tau_B}$ along $Z'$ axis and $K_{\psi_B}, K_{\psi_C}$ along $X'$ axis.

Using this information the expressions for the moments are derived next. To derive the expressions for the restoring moments of the springs, due to blade motion, the total angular displacement of the blade has to be decomposed into components along the directions of the spring restrained hinges. The rotational (angular) displacement components are then multiplied by the corresponding spring
stiffness to yield the appropriate moments about the hinges. Finally, the various
moment vectors are expressed in terms of the component acting along the undeformed
blade axes X, Y and Z respectively. These expressions are then compared with the
results obtained in previous studies.

To arrive at the deformed blade orientation from the undeformed position a
specific sequence of rotations are followed, namely flap, lag and torsion. The unit
vectors in the directions of the undeformed and deformed blade coordinate systems
are related by an Euler angle transformation which is derived below.

The flap rotation is assumed to take place first, thus the X,Y,Z axes system
is rotated through an angle (-\(\beta\)), in the clockwise direction, about Y-axis represent-
ing the flap hinge of the blade shown in Fig. A.3. Components of the angular
displacement (-\(\beta\)) along X', Y', Z' axes can be obtained using the transformation

\[
\begin{bmatrix}
\hat{e}_x' \\
\hat{e}_y' \\
\hat{e}_z'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\theta & \sin\theta \\
0 & -\sin\theta & \cos\theta
\end{bmatrix}
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix}
\]  

(A.1)

where \(\hat{e}_x', \hat{e}_y', \hat{e}_z'\) and \(\hat{e}_x, \hat{e}_y, \hat{e}_z\) are unit vectors along X', Y', Z' and X, Y, Z axes
respectively.

Components of (-\(\beta\)) along X', Y', Z' axes are along

\[
\begin{align*}
X' &= 0 \\
Y' &= -\beta\cos\theta \\
Z' &= -\beta\sin\theta
\end{align*}
\]  

(A.2)

Due to this rotation (-\(\beta\)), the coordinate system is rotated to a new position
X, Y, Z shown in Fig. A.3. The transformation of unit vectors between the two
systems is given by

\[
\begin{bmatrix}
\hat{e}_x \\
\hat{e}_y \\
\hat{e}_z
\end{bmatrix} =
\begin{bmatrix}
\cos\beta & 0 & -\sin\beta \\
0 & 1 & 0 \\
\sin\beta & 0 & \cos\beta
\end{bmatrix}
\begin{bmatrix}
\hat{e}_{x1} \\
\hat{e}_{y1} \\
\hat{e}_{z1}
\end{bmatrix}
\]  

(A.3)

The lag rotation is assumed to take place next, the system X, Y, Z
is rotated about Z_1 axis through angle (\(\zeta\)), in the counter clock-wise direction,
representing lead-lag motion of the blade, shown in Fig. A.4. Equation (A.3) is
used to obtain the components of the angular displacement '\(\zeta\)' along X,Y,Z axes.
These components are

\[
\begin{align*}
\text{along } X &= -\zeta \sin \beta \\
Y &= 0 \\
Z &= \zeta \cos \beta
\end{align*}
\] (A.4)

These components along \(X, Y, Z\) are transformed into components along the direction of the spring axes \(X', Y', Z'\) using equation (A.1). These components are:

\[
\begin{align*}
\text{along } X' &= -\zeta \sin \beta \\
Y' &= \zeta \cos \beta \sin \theta \\
Z' &= \zeta \cos \beta \cos \theta
\end{align*}
\] (A.5)

Next the \(X_1, Y_1, Z_1\) axes system is rotated to a new position \(X_2, Y_2, Z_2\) shown in Fig. A4. The relation between the unit vectors along \(X_2, Y_2, Z_2\) and \(X_1, Y_1, Z_1\) systems is given by

\[
\begin{bmatrix}
\hat{e}_{x1} \\
\hat{e}_{y1} \\
\hat{e}_{z1}
\end{bmatrix} =
\begin{bmatrix}
\cos \zeta & -\sin \zeta & 0 \\
\sin \zeta & \cos \zeta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{e}_{x2} \\
\hat{e}_{y2} \\
\hat{e}_{z2}
\end{bmatrix}
\] (A.6)

Combining equations (A.6) and (A.3), the transformation of unit vectors between \(X_2, Y_2, Z_2\) and \(X, Y, Z\) systems; given below is obtained.

\[
\begin{bmatrix}
\hat{e}_{x} \\
\hat{e}_{y} \\
\hat{e}_{z}
\end{bmatrix} =
\begin{bmatrix}
\cos \beta \cos \zeta & -\cos \beta \sin \zeta & -\sin \beta \\
\sin \zeta & \cos \zeta & 0 \\
\sin \beta \cos \zeta & -\sin \beta \sin \zeta & \cos \beta
\end{bmatrix}
\begin{bmatrix}
\hat{e}_{x2} \\
\hat{e}_{y2} \\
\hat{e}_{z2}
\end{bmatrix}
\] (A.7)

Finally the torsional rotation is assumed to take place. To represent torsion, the blade is rotated through an angle \(\phi\) in the counter clockwise direction along the \(X_2\) axis, as shown in Fig. A5. The components of the angular displacement \(\phi\) along \(X, Y, Z\) system are obtained using equation (A.7). These components are respectively:

\[
\begin{align*}
\text{along } X &= \phi \cos \beta \cos \zeta \\
Y &= \phi \sin \zeta \\
Z &= \phi \sin \beta \cos \zeta
\end{align*}
\] (A.8)
These components are again transformed in the X', Y', Z' directions using equation (A.1). The corresponding components are

\[
\begin{align*}
\text{along } X' &= \phi \cos \beta \cos \zeta \\
\text{along } Y' &= \phi \sin \zeta \cos \theta + \phi \sin \theta \sin \beta \cos \zeta \\
\text{along } Z' &= -\phi \sin \zeta \sin \theta + \phi \cos \theta \sin \beta \cos \zeta \\
\end{align*}
\]  
(A.9)

The total twist of the various springs oriented along X', Y', Z' axes is obtained by adding equations (A.2), (A.5) and (A.9). The various components can be identified as:

\[
\begin{align*}
\text{along } X' &= -\zeta \sin \beta + \phi \cos \beta \cos \zeta \\
\text{along } Y' &= -\beta \cos \theta + \zeta \cos \beta \sin \theta + \phi \sin \zeta \cos \theta \\
&\quad + \phi \sin \beta \cos \zeta \sin \theta \\
\text{along } Z' &= \beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \sin \zeta \sin \theta \\
&\quad + \phi \sin \beta \cos \zeta \cos \theta \\
\end{align*}
\]  
(A.10)

In the model shown in Fig. Al, the springs representing the torsional stiffness of the blade and the pitch link stiffness are in series and are along X' axis. These two springs can be combined and can be represented by an equivalent spring of stiffness \( K_\phi = K_B \cdot K_C / (K_B + K_C) \), when \( K_B \) is very large, then \( K_\phi \) becomes \( K_B \). The restoring moments in the springs, due to the flap, lag and torsional rotation of the blade can be written as

\[
\begin{align*}
M_{x'} &= -K_\phi (\zeta \sin \beta + \phi \cos \beta \cos \zeta) \\
M_y &= -K_B (\beta \cos \theta + \zeta \cos \beta \sin \theta + \phi \sin \zeta \cos \theta + \phi \sin \beta \cos \zeta \sin \theta) \\
M_z &= -K_C (\beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \sin \zeta \sin \theta + \phi \sin \beta \cos \zeta \cos \theta) \\
\end{align*}
\]  
(A.11)

Transformation of these moments along the undeformed blade axes (X,Y,Z), yields the torsional, flapping and lead-lag restoring moments, associated with blade motion. These are:
Torsional moment

\[ M_x = -K_\phi \left(-\zeta \sin \beta + \phi \cos \beta \cos \zeta\right) \]

Flapping moment

\[ M_y = -K_\beta_B \cos \theta \left(-\beta \cos \theta + \zeta \cos \beta \sin \theta + \phi \zeta \cos \theta + \phi \beta \sin \theta\right) + K_\zeta_B \sin \theta \left(\beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \zeta \sin \theta + \phi \beta \cos \theta\right) \]

Lead-lag moment

\[ M_z = -K_\phi \sin \theta \left(-\beta \cos \theta + \zeta \cos \beta \sin \theta + \phi \beta \sin \theta\right) - K_\zeta_B \cos \theta \left(\beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \zeta \sin \theta + \phi \beta \cos \theta\right) \]

Assuming \( \beta \) and \( \zeta \) are small angles and making the approximation \( \sin \beta \approx \beta \), \( \sin \zeta \approx \zeta \), \( \cos \beta \approx 1 \) and \( \cos \zeta \approx 1 \), equation (A.12) can be simplified to yield

\[ M_x = -K_\phi \left(-\zeta \beta + \phi\right) \]

\[ M_y = -K_\beta_B \cos \theta \left(-\beta \cos \theta + \zeta \sin \theta + \phi \zeta \cos \theta + \phi \beta \sin \theta\right) + K_\zeta_B \sin \theta \left(\beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \zeta \sin \theta + \phi \beta \cos \theta\right) \]

\[ M_z = -K_\phi \sin \theta \left(-\beta \cos \theta + \zeta \sin \theta + \phi \zeta \cos \theta + \phi \beta \sin \theta\right) - K_\zeta_B \cos \theta \left(\beta \sin \theta + \zeta \cos \beta \cos \theta - \phi \zeta \sin \theta + \phi \beta \cos \theta\right) \]

Rearranging the terms,

\[ M_x = -K_\phi \left(-\zeta \beta + \phi\right) \]

\[ M_y = \left(\beta - \phi \zeta\right) \left(K_\beta_B \cos^2 \theta + K_\zeta_B \sin^2 \theta\right) + \left(\zeta + \phi \beta\right) \left(K_\zeta_B - K_\beta_B \right) \sin \theta \cos \theta \]

\[ M_z = -\left(\zeta + \phi \beta\right) \left(K_\beta_B \sin^2 \theta + K_\zeta_B \cos^2 \theta\right) - \left(\beta - \phi \zeta\right) \left(K_\zeta_B - K_\beta_B \right) \sin \theta \cos \theta \]

Equation (A.14) represents the torsional, flap and lag restoring moments of the springs due to the flap, lag and torsional motion of the blade.

These expressions for the restoring moments acting on the spring restrained model of the blade are compared with results published in previous studies. For completeness these comparisons are carried out for both spring restrained models of a hingeless blade and fully elastic representations of the hingeless blade.
Ormiston and Hodges [Ref. 13] presented moment expressions for a spring restrained model of a rotor blade representing a hingeless blade. They have also taken hub flexibility into account. In order to compare the results obtained in Ref. 13 with the expression obtained in this report, the hub stiffness is allowed to become infinite, thus only the simplified expressions are compared. Recall that the torsional degree of freedom was not considered in Ref. 13. The expressions [Ref. 13] for the restoring moments are:

Flapping moment (Eq. 44 of Ref. 13)

\[ M_y = -\beta(K^\phi_B \cos^2 \theta + K^r_B \sin^2 \theta) - \zeta(K^r_B - K^\theta_B) \sin \theta \cos \theta \]  

Leadb-lag moment (Eq. 45 of Ref. 13)

\[ M_z = -\zeta(K^\phi_B \sin^2 \theta + K^r_B \cos^2 \theta) - \beta(K^r_B - K^\theta_B) \sin \theta \cos \theta \]  

Setting \( \phi = 0 \) in Eqs. (A.14) yields

\[ M_x = K^\phi_B \zeta \]  

\[ M_y = \beta(K^\phi_B \cos^2 \theta + K^r_B \sin^2 \theta) + \zeta(K^r_B - K^\theta_B) \sin \theta \cos \theta \]  

\[ M_z = -\zeta(K^\phi_B \sin^2 \theta + K^r_B \cos^2 \theta) - \beta(K^r_B - K^\theta_B) \sin \theta \cos \theta \]  

(A.15)

By comparing equations (a) and (b) with equation (A.15), we find there is a negative sign in the expression for \( M_y \). The reason for this discrepancy is that the authors of Ref. 13 used different sign convention. In their case, \( M_y \) clockwise is positive and \( M_z \) counter clockwise is also positive. The authors of Ref. 13 have used this sign convention because their objective was to derive a blade model which was a simple analog to the flexible blade equations (hingeless) which were derived by Houbolt and Brooks [Ref. 15] who have employed the same sign convention for \( M_y \) and \( M_z \). The expression of the present study are also compared with the results of an elastic blade model. The moment expressions obtained by Rosen and Friedmann [Ref. 16] for an elastic blade are (Eqs. (15) of Ref. 16)

\[ M_x = G J (\phi_{,x} + v_{,xx} \omega_{,x}) \]  

\[ M_y = -(E_{12} - E_{13}) \sin \theta \cos \theta (v_{,xx} + \phi \omega_{,xx}) \]  

\[ -(E_{12} \sin^2 \theta + E_{13} \cos^2 \theta)(w_{,xx} - \phi v_{,xx}) \]
\[ M_z = (EI_2 - EI_3) \sin \theta \cos \theta (v_{xx} - \phi \omega_{xx}) \]
\[ + (EI_2 \cos^2 \theta + EI_3 \sin^2 \theta) (v_{xx} + \phi \omega_{xx}) \]  
\[ \text{(c)} \]

where \( v, w, \phi \) represent the elastic lead-lag displacement, flap displacement and twist at any section of the blade, and the quantities \( EI_2, EI_3, GJ \) represent lead-lag, flap and torsional stiffness of the blade.

The expressions obtained in Ref. 16 represent the applied moments, thus to obtain the restoring moments Equation (c) have to be multiplied by (-1) and the resulting expressions are compared with Eqs. (A.14). The expressions for \( M_y \) and \( M_z \) are the same, however there is a sign difference in the expression for torsional moment. The expression derived in this study contains a term \((\phi - \zeta \beta)\) while the corresponding term in Ref. 16 is \((\phi_{xx} + \nu_{xx} \omega_{xx})\).

This discrepancy is due to a different sequence of rotation followed by Rosen and Friedmann [Ref. 16]. The sequence of rotations adopted in Ref. 16 was lead-lag, flap and torsion. When a sequence of flap, lead-lag and torsion, such as employed in the present study is used, the results of Ref. 16 are in agreement with the results obtained in this study. Equations (c), representing a hingeless blade, with Eqs. (A.14) corresponding to the rigid, offset hinged spring restrained blade model one can identify a number of equivalence relations, which provide some physical insight. These equivalence relations are given below. For torsion one has
\[ GJ(\phi_{xx} - \nu_{xx} \omega_{xx}) = K_\psi (\phi - \zeta \beta) \]  
\[ \text{(A.16)} \]

where
\[ K_\psi = K_{\phi B} K_{\phi C} / (K_{\phi B} + K_{\phi C}) \]

If the pitch link flexibility is very large, i.e. \( K_{\phi B} \to \infty \) then \( K_\psi \) becomes \( K_{\phi B} \). We shall consider only \( K_{\phi B} \) in Eq. (A.16) because Rosen and Friedmann have not considered the pitch link flexibility. Thus equation (A.16) should be rewritten as
\[ GJ(\phi_{xx} - \nu_{xx} \omega_{xx}) = K_{\phi B} (\phi - \zeta \beta) \]  
\[ \text{(A.17)} \]

Comparison of the various other terms yields
\[ (EI_2 - EI_3) (v_{xx} + \phi \omega_{xx}) = (K_{\tau B} - K_{\beta B}) (\zeta + \beta \phi) \]  
\[ \text{(A.18)} \]
\[ EI_2 (\omega_{xx} - \phi v_{xx}) = K_{\tau B} (\beta - \phi \zeta) \]  
\[ \text{(A.19)} \]
\( E_{I_3}(w_{,xx} - \phi v_{,xx}) = K_{\beta B}(\beta - \phi \zeta) \)  
(A.20)

\[(E_{I_2} - E_{I_3})(w_{,xx} - \phi v_{,xx}) = (K_{\zeta B} - K_{\phi B})(\beta - \phi \zeta) \]  
(A.21)

\[E_{I_2}(v_{,xx} + \phi w_{,xx}) = K_{\zeta B} (\zeta + \phi \beta) \]  
(A.22)

\[E_{I_3}(v_{,xx} + \phi w_{,xx}) = K_{\phi B} (\zeta + \phi \beta) \]  
(A.23)

Setting \( \phi = 0 \) in equations (A.19), (A.20), (A.22) and (A.23) yields

\[E_{I_2} w_{,xx} = K_{\zeta B} \beta \]

\[E_{I_3} w_{,xx} = K_{\phi B} \beta \]

\[E_{I_2} v_{,xx} = K_{\zeta B} \zeta \]

\[E_{I_3} v_{,xx} = K_{\phi B} \zeta \]  
(A.24)

Examination of these expressions reveals that \( K_{\zeta B}, K_{\phi B} \) represent the lead-lag and flap stiffness \( E_{I_2}, E_{I_3} \) respectively and \( \beta, \zeta \) correspond to the curvatures in the elastic blade analysis. Comparison of the appropriate terms in equation (A.17) reveals

\[GJ = K_{\phi B} \]

\[\phi_{,x} = \phi \]

\[w_{,x} = \beta \]  
(A.25)

This comparison indicates that both \( w_{,xx} \) and \( w_{,x} \) can be identified as quantities corresponding to the flapping angle \( \beta \) in the rigid spring restrained blade model. This statement requires further clarification. The comparison outlined above indicates that when dealing with moment terms, such as \( EI w_{,xx} \), \( \beta \) has the role of the curvature. However when
examining the role of $\beta$ in the various transformations relating the unit vectors of the deformed and undeformed states of the blade the role of $\beta$ corresponds to $\mathbf{w}$ in the flexible blade equations. Furthermore it can be seen that $\phi_x$ in an elastic blade corresponds $\phi$ in a rigid blade model, this is due to the fact that the torsional moment in an elastic blade is defined as $GJ\phi_x$ whereas in the spring restrained blade model the torsional elastic restoring moment is $K_\phi \phi$.

A.3 Moment Equations for an Articulated Blade

A typical articulated blade, which has no root springs in flap and lag will not experience elastic restoring moments in about these hinges. Therefore the model proposed for this case, shown in Fig. A6, involves only a torsional spring and no springs for flap and lag. In the model shown in Fig. A6, two torsional springs are provided. The spring $K_\phi_B$ represents the blade torsional stiffness and $K_\phi_C$ represents the control link stiffness. These are aligned along the undeformed blade axis $x$. Since the blade can perform flap and lag motions during operation, the spring $K_\phi_B$ can orient itself along any direction in space provided the pitch link is inboard of flap and lag hinges. The spring $K_\phi_C$ remains along the undeformed $x$-axis. If the pitch link is outboard of leading and flap hinges, then both $K_\phi_B$ and $K_\phi_C$ change their orientation as the blade undergoes flap and lag motion. Therefore one needs to consider two cases: (a) pitch link inboard of flap and lag hinges and (b) pitch link outboard of flap and lag hinges.

Case (a) Pitch Link Inboard of Flap and Lag Hinges

Consider Fig. A.6 numbers 1, 2, 3 refer to three nodes. At node 2, the two springs $K_\phi_B$ and $K_\phi_C$ are connected. The springs are oriented along the undeformed elastic axis ($X$-axis of the coordinate system). In this case, since the pitch link is inboard of flap and lag hinges, only $K_\phi_B$ takes on different orientation as the blade undergoes flap and lag motion. Assume that the blade has undergone a flap deflection ($-\beta$) and a lag deflection ($\zeta$) respectively, as shown in Fig. A7. The new position of the elastic axis is $X_2$ and the deformed coordinate system is $X_2, Y_2, Z_2$. The spring associated with blade torsional stiffness, $K_\phi_B$, is now oriented along the $X_2$ axis and the spring associated with pitch link stiffness, $K_\phi_C$ is oriented along the undeformed axis, i.e. $X$-axis. For this case one may write a stiffness matrix relating moments to angular rotations as indicated in Eq. (A.26).
\[
\begin{pmatrix}
N_x^1 \\
N_y^1 \\
N_z^1 \\
N_x^2 \\
N_y^2 \\
N_z^2 \\
N_x^3 \\
N_y^3 \\
N_z^3
\end{pmatrix}
= \begin{pmatrix}
K_B \cos^2 \Phi \cos^2 \zeta \\
K_B \cos \Phi \cos \zeta \sin \zeta & K_B \sin^2 \zeta & \text{Symmetric} \\
-K_B \cos \zeta \sin \Phi \sin \zeta & -K_B \sin \Phi \sin \zeta \cos \zeta & K_B \sin^2 \zeta \\
-K_B \cos \Phi \cos \zeta \sin \zeta & -K_B \sin \Phi \sin \zeta \cos \zeta & K_B \cos^2 \Phi \cos^2 \zeta + K_C \cos^2 \Phi \cos^2 \zeta \\
-K_B \cos \zeta \sin \Phi \sin \zeta & -K_B \sin \Phi \sin \zeta \cos \zeta & K_B \cos \Phi \cos \zeta \sin \zeta \\
-K_B \cos \Phi \cos \zeta \sin \zeta & -K_B \sin \Phi \sin \zeta \cos \zeta & K_B \sin \Phi \sin \zeta \cos \zeta \cos \zeta \\
0 & 0 & 0 & -K_C & 0 & 0 & K_C \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_x^1 \\
\phi_y^1 \\
\phi_z^1 \\
\phi_x^2 \\
\phi_y^2 \\
\phi_z^2 \\
\phi_x^3 \\
\phi_y^3 \\
\phi_z^3
\end{pmatrix}
\]

(A.26)
This stiffness matrix relates the generalized displacements (rotations along X, Y, Z directions) and the moments at nodes 1, 2, 3. In equation (A.26), \( M_{x,y,z}^i \) represents the moments along X, Y, Z axes at node i and \( \phi_{x,y,z}^i \) represents the rotations along X, Y, Z axes at node i.

The equivalent stiffnesses of the system in torsion are obtained from the solution of this matrix equation. The conditions under which this equation is solved are

\[
\phi_x^3 = \phi_y^3 = \phi_z^3 = 0 \quad \text{Node 3 is fixed}
\]

and

\[
M_x^2 = M_y^2 = M_z^2 = 0 \quad \text{No external moment at node 2}
\]

When trying to solve the matrix equation under these conditions the individual equations for \( M_x^i, M_y^i, M_z^i \) become redundant. This redundancy is due to the rigid body degrees of freedom being included in the model. The physical reason for this redundancy is due to the fact that the moment along the Y and Z axes cannot be resisted by this spring model. In order to overcome this difficulty, it is assumed that the flap and lead-lag angles are very small while the spring \( K_y^B \) is oriented along the undeformed elastic axis, i.e., X-axis. The springs \( K_y^B \) and \( K_y^C \) are in series and are oriented along X-axis and can be combined. For this model, the relation between the torsional moment and the twist \( \phi \) of the blade becomes

\[
M_x = \frac{K_y^B K_y^C}{K_y^B + K_y^C} \phi
\]  

\[
= K_y \phi \quad \text{(A.27)}
\]

Case (b) Pitch Link Outboard of Flap-Lag Hinges

When the pitch link is outboard of the flap-lag hinges both the springs \( K_y^B \) and \( K_y^C \) are always oriented in the same direction, from which the following relation between torsional moment and elastic twist is obtained.

\[
M_x = \frac{K_y^B K_y^C}{K_y^B + K_y^C} \phi \quad \text{(A.28)}
\]

Therefore for an articulated blade, notwithstanding whether the flap and lag hinges are inboard or outboard of the pitch link, the relation between the torsional moment and the elastic twist is given by

\[
M_x = \frac{K_y^B K_y^C}{K_y^B + K_y^C} \phi
\]

\[
= K_y \phi \quad \text{(A.29)}
\]
Combining this result with the result for a hingeless blade, discussed previously, the general elastic restoring moment expressions become:

**Torsion:**
\[ M_x = -K_{\phi_1} (\phi - \zeta \beta) - K_{\phi_2} \phi \]

**Flap:**
\[ M_y = (\beta - \phi \xi)(K_B \cos^2 \theta + K_{\zeta_B} \sin^2 \theta) + (\phi \beta + \zeta)(K_{CB} - K_{\phi B}) \sin \theta \cos \theta \]

**Lead-lag:**
\[ M_z = -(\xi + \phi \beta)(K_B \sin^2 \theta + K_{\zeta_B} \cos^2 \theta) - (\beta - \phi \xi)(K_{CB} - K_{\phi B}) \sin \theta \cos \theta \]

For a **hingeless blade**
\[ K_{\phi_2} = 0 \quad \text{and} \quad K_{\phi_1} = \frac{K_{\phi_C} K_{\phi_B}}{K_{\phi_B} + K_{\phi_C}} \]

for an **articulated blade**
\[ K_{\phi_1} = 0, \quad K_{\phi_B} = 0, \quad K_{\phi_C} = 0 \]

and
\[ K_{\phi_2} = \frac{K_{\phi_C} K_{\phi_B}}{K_{\phi_B} + K_{\phi_C}} \]

**A.4 Elastic Restoring Moments on a Rigid Blade with Root Springs and Hub Flexibility**

In this section, the elastic restoring moments due to the root springs and due to the hub flexibility are derived. The model for this system based on equivalent springs is illustrated in Fig. A8. There are two sets of orthogonal springs, springs with stiffness \( K_{\phi B} \), \( K_{\zeta B} \), and \( K_{\phi B} \) which represent the blade stiffness in flap, lead-lag, and torsion, respectively. The spring constant \( K_{\phi C} \) represents the stiffness of the pitch link system and \( K_{\zeta B} \) and \( K_{\phi B} \) are the stiffnesses of the hub in flap and lead-lag directions. The hub is assumed to be torsionally rigid. The \( X,Y,Z \) axis system represents the undeformed blade coordinates with the \( X \)-axis oriented along the elastic axis. The system \( X',Y',Z' \) corresponds to the \( X,Y,Z \) axis system after a counter clockwise rotation about the \( X \)-axis by an angle \( \theta \). The angle \( \theta \) represents the collective pitch of the blade.
The orientation of the various springs is as follows: \( K_H \) is along Z-axis, \( K_R \) is along Y-axis, \( K_B \) is along \( Y' \)-axis, \( K_C \) is along \( Z' \)-axis, \( \phi_B \) and \( \phi_B \) are along X-axis (see Fig. A8). The numbers 1, 2, 3 in Fig. A8 refer to the three nodes. The blade spring system is attached to the hub spring system at node 2.

The relation between the moments and the angular displacements for this equivalent spring system describing the combined blade and hub is given by the matrix equation (A.31), on the next page.

In equation (A.31) \( M_x^i, M_y^i, M_z^i \) refer to the elastic moments at node \( i \) in the X, Y, Z directions, respectively and \( \phi_x^i, \phi_y^i, \phi_z^i \) refer to the angular displacements at node \( i \) in the X, Y, Z directions, respectively. The various moments are obtained in terms of the angular displacements by solving this matrix equation subject to the following conditions:

When the hub is fixed, the angular displacements at node 3 are zero and then

\[
\phi_x^3 = \phi_y^3 = \phi_z^3 = 0 \quad (A.32)
\]

Since there is no external moment at node 2,

\[
M_x^2 = M_y^2 = M_z^2 = 0 \quad (A.33)
\]

Solving the matrix equation, Eq. (A.31) subject to the conditions given in equations (A.32) and (A.33) and recognizing that \( M_x^2 = M_y^2 = M_z^2 = 0 \) one obtains

\[
-(K_H \cos^2 \theta + K_B \sin^2 \theta) \phi_x^1 - (K_B - K_C) \sin \theta \cos \phi_x^1 =
\]

\[
-(K_H + K_C \cos^2 \theta + K_B \sin^2 \theta) \phi_z^2 - (K_B - K_C) \sin \theta \cos \phi_z^2 \quad (A.34)
\]

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\[ \begin{bmatrix}
H_x^1 \\
H_y^1 \\
H_x^2 \\
H_y^2 \\
H_x^3 \\
H_y^3 \\
\end{bmatrix} = \begin{bmatrix}
K_B \cos^2 \theta + K_B \sin^2 \theta & (K_B - K_C) \sin \theta \cos \theta & 0 & -(K_B \cos^2 \theta + K_B \sin^2 \theta) & -(K_B - K_C) \sin \theta \cos \theta & 0 & 0 & 0 \\
K_B \sin^2 \theta + K_B \cos^2 \theta & 0 & -(K_B - K_C) \sin \theta \cos \theta & -(K_B \sin^2 \theta + K_B \cos^2 \theta) & 0 & 0 & 0 \\
K_B \sin^2 \theta + K_B \cos^2 \theta & 0 & 0 & -(K_B \sin^2 \theta + K_B \cos^2 \theta) & 0 & 0 & 0 \\
K_B \sin^2 \theta + K_B \cos^2 \theta & 0 & 0 & 0 & -(K_B \sin^2 \theta + K_B \cos^2 \theta) & 0 & 0 & 0 \\
K_B + K_B \cos^2 \theta + K_B \sin^2 \theta & 0 & 0 & 0 & -(K_B \sin^2 \theta + K_B \cos^2 \theta) & 0 & 0 & 0 \\
K_B + K_B \cos^2 \theta + K_B \sin^2 \theta & 0 & 0 & 0 & 0 & -K_B \sin^2 \theta - K_B \cos^2 \theta & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
\phi_x \\
\phi_y \\
\phi_x \\
\phi_y \\
\phi_x \\
\phi_y \\
\end{bmatrix} \]

(A.31)
\[-(K_B - K_r) \sin \theta \cos \theta \phi_z^1 - (K_B \cos^2 \theta + K_r \sin^2 \theta) \phi_y^1 = \]
\[-\phi_z^2 (K_B - K_r) \sin \theta \cos \theta - (K_B + K_B \cos^2 \theta + K_r \sin^2 \theta) \phi_y^2 \]
\[-K_{\phi_B} \phi_x^1 = -(K_{\phi_B} + K_{\phi_C}) \phi_x^2 \tag{A.36} \]

From Eqs. (A.34), (A.35) and (A.36) \( \phi_x^2, \phi_y^2 \) and \( \phi_z^2 \) can be expressed in terms of \( \phi_x^1, \phi_y^1, \phi_z^1 \), thus

\[ \begin{bmatrix} \phi_x^2 \\ \phi_y^2 \\ \phi_z^2 \end{bmatrix} = \begin{bmatrix} K_B + K_B \cos^2 \theta & -(K_B - K_r) \\ +K_r \sin^2 \theta & \sin \theta \cos \theta \\ -(K_B - K_r) & K_r + K_r \cos^2 \theta \end{bmatrix} \begin{bmatrix} (K_B \cos^2 \theta + K_B \sin^2 \theta) \phi_z^1 \\ -(K_B - K_r) \sin \theta \cos \theta \\ -(K_B - K_r) \sin \theta \cos \theta \end{bmatrix} \begin{bmatrix} \frac{1}{\Delta} \end{bmatrix} \tag{A.38} \]

where

\[ \Delta = (K_B + K_r \cos^2 \theta + K_B \sin^2 \theta) (K_B + K_B \cos^2 \theta + K_r \sin^2 \theta) - (K_B - K_r)^2 \sin^2 \theta \cos^2 \theta \]
\[ = K_B \cos^2 \theta + K_r \sin^2 \theta + K_B \sin^2 \theta + K_r \cos^2 \theta + K_B \cos^2 \theta + K_r \sin^2 \theta \tag{A.39} \]

From Equation (A.38)

\[ \phi_z^2 = \frac{1}{\Delta} \left\{ [K_B (K_r \cos^2 \theta + K_B \sin^2 \theta) + K_{\phi_B} K_B] \phi_z^1 + K_{\phi_B} (K_B - K_r) \sin \theta \cos \theta \phi_y^1 \right\} \tag{A.40} \]
and
\[ \phi^2_y = \frac{1}{\Delta} \left\{ K_{\zeta H_B} (K_B - K_C) \sin \delta \cos \phi^1 + [K_{\zeta H_B} (K_B \cos^2 \theta + K_{\zeta B} \sin^2 \theta) + K_{\phi B} K_{\zeta B}] \phi^1 \right\} \] (A.41)

Substituting for \( \phi^2_y, \phi^2_z, \phi_x \) in terms of \( \phi^1_1, \phi^1_2, \phi^1_3 \) in equation (A.31), and writing the expressions for \( M^1_z, M^1_y, M^1_x \), one obtains the torsional moment
\[ M^1_x = \frac{K_{\phi B} K_{\phi C}}{K_{\phi B} + K_{\phi C}} \phi^1_3 \] (A.42)

The flap moment can be written as
\[ M^1_y = \frac{\phi^1_1}{\Delta} \left\{ K_{\zeta H_B} K_{\phi H} (K_B \cos^2 \theta + K_{\zeta B} \sin^2 \theta) + K_{\phi B} K_{\zeta B} \right\} \]
\[ - \frac{\phi^1_1}{\Delta} (K_{\zeta B} - K_{\phi B}) K_{\zeta H_B} \phi^1_3 \sin \delta \cos \theta \] (A.43)

The lead-lag moment can be expressed as
\[ M^1_z = \frac{\phi^1_2}{\Delta} \left\{ K_{\zeta H_B} K_{\phi H} (K_B \cos^2 \theta + K_{\zeta B} \sin^2 \theta) + K_{\phi B} K_{\zeta B} \right\} \]
\[ - \frac{\phi^1_1}{\Delta} (K_{\zeta B} - K_{\phi B}) K_{\zeta H_B} \phi^1_3 \sin \delta \cos \theta \] (A.44)

In the last three equations \( \phi^1_1, \phi^1_2, \phi^1_3 \) represents angular displacements given at node 1 along X, Y, Z axis. These angular displacements can be related to the flap, lead-lag and torsional rotations of the blade. The flap angle \( -\beta \) is about the undeformed Y-axis, the lead-lag angle \( \phi \) is about the Z axis after it has undergone flapping deformation and the torsion \( \zeta \) is along X-axis after the flap and lag deformations have taken place. Resolving these angular displacements along X, Y, Z axis, using equations (A.4) and (A.8) together with the fact that \( -\beta \) is along Y-axis, the components of the angular displacement (or rotation) about the X, Y and Z axis are
\[ \phi^1_x = -\zeta \sin \beta + \phi \cos \beta \cos \zeta \]
\[ \phi^1_y = -\beta + \phi \sin \zeta \]
\[ \phi^1_z = \zeta \cos \beta + \phi \sin \beta \cos \zeta \] (A.45)

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Introducing the small angle assumptions for $\beta$, $\zeta$ and $\phi$, equations (A.45) become

\[
\begin{align*}
\phi^1 &= - \zeta \beta + \phi \\
\phi^1 &= - \beta + \phi \zeta \\
\phi^1 &= \zeta + \phi \beta \\
\end{align*}
\]  

Substituting these in equations (A.42), (A.43), and (A.44), the elastic moments are obtained

\[
\begin{align*}
M^1_z &= \frac{(\zeta+\phi \beta)}{\Delta} \left\{ K_{\alpha H} K_{\beta H} (K_{\gamma B} \cos^2 \theta + K_{\beta B} \sin^2 \theta) + K_{\alpha H} K_{\alpha H} K_{\gamma B} \right\} \\
&+ \frac{(\beta-\phi \zeta)}{\Delta} (K_{\alpha B} - K_{\beta B}) K_{\alpha H} K_{\beta H} \sin \theta \cos \theta \\
M^1_y &= - \frac{(\beta-\phi \zeta)}{\Delta} \left\{ K_{\gamma H} K_{\alpha H} (K_{\beta B} \cos^2 \theta + K_{\gamma B} \sin^2 \theta) + K_{\beta B} K_{\gamma B} \right\} \\
&- \frac{(\zeta+\phi \beta)}{\Delta} (K_{\gamma B} - K_{\beta B}) K_{\gamma H} K_{\beta H} \sin \theta \cos \theta \\
M^1_x &= \left( \frac{K_{\phi B} K_{\phi C}}{K_{\phi B} + K_{\phi C}} \right) (\phi - \zeta \beta) \\
\end{align*}
\]  

Equations (A.47), (A.48), and (A.49) can be written in an alternative form, i.e. in terms of equivalent stiffness in flap, lag and torsion respectively.

Defining

\[
\begin{align*}
K_{\phi} &= \frac{K_{\phi} K_{\phi C}}{K_{\phi} + K_{\phi C}} \\
K_{\beta} &= \frac{K_{\beta} K_{\beta B}}{K_{\beta} + K_{\beta B}} \\
K_{\zeta} &= \frac{K_{\zeta} K_{\zeta B}}{K_{\zeta} + K_{\zeta B}} \\
\end{align*}
\]  

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equations (A.47), (A.48), (A.49) with the superscript 1 deleted, can be written as

\[ M_z = \frac{(z+\phi \beta)}{\Delta'} \{ K_\zeta - R(K_\zeta - K_B) \sin^2 \theta \} + \frac{(\beta-\phi \zeta)}{\Delta'} R(K_\zeta - K_B) \sin \theta \cos \theta \] (A.50)

\[ M_y = -\frac{(\beta-\phi \zeta)}{\Delta'} \{ K_B + R(K_\zeta - K_B) \sin^2 \theta \} - \frac{(z+\phi \beta)}{\Delta'} R(K_\zeta - K_B) \sin \theta \cos \theta \] (A.51)

\[ M_z = K_\phi (\phi - \zeta \beta) \] (A.52)

where

\[ R = \frac{1/K_B - 1/K_\zeta}{(1/K_B + 1/K_\zeta) - (1/K_B + 1/K_\zeta)} \]

and \( \Delta' = 1 + R (1 - R) \sin^2 \theta (K_\zeta - K_B)^2 / K_B K_\zeta \)

When the hub stiffness are very large \( R \) becomes unity and the moment equations reduce to those obtained previously. When the blade stiffness is very large, \( R \) reduces to zero. These moment expressions have to be multiplied by \((-1)\) to obtain the restoring moments of the springs.

Peters [Ref. 14] has given the moment expressions for a spring model including hub flexibility. The torsional degree of freedom was not considered in Ref. 14, thus when substituting \( \phi = 0 \), equations (A.50) - (A.51) reduce to the equations given by Peters.

Combining the results for an articulated blade and for an elastic blade, general expressions for moment are given below.

The restoring moment in flap is

\[ M_y = \frac{(\beta-\phi \zeta)}{\Delta'} \{ K_B + R(K_\zeta - K_B) \sin^2 \theta \} + \frac{(z+\phi \beta)}{\Delta'} R(K_\zeta - K_B) \sin \theta \cos \theta \] (A.53)

The restoring lead-lag moment is

\[ M_z = -\frac{(z+\phi \beta)}{\Delta'} \{ K_\zeta - R(K_\zeta - K_B) \sin^2 \theta \} - \frac{(\beta-\phi \zeta)}{\Delta'} R(K_\zeta - K_B) \sin \theta \cos \theta \] (A.54)

The restoring torsional moment is

\[ M_x = -K_\phi (\phi - \zeta \beta) - K_\phi \phi \] (A.55)
For an articulated blade

\[ K_B = K_\zeta = K_\phi_1 = 0 \text{ and } K_\phi_2 = \frac{K_\phi_1 K_B}{K_\phi_1 + K_B} \]

For a hingeless blade \( K_\phi_2 = 0 \)

\[ K_B = \frac{K_\phi_1 K_B}{K_\phi_1 + K_B} ; \quad K_\zeta = \frac{K_\phi_1 K_\zeta}{K_\phi_1 + K_\zeta} ; \quad K_\phi_1 = \frac{K_\phi_1 K_\phi_2}{K_\phi_1 + K_\phi_2} \]

\[ R = \frac{1/K_B - 1/K_\zeta}{(1/K_B + 1/K_\zeta) - (1/K_\phi_1 + 1/K_\phi_2)} \]

and \( \Delta' = 1 + R (1 - R) \sin^2 \theta \left( K_\zeta - K_\phi \right)^2/K_\zeta K_\phi \)
Figure A1. Equivalent Spring Restrained Blade Model
Figure A2. Undeformed Axes

Figure A3. Flap Angle

Figure A4. Lead-Lag Angle

Figure A5. Torsion Angle
Figure A6. Articulated Blade Model

Figure A7. Orientation of the Deformed Blade After Flap and Lag Motion
Figure A8. Equivalent Spring Restrained Blade Model With Hub Flexibility
This report presents a set of governing coupled differential equations for a model of a hybrid aircraft. The model consists of multiple rotor systems connected by an elastic interconnecting structure, with options to add any combination of or all of the following components; i.e., thrusters, a buoyant hull, and an underslung weight. The dynamic equations are written for the individual blade with hub motions, for the rigid body motions of the whole model, and also for the flexible modes of the interconnecting structure. One of the purposes of this study is to serve as the basis of a numerical study aimed at determining the aeroelastic stability and structural response characteristics of a Hybrid Heavy Lift Airship (HHLA). It is also expected that the formulation may be applicable to analyzing stability and responses of dual rotor helicopters such as a Heavy Lift Helicopter (HLH). Furthermore, the model is capable of representing coupled rotor/body aeromechanical problems of single rotor helicopters.

17. Key Words (Suggested by Author(s))
- Rotary wing aeroelasticity
- Rotor dynamics
- Multirotor systems
- Aeromechanical stability

18. Distribution Statement
Unclassified - Unlimited

Subject Category 08

19. Security Classif. (of this report)
Unclassified

20. Security Classif. (of this page)
Unclassified

21. No. of Pages
145

22. Price
A07