84-32405

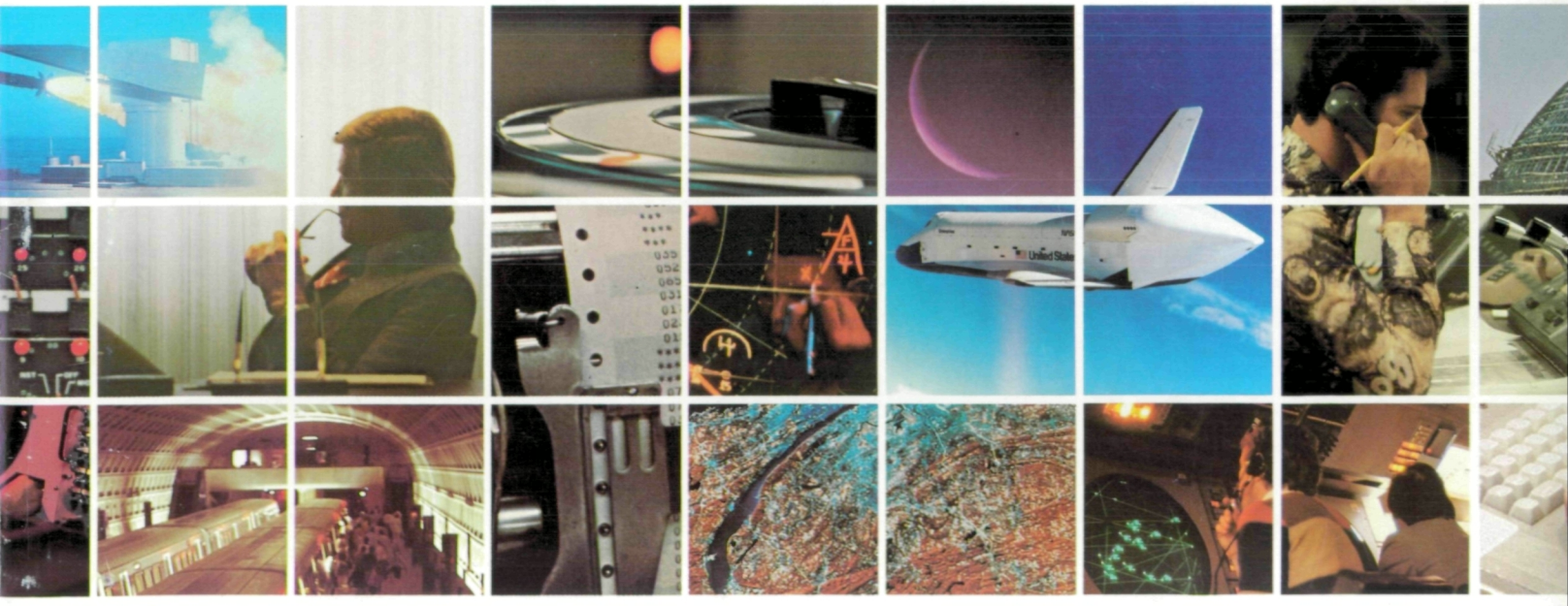
CSC/TM-84/6083

FLIGHT EXPERIMENT DEMONSTRATION SYSTEM (FEDS) MATHEMATICAL SPECIFICATION

Prepared for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Goddard Space Flight Center
Greenbelt, Maryland

CONTRACT NAS 5-27888
Task Assignment 42000

JULY 1984



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By

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Under

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ABSTRACT

This document provides computational models for the Flight Experiment Demonstration System (FEDS), developed at the Goddard Space Flight Center System Technology Laboratory (STL), Code 580. FEDS is a modification of the Automated Orbit Determination System developed at the STL during 1981 and 1982.

9681

TABLE OF CONTENTS

Section 1 - Introduction. 1-1

Section 2 - Coordinate Systems, Transformations, and Time Systems. 2-1

2.1 Coordinate Systems and Transformations 2-1

2.1.1 True Equator and Equinox of Epoch (Inertial) System 2-1
2.1.2 Earth-Fixed System. 2-3
2.1.3 Earth-Fixed-to-TOE Transformation 2-4
2.1.4 Computation of the Greenwich Sidereal Time and Hour Angle 2-6

2.2 Time Systems 2-8
2.3 Time Representations in FEDS 2-9

Section 3 - Orbit Propagation Models. 3-1

3.1 Runge-Kutta Starter. 3-1

3.1.1 Runge-Kutta Integration Algorithm 3-2
3.1.2 Initial Computation of the Ordinate First and Second Sums 3-4

3.2 Multistep Integration. 3-7

3.2.1 Prediction and Correction Formulas. . . . 3-8
3.2.2 PECE* Algorithm for Equations of Motion. 3-12
3.2.3 Corrector-Only Algorithm for Variational Equations. 3-17
3.2.4 State Transition Matrix Computation . . . 3-20

3.3 Multistep Interpolation. 3-21
3.4 Equations of Motion for State Variables. 3-24

3.4.1 Earth, Solar, and Lunar Point Mass Accelerations 3-26
3.4.2 Nonspherical Gravitational Acceleration. 3-35
3.4.3 Atmospheric Drag Acceleration 3-40
3.4.4 Solar Radiation Pressure Acceleration . . 3-44

3.5 Variational Equations. 3-46

TABLE OF CONTENTS (Cont'd)

Section 4 - Data Processing for Transponder Interface 4-1

4.1 Reduction of TDRSS Doppler Data. 4-1
4.2 Creation of Frequency Control Words. 4-2

Section 5 - Estimation Logic. 5-1

5.1 Initial Summation of Batch Estimation Matrices . . 5-3
5.2 State Correction and Inner Edit. 5-5
5.3 End-of-Iteration Testing 5-6
5.4 Preliminary Edit Criteria. 5-8
5.5 Slide Precomputation 5-9

Section 6 - Observation Models. 6-1

6.1 Measurement Equations. 6-1
6.2 Partial Derivatives of Measurements With Respect to Target State. 6-5
6.3 Frequency Bias Effects on Observation Modeling . . 6-6
6.4 Miscellaneous Models Required for Observation Modeling 6-7

6.4.1 Newton-Raphson Light-Time Corrector . . . 6-8
6.4.2 Tropospheric Refraction Corrections . . . 6-9

References

LIST OF ILLUSTRATIONS

Figure

2-1	Geocentric Coordinate System	2-2
2-2	Earth-Fixed Coordinate System.	2-4
2-3	TOE-to-Earth-Fixed Coordinate Rotation	2-5
3-1	Cylindrical Shadow Model	3-46
6-1	Signal Geometry for Doppler Observation. . . .	6-2

LIST OF TABLES

Table

2-1	TJD-to-Calendar-Date Conversion.	2-11
3-1	Coefficients for Runge-Kutta 3(4+) Integrator	3-5
3-2	Adams-Bashforth 11th-Order Predictor Coefficients	3-9
3-3	Adams-Moulton 11th-Order Corrector Coefficients	3-10
3-4	Adams-Moulton 7th-Order Corrector Coefficients	3-11
3-5	Störmer 12th-Order Predictor Coefficients. . .	3-13
3-6	Cowell 12th-Order Corrector Coefficients . . .	3-14
3-7	Cowell 8th-Order Corrector Coefficients. . . .	3-15
3-8	Series for λ_M.	3-31
3-9	Series of θ_M	3-33
3-10	Series for κ_M.	3-34

9681

SECTION 1 - INTRODUCTION

This document provides computational models for the Flight Experiment Demonstration System (FEDS), developed at the Goddard Space Flight Center (GSFC) System Technology Laboratory (STL), Code 580. FEDS is a modification of the Automated Orbit Determination System (AODS), developed at the STL during 1981 and 1982. The purpose of FEDS is to demonstrate, in a simulated spacecraft environment, the feasibility of using microprocessors to perform onboard orbit determination with limited ground support.

During the demonstration, FEDS will execute on the STL's PDP-11/23 microcomputer, located at a remote site. FEDS will be connected to a transponder via a communications link and to the AODS Environment Simulator for Prototype Testing (ADEPT), modified to support FEDS. The transponder will provide observation data, accumulated in real time from a Tracking and Data Relay Satellite System (TDRSS) signal, to FEDS for use in the orbit determination process. ADEPT, executing on the PDP-11/70, will provide all other external information required by FEDS.

The models presented in this document were largely taken from the AODS System Description, Section 5 (Reference 1). In most cases, only the algorithms are presented here; no attempt is made to derive models or show how they are integrated into FEDS. The reader should refer to the AODS System Description, Section 3, for a logical description of the corresponding tasks.

The coordinate systems, transformations, and time systems used throughout FEDS are described in Section 2. The orbit propagation algorithms, including integration and interpolation algorithms, the equations of motion, and the variational

equations, are described in Section 3. Algorithms for reduction of raw observation data and production of frequency control words are given in Section 4, and the estimation algorithms and observation models are presented in Sections 5 and 6, respectively.

SECTION 2 - COORDINATE SYSTEMS, TRANSFORMATIONS, AND TIME SYSTEMS

This section contains a description of the coordinate systems, transformations, and time systems used in FEDS. It also contains the algorithm for the Greenwich hour angle calculation, which is used in transformations throughout the system.

2.1 COORDINATE SYSTEMS AND TRANSFORMATIONS

In FEDS, the propagation of the satellite's state vector and state transition matrix is performed in geocentric rectangular coordinates referenced to a true Equator and equinox of epoch (TOE) coordinate frame. The satellite's acceleration vector and the partial derivatives of the acceleration vector are also expressed in this geocentric system. This system is obtained by freezing the true Equator and equinox of date (TOD) system at a specified epoch (reference time). The tracking measurements are also computed in the TOE system. However, the coordinates of the ground-fixed antenna are expressed in Earth-fixed coordinates and must be transformed to the TOE system. Each of these coordinate systems is defined in the following subsections along with the required transformations.

2.1.1 TRUE EQUATOR AND EQUINOX OF EPOCH (INERTIAL) SYSTEM

The geocentric TOE system is defined as follows:

Origin	Center of the Earth
Reference plane	Equatorial plane of the Earth, perpendicular to the Earth's spin axis at epoch
Principal direction	True vernal equinox of epoch

The equinox is defined as the intersection of the planes of the Earth's Equator and the ecliptic. The Equator is defined as being normal to the Earth's instantaneous spin

axis. The ecliptic is the Earth-Sun orbital plane. The TOE system is equivalent to the TOD system associated with the specified epoch date. However, the TOD system changes with time because of the motion of the equinox, which is due, in turn, to the combined motions of the two planes (the Equator and the ecliptic) that define it.

The motion of the Earth's spin axis or of the Earth's Equator is due to the gravitational attraction of the Sun and the Moon on the Earth's equatorial bulge. This motion consists of lunisolar precession and nutation. The motion of the ecliptic is due to the planets' gravitational pull on the Earth and consists of a slow rotation of the ecliptic. This motion is known as planetary precession.

The rectangular Cartesian coordinates (see Figure 2-1) associated with the TOE coordinate system are defined with respect to the following axes:

x-axis = principal direction

y-axis = normal to the x and z axes to form a right-hand system

z-axis = normal to the equatorial plane of epoch in the direction of the Earth's spin axis

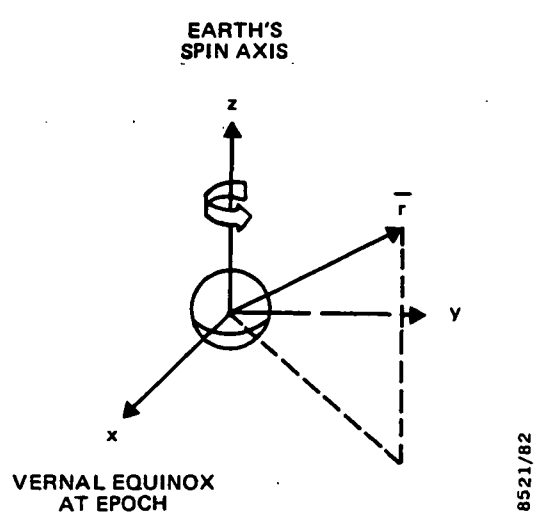

Figure 2-1. Geocentric Coordinate System

Quantities $\vec{r}$, x, y, and z designate the position vector and Cartesian coordinates referred to the TOE frame.

2.1.2 EARTH-FIXED SYSTEM

The Earth-fixed coordinate system is defined as follows:

Origin	Center of the Earth
Reference plane	Equatorial plane of the Earth, perpendicular to the adopted polar geographic axis
Principal direction	Intersection of the Greenwich meridian with the reference plane

The Earth's axis of figure (i.e., principal moment of inertia) is not coincident with the Earth's instantaneous spin axis. It moves with respect to the latter, causing the polar motion effect. Therefore, the motion of the spin axis pole is given with respect to the pole at some established epoch. The pole at the established epoch is referred to as the adopted geographic pole and corresponds to the Earth-fixed z-axis, z_b. The adopted geographic pole is the mean pole of 1903.0, which is consistent with that used by the International Polar Motion Service.

The Greenwich meridian is the plane containing the adopted polar axis that passes through the former Royal Observatory at Greenwich, England.

The rectangular Cartesian coordinates (see Figure 2-2) associated with the Earth-fixed coordinate system are defined with respect to the following axes:

x_b-axis = principal direction

y_b-axis = normal to the x_b and z_b axes to form a right-hand system

z_b-axis = axis along the vector passing through the adopted geographic pole

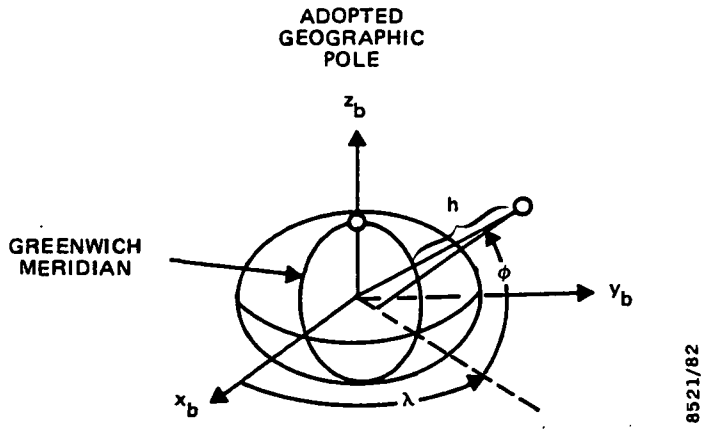

Figure 2-2. Earth-Fixed Coordinate System

2.1.3 EARTH-FIXED-TO-TOE TRANSFORMATION

The exact transformation that relates the TOE coordinates to the Earth-fixed coordinates accounts for three separate effects. The first relates the true vernal equinox of epoch to the true vernal equinox of date (i.e., accounts for precession and nutation effects). The second effect relates the true equinox of date of the Greenwich meridian of the rotating Earth by means of the angle, α_g, the true right ascension of Greenwich (see Figure 2-3). The third effect, called polar motion, accounts for the fact that the pole of the Earth-fixed axis, z_b, does not coincide with the Earth's instantaneous spin axis, the pole of the TOD geocentric axes.

In FEDS, the first and third effects are assumed to be zero. This is equivalent to assuming that the TOE z-axis is coincident with both the instantaneous spin axis and the Earth-fixed z_b-axis. The transformation from the TOE to the Earth-fixed coordinate system reduces to a rotation

about the TOE z-axis through the true right ascension of Greenwich, α_g, yielding

$$\vec{r}_b = B_1 \vec{r} \tag{2-1}$$

where

$$B_1 = \begin{bmatrix} \cos \alpha_g & \sin \alpha_g & 0 \\ -\sin \alpha_g & \cos \alpha_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-2}$$

Computation of the true right ascension of Greenwich, α_g, is discussed in Section 2.1.4.

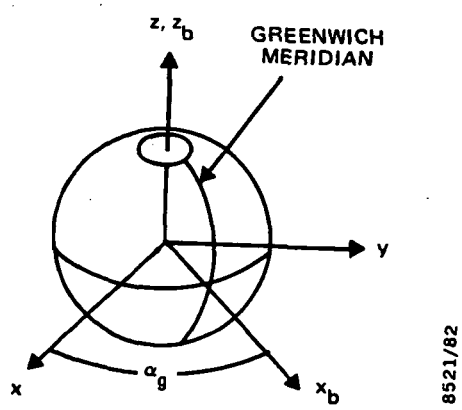

Figure 2-3. TOE-to-Earth-Fixed Coordinate Rotation

The Earth-fixed coordinates are transformed into TOE coordinates as follows:

$$\vec{r} = B_1^T \vec{r}_b \tag{2-3}$$

Differentiation yields the velocity transformation

$$\dot{\vec{r}} = B_1^T \dot{\vec{r}}_b + \dot{B}_1^T \vec{r}_b \tag{2-4}$$

where

$$\dot{B}_1 = \begin{bmatrix} -\sin \alpha_g & \cos \alpha_g & 0 \\ -\cos \alpha_g & -\sin \alpha_g & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\alpha}_g \tag{2-5}$$

and where α_g (the rate of change of α_g) is considered to be constant.

2.1.4 COMPUTATION OF THE GREENWICH SIDEREAL TIME AND HOUR ANGLE

The TOD right ascension of Greenwich, α_g, is measured easterly from the true vernal equinox to Greenwich. A related quantity is the Greenwich hour angle (GHA), also called the true Greenwich sidereal time, which is measured westerly in the plane of the Equator from Greenwich to the true vernal equinox. Thus, although their definitions differ, the right ascension of Greenwich, α_g, and the Greenwich sidereal time and hour angle are equal in magnitude. In FEDS software, the approximation is made that the TOE Greenwich sidereal time is equal to the TOD Greenwich sidereal time.

The true Greenwich sidereal time, α_g, is obtained from the mean Greenwich sidereal time, α_{GM}, by applying a correction, ΔH, due to nutation in longitude and obliquity as follows:

$$\alpha_g = \alpha_{GM} + \Delta H \tag{2-6}$$

The mean Greenwich sidereal time is calculated using the following equation:

$$\alpha_{GM} = 100\overset{\circ}{.}075542 + 0\overset{\circ}{.}98564735\ \Delta d + (2\overset{\circ}{.}9015 \times 10^{-13})\ \Delta d^2 + \omega_e f_d \tag{2-7}$$

where ω_e is the angular velocity (degrees per second) of the Earth and is given by

$$\omega_e = \frac{0\overset{\circ}{.}0041780742}{1.0 + (5.21 \times 10^{-13})\ \Delta d} \tag{2-8}$$

where Δd = number of whole days elapsed from atomic time (A.1) January 0, 1950

f_d = fraction of days measured from midnight (seconds)

If the input request time is expressed in terms of the A.1 modified Julian date (i.e., days since 2430000), DJO, then

$$DJO - 3282.5 = \Delta d + f_d \tag{2-9}$$

The nutation term, ΔH, is computed by evaluating a ninth-order nutation polynomial derived from the GTDS solar-lunar-planetary ephemeris (SLP) file (Reference 2, Section 3.6)

$$\Delta H = P_1 + \sum_{i=2}^{10} P_i (\Delta t)^{i-1} \tag{2-10}$$

where P_i = coefficient of the ith term of the nutation polynomial that covers the request time, DJO

Δt = DJO - (PDELHT + 10) (days)

PDELHT = uplink start of polynomial fit (modified Julian date in A.1 days from start of SLP file)

Since each nutation polynomial covers a 20-day span, two polynomials are used in FEDS to provide for continuity over the estimation data arc. Only the observation modeling routines require the addition of the nutation term described earlier.

Subsequent requests to compute α_g at another time, t, may use a reference time and an associated $\alpha_{g_{ref}}$ as follows:

$$\alpha_g = \alpha_{g_{ref}} + \omega(t - t_{ref}) \tag{2-11}$$

where $\alpha_{g_{ref}}$ = reference GHA computed at time t_{ref}

ω = rotation rate of the Earth

2.2 TIME SYSTEMS

The basic time system used in FEDS is the A.1 system. Because time advances at a constant rate in A.1 time, simulation time can be compared directly with system clock time (in seconds from reference) for system control and scheduling purposes. All input time tags are in the Universal Time Coordinated (UTC) system and are converted to A.1 time on input as follows:

$$t_{A.1} = t_{IN} + a_{i1} \tag{2-12}$$

where $t_{A.1}$ = internal time (A.1)

t_{IN} = input time (UTC)

a_{i1} = constant[1] adjustment (UTC to A.1) for time period surrounding t_{IN}

After output, all time tags are converted from A.1 to UTC as follows:

$$t_{OUT} = t_{A.1} - a_{i1} \tag{2-13}$$

where $t_{A.1}$ = internal time (A.1)

t_{OUT} = output time (UTC)

a_{i1} = constant[1] adjustment (UTC to A.1) for time period surrounding t_{OUT}

One other time system is used in FEDS. The A.1 time is converted to ephemeris time (ET) during the computation of the Sun and the Moon positions in the orbit propagator as follows:

$$t_{ET} = t_{A.1} + \frac{32.15}{86400} \text{ days} \tag{2-14}$$

2.3 TIME REPRESENTATIONS IN FEDS

Time is represented in three basic ways in FEDS. When the START command is received, a time message is requested from an external Parallel Grouped Binary Time Code 5 (PB5) generator. This message is used to establish a simulation reference time. All input time tags are converted from the input form of YYMMDDHHMMSS.SS (UTC) to seconds from reference (A.1). All time parameters are kept and measured in FEDS in terms of seconds from reference. System clock time is also measured in seconds from reference for controlling real-time processing.

---

[1]The standard conversion polynomial is $t_{A.1} = t_{UTC} + a_{1i} + a_{2i}\,\Delta t + a_{3i}\,\Delta t^2$. However, since a_{2i} and a_{3i} are zero for the times FEDS will use, they were omitted to save memory.

The PB5 time code used in FEDS comprises three groups, totaling 41 bits: truncated Julian's Day (TJD), seconds of day, and milliseconds. TJD is defined as the number of whole days since midnight (UTC) of May 5, 1968, and is computed as follows:

$$TJD = \text{Julian Day Number} - 244000.5$$

The TJD-to-calendar date conversion is presented in Table 2-1.

Modified Julian dates are also used throughout FEDS. A modified Julian date is defined as follows in FEDS:

$$DJO = \text{days since } 2430000 \text{ in A.1 time}$$

The reference time is also stored in modified Julian date form for quick computation of days since reference time.

Table 2-1. TJD-to-Calendar-Date Conversion

TJD	Date (Year, Month, and Day)
0	680524*
222	690101
587	700101
952	710101
1317	720101*
1683	730101
2048	740101
2413	750101
2778	760101*
3144	770101
3509	780101
3874	790101
4239	800101*
4605	810101
4970	820101
5335	830101
5700	840101*
6066	850101
6431	860101
6796	870101
7161	880101*
7527	890101
7892	900101
8257	910101
8622	920101*
8988	930101
9353	940101
9718	950101
0	951010
⋮	⋮

*Denotes Leap Year

SECTION 3 - ORBIT PROPAGATION MODELS

Orbit propagation is performed in FEDS using a multistep integration process (with a single-step starter). The propagator integrates one full fixed-length step at a time and uses a multistep interpolator to obtain the spacecraft state or state transition matrix at the request time.

Because the FEDS orbit propagator is required to propagate high-altitude geosynchronous orbits (TDRS) and low-altitude drag-perturbed orbits (user spacecraft), various forces are modeled in the equations of motion, including atmospheric drag, up to a 15-by-15 geopotential field, gravitational effects of the Earth, the Sun, and the Moon, and solar radiation pressure. However, due to timing and memory constraints, only the gravitational effect of the Earth and zonal coefficients, X_{J_2}, X_{J_3}, and X_{J_4}, are used in the variational equations.

The computational models for the multistep integrator and interpolator, the single-step integrator (starter), the equations of motion, and the variational equations are given in the following subsections.

3.1 RUNGE-KUTTA STARTER

The multistep method avoids the multiple function evaluations at each integration step that are required by the Runge-Kutta single-step method, but it is not self-starting. Starting from an initial position and velocity, the Runge-Kutta method is used to build the required starting array for the Cowell equations of motion and variational equations. After the Runge-Kutta integrator has entered 10 equally spaced accelerations in the backpoints table, the initial ordinate first and second sums are computed. The Runge-Kutta algorithm and the initial computation of the

first and second ordinate sums are presented in the following subsections.

3.1.1 RUNGE-KUTTA INTEGRATION ALGORITHM

The Runge-Kutta method is a single-step, self-starting numerical integration technique by which the value of the dependent variable at some future time, $t_1 + \tau$, where τ is the starter integration step size ($\tau = h/6$, where h is the multistep integration step size), can be calculated from a weighted summation formula and the value of the dependent variable at t_1. The integrator algorithm is as follows.

Given a first-order differential equation of the form

$$\frac{dx}{dt} = f(t, x)$$

and an initial value

$$x_1 = x(t_1)$$

the dependent variable, x, at time $t_1 + \tau$ is as follows:

$$x(t_1 + \tau) = x_1 + \tau \sum_{j=1}^{N} w_j f_j \qquad (3\text{-}1)$$

where

$$f_1 = f(t_1, x_1) \qquad (3\text{-}2)$$

and

$$f_j = f\left(t_j + \tau\gamma_j,\ x_1 + \tau \sum_{i=1}^{j-1} \beta_{ji} f_i\right) \tag{3-3}$$

The derivatives f are evaluated at n subintervals from t_1 to $t_1 + \tau$. The location of these subintervals is defined by coefficients γ_j. The summation is weighted by β_{ji} and w_j. The number of derivative evaluations needed is N, and the error made in the integration is of the order of τ^{p+1}, where p is the order of the method, which is not necessarily equal to N.

Runge-Kutta methods have been developed by comparing two integrators of different order in which the lower order method derivatives are a subset of those used for the higher order. An estimate of the truncation error made from using the lower order integration method can be obtained by differencing the two solutions of orders p and q (where $q < p$) to obtain

$$E(t_1 + \tau) = \tau \sum_{j=1}^{N} \delta_j f_j \tag{3-4}$$

where N is the number of derivative evaluations for the solution of order p. Equation (3-4) can be used to compute the relative truncation error, E_{rel}, at each integration step as follows:

$$E_{rel} = \frac{E(t_1 + \tau)}{|x\ (t_1 + \tau)|} \tag{3-5}$$

Runge-Kutta integration formulas are available for which the integration constants have been chosen so as to minimize the truncation error of the higher order method and to use all free parameters in the equations of condition to optimize the integrator. This approach produces better integrators in that there are fewer function evaluations for the accuracy obtained than for Runge-Kutta integrators derived without such considerations.

The integrator used in FEDS is a fixed-step integrator derived under these conditions. The integration coefficients (see Table 3-1 drawn from Reference 3) require five derivative evaluations and produce a solution accurate to order p, where p is greater than four but less than five. The error coefficients, δ_j, are consistent with the difference of solutions of orders p and q, where q equals three. For this reason, the algorithm is referred to as a Runge-Kutta 3(4+) method.

3.1.2 INITIAL COMPUTATION OF THE ORDINATE FIRST AND SECOND SUMS

The equations for computing the initial ordinate first and second sums for state integration are as follows:

$$I_{S_{n-1}} = \frac{\dot{x}_n}{h} - \sum_{i=0}^{10} \beta^*_{i+1} \ddot{x}_{n-i+1} \tag{3-6}$$

$$II_{S_{n-1}} = \frac{x_n}{h^2} - \sum_{i=0}^{10} \alpha^*_{i+2} \ddot{x}_{n-i+1} \tag{3-7}$$

where $n = 10$

x_n and $\dot{x}_n$ = position and velocity vectors at time t_n integration

Table 3-1. Coefficients for Runge-Kutta 3(4+) Integrator

INDEX j	COEFFICIENTS		
	α_j	C_j	δ_j*
1	0	1.0888304916410344 D-1	−4.9939143641338931 D-3
2	2/7	2.4547988976401333 D-1	6.8461187539313607 D-2
3	7/15	3.3907286031851855 D-1	−1.0922264375986756 D-1
4	35/38	4.4251397222216452 D-1	1.8170514205348768 D-1
5	1	−1.3594977146879983 D-1	−1.3594977146879983 D-1

8521/82

*δ_j COEFFICIENTS ARE INCLUDED HERE TO SUPPORT DISCUSSION; THEY ARE NOT USED IN AODS.

INDEX	COEFFICIENTS $\beta_{j,i}$			
j \ i	1	2	3	4
1	–	–	–	–
2	2.8571428571428571 D-1	–	–	–
3	−9.3777509730408418 D-2	5.6044417639707508 D-1	–	–
4	6.0836163032387864 D-1	−1.2040651669594905 D+0	1.5167561682145592 D+0	–
5	1.4613130509525770 D+0	−3.8111651289355121 D+0	3.6068242907277299 D+0	−2.5697221274479485 D-1

8521/82

h = integration step size

$\ddot{x}$ = accelerations stored in the backpoints table

β_i^* = Adams-Moulton correction coefficients

α_i^* = Cowell correction coefficients

These equations are derived by inverting the Adams-Moulton and Cowell correction formulas given in Section 3.2 (Equations (3-15) and (3-17)).

The following equations are used to update the sums to time t_n:

$$^{I}S_n = {}^{I}S_{n-1} + \ddot{x}_n \tag{3-8}$$

$$^{II}S_n = {}^{II}S_{n-1} + {}^{I}S_n \tag{3-9}$$

The equations for computing the initial ordinate first and second sums for the partial derivative integration are as follows:

$$^{I}P_{n-1} = \frac{\dot{Y}}{h} - \sum_{i=0}^{10} \beta_{i+1}^* \ddot{Y}_{n+1-i} \tag{3-10}$$

$$^{II}P_{n-1} = \frac{Y}{h^2} - \sum_{i=0}^{10} \alpha_{i+2}^* \ddot{Y}_{n+1-i} \tag{3-11}$$

where $n = 10$

$\dot{Y}$ = partial derivatives of the velocity at t_n

Y = partial derivatives of the position at t_n

$\ddot{Y}$ = partial derivatives of the accelerations stored in the backpoints table

h, α_i^*, β_i^* = the values defined previously

These equations are derived by inverting the Adams-Moulton and Cowell correction formulas given in Section 3.2.1.

The following equations are used to update the sums to time t_n:

$$I_{P_n} = I_{P_{n-1}} + \ddot{Y} \tag{3-12}$$

$$II_{P_n} = II_{P_{n-1}} + I_{P_n} \tag{3-13}$$

3.2 MULTISTEP INTEGRATION

Multistep integration methods minimize the number of derivative evaluations required to produce a given accuracy at the end of the requested interval of integration. Since, in general, the major cost in computing an orbit is the evaluation of the complex force function (see Section 3.4), multistep algorithms appear to be most efficient. FEDS uses a 12th-order, fixed-step, multistep integrator. A predict/evaluate derivative/correct/pseudoevaluate (PECE*) algorithm is used to integrate the equations of motion, and a corrector-only algorithm is used to integrate the variational equations. The summed ordinate predictor/corrector formulas are given in Section 3.2.1, and the algorithms for integrating the equations of motion and the variational equations are described in Sections 3.2.2 and 3.2.3, respectively. Finally, the algorithm for computing the state transition matrix for mapping the partial derivatives is given in Section 3.2.4.

3.2.1 PREDICTION AND CORRECTION FORMULAS

In FEDS, the Adams-Bashforth predictor and the Adams-Moulton corrector methods are used to integrate first-order equations of motion and the velocity vector. The Störmer predictor and the Cowell corrector methods are used for second-order equations of motion of the position vector.

The Adams-Bashforth prediction formula used to predict velocity is as follows:

$$\dot{x}_{n+1} = h \left[{}^{I}S_n + \sum_{i=0}^{k} \beta_i' \, \ddot{x}_{n-1} \right] \tag{3-14}$$

where n is the number of backpoints available and k+1 is the order of the integrator. The Adams-Bashforth 11th-order coefficients, β_i', are given in Table 3-2.

The Adams-Moulton correction formula used to correct velocity is as follows:

$$\dot{x}_{n+1} = h \left[{}^{I}S_n + \sum_{i=0}^{k} \beta_{i+1}^{*} \, \ddot{x}_{n+1-i} \right] \tag{3-15}$$

where n is the number of backpoints in the table and k+1 is the order of the integrator. The Adams-Moulton correction coefficients, β_i^*, i.e., the 11th-order coefficients for equations of motion and the 7th-order coefficients for variational equations, are given in Tables 3-3 and 3-4, respectively.

Table 3-2. Adams-Bashforth 11th-Order Predictor Coefficients

$$\beta'_1 = 3.451988400456282$$

$$\beta'_2 = -13.81683726526174$$

$$\beta'_3 = 35.43365334896585$$

$$\beta'_4 = -60.83539672518839$$

$$\beta'_5 = 72.18373924011945$$

$$\beta'_6 = -59.70768131463444$$

$$\beta'_7 = 33.93953914141414$$

$$\beta'_8 = -12.67749704385121$$

$$\beta'_9 = 2.808681814424002$$

$$\beta'_{10} = -0.2801895964439367$$

Table 3-3. Adams-Moulton 11th-Order Corrector Coefficients

$$\beta_1^* = 0.2801895964439367$$

$$\beta_2^* = 0.6500924360169152$$

$$\beta_3^* = -1.208305425284592$$

$$\beta_4^* = 1.810901775693442$$

$$\beta_5^* = -1.99558147196168$$

$$\beta_6^* = 1.575960936247395$$

$$\beta_7^* = -0.8678660614077281$$

$$\beta_8^* = 0.3167875681417348$$

$$\beta_9^* = -0.06896520387405804$$

$$\beta_{10}^* = 0.006785849984634707$$

Table 3-4. Adams-Moulton 7th-Order Corrector Coefficients

$$\beta_1^* = 0.3155919312169312$$

$$\beta_2^* = 0.3921792328042328$$

$$\beta_3^* = -0.3760251322751323$$

$$\beta_4^* = 0.2440806878306878$$

$$\beta_5^* = -0.09009589947089947$$

$$\beta_6^* = 0.01426917989417989$$

The Störmer prediction formula used to predict position is as follows:

$$x_{n+1} = h^2 \left[II_{S_n} + \sum_{i=0}^{k} \alpha'_{i+2} \ddot{x}_{n-1} \right] \tag{3-16}$$

where n is the number of backpoints in the table and k+2 is the order of the integrator. The 12th-order Störmer prediction coefficients, α'_i, are given in Table 3-5.

The Cowell correction formula used to correct position is as follows:

$$x_{n+1} = h^2 \left[II_{S_n} + \sum_{i=0}^{k} \alpha^*_{i+2} \ddot{x}_{n+1-i} \right] \tag{3-17}$$

where n is the number of backpoints in the table and k+2 is the order of the integrator. The Cowell correction coefficients, α^*_i, i.e., 12th-order coefficients for equations of motion and 8th-order coefficients for variational equations, are given in Tables 3-6 and 3-7, respectively.

3.2.2 PECE* ALGORITHM FOR EQUATIONS OF MOTION

The concept of pseudoevaluation is introduced as a device to help stabilize the numerical integration at little or no cost in computation. It is recognized that in a predictor-corrector scheme, the numerical stability region is proportional to the number of derivative evaluations within a given step (Reference 4) and that for systems of the form, $X = f(X) + \varepsilon g(X)$, where ε is a small parameter, the stability region is mainly influenced by the f(X) term.

The idea, then, is to introduce into a predictor-corrector algorithm, which is designed to solve the system above, a

Table 3-5. Störmer 12th-Order Predictor Coefficients

$$\alpha'_2 = 0.7093330001402910$$

$$\alpha'_3 = -2.949775927679574$$

$$\alpha'_4 = 7.565365635521886$$

$$\alpha'_5 = -12.95853327421036$$

$$\alpha'_6 = 15.34411576078243$$

$$\alpha'_7 = -12.67150744799182$$

$$\alpha'_8 = 7.193720363355780$$

$$\alpha'_9 = -2.684381939434023$$

$$\alpha'_{10} = 0.594237726972102$$

$$\alpha'_{11} = -0.05924056412337662$$

Table 3-6. Cowell 12th-Order Corrector Coefficients

$$\alpha_2^* = 0.05924056412337662$$

$$\alpha_3^* = 0.1169273589065256$$

$$\alpha_4^* = -0.2839505421276255$$

$$\alpha_5^* = 0.4564979407166907$$

$$\alpha_6^* = -0.5180148083012666$$

$$\alpha_7^* = 0.4154936016915184$$

$$\alpha_8^* = -0.2309889820827321$$

$$\alpha_9^* = 0.08485266855058522$$

$$\alpha_{10}^* = -0.01855655388207472$$

$$\alpha_{11}^* = 0.001832085738335738$$

Table 3-7. Cowell 8th-Order Corrector Coefficients

$$\alpha_2^* = 0.06820436507936508$$

$$\alpha_3^* = 0.05115740740740741$$

$$\alpha_4^* = -0.07000661375661376$$

$$\alpha_5^* = 0.05019841269841270$$

$$\alpha_6^* = -0.01936177248677249$$

$$\alpha_7^* = 0.003141534391534392$$

pseudoevaluation, i.e., a partial evaluation of X, where f(X) is recomputed using the latest corrected value of X, and g(X) is reused based on a previous value of X. For example, assume that the equations to be integrated have the form

$$\ddot{\vec{r}} = \frac{-\mu\vec{r}}{r^3} + P(t, \vec{r}, \dot{\vec{r}}) \tag{3-18}$$

where the first term represents the primary attracting body acting on the satellite. Assuming that the accelerations and sums are known, then the iterative algorithm to advance to time t_{n+1} is

1. <u>Predict</u>. Using Equations (3-14) and (3-16), predict values (denoted by superscript p)

$$\vec{r}^{(p)}(t_{n+1}) = \left[x_{n+1}^{(p)}, y_{n+1}^{(p)}, z_{n+1}^{(p)}\right] \tag{3-19}$$

$$\dot{\vec{r}}^{(p)}(t_{n+1}) = \left[\dot{x}_{n+1}^{(p)}, \dot{y}_{n+1}^{(p)}, \dot{z}_{n+1}^{(p)}\right] \tag{3-20}$$

2. <u>Evaluate</u>. Using Equation (3-18), evaluate

$$\ddot{\vec{r}}(t_{n+1}) = \frac{-\mu\vec{r}_{n+1}^{(p)}}{r_{n+1}^{(p)3}} + P\left(t_{n+1}, \vec{r}_{n+1}^{(p)}, \dot{\vec{r}}_{n+1}^{(p)}\right) \tag{3-21}$$

3. <u>Correct</u>. Using Equations (3-15) and (3-17), obtain the improved values (denoted by superscript c), $\vec{r}_{n+1}^{(c)}$ and $\dot{\vec{r}}_{n+1}^{(c)}$.

4. <u>Pseudoevaluate</u>. Compute the acceleration

$$\ddot{\vec{r}}(t_{n+1}) = \frac{-\mu \vec{r}^{(c)}_{n+1}}{r^{(c)3}_{n+1}} + P\left(t_{n+1}, \vec{r}^{(p)}_{n+1}, \dot{\vec{r}}^{(p)}_{n+1}\right) \tag{3-22}$$

where the P term is obtained from step 2.

5. <u>Update Sums</u>. Compute the updated sums

$$I_{S_{n+1}} = I_{S_n} + \ddot{\vec{r}}(t_{n+1}) \tag{3-23}$$

$$II_{S_{n+1}} = II_{S_n} + I_{S_{n+1}} \tag{3-24}$$

The computational cycle of steps 1 through 5 may then be repeated with n = n + 1.

3.2.3 CORRECTOR-ONLY ALGORITHM FOR VARIATIONAL EQUATIONS

In the Cowell formulation, the position and velocity partial derivatives of the satellite motion with respect to any parameter appearing in the acceleration model in Equation (3-18) or state (dynamic parameters) may be obtained by the numerical integration of a system of equations of the form

$$\ddot{Y} = A(t)\ Y + B(t)\ \dot{Y} + C(t) \tag{3-25}$$

from initial conditions at time t_0 given by

$$Y(t_0) = \frac{\partial \vec{r}(t_0)}{\partial \vec{p}}, \quad \dot{Y}(t_0) = \frac{\partial \dot{\vec{r}}(t_0)}{\partial \vec{p}} \tag{3-26}$$

where

$$A(t) = \left[\frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{r}}\right]_{3x3} \tag{3-27}$$

$$B(t) = \left[\frac{\partial \ddot{\vec{r}}(t)}{\partial \dot{\vec{r}}}\right]_{3x3} \tag{3-28}$$

$$C(t) = \left[\frac{\partial \ddot{\vec{r}}(t)}{\partial \vec{p}}\right] \quad \text{3 x } \ell \text{ matrix of acceleration partial derivatives} \tag{3-29}$$

$$Y(t) = \left[\frac{\partial \vec{r}(t)}{\partial \vec{p}}\right] \quad \text{3 x } \ell \text{ matrix of position partial derivatives} \tag{3-30}$$

$$\dot{Y}(t) = \left[\frac{\partial \dot{\vec{r}}(t)}{\partial \vec{p}}\right] \quad \text{3 x } \ell \text{ matrix of velocity partial derivatives} \tag{3-31}$$

Vector $\vec{p}$ contains the parameters in the acceleration model to be estimated.

The components of matrices A, B, and C are developed in Section 3.5.

Optionally, the components of $\vec{p}$ correspond to the spacecraft's position and velocity at epoch and are expressed in true of date Cartesian coordinates. Since the first six elements of $\vec{p}$ are the state vectors, the first six columns of C are zero. The model parameter, drag, enters into $P(t, \vec{r}, \dot{\vec{r}})$ of Equation (3-18) linearly, so that the computation of C(t) is simplified by retaining many of the quantities used in the computation of $\ddot{\vec{r}}(t)$.

The integration of system Equation (3-25) is performed by utilizing the corrector-only formula as follows. Assume

that the satellite position and velocity, $\vec{r}(t_{n+1})$ and $\dot{\vec{r}}(t_{n+1})$, the matrices, $\ddot{Y}_{n+1}$ for step i, where i = 0, 1, 2, ..., k, and the summation matrices, ${}^{I}P_n$ and ${}^{II}P_n$ (3-by-ℓ, are known; the algorithm to advance Y to time t_{n+1} is

1. Compute matrices $A(t_{n+1})$, $B(t_{n+1})$, and $C(t_{n+1})$, which depend only on t_{n+1}, $\vec{r}_{n+1}$, and $\dot{\vec{r}}_{n+1}$.
2. Compute the 6-by-6 matrix $[I - H]^{-1}$

$$H = \begin{bmatrix} h^2\alpha_0^* A_{n+1} & h^2\alpha_0^* B_{n+1} \\ h\beta_0^* A_{n+1} & h\beta_0^* B_{n+1} \end{bmatrix} \tag{3-32}$$

   where α_0^* and β_0^* are the corrector coefficients of Equations (3-15) and (3-17), and h is the step size.

3. Form the 3-by-ℓ matrices, X_n and V_n, as follows:

$$X_n = h^2 \left[{}^{II}P_n + \sum_{i=1}^{k} \alpha_i^* \ddot{Y}_{n+1-i} + \alpha_0^* C_{n+1} \right] \tag{3-33}$$

$$V_n = h \left[{}^{I}P_n + \sum_{i=1}^{k} \beta_i^* \ddot{Y}_{n+1-i} + \beta_0^* C_{n+1} \right] \tag{3-34}$$

4. Compute the required position and velocity partial derivatives, Y_{n+1} and $\dot{Y}_{n+1}$, by the matrix equation

$$\begin{bmatrix} Y_{n+1} \\ \dot{Y}_{n+1} \end{bmatrix}_{6x\ell} = [I - H]^{-1}_{6x6} \begin{bmatrix} X_n \\ V_n \end{bmatrix}_{6x\ell} \tag{3-35}$$

5. Update acceleration and sums by

$$\ddot{Y}_{n+1} = A_{n+1} Y_{n+1} + B_{n+1} \dot{Y}_{n+1} + C_{n+1} \tag{3-36}$$

$$^{I}P_{n+1} = {}^{I}P_{n} + \ddot{Y}_{n+1} \tag{3-37}$$

$$^{II}P_{n+1} = {}^{II}P_{n} + {}^{I}P_{n+1} \tag{3-38}$$

and complete the cycle. After computing $\vec{r}_{n+2}$ and $\dot{\vec{r}}_{n+2}$, steps 1 through 5 may be repeated with n = n + 1.

3.2.4 STATE TRANSITION MATRIX COMPUTATION

In FEDS, the orbit propagator computes (if requested) the state transition matrix that is used in the observation model to map the observation partial derivatives with respect to the solve-for parameters from the observation time to the epoch time. Since the orbit propagator is always restarted at epoch each time the epoch is changed, the state transition matrix is as follows:

$$\Phi(t_n, t_0) = \begin{bmatrix} Y \\ \dot{Y} \end{bmatrix}_{6x\ell} \tag{3-39}$$

where t_n = request time

t_0 = epoch

Y = partial derivatives of the position at time t_n with respect to the solve-for parameters (see Equation (3-30))

$\dot{Y}$ = partial derivatives of the velocity at time t_n with respect to the solve-for parameters (see Equation (3-31))

3.3 MULTISTEP INTERPOLATION

The multistep interpolator uses the first and second ordinate sums and the 10 backpoints (accelerations) computed during multistep integration to compute the spacecraft position and velocity and the associated partial derivatives (if desired) at a request time.

The multistep interpolation algorithm is as follows:

1. Compute constants $\gamma_i'(0)$ and $\gamma_i''(0)$ to be used in later calculations as follows:

$$\gamma_0'(0) = 1$$

$$\gamma_i'(0) = -\sum_{j=0}^{i-1} \frac{1}{i - j + 1} \gamma_j'(0) \qquad (3\text{-}40)$$

$$\gamma_i''(0) = \sum_{j=0}^{i} \gamma_j'(0)\ \gamma_{i-j}'(0) \qquad (3\text{-}41)$$

where $i = 1, 2, \ldots, 10$.

2. Let $s = \frac{t - t_n}{h}$ in the following steps

where t_n = time associated with most recent entry in the backpoints table, x_n

t = request time

h = step size

3. Compute $\gamma_i(s)$, where $i = 0, 1, \ldots, 12$:

$$\gamma_0(s) = 1$$

$$\gamma_i(s) = \frac{s + i - 1}{i} \gamma_{i-1}(s) \qquad (i = 1, 2, \ldots, 12) \qquad (3\text{-}42)$$

4. Compute $\gamma_i'(s)$, where $i = 0, 1, \ldots, 11$:

$$\gamma_0'(s) = 1$$

$$\gamma_i'(s) = \sum_{j=0}^{i} \gamma_j'(0)\, \gamma_{i-j}(s) \qquad (i = 1, 2, \ldots, 11) \qquad (3\text{-}43)$$

5. Compute $\delta_i'(s)$, where $i = 0, 1, \ldots, 10$:

$$\delta_0'(s) = \sum_{j=1}^{11} \gamma_i'(s)$$

$$\delta_i'(s) = (-1)^i \left[\gamma_{i+1}'(s) + \sum_{j=i+2}^{11} \binom{j-1}{i} \gamma_j'(s) \right] \qquad (3\text{-}44)$$

$$(i = 1, 2, \ldots, 9)$$

$\binom{m}{i}$ is computed as $\frac{m!}{(m-i)!\,i!}$

$$\delta_{10}'(s) = \gamma_{11}'(s)$$

6. Compute the velocity as follows:

$$\dot{x}(t) = \dot{x}(t_n + sh) = h \left[{}^{I}S_n + \sum_{i=0}^{10} \delta_i'(s)\, \ddot{x}_{n-i} \right] \qquad (3\text{-}45)$$

where ${}^{I}S_n$ = first ordinate sums

$\ddot{x}$ = accelerations in the backpoints table

n = number of accelerations in the backpoints table

and s, h, and $\delta_i'(s)$ have been defined previously.

7. Compute $\gamma_i''(s)$, where i = 0, 1, ..., 12:

$$\gamma_0''(s) = 1$$

$$\gamma_i''(s) = \sum_{j=0}^{i} \gamma_j''(0)\ \gamma_{i-j}(s) \qquad (i = 1, 2, \ldots, 12) \qquad (3\text{-}46)$$

8. Compute $\delta_i''(s)$, where i = 0, 1, ..., 10:

$$\delta_0''(s) = \sum_{j=2}^{12} \gamma_j''(s)$$

$$\delta_i''(s) = (-1)^i \left[\gamma_{i+1}''(s) + \sum_{j=i+3}^{12} \binom{j-2}{i} \gamma_j''(s) \right] \qquad (3\text{-}47)$$

$$(i = 1, 2, \ldots, 9)$$

$$\delta_{10}''(s) = \gamma_{12}''(s)$$

9. Compute the position as follows:

$$x(t) = x(t_n + sh) = h^2 \left[{}^{II}S_n + (s - 1)\ {}^{I}S_n + \sum_{i=0}^{10} \delta_i''(s)\ \ddot{x}_{n-1} \right] \qquad (3\text{-}48)$$

where ${}^{I}S_n$ and ${}^{II}S_n$ = first and second ordinate sums

$\ddot{x}$ = accelerations in the backpoints table

n = number of entries in the backpoints table

and s, h, and $\delta_i''(s)$ have been defined previously.

The formulas for interpolating the partial derivatives are obtained by substituting $\partial\ddot{x}_{n-1}/\partial p$ for $\ddot{x}_{n-1}$, ${}^{I}P_n$ for ${}^{I}S_n$, and ${}^{II}P_n$ for ${}^{II}S_n$ in Equations (3-45) and (3-48).

3.4 EQUATIONS OF MOTION FOR STATE VARIABLES

The orbital equations of motion, expressed in Cartesian coordinates, are

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} \tag{3-49}$$

where $\vec{r}$ = satellite position vector in the TOE coordinate frame

$\vec{a}$ = total acceleration vector in the TOE coordinate frame

In FEDS, this set of three second-order differential equations is transformed to an equivalent set of six first-order differential equations:

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \dot{\vec{r}} \\ \frac{d\dot{\vec{r}}}{dt} &= \vec{a} \end{aligned} \tag{3-50}$$

where $\dot{\vec{r}}$ is the satellite velocity expressed in the TOE coordinate frame.

For the total FEDS state vector

$$\vec{X} = \begin{bmatrix} \vec{r} \\ \dot{\vec{r}} \\ C_D \\ \vec{b} \end{bmatrix} \tag{3-51}$$

where C_D = atmospheric drag constant

$\vec{b}$ = clock bias coefficients (b_1, b_2, and b_3)

the equations of motion can be expressed as

$$\dot{\vec{X}} = F(\vec{X}, t) \tag{3-52}$$

where

$$F(\vec{X}, t) = \begin{bmatrix} \dot{\vec{r}} \\ \vec{a} \\ 0 \\ 0 \end{bmatrix}$$

The total acceleration of the satellite includes the following components:

- Gravitational acceleration of the satellite due to the Earth's mass ($\vec{a}_E$) and the solar and lunar masses ($\vec{a}_S$ and $\vec{a}_M$, respectively)
- Gravitational acceleration of the satellite due to the nonsphericity of the Earth's gravitational potential ($\vec{a}_{NS}$)
- Satellite acceleration due to atmospheric drag forces ($\vec{a}_D$)

- Satellite acceleration due to solar radiation pressure ($\vec{a}_{SR}$)

The total acceleration, a, is expressed in terms of these components as

$$\vec{a} = \vec{a}_E + \vec{a}_S + \vec{a}_M + \vec{a}_{NS} + \vec{a}_D + \vec{a}_{SR} \tag{3-53}$$

All or any subset of these effects can be included in the acceleration vector, which is used in constructing the equations of motion. These accelerations are discussed in the following subsections.

3.4.1 EARTH, SOLAR, AND LUNAR POINT MASS ACCELERATIONS

To first order, the gravitational attraction of a body of mass m can be approximated as that arising from a dimensionless particle of mass m located at the center of mass of the body. Three point mass terms are included in the acceleration model of FEDS: the Earth's central body term and the perturbing accelerations from the Sun and the Moon.

The acceleration experienced by a satellite attracted by n point masses, expressed in an inertial coordinate system, is

$$\frac{d^2\tilde{r}}{dt^2} = - \sum_{k=1}^{n} \frac{\mu_k}{r_{kp}^3} \vec{r}_{kp} \tag{3-54}$$

where $\tilde{r}$ = vector from the center of an inertial coordinate system to the satellite

μ_k = product of the universal gravitational constant and the mass of the kth point mass

$\vec{r}_{kp}$ = vector from the kth point mass to the satellite

r_{kp} = magnitude of the vector, $\vec{r}_{kp}$

In FEDS, the motion of the satellite is referenced to the Earth's position; i.e., an Earth-centered TOE coordinate

system is used. The desired form for the acceleration is obtained by subtracting the acceleration acting on the reference body

$$\frac{d^2\tilde{r}_E}{dt^2} = \sum_{\substack{k=1 \\ k\neq E}}^{n} \frac{\mu_k}{r_k^3} \vec{r}_k \tag{3-55}$$

from each side of Equation (3-54) to obtain

$$\frac{d^2(\tilde{r} - \tilde{r}_E)}{dt^2} = - \sum_{k=1}^{n} \frac{\mu_k}{r_{kp}^3} \vec{r}_{kp} - \sum_{\substack{k=1 \\ k\neq E}}^{n} \frac{\mu_k}{r_k^3} \vec{r}_k \tag{3-56}$$

For the case in which the reference central body is the Earth and the perturbing point masses are the Sun and the Moon, Equation (3-56) becomes

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}_E + \vec{a}_S + \vec{a}_M \tag{3-57}$$

where

$$\vec{a}_E = - \frac{\mu_E \vec{r}}{r^3} \tag{3-58}$$

$$\vec{a}_S = \mu_S \left(\frac{\vec{r}_S - \vec{r}}{|\vec{r}_S - \vec{r}|^3} - \frac{\vec{r}_S}{|\vec{r}_S|^3} \right) \tag{3-59}$$

$$\vec{a}_M = \mu_M \left(\frac{\vec{r}_M - \vec{r}}{|\vec{r}_M - \vec{r}|^3} - \frac{\vec{r}_M}{|\vec{r}_M|^3} \right) \tag{3-60}$$

The quantities μ_E, μ_S, and μ_M are the gravitational constants of the Earth, the Sun, and the Moon, respectively, and $\vec{r}_S$ and $\vec{r}_M$ are the position vectors of the Sun and the Moon, referenced to the Earth.

The positions of the Sun and the Moon are determined by evaluating series expansions in the latitude and longitude of the Sun and the Moon, where the series are truncated from those of Brown's lunar theory by Woodard (Reference 5, pages 52 through 62). The Sun's position components are (x_S, y_S, z_S), and the Moon's position components are (x_M, y_M, z_M).

For the Sun, the components are related to the orbital elements by

$$\frac{x_S}{r_S} = \cos \lambda_S$$

$$\frac{y_S}{r_S} = \cos \bar{\varepsilon} \sin \lambda_S \qquad (3\text{-}61)$$

$$\frac{z_S}{r_S} = \sin \bar{\varepsilon} \sin \lambda_S$$

For the Moon, the components are related to the orbital elements by

$$\frac{x_M}{r_M} = \cos \theta_M \cos \lambda_M$$

$$\frac{y_M}{r_M} = \cos \theta_M \sin \lambda_M \cos \bar{\varepsilon} - \sin \theta_M \sin \bar{\varepsilon} \qquad (3\text{-}62)$$

$$\frac{z_M}{r_M} = \cos \theta_M \sin \lambda_M \sin \bar{\varepsilon} + \sin \theta_M \cos \bar{\varepsilon}$$

where $\bar{\varepsilon}$ = mean obliquity of the ecliptic

λ_S = solar longitude

λ_M = lunar ecliptic longitude

θ_M = lunar ecliptic latitude.

The distances r_M and r_S are found from evaluation of the terms d_M/r_M and d_S/r_S (the ratios of the mean to the true geocentric distances) as follows:

$$\frac{d_M}{r_M} = 1 + \kappa_M$$

$$\frac{d_S}{r_S} = 1 + \kappa_S \tag{3-63}$$

where d_M = mean lunar geocentric distance (384399.06 kilometers)

d_S = mean solar geocentric distance (149497971.0 kilometers)

κ_M, κ_S = series presented in this subsection

The time ΔD (the elapsed time since 1900.00 in units of 10^{-4} days) in the series expansion is given by

$$\Delta D = 3.6525\ T_c \tag{3-64}$$

where T_c, the number of Julian centuries since 1900.00, is

$$T_c = \frac{JD - 2415020.0}{36525.0} \tag{3-65}$$

where JD is the full Julian date.

The mean obliquity of the ecliptic, $\bar{\varepsilon}$, is as defined as follows:

$$\bar{\varepsilon} = 23\overset{\circ}{.}4522940 - 0\overset{\circ}{.}0130125\ T_c \tag{3-66}$$

The solar longitude, λ_S, is

$$\lambda_S = G_S + \ell_S + 2e_S \sin(\ell_S) \tag{3-67}$$

where

$$\begin{aligned} G_S &= 281\overset{\circ}{.}220833 + 4\overset{\circ}{.}70684\ \Delta D \\ \ell_S &= 358\overset{\circ}{.}475845 + 98560\overset{\circ}{.}0267\ \Delta D \\ e_S &= 0.01675104 - 1.1444 \times 10^{-5}\ \Delta D \end{aligned} \tag{3-68}$$

In Equation (3-67), G_S is the longitude of perigee of the solar orbit; ℓ_S is the solar mean anomaly; and e_S is the eccentricity of the solar orbit.

The series κ_S is given by

$$\kappa_S = e_S \cos(\ell_S) \tag{3-69}$$

The lunar ecliptic longitude, λ_M, is determined by subtracting a series in (D_M, F_M, ℓ_M, ℓ_S) (presented in Table 3-8) from L_M (the mean longitude of the Moon, measured in the ecliptic plane from the mean equinox of date to the mean ascending node of the lunar orbit, and then along the orbit):

$$\lambda_M = L_M - \text{series}$$

Table 3-8. Series for λ_M

Coefficient (radians)		Argument Multiple of ℓ_M	F_M	D_M	ℓ_S
-0.000607	sin	0	0	1	0
0.11490	sin	0	0	2	0
-0.000267	sin	0	2	-2	0
-0.001996	sin	0	2	0	0
-0.000801	sin	1	2	-2	0
-0.003238	sin	1	0	0	0
-0.000118	sin	1	0	2	0
0.000138	sin	1	0	-2	-1
0.000716	sin	1	0	0	-1
0.000192	sin	1	-2	0	0
-0.000186	sin	1	0	-4	0
-0.022236	sin	1	0	-2	0
0.10976	sin	1	0	0	0
0.000931	sin	1	0	2	0
-0.000219	sin	1	2	0	0
-0.000999	sin	1	0	-2	1
-0.000532	sin	1	0	0	1
-0.000149	sin	2	0	-4	0
-0.001026	sin	2	0	-2	0
0.003728	sin	2	0	0	0
0.000175	sin	3	0	0	0

The lunar ecliptic latitude, θ_M, is directly determined from a series in (D_M, F_M, ℓ_M, ℓ_S) and is presented in Table 3-9. The series κ_M is determined from the series in (D_M, F_M, ℓ_M, ℓ_S) and is presented in Table 3-10. In these series,

$$L_M = 270^\circ 26' 02''.99 + (480960^\circ.0\ 52' 59''.31) T_c - 4''.08\ T_c^2$$

$$\ell_M = 296^\circ 06' 16''.59 + (477000^\circ.0 + 198^\circ 50' 56''.79) T_c + 33''.09\ T_c^2$$

$$\ell_S = 358^\circ 28' 33''.00 + (35640^\circ.0 + 359^\circ 02' 59''.10) T_c \qquad (3\text{-}70)$$

$$F_M = 11^\circ 15' 03''.20 + (483120^\circ.0 + 82^\circ 01' 30''.54) T_c - 11''.56\ T_c^2$$

$$D_M = 350^\circ 44' 14''.95 + (444960^\circ.0 + 307^\circ\ 06' 51''.18) T_c - 5''.17\ T_c^2$$

where F_M = argument of latitude of the Moon
D_M = mean elongation of the Moon from the Sun

As noted previously, the series are summarized in Tables 3-8 through 3-10. These tables give the coefficients of each trigonometric term and the angles that appear in that term. The trigonometric terms contain only integer multiples of the four angles. The first term in θ_M is

$$\theta_M = 0.089503 \sin(F_M) \text{ radians}$$

Table 3-9. Series of θ_M

Coefficient (radians)		Argument Multiple of ℓ_M	F_M	D_M	ℓ_S
0.089503	sin	0	1	0	0
0.000569	sin	0	1	2	0
-0.003023	sin	0	1	-2	0
-0.000144	sin	0	1	-2	1
0.004897	sin	1	1	0	0
-0.000807	sin	1	1	-2	0
0.004847	sin	1	-1	0	0
-0.000967	sin	1	-1	-2	0
0.00301	sin	2	1	0	0
0.000154	sin	2	-1	0	0
0.000161	sin	1	-1	2	0

Table 3-10. Series for κ_M

Coefficient (radians)		Argument Multiple of ℓ_M	F_M	D_M	ℓ_S
0.0082488	cos	0	0	2	0
0.0005604	cos	0	0	-2	1
0.0003369	cos	1	0	0	-1
-0.0002086	cos	1	-2	0	0
0.0100247	cos	1	0	-2	0
0.0545008	cos	1	0	0	0
0.0009017	cos	1	0	2	0
0.0004219	cos	1	0	-2	1
-0.0002773	cos	1	0	0	1
0.0029700	cos	2	0	0	0
0.0001817	cos	3	0	0	0

The formulas for the lunar position include the leading terms of Brown's lunar theory as summarized by Woodard (Reference 5). All perturbation terms with amplitudes greater than 50 kilometers are included (22 terms in the ecliptic longitude, 11 terms in the ecliptic latitude, and 11 terms in the distance); this achieves an overall positional accuracy of 1 arc-minute (0.005 radian or 200 kilometers).

3.4.2 NONSPHERICAL GRAVITATIONAL ACCELERATION

The inertial acceleration vector resulting from nonspherical gravitational effects is given by the gradient of the nonspherical terms in the geopotential function, ψ_{NS}

$$\vec{a}_{NS} = \psi_{NS} \tag{3-71}$$

Expressed in terms of spherical polar coordinates,

- r = magnitude of the vector from the Earth's center of mass to the satellite
- ϕ = geocentric latitude
- λ = geocentric longitude (measured east from the prime meridian)

The nonspherical geopotential, ψ_{NS}, is given by

$$\begin{aligned}\psi_{NS}(r, \phi, \lambda) = \frac{\mu_E}{r} \sum_{n=2}^{\infty} C_n^0 \left(\frac{R_e}{r}\right)^n P_n^0(\sin \phi) \\ + \frac{\mu_E}{r} \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left[\left(\frac{R_e}{r}\right)^n P_n^m(\sin \phi) \right. \\ \left. * \left(S_n^m \sin m\lambda + C_n^m \cos m\lambda\right)\right]\end{aligned} \tag{3-72}$$

where μ_E = gravitational constant of the Earth
R_e = Earth's equatorial radius

$P_n^m(\sin\phi)$ = associated Legendre function

S_n^m, C_n^m = harmonic coefficients (zonal harmonics for $m = 0$, sectorial harmonics for $m = n$, and tesseral harmonics for $n > m \neq 0$)

(<u>NOTE</u>: $J_n = -C_n^0$, where J are the zonal coefficients

The first and second terms are the nonspherical potential due to the sum of zonal and tesseral harmonics, respectively. The term $n = 1$ is not present, since the origin of the coordinate system is placed at the center of mass of the Earth.

Expressing the gradient in Earth-fixed coordinates, $\vec{r}_b = (x_b, y_b, z_b)$ (see Section 2.1.2), the form for the inertial acceleration vector is obtained as follows:

$$\vec{a}_{b_{NS}} = \begin{bmatrix} \ddot{x}_{b_{NS}} \\ \ddot{y}_{b_{NS}} \\ \ddot{z}_{b_{NS}} \end{bmatrix} = \frac{\partial\psi_{NS}}{\partial r}\left(\frac{\partial r}{\partial \vec{r}_b}\right)^T + \frac{\partial\psi_{NS}}{\partial\phi}\left(\frac{\partial\phi}{\partial\vec{r}_b}\right)^T + \frac{\partial\psi_{NS}}{\partial\lambda}\left(\frac{\partial\lambda}{\partial\vec{r}_b}\right)^T \tag{3-73}$$

where $\ddot{x}_{b_{NS}}$, $\ddot{y}_{b_{NS}}$, and $\ddot{z}_{b_{NS}}$ are the components of the inertial acceleration of the spacecraft, expressed in the Earth-fixed coordinate system and not the acceleration with respect to the Earth-fixed coordinate system. Thus, it is necessary to transform these components into the TOE coordinate system in which the orbital equations of motion are expressed.

This transformation, which is discussed in Section 2.1.2, is given by

$$\vec{r}_{NS} = B_1^T \vec{a}_{b_{NS}} \tag{3-74}$$

where the B_1 rotation matrix is defined in Equation (2-2), and it is assumed that the geographic pole axis, z_b, is aligned with the spin axis, z, of the TOE coordinate system. This rotation is equivalent to replacing (r_b, x_b, y_b, z_b) in Equation (3-73) by (r, x, y, z), the TOE components, and calculating the longitude and latitude as follows:

$$\begin{aligned} \lambda &= \alpha - \alpha_g \\ \phi &= \sin^{-1}\left(\frac{z}{r}\right) \end{aligned} \tag{3-75}$$

where α = right ascension of the spacecraft [$\alpha = \tan^{-1}(y/x)$]

α_g = right ascension of Greenwich

Equation (3-73) can be written as

$$\vec{a}_{NS} = \begin{bmatrix} \ddot{x}_{NS} \\ \ddot{y}_{NS} \\ \ddot{z}_{NS} \end{bmatrix} \tag{3-76}$$

$$= \frac{\partial \psi_{NS}}{\partial r} \left(\frac{\partial r}{\partial \vec{r}}\right)^T + \frac{\partial \psi_{NS}}{\partial \phi} \left(\frac{\partial \phi}{\partial \vec{r}}\right)^T + \frac{\partial \psi_{NS}}{\partial \lambda} \left(\frac{\partial \lambda}{\partial \vec{r}}\right)^T$$

The partial derivatives of the nonspherical portion of the Earth's potential with respect to r, ϕ, and λ are given by

$$\frac{\partial \psi_{NS}}{\partial r} = -\frac{1}{r}\frac{\mu_E}{r}\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n (n+1)\sum_{m=0}^{n}\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) P_n^m(\sin\phi)$$

$$\frac{\partial \psi_{NS}}{\partial \phi} = \frac{\mu_E}{r}\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n}\left\{\left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) * \left[P_n^{m+1}(\sin\phi) - m\tan\phi\, P_n^m(\sin\phi)\right]\right\} \tag{3-77}$$

$$\frac{\partial \psi_{NS}}{\partial \lambda} = \frac{\mu_E}{r}\sum_{n=2}^{\infty}\left(\frac{R_e}{r}\right)^n \sum_{m=0}^{n} m\left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right) P_n^m(\sin\phi)$$

The Legendre functions and the terms cos $m\lambda$, sin $m\lambda$, and m tan ϕ are computed via recursion formulas, as follows:

$$P_n^0(\sin\phi) = \left[(2n-1)\sin\phi\, P_{n-1}^0(\sin\phi) - (n-1)\, P_{n-2}^0(\sin\phi)\right]/n$$

$$P_n^m(\sin\phi) = P_{n-2}^m(\sin\phi) + (2n-1)\cos\phi\, P_{n-1}^{m-1}(\sin\phi) \quad (m \neq 0;\ m < n) \tag{3-78}$$

$$P_n^m(\sin\phi) = (2n-1)\cos\phi\, P_{n-1}^{n-1}(\sin\phi) \quad (m \neq 0;\ m = n)$$

where

$$P_0^0(\sin\phi) = 1; \quad P_1^0(\sin\phi) = \sin\phi; \quad P_1^1(\sin\phi) = \cos\phi$$

$$\sin m\lambda = 2\cos\lambda\sin(m-1)\lambda - \sin(m-2)\lambda$$

$$\cos m\lambda = 2\cos\lambda\cos(m-1)\lambda - \cos(m-2)\lambda$$

$$m\tan\phi = [(m-1)\tan\phi)] + \tan\phi$$

The partial derivatives of r, ϕ, and λ, with respect to x, y, and z, are computed from the expressions

$$\frac{\partial r}{\partial \vec{r}} = \frac{\vec{r}^T}{r}$$

$$\frac{\partial \phi}{\partial \vec{r}} = \frac{1}{\sqrt{(x^2 + y^2)}}\left[-\frac{z\vec{r}^T}{r^2} + \frac{\partial z}{\partial \vec{r}}\right] \tag{3-79}$$

$$\frac{\partial \lambda}{\partial \vec{r}} = \frac{1}{(x^2 + y^2)}\left[x\frac{\partial y}{\partial \vec{r}} - y\frac{\partial x}{\partial \vec{r}}\right]$$

where the row vectors (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively, are given by

$$\frac{\partial x}{\partial \vec{r}}, \quad \frac{\partial y}{\partial \vec{r}}, \quad \frac{\partial z}{\partial \vec{r}}$$

Substituting Equations (3-79) into Equation (3-73) yields

$$\ddot{x}_{NS} = \left(\frac{1}{r}\frac{\partial\psi_{NS}}{\partial r} - \frac{z}{r^2\sqrt{x^2+y^2}}\frac{\partial\psi_{NS}}{\partial\phi}\right)x - \left(\frac{1}{(x^2+y^2)}\frac{\partial\psi_{NS}}{\partial\lambda}\right)y$$

$$\ddot{y}_{NS} = \left(\frac{1}{r}\frac{\partial\psi_{NS}}{\partial r} - \frac{z}{r^2\sqrt{x^2+y^2}}\frac{\partial\psi_{NS}}{\partial\phi}\right)y + \left(\frac{1}{x^2+y^2}\frac{\partial\psi_{NS}}{\partial\lambda}\right)x \qquad (3\text{-}80)$$

$$\ddot{z}_{NS} = \left(\frac{1}{r}\frac{\partial\psi_{NS}}{\partial r}\right)z + \frac{\sqrt{x^2+y^2}}{r^2}\frac{\partial\psi_{NS}}{\partial\phi}$$

3.4.3 ATMOSPHERIC DRAG ACCELERATION

Atmospheric drag acceleration is modeled as a drag force in the direction of the relative wind vector acting on a satellite of constant surface area. The velocity of the satellite relative to the atmosphere is computed in the inertial coordinate system by subtracting the motion of the atmosphere, assumed to rotate with the Earth, from that of the satellite:

$$\vec{V}_{rel} = \dot{\vec{r}} - \vec{\omega} \times \vec{r} \qquad (3\text{-}81)$$

The Earth's rotation vector, $\vec{\omega}$, is directed along the Earth's instantaneous spin axis with a magnitude equal to the rotation rate of the Earth and components $(\omega_1, \omega_2, \omega_3)$.

In the TOE coordinate system, ignoring the effects of precession and nutation, the z-axis is aligned with the north polar spin axis such that ω_1 and ω_2 are equal to zero. In addition, ω is assumed to be constant. Therefore, in the TOE system, Equation (3-81) reduces to

$$\vec{V}_{rel} = \begin{bmatrix} \dot{x} + \omega y \\ \dot{y} - \omega x \\ \dot{z} \end{bmatrix} \tag{3-82}$$

For the case of a spherical satellite, the atmospheric drag acceleration is computed as

$$\vec{a}_D = -\frac{1}{2}\left(\frac{C_D A}{m}\right) \rho_0(h)\, \vec{V}_{rel}\, |\vec{V}_{rel}| \tag{3-83}$$

where C_D = aerodynamic force coefficient, which is an adjustable parameter

A = surface area of the satellite

m = mass of the satellite

$\rho_0(h)$ = altitude density function computed from the atmospheric drag model

Nominally, for a spherical satellite, the aerodynamic force coefficient, C_D, is equal to 1.0. In order to absorb an error in any of the above terms, C_D is an adjustable parameter.

In FEDS, the altitude density function, $\rho_0(h)$, is modeled using a Harris-Priester atmospheric model. Harris and Priester determined the physical properties of the upper atmosphere theoretically by solving the heat conduction equation under quasi-hydrostatic conditions (see References 6 through 8). Approximations for fluxes from the

extreme ultraviolet and corpuscular heat sources were included, but the model averages the semiannual and seasonal-latitudinal variations and does not attempt to account for the extreme ultraviolet 27-day effect.

The atmospheric model presented here is a modification of the Harris-Priester concept. The modification attempts to account for the diurnal bulge by including a cosine variation between a maximum density profile at the apex of the diurnal bulge (which is located approximately 30 degrees east of the subsolar point) and a minimum density profile at the antapex of the diurnal bulge. Discrete values of the maximum- and minimum-density altitude profiles, corresponding to mean solar activity, are stored in tabular form as $\rho_{max}(h_i)$ and $\rho_{min}(h_i)$, respectively. Different maximum and minimum profiles are available for different levels of solar activity (Reference 2). Exponential interpolation is used between entries; i.e., the minimum and maximum densities, ρ_{min} and ρ_{max}, are given by

$$\rho_{min}(h) = \rho_{min}(h_i) \exp\left(\frac{h_i - h}{H_{min}}\right)$$

$$\rho_{max}(h) = \rho_{max}(h_i) \exp\left(\frac{h_i - h}{H_{max}}\right) \qquad (3\text{-}84)$$

where $(h_i \leq h \leq h_{i+1})$ and the respective scale heights, H_{min} and H_{max}, are given by

$$H_{min} = \frac{h_i - h_{i+1}}{\ln[\rho_{min}(h_i+1)/\rho_{min}(h_i)]}$$

$$H_{max} = \frac{h_i - h_{i+1}}{\ln[\rho_{max}(h_i+1)/\rho_{max}(h_i)]} \tag{3-85}$$

A good approximation (neglecting polar motion) for the satellite height, h, is given by

$$h = r - R_E \tag{3-86}$$

where R_E is the mean radius of the Earth, given as

$$R_E = \frac{R_e(1 - f_E)}{\sqrt{1 - (2f_E - f_E^2)\cos^2\delta}} \tag{3-87}$$

and r = magnitude of the satellite position vector

R_e = equatorial radius of the Earth

f_E = Earth's flattening coefficient

δ = declination of the satellite (it is assumed that δ equals the geocentric latitude of the subsatellite point)

If the density is assumed to be maximum at the apex of the bulge, then the cosine variation between the maximum and minimum density profiles is

$$\rho_0(h) = \rho_{min}(h) + [\rho_{max}(h) - \rho_{min}(h)]\cos^n\left(\frac{\gamma}{2}\right) \tag{3-88}$$

where γ is the angle between the satellite position vector and the apex of the diurnal bulge.

The cosine function in Equation (3-88) can be determined directly as

$$\cos^n \frac{\gamma}{2} = \left[\frac{1 + \cos\gamma}{2}\right]^{n/2} = \left[\frac{1}{2} + \frac{\vec{r} \cdot \vec{U}_B}{2r}\right]^{n/2} \qquad (3\text{-}89)$$

where $\vec{r}$ = satellite position vector (TOE coordinates)

$\vec{U}_B$ = unit vector directed toward the apex of the diurnal bulge (TOE coordinates)

For FEDS, n has been assigned the value of 6.

Vector $\vec{U}_B$ has the following components:

$$\begin{aligned} U_{B_x} &= \cos\delta_S \cos(\alpha_s + \bar{\lambda}) \\ U_{B_y} &= \cos\delta_S \sin(\alpha_s + \bar{\lambda}) \\ U_{B_z} &= \sin\delta_S \end{aligned} \qquad (3\text{-}90)$$

where δ_S = declination of the Sun

α_S = right ascension of the Sun

$\bar{\lambda}$ = lag angle between the Sun line and the apex of the diurnal bulge (approximately 30 degrees)

3.4.4 SOLAR RADIATION PRESSURE ACCELERATION

The model for acceleration $\vec{a}$, due to direct solar radiation acting on a satellite, is

$$\vec{a}_{SR} = \nu P_S R^2_{SUN} \frac{C_R A}{m} \frac{\vec{r}_{vs}}{r_{vs}} \qquad (3\text{-}91)$$

where ν = an eclipse factor such that
$\nu = 0$ if the satellite is in shadow (umbra)
$\nu = 1$ if the satellite is in sunlight
$0 < \nu < 1$ if the satellite is in penumbra

The vector from the Sun to the satellite is given by

$$\vec{r}_{vs} = \vec{r} - \vec{r}_s \tag{3-92}$$

where P_S = mean solar flux at one astronomical unit, divided by the speed of light
$\vec{r}_S$ = position vector of the Sun
$\vec{r}$ = position vector of the satellite
R_{SUN} = one astronomical unit
C_R = solar radiation pressure constant
A = surface area of the satellite
m = mass of the satellite
r_{vs} = magnitude of the vector $\vec{r}_{vs}$

The solar radiation pressure constant is given by

$$C_R = 1 + \eta \tag{3-93}$$

where η is the reflectivity of the surface. All of the factors listed above, except r_{vs}, are constant for a given satellite.

A simple cylindrical shadow model is used to determine the eclipse factor. From Figure 3-1, it is apparent that the satellite is in sunlight ($\nu = 1$) if the vector product ($\vec{r} \cdot \vec{U}_S$) is greater than zero, where $\vec{r}$ is the satellite position vector relative to the Earth, and $\vec{U}_S$ is the solar position unit vector relative to the Earth.

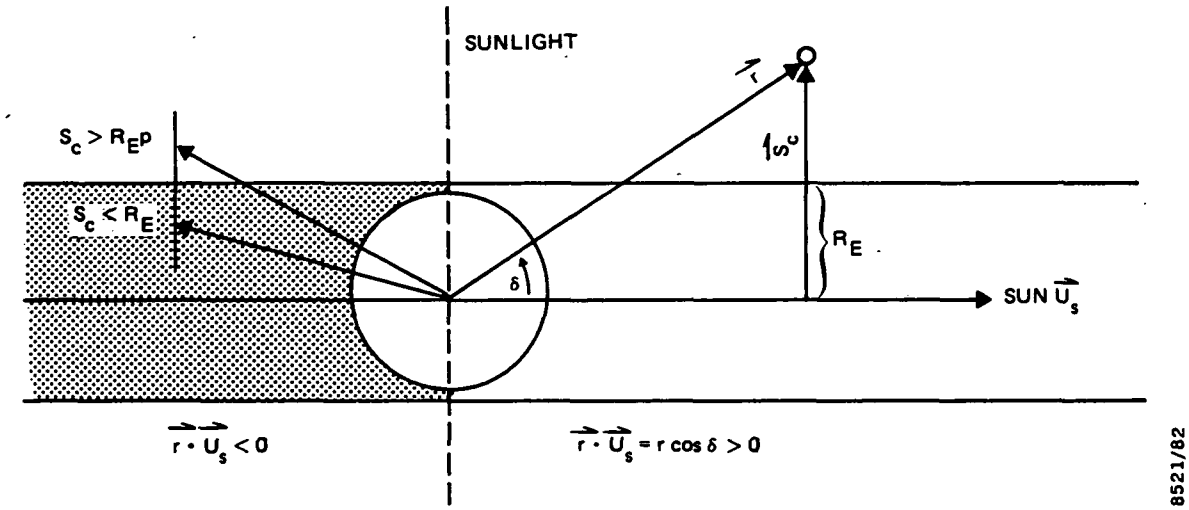

Figure 3-1. Cylindrical Shadow Model

If this product is less than zero and the vector to the satellite along the normal to the solar unit vector

$$\vec{S}_C = \vec{r} - (\vec{r} \cdot \vec{U}_S)\ \vec{U}_S \tag{3-94}$$

has a magnitude less than the Earth's radius, then the satellite is in shadow (i.e., ν equals 0); otherwise, it is assumed that the satellite is in sunlight and ν equals 1.

3.5 VARIATIONAL EQUATIONS

The variational equations are differential equations describing the rate of change of state parameters. The variational factors are used to solve the following equation (see Section 3.2.3):

$$\ddot{Y} = A(t)\ Y + B(t)\ \dot{Y} + C(t) \tag{3-95}$$

where

$$A(t) = \frac{\partial \vec{a}}{\partial \vec{r}}$$

$$B(t) = \frac{\partial \vec{a}}{\partial \dot{\vec{r}}} = \frac{\partial \vec{a}_D}{\partial \dot{\vec{r}}} \tag{3-96}$$

$$C(t) = \frac{\partial \vec{a}}{\partial C_D} = \frac{\partial \vec{a}_D}{\partial C_D}$$

To reduce computation in FEDS, the acceleration vector used in the variation equations is reduced to the following: the Earth's central body acceleration, $\vec{a}_E$, the zonal terms x_{J_2}, x_{J_3}, x_{J_4} in the nonspherical gravitational acceleration, $\vec{a}_Z$, and the atmospheric drag acceleration, $\vec{a}_D$. Using this reduced acceleration model, A(t) can be expressed as

$$A(t) = \frac{\partial \vec{a}_E}{\partial \vec{r}} + \frac{\partial \vec{a}_Z}{\partial \vec{r}} + \frac{\partial \vec{a}_D}{\partial \vec{r}} \tag{3-97}$$

The Earth's central body acceleration is defined in Equation (3-58). The partial derivatives with respect to the position vector are given by

$$\frac{\partial \vec{a}_E}{\partial \vec{r}} = \frac{\mu_E I}{r^3} + \frac{3\mu_E \vec{r}\,\vec{r}^T}{r^5}$$

$$= -\frac{\mu_E I}{r^3} + \frac{3\mu_E}{r^5} \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} \tag{3-98}$$

The nonspherical gravitation acceleration is defined in Equation (3-76). The zonal terms in the nonspherical potential, ψ_{NS}, have m equal to zero, such that the partial derivatives of these terms with respect to the longitude, λ, are zero. In this case, Equation (3-76) reduces to

$$\vec{a}_Z = \frac{\partial \psi_Z}{\partial r} \left(\frac{\partial r}{\partial \vec{r}}\right)^T + \frac{\partial \psi_Z}{\partial \phi} \left(\frac{\partial \phi}{\partial \vec{r}}\right)^T \tag{3-99}$$

where the zonal potential, ψ_Z, is given by

$$\psi_Z = \frac{\mu_E}{r} \sum_{n=2}^{\infty} C_n^0 \left(\frac{R_e}{r}\right)^n P_n^0(\sin \phi) \tag{3-100}$$

and

$$\begin{aligned} \frac{\partial \psi_Z}{\partial r} &= -\frac{1}{r} \frac{\mu_E}{r} \sum_{n=2}^{\infty} \left(\frac{R_e}{r}\right)^n (n+1)\, C_n^0\, P_n^0(\sin \phi) \\ \frac{\partial \psi_Z}{\partial \phi} &= \frac{\mu_E}{r} \sum_{n=2}^{\infty} \left(\frac{R_e}{r}\right)^n C_n^0\, P_n^1(\sin \phi) \end{aligned} \tag{3-101}$$

The partial derivatives of $\vec{a}_Z$ with respect to $\vec{r}$ are obtained by differentiating Equation (3-99) as follows:

$$\begin{aligned} \frac{\partial \vec{a}_Z}{\partial \vec{r}} = {} & \frac{\partial}{\partial \vec{r}} \left(\frac{\partial \psi_Z}{\partial r}\right) \left(\frac{\partial r}{\partial \vec{r}}\right)^T + \frac{\partial}{\partial \vec{r}} \left(\frac{\partial \psi_Z}{\partial \phi}\right) \left(\frac{\partial \phi}{\partial \vec{r}}\right)^T \\ & + \frac{\partial \psi_Z}{\partial r} \left(\frac{\partial^2 r}{\partial \vec{r}^2}\right)^T + \frac{\partial \psi_Z}{\partial \phi} \left(\frac{\partial^2 \phi}{\partial \vec{r}^2}\right)^T \end{aligned} \tag{3-102}$$

The required partial derivatives of ψ_z are as follows:

$$\frac{\partial}{\partial \vec{r}}\left(\frac{\partial \psi_Z}{\partial r}\right) = \frac{\partial^2 \psi_Z}{\partial r^2}\left(\frac{\partial r}{\partial \vec{r}}\right)^T + \frac{\partial^2 \psi_Z}{\partial r \partial \phi}\left(\frac{\partial \phi}{\partial \vec{r}}\right)^T$$

$$\frac{\partial}{\partial \vec{r}}\left(\frac{\partial \psi_Z}{\partial \phi}\right) = \frac{\partial^2 \psi_Z}{\partial \phi \partial r}\left(\frac{\partial r}{\partial \vec{r}}\right)^T + \frac{\partial^2 \psi_2}{\partial \phi^2}\left(\frac{\partial \phi}{\partial \vec{r}}\right)^T \qquad (3\text{-}103)$$

To minimize computations, the symmetry property of the second partial derivatives of ψ_Z is utilized as shown below. These partial derivatives are obtained by differentiating Equation (3-100) as follows:

$$\frac{\partial^2 \psi_Z}{\partial r^2} = \frac{\mu_E}{r^3} \sum_{n=2}^{\infty} \left(\frac{R_e}{r}\right)^n (n + 2)\ (n + 1)\ C_n^0\ P_n^0(\sin \phi)$$

$$\frac{\partial^2 \psi_Z}{\partial r \partial \phi} = \frac{\partial^2 \psi_Z}{\partial \phi \partial r} = -\frac{\mu_E}{r^2} \sum_{n=2}^{\infty} \left(\frac{R_e}{r}\right)^n (n + 1)\ C_n^0\ P_n^1(\sin \phi) \qquad (3\text{-}104)$$

$$\frac{\partial^2 \psi_Z}{\partial \phi^2} = \frac{\mu_E}{r} \sum_{n=2}^{\infty} \left(\frac{R_e}{r}\right)^n C_n^0 \left[P_n^2(\sin \phi) - \tan \phi\ P_n^1(\sin \phi)\right]$$

where

$$P_n^2(\sin \phi) = 2 \tan \phi\ P_n^1(\sin \phi) - n(n + 1)\ P_n^0(\sin \phi) \qquad (3\text{-}105)$$

The partial derivatives of r and ϕ with respect to $\vec{r}$ are given in Equations (3-79). The required second partial derivatives of r and ϕ with respect to $\vec{r}$ are obtained by differentiating Equations (3-79) with respect to $\vec{r}$, yielding

$$\frac{\partial^2 r}{\partial r^2} = \frac{1}{r}\left[I - \frac{\vec{r}\vec{r}^T}{r^2}\right] \tag{3-106}$$

$$\begin{aligned}\frac{\partial^2 \phi}{\partial \vec{r}^2} = & -\frac{1}{(x^2 + y^2)^{3/2}}\left[\left(\frac{\partial z}{\partial \vec{r}}\right)^T - \frac{z\vec{r}}{r^2}\right] \\ & * \left[x\left(\frac{\partial x}{\partial \vec{r}}\right) + y\left(\frac{\partial y}{\partial \vec{r}}\right)\right] \\ & - \frac{1}{\sqrt{r^2 \quad x^2 + y^2}}\left[\vec{r}\left(\frac{\partial z}{\partial \vec{r}}\right) + zI - \frac{2z}{r^2}\vec{r}\vec{r}^T\right]\end{aligned} \tag{3-107}$$

where $\partial x/\partial \vec{r}$, $\partial y/\partial \vec{r}$, and $\partial z/\partial \vec{r}$ are (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.

Taking into account the symmetry properties of the second partial derivatives of r, ϕ, and λ

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$$

$$\frac{\partial^2}{\partial x \partial z} = \frac{\partial^2}{\partial z \partial x}$$

$$\frac{\partial^2}{\partial y \partial z} = \frac{\partial^2}{\partial z \partial y}$$

and the fact that the potential function, ψ, satisfies Laplace's equation, $\Delta^2\psi = 0$; therefore

$$\frac{\partial^2\psi_z}{\partial x^2} = -\left(\frac{\partial^2\psi_z}{\partial y^2} + \frac{\partial^2\psi_z}{\partial z^2}\right) \tag{3-108}$$

reduces the amount of computation required.

The atmospheric drag acceleration is defined in Equations (3-83) and (3-84). The partial derivatives of $\vec{a}_D$ are as follows:

$$\frac{\partial\vec{a}_D}{\partial\vec{r}} = -\frac{\partial\vec{a}_D}{\partial\vec{V}_{rel}}\frac{\partial\vec{V}_{rel}}{\partial\vec{r}} + \frac{\vec{a}_D}{\rho_o(h)}\frac{\partial\rho_o(h)}{\partial\vec{r}} \tag{3-109a}$$

$$\frac{\partial\vec{a}_D}{\partial\dot{\vec{r}}} = \frac{\partial\vec{a}_D}{\partial\vec{V}_{rel}}\frac{\partial\vec{V}_{rel}}{\partial\dot{\vec{r}}} = \frac{\partial\vec{a}_D}{\partial\vec{V}_{rel}} \tag{3-109b}$$

$$\frac{\partial\vec{a}_D}{\partial C_D} = \frac{\vec{a}_D}{C_D} \tag{3-109c}$$

Making use of Equations (3-82) and (3-83) yields

$$\frac{\partial\vec{V}_{rel}}{\partial\vec{r}} = \begin{bmatrix} 0 & +\omega & 0 \\ -\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\vec{a}_D}{\partial\vec{V}_{rel}} = -\frac{1}{2}\frac{C_D A}{m}\rho\left(\frac{\vec{V}_{rel}\,\vec{V}_{rel}^{\,T}}{|V_{rel}|} + |\vec{V}_{rel}|\,I\right) \tag{3-110}$$

The partial derivatives of the density with respect to position are derived from Equation (3-88) as follows:

$$\frac{\partial \rho_o(h)}{\partial \vec{r}} = \left\{ \left[1 - \cos^n \left(\frac{\gamma}{2}\right) \right] \frac{\partial \rho_{min}(h)}{\partial h} + \cos^n \left(\frac{\gamma}{2}\right) \frac{\partial \rho_{max}(h)}{\partial h} \right\} \frac{\partial h}{\partial \vec{r}} - \frac{n}{2} \cos^{n-1} \left(\frac{\gamma}{2}\right) \sin \left(\frac{\gamma}{2}\right) * \left[\rho_{max}(h) - \rho_{min}(h)\right] \frac{\partial \gamma}{\partial \vec{r}} \tag{3-111}$$

where

$$\frac{\partial \rho_{min}(h)}{\partial h} = - \frac{\rho_{min}(h)}{H_{min}} \tag{3-112}$$

$$\frac{\partial \rho_{max}(h)}{\partial h} = - \frac{\rho_{max}(h)}{H_{max}}$$

$$\frac{\partial \gamma}{\partial r} = \frac{1}{\sin \gamma} \left[\left(\frac{\vec{r} \cdot \vec{U}_B}{r^3} \right) \vec{r} - \frac{\vec{U}_B}{r} \right] \tag{3-113}$$

The partial derivative of the height with respect to $\vec{r}$ is obtained by differentiating Equation (3-86), yielding

$$\frac{\partial h}{\partial \vec{r}} = \frac{\vec{r}}{r} - R_e \left\{ \frac{(1 - f_E) \left(2f_E - f_E^2\right) \cos \delta}{\left[1 - \left(2f_E^2 - f_E^2\right) \cos^2 \delta\right]^{3/2}} \right\} \frac{\partial (\cos \delta)}{\partial \vec{r}} \tag{3-114}$$

where

$$\frac{\partial(\cos\ \delta)}{\partial\vec{r}} = \frac{1}{r^4\ \cos\ \delta} \begin{bmatrix} xz^2 \\ yz^2 \\ -z(x^2 + y^2) \end{bmatrix} \qquad (3\text{-}115)$$

SECTION 4 - DATA PROCESSING FOR TRANSPONDER INTERFACE

This section contains algorithms used in processing data received from and to be sent to the transponder, including algorithms to convert the raw Doppler data message into engineering units and to create a frequency control word. Communication with the transponder is accomplished using an Intel 8086 microprocessor interface provided by Code 530. FEDS forms data to be sent to the transponder into a message and transmits it to the microprocessor via a standard RS232 terminal port. The microprocessor then translates the message into pulses to be sent to the various connectors on the transponder. For data traveling from the transponder to FEDS, the microprocessor collects the pertinent information from the transponder ports, forms a message, and transmits to FEDS again via a standard terminal port.

4.1 REDUCTION OF TDRSS DOPPLER DATA

In FEDS, the only observation data collected are one-way TDRSS Doppler measurements. The raw measurement consists of a nondestruct Doppler count of a nominal bias frequency added to a Doppler sample over a fixed time interval. The count is cumulative because the accumulator is reset only between passes, not between successive measurements.

The one-way Doppler measurement is performed by transmitting a signal from a ground transmitting station to a forward-link TDRS. The TDRS coherently translates the signal to the tracking frequency of the user transponder. The transponder then accumulates a Doppler measurement for transmission to FEDS.

Although all measurements should be valid due to the resetting of the accumulator and interception by the microprocessor interface of measurements with invalid lock flags, the

first measurement of a tracking pass is ignored. The rejection of the first measurement should not significantly affect the results of the estimation because there is expected to be an abundance of observation data. The following algorithm is used to preprocess all Doppler observations:

$$f_{D_O}(t_i) = \left(\frac{\Delta N_i}{M} - f_B\right) \frac{1}{K} \cdot \frac{f_u \cdot 17}{2^{20} \cdot 240} \tag{4-1}$$

where $f_{D_O}(t_i)$ = observed Doppler shift (hertz) averaged over the time interval between t_{i-1} and t_i

N_i = value of the accumulator (counts) at time t_i

$\Delta N_i = N_i - N_{i-1}$ = change in the value of the accumulator (counts) between t_{i-1} and t_i

M = number of Doppler samples added to the accumulator during the time interval between t_{i-1} and t_i (nominally 40000)

f_B = frequency bias (nominally $2^{21} = 2097152$)

K = rational multiplier of the Doppler sample (nominally 1/4)

f_u = frequency of the oscillator used by the transponder in forming a Doppler sample (nominally 19.056392 megahertz)

During a nominal pass, the Doppler accumulator is expected to overflow at least once. When overflow is detected, ΔN_i, used in the preprocessing of the Doppler observation, will be adjusted as follows:

$$\Delta N_i = N_i + 2^{40} - N_{i-1} \tag{4-2}$$

4.2 CREATION OF FREQUENCY CONTROL WORDS

The frequency control word is produced by FEDS and transmitted to the transponder to enable acquisition of the TDRSS signal. FEDS outputs an initial control word at a fixed timespan (nominally 60 seconds) prior to the beginning of a

tracking interval and will provide an updated control word at a fixed frequency (nominally one control word every 6 seconds) until acquisition occurs or until the end of the scheduled tracking pass. Output of control words will resume if the tracking signal is lost before the scheduled end of the pass.

The first bit of the frequency control word is not used by FEDS. The second bit is always set to 1. The remaining 14 bits of the control word are a signed binary (two's complement) number computed as follows:

$$CW(t_i) = \frac{f_d(t_i) \cdot 2^{16} \cdot 3121}{f_{ref}} \qquad (4\text{-}3)$$

where $CW(t_i)$ = data portion of the frequency control word ($-8196 \leq CW \leq 8195$) for output at time t_i

$f_d(t_i)$ = predicted Doppler shift at time t_i

f_{ref} = reference frequency

SECTION 5 - ESTIMATION LOGIC

Reference 9 provides the basis for the following section, which describes the estimation technique used in FEDS: a sliding batch differential correction (SBDC) algorithm. The procedure uses a classical weighted least-squares method solved by Newton iteration. The SBDC moves in discrete steps along the observation data, performing a state estimation at each step (slide). The output solve-for state for each slide becomes the a priori input state for the following slide. The major assumptions underlying this approach are as follows:

1. Given moderate changes in the solve state from iteration to iteration, the measurement partial derivatives do not vary enough to affect the DC process. Thus, the partial derivatives are computed for the first iteration of each slide and are held constant for subsequent iterations at which moderate state changes occur.

2. If the a priori state is reasonably close to the actual state, all of the spurious measurements can be edited during the first DC iteration. Thus, an iterative residual edit loop is always performed for the first DC iteration and, on subsequent iterations, is done only when the linearity constraint is violated.

3. Idle time in the system should be used to advantage. The algorithm provides a precompute phase that initiates the next DC slide before all of the measurement data is available.

4. The solutions are not noise dominated; as a result, data can be sampled down to produce a reduced observation set that decreases the data storage requirements.

The following subsections describe the SBDC algorithm as implemented in FEDS; this algorithm resembles the estimator used in GTDS (Reference 2).

This description is not meant to be a rigorous derivation of the techniques used in FEDS. The first three subsections describe one complete slide of the estimator, and later subsections cover the preliminary edit criteria (Section 5.4) and the special-case logic used to complete a partially precomputed slide (Section 5.5).

Each slide consists of one or more iterations through the measurement data to determine a state vector, $\vec{X}$, that best fits the data in a least-squares sense. The state vector, $\vec{X}$, that is solved for contains a maximum of 10 parameters (the first 6 are mandatory, whereas the last 4 are optional):

$$\vec{X} = [\vec{r}, \dot{\vec{r}}, C_D, \vec{b}]$$

where $\vec{r}$ = user spacecraft position vector in inertial Cartesian coordinates

$\dot{\vec{r}}$ = user spacecraft velocity vector in inertial Cartesian coordinates

C_D = coefficient of drag term

$\vec{b}$ = user spacecraft clock error terms (second-order polynomial)

Each iteration can be divided into three functional areas:

- An initial summation of the normal matrix and associated batch estimation matrices
- An inner iterative process that consists of the computation of the state correction vector followed by an adjustment to the matrices previously accumulated
- End-of-iteration testing for slide convergence or divergence and linearity violations

Depending on the result, the slide may either terminate with a new solution (converge) or terminate with no new solution (diverge) or continue by performing another iterative cycle.

5.1 INITIAL SUMMATION OF BATCH ESTIMATION MATRICES

The first functional area of the iterative process performs one complete pass through the measurement data. The computed measurements from the observation model (Section 6) are used to form the measurement residuals, which are then subjected to a preliminary edit procedure (Section 5.4). Partial derivatives are computed for the nonedited measurements, and the normal matrix is formed. To facilitate the description of this algorithm, the following standard indexing has been adopted.

Parameter	Meaning
Subscript i	Iteration-dependent variable
Subscript j	Observation-record-dependent variable

The initial summation is performed by processing each available observation record in the following manner:

1. Compute measurement residuals (o - c):

$$(o - c)_j = o_j - c_j \tag{5-1}$$

where o_j = preprocessed observed measurement (Section 4)
c_j = computed measurement (Section 6)

2. Perform preliminary editing based on maximum allowable residual test and, for new measurements only, check the measurement geometry (Section 5.4).

3. For the nonedited measurements, compute the partial derivatives, a_j, the observation at time t_j, with respect to the solve-for variables, X, at epoch time t_0:

$$a_j = \frac{\partial o_j}{\partial X_0} \tag{5-2}$$

$$= \begin{bmatrix} \frac{\partial f_o(t_j)}{\partial \vec{r}(t_j)} & \frac{\partial \vec{f}_o(t_j)}{\partial \dot{\vec{r}}(t_j)} & \frac{\partial \vec{f}_o(t_j)}{\partial C_D} & \frac{\partial \vec{f}_o(t_j)}{\partial \vec{b}} \end{bmatrix} * \begin{bmatrix} \frac{\partial \vec{r}(t_j)}{\partial \vec{X}_0} \\ \frac{\partial \dot{\vec{r}}(t_j)}{\partial \vec{X}_0} \\ \frac{\partial C_D(t_j)}{\partial \vec{X}_0} \\ \frac{\partial \vec{b}}{\partial \vec{X}_0} \end{bmatrix}$$

where $\vec{f}_o$ = the observation equation used to compute c_j

The first matrix on the right is explicitly determined from the observation equations in Section 6. The second matrix is obtained by integrating the variational equations (Section 3.5). The values of a_j are used to form a single row of the F-matrix.

4. The normal matrix, F^TWF, and two other estimation matrices, $F^TWF(o - c)$ and Q, are then updated with current measurement information as follows:

$$w = \frac{1}{\sigma^2} \tag{5-3}$$

where σ equals the measurement standard deviation.

$$F^T WF_{i,j} = F^T WF_{i,j-1} + (a_j^T * w * a_j) \tag{5-4}$$

$$F^T WF(o - c)_{i,j} = F^T WF(o - c)_{i,j-1} + a_j * w * (o - c)_j \tag{5-5}$$

$$Q_{i,j} = \frac{Q_{i,j-1} + (o - c)_j^2}{\sigma} \tag{5-6}$$

The total number of observations used in the summations, n, is also incremented. Only the upper half and diagonal of the symmetric $F^T WF$ matrix are stored.

5.2 STATE CORRECTION AND INNER EDIT

1. The correction to the state, $\Delta\vec{X}_i$, is formed by solving the system $(F^T WF)_i \, \Delta\vec{X}_i = F^T WF(o - c)_i$ as follows:

$$\Delta\vec{X}_i = (F^T WF)_i^{-1} \, F^T WF(o - c)_i \tag{5-7}$$

where $F^T WF_i$ is inverted using the Schur identity method of partitioning (Reference 2, Section 8.6.1).

2. Estimation statistics are computed for later use. The standard error fit is as follows:

$$S_i = \sqrt{\frac{Q_i - \left[\Delta\vec{X}_i * F^T WF(o - c)_i\right]}{n - 1}} \tag{5-8}$$

The residual root mean square is as follows:

$$RMS_i = \sqrt{\frac{Q_i}{n - 1}} \tag{5-9}$$

3. Inner loop editing is performed on the first iteration or if the linearity constraint was violated on the previous iteration (Section 5.3) as follows:

Predicted residuals, r_j, are computed and tested

$$r_j = \frac{1}{\sigma^2} [(o - c)_j - (a_j) (\Delta\vec{X}_i)] \qquad (5\text{-}10)$$

Given m, an input scaling parameter, each r_j is compared with $m_i S_i$. If the r_j is greater, the observation is edited, and its contribution is subtracted from the $F^T WF_i$, $F^T WF(o - c)_i$, and Q_i matrices. The total number of observations used, n, is also decremented.

4. If no observations are edited, inner editing is terminated. Otherwise, new S_i and RMS_i statistics are computed based upon the modified sums and a new state correction vector, $\Delta\vec{X}_i$, is obtained. Inner editing will terminate at this point if either the maximum number of inner loops has been performed or if $S_i - S_{i-1}/S_i \leq E$, where E is the standard error fit input tolerance. Otherwise, the inner edit loop is repeated until one of the above conditions is satisfied. The last state correction vector computed during this process becomes the state correction vector, $\Delta\vec{X}_i$, for this iteration.

5.3 END-OF-ITERATION TESTING

Various tests are made at the end of each iteration to determine whether the solution state has been attained and, if not, whether the correction has violated linearity constraints. The convergence/divergence tests are performed in the following order:

1. Divergence indicators from previous operations are checked for one of three conditions: error return from inversion of the normal matrix, observation timespan less than

the minimum required, or all observations in the latest pass have been edited.

2. Convergence checking is performed next. The slide converges if both of the following are true:

$$|\Delta P_i| = \sqrt{\sum_{\ell=1}^{3} \Delta X_{i\ell}^2} < conv_1 \qquad (5\text{-}11a)$$

$$|\Delta V_i| = \sqrt{\sum_{\ell=4}^{6} \Delta X_{i\ell}^2} < conv_2 \qquad (5\text{-}11b)$$

or if

$$\left| \frac{RMS_i - S_i}{S_i} \right| < conv_3 \qquad (5\text{-}11c)$$

where $conv_1$, $conv_2$, and $conv_3$ are uplinked convergence tolerances.

3. The slide diverges if i = 1 and either of the following is true:

$$|\Delta P_i| > div_1 \qquad (5\text{-}12a)$$

$$|\Delta V_i| > div_2 \qquad (5\text{-}12b)$$

where div_1 and div_2 are uplinked divergence tolerances.

The slide diverges if i > 1 and either of the following is true:

$$|\Delta P_i| > m_2 \, |\Delta P_{i-1}| \qquad (5\text{-}12c)$$

$$|\Delta V_i| > m_2 \, |\Delta V_{i-1}| \qquad (5\text{-}12d)$$

where m_2 is a scaling factor.

4. If the maximum number of iterations has been performed, a test for reduced convergence is made by multiplying $conv_1$, $conv_2$, and $conv_3$ by 3 and checking for convergence based on the new tolerances. Otherwise, the slide diverges.

5. If neither convergence or divergence has occurred, a linearity check is made and another complete iteration is performed. The linearity test determines whether or not new partial derivatives, a_j, are to be computed and the inner loop edit performed during the next iteration. The linearity constraint is satisfied (i.e., the test succeeds) if $|\Delta r_i| < e_1$ and $|\Delta \dot{r}_i| < e_2$, where e_1 and e_2 are uplink linearity tolerances.

If the test just described succeeds, no new partial derivatives are computed, and the partial derivatives from the previous iterations are used to compute the $[F^T WF(o-c)]$ matrix while the normal matrix remains unchanged.

5.4 PRELIMINARY EDIT CRITERIA

During the initial processing of the measurement data described in Section 5.1, two edit checks are performed to detect and remove anomalous data from the estimation process. The first simply verifies that the measurement residuals, $(o - c)_j$, fall within acceptable limits. The residual is compared with the maximum allowed, and the observation is edited if the residual exceeds the maximum. The next edit check is based on the measurement geometry between the user spacecraft and the TDRS spacecraft.

For the geometry edit test, two parameters are computed: H, the height of the TDRS-to-user ray path segment, and C, the central angle formed between two vectors (the Earth center-to-TDRS spacecraft and the Earth center-to-user spacecraft). These parameters are compared with two uplink parameters, H_0 and C_0; the measurement is edited only if $H < H_0$ and $C > C_0$, where H and C are defined by the following:

$$\begin{aligned} H &= |\vec{P}_{user}| - R_e \qquad (\beta > 90^\circ) \\ &= |\vec{P}_{user}| \sin \beta - R_e \qquad (\beta < 90^\circ) \end{aligned} \tag{5-13}$$

where

$$\beta = \cos^{-1} \frac{[\vec{P}_{user} \cdot (\vec{P}_{user} - \vec{P}_{TDRS})]}{|\vec{P}_{user}| \; |\vec{P}_{user} - \vec{P}_{TDRS}|} \tag{5-14}$$

where $\vec{P}_{user}$ = user satellite position (appropriate geocentric coordinates)

$\vec{P}_{TDRS}$ = appropriate TDRS position (appropriate geocentric coordinates)

R_e = equatorial radius of the Earth

<u>NOTE</u>: β is in units of degrees with $0^\circ \leq \beta < 180^\circ$.

$$C = \cos^{-1} \left\{ \frac{[\vec{P}_{TDRS} \cdot \vec{P}_{user}]}{[|\vec{P}_{TDRS}| \; |\vec{P}_{user}|]} \right\} \tag{5-15}$$

where $0^\circ \leq C \leq 180^\circ$

5.5 SLIDE PRECOMPUTATION

During precomputation of the next slide, an arbitrary epoch is chosen by advancing the epoch of the previous slide by a

fixed Δt. Therefore, the measurement partial derivatives computed during the precompute phase of estimation are referenced to that arbitrary epoch. When additional measurements are made available to the estimator and when the actual estimation process begins as described in Section 5.1, a new epoch may be chosen. If the epoch used during precomputation is earlier than the new observation timespan, the epoch for the slide is set to the latest observation time. If this is done, the partial derivatives are no longer referenced to the proper epoch time. To advance the partial derivatives to the new epoch, the measurement partial derivatives are postmultiplied by the state transition matrix, which maps the Cartesian state from the old to new epochs. The updated partial derivatives are then used during the estimation process. If the epoch used during precomputation is later than the final new observation, estimation is referenced to the precomputation epoch.

SECTION 6 - OBSERVATION MODELS

This section contains the mathematical specifications for the observation models in FEDS. Section 6.1 contains the observation model for one-way TDRSS Doppler measurements. Section 6.2 contains the algorithm for computing the partial derivatives of the measurements with respect to the target satellite state. Section 6.3 contains miscellaneous models required to solve the observation and observation partial derivative equations.

6.1 MEASUREMENT EQUATIONS

A monochromatic source of electromagnetic radiation in motion with respect to the observer experiences a phenomenon referred to as Doppler shift. It can be approximated by the following relation:

$$f = f_{ref} \frac{c}{c + V_{rel}} \tag{6-1}$$

where f = observed, Doppler-shifted frequency

f_{ref} = transmitted, reference frequency (no relative motion between source and observer)

V_{rel} = line-of-sight component of the relative velocity between the source and the observer

c = speed of light

For FEDS, observation measurements have the geometry shown in Figure 6-1. The figure shows a series of nodes that are connected by legs. The nodes are the end points of the signal path (i.e., TDRSS spacecraft, White Sands Ground Tracking Station (WSGT), FEDS at GSFC, and a fictitious target spacecraft); the legs are the signal paths between the nodes. The arrows on the legs represent the direction of signal propagation.

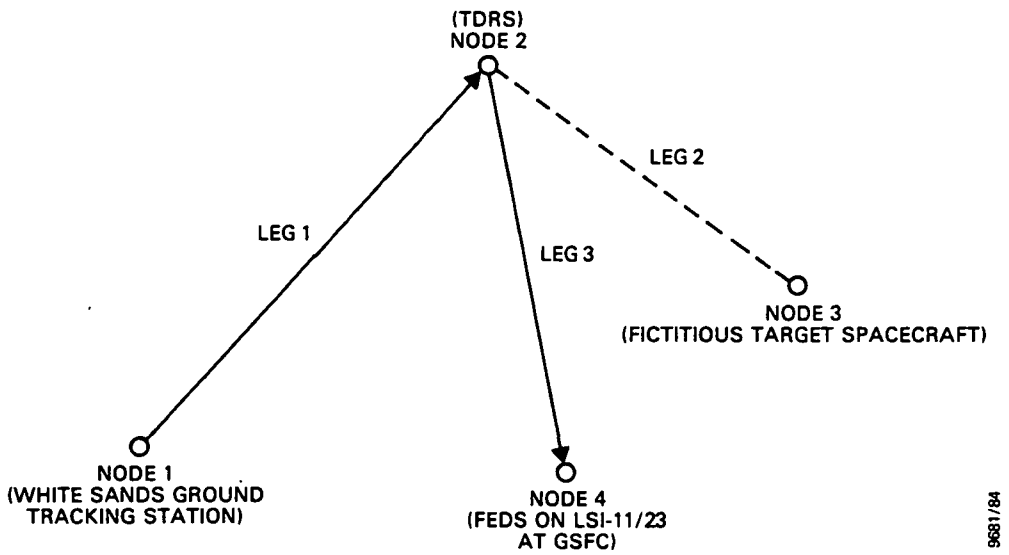

Figure 6-1. Signal Geometry for Doppler Observation

During the demonstration, WSGT will transmit at a frequency given by

$$f_t(t - \delta t_3 - \delta t_1) = \frac{1}{\alpha_1(t - \delta t_3)\ \alpha_2(t - \delta t_3 + \delta t_2)} f_{ref} \quad (6\text{-}2)$$

where $f_t(t)$ = frequency of signal transmitted at time t

f_{ref} = demonstration reference frequency (nominally 2287.5 megahertz)

$\alpha_i(t)$ = Doppler shift due to leg i at receive time t

δt_i = time of signal propagation for leg i

It should be noted that leg 1 and leg 3 signal propagation times are affected by transmission through the atmosphere. The propagation time is given by the following relation:

$$\delta t_i = \Delta \ell_i + \Delta RC_i \tag{6-3}$$

where $\Delta \ell_i$ = time of signal propagation in a vacuum for leg i (light-time correction; see Section 6.3.1)

ΔRC_i = refraction correction associated with leg i, applied only for ground-to-space or space-to-ground legs (see Section 6.3.2)

WSGT will compute $\alpha_i(t)$ based on predicted fictitious spacecraft ephemeris, actual ground antenna position, and real-time TDRS solutions. Equation (6-3) indicates that the signal would arrive at a fictitious target spacecraft at a nearly constant frequency.

The signal will arrive at the experiment transponder at a frequency given by

$$f_G(t) = \alpha_1(t - \delta t_3)\ \alpha_3(t)\ f_t(t - \delta t_3 - \delta t_1) \tag{6-4}$$

where $f_G(t)$ is the frequency received by the transponder located at GSFC. Combining Equations (6-2) and (6-4), the received frequency becomes

$$f_G(t) = \frac{\alpha_3(t)}{\alpha_2(t - \delta t_3 + \delta t_2)}\ f_{ref} \tag{6-5}$$

Substituting Equation (6-1) into Equation (6-5) twice and dropping negligible terms, the instantaneous Doppler shift is obtained as

$$f_D(t) = f_G(t) - f_{ref} = \frac{f_{ref}}{c} \left(\dot{\rho}_2 - \dot{\rho}_3\right) \tag{6-6}$$

where $f_D(t)$ = the instantaneous Doppler shift at time t

$$\dot{\rho}_i = \frac{d\rho_i}{dt} \tag{6-7}$$

The various leg distances, ρ_i, are defined as follows:

$$\rho_3 = |\vec{r}_G(t) - \vec{r}_T(t - \delta t_3)| \tag{6-8a}$$

$$\rho_2 = |\vec{r}_f(t - \delta t_3 + \delta t_2) - \vec{r}_T(t - \delta t_3)| \tag{6-8b}$$

where $\vec{r}_G(t)$ = position vector of the transponder at time t
$\vec{r}_T(t)$ = position vector of TDRSS satellite at time t
$\vec{r}_f(t)$ = simulated position vector of the fictitious target satellite at time t

Averaging over the interval Δt, the Doppler observation is computed as

$$\bar{F}_D(t) = \frac{\int_{t-\Delta t}^{t} f_D(t)\, dt}{\Delta t} = \frac{f_{ref}}{c\Delta t} \quad \Delta\rho_2 - \Delta\rho_3 \tag{6-9}$$

where the change in leg length, $\Delta\rho_i$, is defined as

$$\Delta\rho_i = \rho_i(t) - \rho_i(t - \Delta t) \tag{6-10}$$

6.2 PARTIAL DERIVATIVES OF MEASUREMENTS WITH RESPECT TO TARGET STATE

In general, the partial derivatives of an observation are computed as follows:

$$\frac{\partial f(t_R)}{\partial \vec{X}_3(t_o)} = \frac{\partial f(t_R)}{\partial \vec{X}_3(t_R - \delta t_3 + \delta t_2)} \cdot \Phi_3\ (t_R - \delta t_3 + \delta t_2,\ t_o) \tag{6-11}$$

where $f(t)$ = a measurement at time tag t

$\vec{X}_3(t)$ = state vector of node 3 at time t

$\Phi_3(t,\ t_o) = \partial\vec{X}_3(t)/\partial\vec{X}_3(t_o)$ = state transition matrix

t_o = epoch time

t_R = observation time tag

The Doppler measurement partial derivatives are obtained from Equation (6-9) as follows:

$$\frac{\partial \bar{F}_D(t_R)}{\partial \vec{X}_3(t_o)} = \frac{f_{ref}}{c\Delta t}\left(\frac{\partial \Delta\rho_2}{\partial \vec{X}_3(t_o)} - \frac{\partial \Delta\rho_3}{\partial \vec{X}_3(t_o)}\right) \tag{6-12}$$

Since leg 3 is independent of the target state,

$$\frac{\partial \Delta\rho_3}{\partial \vec{X}_3(t_o)} = 0 \tag{6-13}$$

Equation (6-12) reduces to

$$\frac{\partial \bar{F}_D(t_R)}{\partial \vec{X}_3(t_o)} = \frac{f_{ref}}{c\Delta t}\left(\frac{\partial \Delta\rho_2}{\partial \vec{X}_3(t_o)}\right) \tag{6-14}$$

where

$$\frac{\partial \Delta\rho_2}{\partial \vec{X}_3(t_o)} = \begin{bmatrix} \dfrac{\partial \rho_2(t_R)}{\partial \vec{X}_3(t_R - \delta t_3 + \delta t_2)} \\ 0 \\ 0 \\ 0 \end{bmatrix} * \Phi(t - \delta t_3 + \delta t_2, t_o)$$

$$- \begin{bmatrix} \dfrac{\partial \rho_2(t_R - \Delta t)}{\partial \vec{X}_3(t_R - \Delta t - \delta t_3 + \delta t_2)} \\ 0 \\ 0 \\ 0 \end{bmatrix} * \Phi(t_R - \Delta t - \delta t_3 + \delta t_2, t_o) \qquad (6\text{-}15)$$

with

$$\frac{\partial \rho_2(t)}{\partial \vec{X}_3(t - \delta t_3 + \delta t_2)} = \frac{\vec{r}_f(t - \delta t_3 + \delta t_2) - \vec{r}_T(t - \delta t_3)}{\rho_2} \qquad (6\text{-}16)$$

6.3 FREQUENCY BIAS EFFECTS ON OBSERVATION MODELING

In FEDS, frequency biases are modeled as a second-order polynominal given by

$$\Delta F(t) = \Delta F(t_e) + \Delta\dot{F}(t_e)(t - t_3) + \Delta\ddot{F}(t_e)(t - t_e)^2 \qquad (6\text{-}17)$$

where $\Delta F(t)$ = frequency bias at time t

t_e = epoch time

$\Delta\dot{F}(t_e)$ = frequency drift at epoch

$\Delta\ddot{F}(t_e)$ = frequency acceleration at epoch

When frequency biases are included in the transmitted frequency, Equation (6-5) becomes

$$f_G(t) = \frac{\alpha_3(t)}{\alpha_2(t + \delta t_3 + \delta t_2)} f_{ref} + \alpha_3(t)\alpha_1(t - \delta t_3)\ \Delta F(t) \qquad (6\text{-}18)$$

Since $\alpha_3(t)$ and $\alpha_1(t - \delta t_3)$ are near unity, the Doppler shift can be obtained from Equation (6-6) as follows:

$$f_D(t) = \frac{f_{ref}}{c} (\dot{\rho}_2 - \dot{\rho}_3) + \Delta F(t) \tag{6-19}$$

The averaged Doppler over the interval Δt is then

$$\bar{F}_D(t) = \frac{f_{ref}}{c\Delta t} (\Delta\rho_2 - \Delta\rho_3) + \Delta F(t) - \frac{\Delta\dot{F}(t)\Delta t}{2} + \frac{\Delta\ddot{F}(t)(\Delta t)^2}{6} \tag{6-20}$$

Partial derivatives with respect to frequency biases can be obtained by differentiating Equation (6-20), as follows:

$$\frac{\partial F_D(t)}{\partial \Delta F(t_e)} = 1 \tag{6-21}$$

$$\frac{\partial F_D(t)}{\partial \Delta\dot{F}(t_e)} = \frac{(t - t_e)^2 - (t - \Delta t - t_e)^2}{2\ \Delta t} \tag{6-22}$$

$$\frac{\partial F_D(t)}{\partial \Delta\ddot{F}(t_e)} = \frac{(t - t_e)^3 - (t - \Delta t - t_e)^3}{6\ \Delta t} \tag{6-23}$$

6.4 MISCELLANEOUS MODELS REQUIRED FOR OBSERVATION MODELING

The miscellaneous models required to compute both the observation and the observation partial derivatives are

- Newton-Raphson light-time corrector
- Tropospheric refraction corrections

6.4.1 NEWTON-RAPHSON LIGHT-TIME CORRECTOR

The Newton-Raphson iterative method for calculating the time at a transmitting node, given the receiving node's time, (backward light-time correction) is

$$t'_{T_n} = t_{T_n} + \frac{c\left(t_{R_{n+1}} - t_{T_n}\right) - \left|\vec{r}_{n+1}\left(t_{R_{n+1}}\right) - \vec{r}_n\left(t_{T_n}\right)\right|}{c - \hat{u}_{n,n+1} * \dot{\vec{r}}\left(t_{T_n}\right)} \qquad (6\text{-}24)$$

where t'_{T_n} = new estimate of the transmission time from node n

t_{T_n} = previous estimate of the transmission time from node n $\left(\text{initialized to time } t_{R_{n+1}}\right)$

$t_{R_{n+1}}$ = receive time at node n + 1

$\vec{r}_n(t)$ = position vector at time t

$\hat{u}_{n,n+1}$ = unit vector between nodes n and n + 1 (leg n)

$\dot{\vec{r}}_n(t)$ = velocity vector of node n at time t

c = speed of light

Ignoring negligible terms, Equation (6-24) can be reduced to

$$t'_{T_n} = t_{R_{n+1}} - \frac{\left|\vec{r}_{n+1}\left(t_{R_{n+1}}\right) - \vec{r}_n\left(t_{T_n}\right)\right|}{c} \qquad (6\text{-}25)$$

or

$$\delta t_N = t'_{T_n} - t_{R_{n+1}} = -\frac{\left|\vec{r}_{n+1}\left(t_{R_{n+1}}\right) - \vec{r}_n\left(t_{T_n}\right)\right|}{c} = \frac{-P_N}{c} \qquad (6\text{-}26)$$

The equation is solved N times until $\left|P_N - P'_N\right| <$ meter.

6.4.2 TROPOSPHERIC REFRACTION CORRECTIONS

The general approach to the problem of refraction correction of measurements is to obtain the most accurate corrections possible that are consistent with rapid computations. To preserve the raw tracking data in its original form, refraction corrections are applied to the computed values of the measurements. Furthermore, the corrections are applied only to range. The Doppler observation is corrected by relating it to corrected range values. The method of computing tropospheric refraction presented here was taken from Reference 10.

Measurement Corrections. The refraction-corrected, computed measurements are obtained as follows:

$$R_{CC} = R_C + \Delta R_T \tag{6-27}$$

where R_{CC} = corrected, computed range

R_C = computed distance between two nodes (uncorrected range)

ΔR_T = refraction correction

The Doppler shift is corrected for refraction effects by using the corrected range in Equation (6-9).

Tropospheric Correction. Tropospheric refraction corrections to range are assumed to be functions of two variations: elevation angle and surface refractivity. Range correction is tabulated for nominal surface refractivity, indexed by elevation angle. The tabulated values were computed by a ray tracing algorithm presented in Reference 10.

The standard value of surface refractivity was chosen to be 340 N units, because it is the average value of the range of surface refractivities encountered at the stations (280 N to

400 N units). The approximation for range correction is in terms of variation of the surface refractivity from the standard. The method, then, is an algorithm that produces tropospheric refraction correction to range, and these refraction corrections agree with ray tracing results to within the specified design criteria.

The refraction correction to range is therefore obtained by evaluating

$$\Delta R_T = \overline{\Delta R}_T + \left(\frac{\partial \overline{\Delta R}_T}{\partial N}\right) \Delta N + \frac{1}{2} \left(\frac{\partial^2 \overline{\Delta R}_T}{\partial N^2}\right) \Delta N^2 \qquad (6\text{-}28)$$

where $\overline{\Delta R}_T$ = interpolated range correction for the standard surface refractivity

$\partial \overline{\Delta R}_T / \partial N$ = first-order partial derivatives of the range correction with respect to change in surface refractivity from the standard

$\partial^2 \overline{\Delta R}_T / \partial N^2$ = second-order partial derivative of the range correction with respect to change in surface refractivity from the standard; the superscript bar denotes values for the standard refractivity obtained through interpolation

The method of computing the tropospheric refraction corrections for the range proceeds as follows. For the specified elevation angle, the necessary parameters are obtained from the tabulated data by numerical spline interpolation. (The spline interpolation method is the mathematical analog of the draftsman's mechanical spline, which is a long, very flexible, slender device used to pass a smooth curve through many data points. The technique is presented in Reference 11.) Equation (6-28) is evaluated to obtain the refraction correction to the range.

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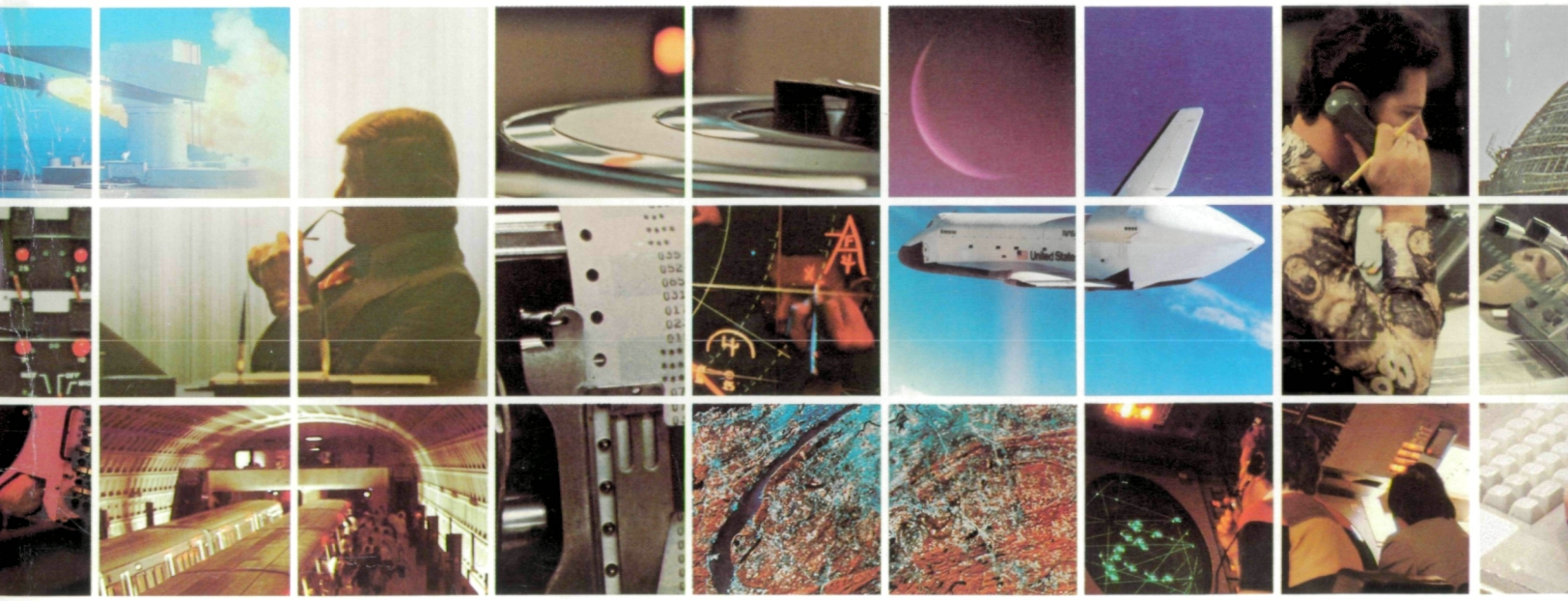
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