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THE
ALLIED
BENDIX
CORPORATION

GUIDANCE
SYSTEMS
DIVISION

TETERBORO
NEW JERSEY  07603

MODULAR DESIGN
ATTITUDE CONTROL
SYSTEM

FINAL REPORT
5 OCTOBER 1964

PREPARED FOR:

GEORGE C. MARSHALL
SPACE FLIGHT CENTER
HUNTSVILLE, ALABAMA

NASA CONTRACT NO.
NASS-33979
EXHIBIT D

OCTOBER 1, 1963
to
AUGUST 31, 1964

APPROVED BY:

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SYSTEMS DESIGN

PREPARED BY:

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FOREWORD

This final report is submitted in accordance with “Scope of Work, Exhibit D” for Contract NAS8-33979. The study was directed from the Guidance Systems Division (GSD) of the Allied Bendix Corporation. The engineering manager at this location was Mr. Joel Levinthal. Most of the analytical effort in support of this project was provided by Dr. Frederick Chichester, who wrote all sections of this report. Most of the algebraic development for this report was verified using the Symbolic Manipulation Program (SMP) software package on a VAX-11 computer by Mr. Alan Reynolds who also prepared this report using the TEX typesetting software package on the same computer. The guidance of Dr. Henry B. Waites and Mr. Stan Carroll of MSFC during the course of this study is gratefully acknowledged.
ABSTRACT

The problem of applying modular attitude control to a rigid body - flexible suspension model of a flexible spacecraft with some state variables inaccessible was addressed by developing a sequence of single-axis models and generating a series of reduced state linear observers of minimum order to reconstruct those scalar state variables that were inaccessible. The specific single-axis models treated consisted of two, three, four and five rigid bodies, respectively, interconnected by a flexible shaft passing through the mass centers of the bodies. Reduced state linear observers of all orders up to one less than the total number of scalar state variables were generated for each of the four single-axis models cited. Each of the single-axis models was then transformed to a corresponding modal model to which modal damping was added. Each of the damped modal models was written in state variable form. With the assumption that at least one of the scalar modal state variables was accessible, reduced state linear observers were developed for synthesizing the inaccessible modal state variables for each modal model.
SECTION 1
INTRODUCTION

This report is submitted in compliance with the Scope of Work under contract NAS8-33979. The period of performance covered by the contract is from October 1, 1983 to August 31, 1984. The submission and approval of this report constitute the successful completion of the "Exhibit D" portion of the contract.

This report is a sequel to five others, two of them previously submitted under a different contract number. The two prior reports, under a different contract number, references (1-1) and (1-2), were submitted in October 1978 and September, 1979 and covered the periods from July 27, 1977, to July 27, 1978, and from August 20, 1978, to August 26, 1979, respectively, in compliance with "Exhibit A" of contract NAS8-3260. Three prior final reports were prepared under contract NAS-33979. Reference (1-3) was submitted on March 8, 1982, and covered the period from August 15, 1980, to October 15, 1981, in compliance with "Exhibit A" of the contract. Reference (1-4) was submitted on March 18, 1983, and covered the period from October 16, 1981, to October 31, 1982, in compliance with "Exhibit B". Reference (1-5) was submitted on January 24, 1984, and covered the period from November 1, 1982, to September 30, 1983, in compliance with "Exhibit C".

1.1 OBJECTIVE

The sections that follow summarise the effort expended on the Modular Design Attitude Control System Study contract, from October 1, 1983, to August 31, 1984. In prior applications of modular attitude control to rigid body-flexible suspension approximations of the rotational dynamics of prototype flexible spacecraft, it was assumed that all of the scalar state variables of the linearised models were accessible for measurement and/or control. Actual spacecraft to be controlled almost never satisfy such a broad condition. Therefore, the principal objective of the development of modular attitude control, completed August 31, 1984, was the generation of a series of linear observers to support the application of control to state variable models of flexible spacecraft with damping for which one or more state variables are inaccessible.

1.2 SCOPE

Study effort was concentrated in four main areas:

A. Development of a series of single axis state variable models of flexible spacecraft with damping to be utilised in the comparison of different approaches to the development of modular attitude control systems. These models consisted of two, three, four, or five rigid bodies serially connected by a flexible suspension in such a way that motion was restricted to rotation about a common axis through the mass centers of the bodies.

B. Generation of reduced state linear observers for each damped single axis model developed in Task A corresponding to various numbers and distributions of inaccessible state variables following the approaches presented in Luenberger (1-6), (1-7), (1-8), and Sage (1-9).

C. Transformation of the undamped versions of the single axis models developed in Task A to their corresponding modal models with modal damping following the approach presented in Thomson (1-10).

D. Generation of reduced state linear observers for each modal model developed in Task C with various numbers of inaccessible modal state variables utilising direct matrix products as described in Lancaster (1-11).

1.3 GENERAL

This report is comprised of seven sections. Sections 2 through 5 describe the development of the two-, three-, four- and five-body single-axis state variable models, respectively, of a prototype flexible spacecraft with damping and the generation of the minimum order reduced state linear observers for the reconstruction of inaccessible scalar state variables of these models. Section 6 begins with the transformation of the single-axis models of Sections 2 through 5 to modal forms to which modal damping is added and concludes with the development of reduced state linear observers for these models when one or more modal state variables are inaccessible. Section 7 lists a number of conclusions and recommendations drawn from generation of linear observers for the series of single-axis state variable models described above. References are listed at the end of each section.
The original RFQ requested that the International System of units (designated as SI) be used in the program and in any reporting. Torques, moments, angular momentum, moments of inertia and distances, however, are stated in English units since this was the system of units used in presenting all of the vehicle data in the RFQ.
REFERENCES


SECTION 2
DEVELOPMENT OF TWO-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

2.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the two-body single-axis model of a flexible spacecraft with damping shown in Fig. 2-1 may be represented by the following set of equations:

\[ I_1 \ddot{\theta}_1 = -c_1 (\dot{\theta}_1 - \dot{\theta}_2) - h_1 (\theta_1 - \theta_2) + q_1 \]  
\[ I_2 \ddot{\theta}_2 = c_1 (\dot{\theta}_1 - \dot{\theta}_2) + h_1 (\theta_1 - \theta_2) + q_2 \]  

(2-1) \hspace{1cm} (2-2)

where:

- \( I_i \) = rotational inertia of body \( i \); \( i = 1, 2 \)
- \( \theta_i \) = angular displacement of body \( i \)
- \( \dot{\theta}_i \) = angular rate of body \( i \)
- \( q_i \) = torque applied to body \( i \)
- \( h_1 \) = rotational spring coefficient at the interface between the bodies
- \( c_1 \) = rotational damping coefficient at the interface between the bodies

2.2 STATE VARIABLE MODEL

The state variable form of the two-body single-axis model of a flexible spacecraft with damping shown in Fig. 2-1 may be expressed as follows:

\[ \dot{x} = Ax + Bu \]  
\[ x_A = Cx \]  

(2-3) \hspace{1cm} (2-4)

where:

- \( x = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2]^T = [x_1 \ x_2 \ x_3 \ x_4]^T = [x_A^T \ x_A^T]^T \) = state vector
- \( x_A = \) vector of accessible scalar states
- \( x_{\bar{A}} = \) vector of inaccessible scalar states
- \( u = [u_1 \ u_2]^T = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}^T \) = control vector
- \( A = 4 \times 4 \) state vector coefficient matrix
- \( B = 4 \times r \) control vector coefficient matrix (\( r = 1 \) or 2)
- \( C = m \times 4 \) measurement or observation matrix

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{23} & -a_{23}r_1 & a_{28} & a_{28}r_1 \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{41}r_1 & -a_{41} & -a_{41}r_1 \end{bmatrix} \]  

(2-5)

\[ r_1 = \frac{c_1}{h_1} \]  

(2-6)

\[ a_{23} = \frac{h_1}{I_1} \]  

(2-7)

\[ a_{41} = \frac{h_1}{I_2} \]  

(2-8)

\[ B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } r = 2 \]  

(2-9)
FIGURE 2-1
TWO-BODY SINGIF-AXIS MODEL WITH DAMPING
The block diagram corresponding to this model is depicted in Fig. 2-2.

2.3 REDUCED STATE LINEAR OBSERVERS

2.3.1 Introduction

The minimum order (number of scalar state variables) of a reduced state linear observer required to reconstruct the \( p \) inaccessible scalar states of the two-body single-axis model represented by equations (2-8) through (2-9) is \( p = 4 - m \). This reconstruction was accomplished for a given state variable model in three main stages.

1) Synthesising a linear observer of minimum required order (\( p \)).
2) Defining a synthesised variable corresponding to each of the inaccessible state variables of the given state variable model.
3) Expressing each synthesised variable as a function of the state variables of the reduced state observer and the accessible state variables of the given state variable model.

The relationship between the single-axis model and its corresponding reduced state observer is depicted in Fig. 2-3.

The equations for the reduced state observers corresponding to the state variable model of equations (2-3) through (2-9) are the following:

\[
\begin{align*}
\dot{x} &= Dz + Eu + Gy \\
\dot{z} &= Tx \\
E &= TB
\end{align*}
\]

where:

\( D = p \times p \) observer coefficient matrix (assumed diagonal)
\( E = p \times r \) observer control vector coefficient matrix
\( G = p \times m \) observer vector of observed states coefficient matrix
\( T = p \times 4 \) observer weighting matrix

The corresponding block diagram appears in Fig. 2-4.

2.3.2 Observer Synthesis Equations

The equations for synthesising the reduced state linear observers, based on those appearing in Luenberger (1-1), (1-2), (1-3) and Sage (1-4), were written in the following form.

\[
TA - DT = F
\]

\[
F = GC
\]

For

\[
T = \begin{bmatrix}
t_{11} & t_{12} & t_{13} & t_{14} \\
\vdots & \vdots & \vdots & \vdots \\
t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4}
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} \\
\vdots & \vdots & \vdots & \vdots \\
f_{p,1} & f_{p,2} & f_{p,3} & f_{p,4}
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
d_{11} & 0 \\
\vdots & \vdots \\
0 & d_{p,p}
\end{bmatrix}
\]

\( p \) columns of scalar state variables of a reduced state linear observer required to reconstruct the 4-m intrinsic scalar states of the two-body single-axis model represented by equations (2-8) through (2-9) is \( p = 4 - m \). This reconstruction was accomplished for a given state variable model in three main stages.

1) Synthesising a linear observer of minimum required order (\( p \)).
2) Defining a synthesised variable corresponding to each of the inaccessible state variables of the given state variable model.
3) Expressing each synthesised variable as a function of the state variables of the reduced state observer and the accessible state variables of the given state variable model.

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\[
TA - DT = F
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For

\[
T = \begin{bmatrix}
t_{11} & t_{12} & t_{13} & t_{14} \\
\vdots & \vdots & \vdots & \vdots \\
t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4}
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} \\
\vdots & \vdots & \vdots & \vdots \\
f_{p,1} & f_{p,2} & f_{p,3} & f_{p,4}
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
d_{11} & 0 \\
\vdots & \vdots \\
0 & d_{p,p}
\end{bmatrix}
\]
State and Observation Equations:
\[
\dot{x} = Ax + Bu
\]
\[
X_A = Cx
\]
\[ u = \text{vector of scalar inputs to vehicle model} \]
\[ x_A = \text{vector of accessible scalar states of model} \]
\[ z = \text{vector of scalar states of observer} \]
\[ T = \text{observer weighting matrix} \]
\[ \hat{x}_I = \text{vector of reconstructed scalar states of model} \]
\[ \hat{x} = \begin{bmatrix} x_A \\ \hat{x}_I \end{bmatrix} = \text{reconstructed vector of all scalar state variables of vehicle model} \]
Observer Equations:

\[ \dot{z} = Dz + Gz_A + Eu \]

Since \( Gz_A = GCx = Fx \),

\[ \dot{z} = Dz + Fx + Eu \]
and the form of the $A$ matrix given in equation (2-5) the observer synthesis equations reduce to the following general forms.

$$t_{i1} = d_{i1} t_{i2} - a_{41} r_{i1} f_{i4} + f_{i2}$$

$$t_{i2} = -a_{23} r_{i1} t_{i2} + d_{i2} t_{i4} + f_{i4}$$

$$\left[ \begin{array}{cc}
-a_{23} r_{i1} + d_{i2} & a_{41} r_{i1} \\
-a_{41} r_{i1} & -a_{41} r_{i1} + d_{i2}
\end{array} \right] \begin{bmatrix} t_{i3} \\ t_{i4} \end{bmatrix} = \begin{bmatrix} f_{i1} + d_{i2} f_{i3} \\ f_{i2} + d_{i2} f_{i4} \end{bmatrix}$$

Where:

$$d_{i1} = a_{23} r_{i1} + d_{i2}$$

$$d_{i2} = a_{41} r_{i1} + d_{i2}$$

$$p_{i1} = 1 + r_{i1} d_{i2}$$

$$\Delta_{i2} = (a_{23} r_{i1} + d_{i2})(a_{41} r_{i1} + d_{i2}) - a_{23} a_{41} r_{i1}$$

$$\Delta_{i2} = d_{i2} (a_{23} r_{i1} + a_{41} r_{i1} + d_{i2})$$

$$t_{i2} = \frac{(a_{41} r_{i1} + d_{i2})(f_{i1} + d_{i2} f_{i3}) + a_{41} r_{i1} (f_{i3} + d_{i2} f_{i4})}{\Delta_{i2}}$$

$$t_{i4} = \frac{-a_{23} r_{i1} (f_{i1} + d_{i2} f_{i3}) + (a_{23} r_{i1} + d_{i2})(f_{i3} + d_{i2} f_{i4})}{\Delta_{i2}}$$

$$t_{i1} = \frac{-d_{i2} [a_{41} + (a_{23} + a_{41}) r_{i1} d_{i2} + d_{i2}]}{\Delta_{i2}} (f_{i1} + d_{i2} f_{i3}) + f_{i3} + \frac{a_{41} d_{i2}}{\Delta_{i2}} (f_{i3} + d_{i2} f_{i4})$$

$$t_{i3} = \frac{-a_{23} d_{i2} (f_{i1} + d_{i2} f_{i3}) + d_{i2} [a_{23} + (a_{23} + a_{41}) r_{i1} d_{i2} + d_{i2}]}{\Delta_{i2}} (f_{i3} + d_{i2} f_{i4}) + f_{i4}$$

### 2.3.3 Comparison of $T$ Matrices for Damped and Undamped Models

If damping is removed from the model, $r_{i1} \rightarrow 0, d_{i1} \rightarrow d_{i2}, d_{i2} \rightarrow d_{i2}, p_{i1} \rightarrow 1$ and $\Delta_{i2} \rightarrow d_{i2} (a_{23} + a_{41} + d_{i2}) = \Delta_{i2}$.

$$t_{i2} = \frac{(a_{41} + d_{i2})(f_{i1} + d_{i2} f_{i3}) + a_{41} (f_{i3} + d_{i2} f_{i4})}{\Delta_{i2}}$$

$$t_{i4} = \frac{-a_{23} (f_{i1} + d_{i2} f_{i3}) + (a_{23} + d_{i2})(f_{i3} + d_{i2} f_{i4})}{\Delta_{i2}}$$

$$t_{i1} = \frac{-d_{i2} [a_{41} + d_{i2}]}{\Delta_{i2}} (f_{i1} + d_{i2} f_{i3}) - a_{41} (f_{i3} + d_{i2} f_{i4})$$

$$t_{i3} = \frac{d_{i2} [a_{23} (f_{i1} + d_{i2} f_{i3}) - (u_{23} + d_{i2})(f_{i3} + d_{i2} f_{i4})]}{\Delta_{i2}}.$$
1. In the equations expressing \( t_{13} \) and \( t_{14} \), the elements of the even columns of the \( T \) matrix as a function of the \( f_{ij} \), the elements of the \( F \) matrix, the form of each equation remains the same under addition of damping with \( a_{22}p_1 \) and \( a_{41}p_1 \) being substituted for each scalar \( a_{22} \) and \( a_{41} \) appearing in the corresponding equations for the undamped two body model.

2. In the equation expressing \( t_{11} \), the elements of the first column of the \( T \) matrix as a function of the \( f_{ij} \), the elements of the \( F \) matrix, the form of the equation remains the same under addition of damping except that the expression, \( a_{41}p_1 + a_{22}r_1d_{ij} \), appears in the place of \( a_{41} \) in the coefficient of \( (f_{ij} + d_{ij}f_{ij}) \) in the numerator and \( a_{22}p_1 \) and \( a_{41}p_1 \) appear in the place of \( a_{22} \) and \( a_{41} \) respectively in the denominator.

3. In the equation expressing \( t_{10} \), the elements of the third column of the \( T \) matrix as a function of the \( f_{ij} \), the elements of the \( F \) matrix, the form of the equation remains the same under addition of damping except that the expression, \( a_{22}p_1 + a_{41}r_1d_{ij} \), appears in the place of \( a_{22} \) in the coefficient of \( (f_{ij} + d_{ij}f_{ij}) \) in the numerator and \( a_{22}p_1 \) and \( a_{41}p_1 \) appear in the place of \( a_{22} \) and \( a_{41} \) respectively in the denominator.

### 2.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

#### 2.4.1 Introduction

Inaccessibility of a state variable in the model equations (2-3), (2-4) is reflected by a corresponding null column in the observation matrix, \( C \), and a corresponding null column in the \( F \) matrix as implied by equation (2-14). For the generation of reduced order observers for the two body model the number of inaccessible state variables can be 1, 2 or 3.

#### 2.4.2 First Order Observers (\( p = 1 \))

A first order linear observer corresponds to inaccessibility of one of the four scalar state variables of the two body model. The observer equation then reduces to:

\[
i = ds + Eu + Gy,
\]

the \( F \) and \( T \) matrices reduce to:

\[
F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix}
\]

\[
T = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix}
\]

and the observer synthesis equations reduce to the following forms:

\[
t_2 = -\frac{(a_{41}p_1 + d^2)(f_1 + df_2) + a_{41}p_1(f_2 + df_4)}{\Delta'_2}
\]

\[
t_4 = \frac{-a_{22}p_1(f_1 + df_2) + (a_{22}p_1 + d^2)(f_3 + df_4)}{\Delta'_2}
\]

\[
t_1 = \frac{-d(a_{41}p_1 + a_{22}r_1d + d^2)(f_1 + df_2) + f_2 + a_{41}d^2}{\Delta'_2} (f_3 + df_4)
\]

\[
t_8 = \frac{-a_{22}d}{\Delta'_2} (f_1 + df_2) + \frac{d(a_{22}p_1 + a_{41}r_1d + d^2)(f_3 + df_4) + f_4}{\Delta'_2}
\]

\[
p_1 = 1 + r_1d
\]

\[
\Delta'_2 = d^2(a_{22}p_1 + a_{41}p_1 + d^2)
\]

Since this case corresponds to inaccessibility of one state variable, one of the \( f_i \) \( (i = 1, 2, 3, 4) = 0 \).
Suppose $z_4$, the scalar state representing the angular rate of body 2, is inaccessible. Then it is assumed that:

$$
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

(2-42)

for which:

$$
P = \begin{bmatrix}
1 & 2 & 3 & 4
\end{bmatrix}
$$

(2-43)

and $T$ is of the form shown in equation (2-35).

From equations (2-14), (2-42) and (2-43),

$$
C = \begin{bmatrix}
1 & 2 & 3 & 4
\end{bmatrix}
$$

(2-44)

and from equations (2-9), (2-12) and (2-35)

$$
E = \begin{bmatrix}
t_2 & t_4
\end{bmatrix}
$$

(2-45)

This equation corresponds to $r = 2$, control torques applied to both bodies. For control torque applied only to body 1,

$$
E = \begin{bmatrix}
t_2 & 0
\end{bmatrix}
$$

(2-46)

and for control torque applied only to body 2,

$$
E = \begin{bmatrix}
0 & t_4
\end{bmatrix}
$$

(2-47)

The equations for determining the elements of the $T$ matrix reduce to the following forms:

$$
t_2 = -\frac{(a_{41}p_1 + d^2)(f_1 + d_2) + a_{41}p_1 f_2}{\Delta_2'}
$$

(2-48)

$$
t_4 = -\frac{a_{28}p_1(f_1 + f_2) + (a_{28}p_1 + d^2)}{\Delta_2'}
$$

(2-49)

$$
t_1 = \frac{d(a_{41}p_1 + a_{28}p_1 + d^2)(f_1 + d_2) + f_2 + a_{41}d}{\Delta_2'}
$$

(2-50)

$$
t_3 = -\frac{a_{28}d(f_1 + d_2) + d(a_{28}p_1 + a_{41}r_1 + d^2)}{\Delta_2'}
$$

(2-51)

From equations (2-11) and (2-35),

$$
s = t_1 x_2 + t_2 x_3 + t_3 x_4 + t_4 z_4
$$

(2-52)

where $z_4$ is the synthesized $z_4$.

Solving for $z_4$ yields:

$$
z_4 = \frac{1}{t_4}(s - \sum_{i=1}^{3} t_i x_i)
$$

(2-53)
For inaccessibility of $s_1$, $s_2$, or $s_4$, the equations for determining $t_i$, (2-38) through (2-39) are appropriately modified.

2.4.3 Second Order Observers ($p = 2$)

The equation for a linear observer of order two corresponds to two of the four scalar state variables being inaccessible. It is represented here as equation (2-10). If the observer coefficient matrix is assumed to be diagonal in this case it appears as follows:

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$$

(2-54)

Since the observer is of order two,

$$s = [s_1, s_2]^T$$

(2-55)

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix}$$

(2-56)

and,

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix}$$

(2-57)

The specific forms of the equations for generating the elements of $T$ depend upon which two of the scalar states are inaccessible. For each inaccessible state the corresponding columns in the $C$ and $F$ matrices are null.

Example

Corresponding to the angular position and rate, respectively, of body 2, suppose that the scalar states $x_3$ and $x_4$ are inaccessible. Then the equations for generating the elements of the $T$ matrix assume the following forms.

$$t_{i2} = -\frac{(a_{41}p_1 + d_{ii})}{\Delta_{i2}'}(f_{i1} + d_{ii}f_{i2}) \quad i = 1, 2$$

(2-58)

$$t_{i4} = -\frac{a_{ii}p_1}{\Delta_{i2}'}(f_{i1} + d_{ii}f_{i2})$$

(2-59)

$$t_{i1} = -\frac{d_{ii}(a_{41}p_1 + a_{31}p_1d_{ii}) + d_{ii}^2}{\Delta_{i2}'}(f_{i1} + d_{ii}f_{i2}) + f_{i2}$$

(2-60)

$$t_{i0} = -\frac{a_{31}d_{ii}}{\Delta_{i2}'}(f_{i1} + d_{ii}f_{i2})$$

(2-61)

where $p_1$ and $\Delta_{i2}'$ are defined in equations (2-23) and (2-24). From equation (2-11),

$$\begin{bmatrix} t_{i2} & t_{i4} \\ t_{22} & t_{24} \end{bmatrix} \begin{bmatrix} \hat{s}_3 \\ \hat{s}_4 \end{bmatrix} = \begin{bmatrix} s_1 - t_{11}s_1 - t_{12}s_2 \\ s_3 - t_{21}s_1 - t_{22}s_2 \end{bmatrix}$$

(2-62)

where $\hat{s}_3$ and $\hat{s}_4$ are synthesized state variables.

Let

$$\Delta_2 = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix} = t_{12}t_{24} - t_{14}t_{22} \neq 0$$

where:

$$(\Delta_2)_{i,j} = \Delta_2$$

without elements of $i$th row and $j$th column.
\[ \Delta_0 = \frac{(\Delta_3)_{1,1}(s_1 - t_{11}s_1 - t_{12}s_2) - (\Delta_3)_{2,1}(s_2 - t_{21}s_1 - t_{22}s_2)}{\Delta_3} \] (2-63)

\[ \Delta_4 = \frac{-(\Delta_3)_{1,2}(s_1 - t_{11}s_1 - t_{12}s_2) + (\Delta_3)_{2,2}(s_2 - t_{21}s_1 - t_{22}s_2)}{\Delta_3} \] (2-64)

For \( s_0 \) and \( s_4 \) inaccessible, it is assumed that:

\[
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0
\end{bmatrix}
\] (2-65)

\[
\begin{bmatrix}
  f_{11} & f_{12} & 0 & 0 \\
  f_{21} & f_{22} & 0 & 0
\end{bmatrix}
\] (2-66)

From \( F = GC \),

\[
G = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}
\] (2-67)

From \( E = TB \),

\[
E = \begin{bmatrix}
  f_{12} & f_{14} \\
  f_{22} & f_{24}
\end{bmatrix}
\] for \( r = 2 \) (control torques applied to both bodies) (2-68)

\[
E = \begin{bmatrix}
  f_{12} & 0 \\
  f_{22} & 0
\end{bmatrix}
\] for control restricted to body 1 (2-69)

\[
E = \begin{bmatrix}
  0 & f_{14} \\
  0 & f_{24}
\end{bmatrix}
\] for control restricted to body 2 (2-70)

2.4.4 Third Order Observers \((p = 3)\)

The equation for the linear observer of order one less than the system's dimension corresponds to three of the four scalar state variables being inaccessible. It is represented here as equation (2-10).

If the observer coefficient matrix is assumed to be diagonal in this case it appears as follows,

\[
D = \begin{bmatrix}
  d_{11} & 0 & 0 \\
  c & d_{22} & 0 \\
  0 & 0 & d_{33}
\end{bmatrix}
\] (2-71)

Since the observer is of order 3,

\[
\mathbf{x} = \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix}^T
\] (2-72)

\[
F = \begin{bmatrix}
  f_{11} & f_{12} & f_{13} & f_{14} \\
  f_{21} & f_{22} & f_{23} & f_{24} \\
  f_{31} & f_{32} & f_{33} & f_{34}
\end{bmatrix}
\] (2-73)

and,

\[
T = \begin{bmatrix}
  t_{11} & t_{12} & t_{13} & t_{14} \\
  t_{21} & t_{22} & t_{23} & t_{24} \\
  t_{31} & t_{32} & t_{33} & t_{34}
\end{bmatrix}
\] (2-74)

The specific forms of the equations for generating the elements of \( T \) depend upon which three of the scalar states are inaccessible. For each inaccessible state the corresponding columns in the \( C \) and \( F \) matrices are null.
Suppose the scalar states, \( a_2, a_3 \) and \( a_4 \), representing the angular rate of body 1 and the angular position and rate of body 2, are inaccessible. Then the equations for generating the elements of the T matrix assume the following form since \( f_{i1} = f_{i3} = f_{i4} = 0 \) for \( i = 1, 2, 3 \).

\[
t_{i2} = -\frac{(a_{41} p_1 + d_{i2}^2)}{\Delta_{i2}^t} f_{i1} \quad i = 1, 2, 3 \\
t_{i4} = -\frac{a_{42} p_{1} f_{i1}}{\Delta_{i2}^t} \\
t_{i1} = -\frac{d_{i1}(a_{41} p_{1} + a_{42} p_{2} + d_{i1}^2)}{\Delta_{i2}^t} f_{i1} \\
t_{i4} = -\frac{a_{42} d_{i2} f_{i1}}{\Delta_{i2}^t}
\]

where \( p_1 \) and \( \Delta_{i2}^t \) are defined in equations (2-23) and (2-24).

From equation (2-11),

\[
\begin{bmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} \dot{s}_2 \\ \dot{s}_3 \\ \dot{s}_4 \end{bmatrix} = \begin{bmatrix} s_1 - t_{11} x_1 \\ s_2 - t_{21} x_1 \\ s_3 - t_{31} x_1 \end{bmatrix}
\]

where \( \dot{s}_2, \dot{s}_3 \) and \( \dot{s}_4 \) are synthesized state variables.

Let \( \Delta_3 = \begin{vmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{vmatrix} \neq 0 \)

where,

\[
(\Delta_3)_{i,j} = \Delta_3 \text{ without elements of } i \text{th row and } j \text{th column}
\]

\[
\dot{s}_{j+1} = \sum_{i=1}^{3} (-1)^{i+j}(\Delta_3)_{i,j}(s_j - s_1) / \Delta_3 \quad j = 1, 2, 3
\]

For \( a_2, a_3 \) and \( a_4 \) inaccessible, it is assumed that:

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
F = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

From \( F = GC \),

\[
G = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{bmatrix}
\]

From \( E = TB \),

\[
E = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \end{bmatrix} \quad \text{for } r = 2 \text{ (control torques applied to both bodies)}
\]

\[15\]
2.5 REFERENCES


3.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the three-body single-axis model of a flexible spacecraft with damping shown in Fig. 3-1 may be represented by the following set of equations:

\[
\begin{align*}
I_1 \ddot{\theta}_1 &= -c_1(\dot{\theta}_1 - \dot{\theta}_2) - k_1(\theta_1 - \theta_2) + q_1 \\
I_2 \ddot{\theta}_2 &= c_1(\dot{\theta}_1 - \dot{\theta}_2) + k_1(\theta_1 - \theta_2) + c_2(\dot{\theta}_2 - \dot{\theta}_3) + k_2(\theta_2 - \theta_3) + q_2 \\
I_3 \ddot{\theta}_3 &= -c_2(\dot{\theta}_2 - \dot{\theta}_3) - k_2(\theta_2 - \theta_3) + q_3
\end{align*}
\]

where

\[I_i = \text{rotational inertia of body } i; \ i = 1, 2, 3\]
\[\theta_i = \text{angular displacement of body } i\]
\[\dot{\theta}_i = \text{angular rate of body } i\]
\[q_i = \text{torque applied to body } i\]
\[k_j = \text{rotational spring coefficient at interface } j; \ j = 1, 2\]
\[c_j = \text{rotational damping coefficient at interface } j\]

3.2 STATE VARIABLE MODEL

The state variable form of the three-body single-axis model of a flexible spacecraft shown in Fig. 3-1 may be expressed as follows:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
x_A &= Cx
\end{align*}
\]

where:

\[x = [x_1 \ x_2 \ \ldots \ x_6]^T = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2 \ \theta_3 \ \dot{\theta}_3]^T = [x_A^T \ x^T]^T = \text{state vector}\]
\[x_A = m\text{-vector of accessible scalar states}\]
\[x_1 = p\text{-vector of inaccessible scalar states}\]
\[u = [u_1 \ \ldots \ u_r]^T = [\frac{q_1}{I_1} \ \ldots \ \frac{q_r}{I_r}]^T \quad (r = 1, 2 \text{ or } 3)\]
\[C = \text{observation matrix of dimensions } m \times 6, \ m = 1, 2, \ldots, 5 \quad (\text{Minimum dimension of reduced order observer required } = 6 - m).\]

Partitioning of this model by rigid body results in the following forms for its coefficient matrices.

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-a_{22} & -a_{23}r_1 & a_{23} & a_{23}r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
a_{41} & a_{41}r_1 & a_{44} & a_{44} & a_{44} & a_{44}r_2 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & a_{43} & a_{43}r_2 & -a_{43} & -a_{43}r_2
\end{bmatrix}
\]
FIGURE 3-1

THREE-BODY SINGLE-AXIS MODEL WITH DAMPING AT BOTH INTERFACES
\[ a_{22} = \frac{\dot{h}_1}{\dot{t}_1} \]
\[ a_{41} = \frac{\dot{h}_1}{\dot{t}_2} \]  \hspace{1cm} (3-7)
\[ a_{42} = \frac{\dot{h}_2}{\dot{t}_2} \]
\[ a_{43} = -(a_{41} + a_{42}) \]
\[ a_{44} = -(a_{41}t_1 + a_{42}t_2) \]
\[ a_{63} = \frac{\dot{h}_3}{\dot{t}_3} \]
\[ r_j = \frac{\dot{q}_j}{h_j} \quad j = 1, 2 \] \hspace{1cm} (3-8)
\[ B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies)} \] \hspace{1cm} (3-9)

The block diagram corresponding to this model is shown in Fig. 2-2.

### 3.3 REDUCED STATE LINEAR OBSERVERS

#### 3.3.1 Introduction

For the three-body single-axis model represented by equations (3-4) through (3-9), the minimum order of a reduced state linear observer required to generate the inaccessible states is \( p = 6 - m \) \((m = 1, 2, \ldots, 5)\). All of the reduced state linear observers for the three body model may be written in the form represented by equations (2-10) and (2-11) where, in this case, the observer coefficient matrix, \( D \), is assumed to be diagonal and of dimensions \( p \times p \). The corresponding observer weighting matrix is of the following form:

\[ T = \begin{bmatrix} t_{11} & \cdots & t_{16} \\ \vdots & \ddots & \vdots \\ t_{p,1} & \cdots & t_{p,6} \end{bmatrix} \] \hspace{1cm} (3-10)

From equations (2-12), (3-9) and (3-10),

\[ E = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ \vdots & \ddots & \vdots \\ t_{p,2} & t_{p,4} & t_{p,6} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all 3 bodies)} \] \hspace{1cm} (3-11)

\[ F = \begin{bmatrix} f_{11} & \cdots & f_{16} \\ \vdots & \ddots & \vdots \\ f_{p,1} & \cdots & f_{p,6} \end{bmatrix} \] \hspace{1cm} (3-12)

The corresponding observer block diagram appears in Fig 2-4.
3.3.3 Observer Synthesis Equations

From Lambourger (3-1), (3-2), (3-3) and Sage (3-4) the equations for synthesizing the reduced state linear observers for the three-body single-axis model represented by equations (3-4) through (3-9) are given by equations (2-13) and (2-14). With coefficient matrices of the forms listed in 3.3.1 this set of observer synthesis equations reduces to the following forms:

\[
\begin{align*}
\zeta_{13} &= \frac{(A'_{13})_{1,1}(f_{i1} + d_{ii}f_{i3}) - (A'_{13})_{2,1}(f_{i2} + d_{ii}f_{i4}) + (A'_{13})_{3,1}(f_{i3} + d_{ii}f_{i4})}{\Delta'_{13}} \\
&= \frac{[a_{41}p_{11} + d_{ii}^2]([a_{42}p_{32} + d_{ii}^2] + a_{43}p_{42}d_{ii}]}{\Delta'_{13}}(f_{i1} + d_{ii}f_{i3}) \\
&\quad + \frac{a_{41}p_{11}([a_{42}p_{32} + d_{ii}^2] + a_{43}p_{42})}{\Delta'_{13}}(f_{i3} + d_{ii}f_{i4}) + a_{43}p_{42}(f_{i3} + d_{ii}f_{i4}) \\
&\quad + \frac{[a_{43}p_{41}d_{ii}]}{\Delta'_{13}}(f_{i3} + d_{ii}f_{i4}) \\
&= \frac{(A'_{13})_{1,2}(f_{i1} + d_{ii}f_{i3}) - (A'_{13})_{2,2}(f_{i2} + d_{ii}f_{i4}) + (A'_{13})_{3,2}(f_{i3} + d_{ii}f_{i4})}{\Delta'_{13}} \\
&= \frac{a_{32}p_{12}(a_{42}p_{32} + d_{ii}^2)(f_{i1} + d_{ii}f_{i3}) + [a_{32}p_{12} + d_{ii}^2][a_{43}p_{42} + d_{ii}^2]}{\Delta'_{13}}(f_{i3} + d_{ii}f_{i4}) \\
&\quad + \frac{a_{32}p_{12}(a_{43}p_{42} + d_{ii}^2)}{\Delta'_{13}}(f_{i3} + d_{ii}f_{i4}) \\
&= \frac{(A'_{13})_{1,3}(f_{i1} + d_{ii}f_{i3}) - (A'_{13})_{2,3}(f_{i2} + d_{ii}f_{i4}) + (A'_{13})_{3,3}(f_{i3} + d_{ii}f_{i4})}{\Delta'_{13}} \\
&= \frac{a_{23}p_{13}(a_{43}p_{41} + d_{ii}^2)(f_{i1} + d_{ii}f_{i3}) + a_{43}p_{41}p_{23}(a_{43}p_{42} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})}{\Delta'_{13}} \\
&\quad + \frac{[a_{23}p_{13} + d_{ii}^2 + a_{43}p_{41}d_{ii}]}{\Delta'_{13}}(f_{i3} + d_{ii}f_{i4}) \\
&= \frac{d_{ii}[(a_{41}p_{11} + d_{ii}^2)(a_{42}p_{32} + d_{ii}^2) + a_{43}p_{42}d_{ii}]}{\Delta'_{13}} \\
&\quad + \frac{a_{23}p_{13}d_{ii}[(a_{43} + a_{43}p_{41} + d_{ii}^2)]}{\Delta'_{13}}(f_{i1} + d_{ii}f_{i3}) + f_{i2} \\
&\quad + \frac{a_{41}d_{ii}([a_{42}p_{32} + d_{ii}^2](f_{i3} + d_{ii}f_{i4}) + a_{43}p_{42}(f_{i3} + d_{ii}f_{i4})]}{\Delta'_{13}} \quad (3-13)
\end{align*}
\]
E 2.3 Comparison of T Matrices for Damping at Various Interfaces

The observer synthesis equations for the three-body single-axis model with damping were compared with those for the same model without damping. A general form was developed for these equations that encompassed the elements of the observer T matrix for the following conditions with respect to damping in the model.

1. No damping;
2. Damping only at the interface between bodies 1 and 2.
3. Damping only at the interface between bodies 2 and 3.
4. Damping at all three interfaces.

where:

\[ T_{ij} = \begin{cases} 1 & i = j \geq 1, \leq 3 \\ \text{else} & \end{cases} \]

\[ \Delta T_{ij} = \begin{cases} -\alpha_i (\alpha_i + \alpha_j) & i = j = 1, 2 \\ 0 & \text{else} \end{cases} \]

\[ \Delta T_{ij} = \begin{cases} \alpha_i (\alpha_i + \alpha_j) & i = j = 1, 2 \\ 0 & \text{else} \end{cases} \]

\[ \Delta T_{ij} = \begin{cases} (\alpha_i + \alpha_j) & i = j = 1, 2 \\ 0 & \text{else} \end{cases} \]

\[ \Delta T_{ij} = \begin{cases} (\alpha_i + \alpha_j) & i = j = 1, 2 \\ 0 & \text{else} \end{cases} \]

Comparison of T Matrices for Damping at Various Interfaces
3. Damping only at the interface between bodies 2 and 3;
4. Damping at both interfaces.

Elimination of damping at interface \( j \) of the model corresponds to setting \( r_j = 0 \) and \( p_j = 1 \) in the equations for generating the elements of the \( T \) matrix with damping present at both interfaces, equations (3-13) through (3-20). If all damping is removed from the three body model, \( r_j \rightarrow 0, p_j \rightarrow 1, d_{ii1} \rightarrow d_{ii2} \rightarrow d_{ii} \) and \( \Delta_{ii} \rightarrow -d_{ii}^2 \left[ a_{29} a_{45} + a_{28} a_{43} + a_{41} a_{48} + (a_{28} + a_{41} + a_{48} + a_{43}) d_{ii}^2 + d_{ii}^4 \right] = \Delta_{ii}^0. \)

\[
t_{13} = \frac{[a_{44} d_{ii}^2 (a_{63} + d_{ii}^2) + a_{43} d_{ii}^4]}{\Delta_{ij}} (f_{i1} + d_{ii} f_{i3}) \\
+ \frac{a_{41} [a_{63} + d_{ii}^2] (f_{i3} + d_{ii} f_{i4}) + a_{43} (f_{i5} + d_{ii} f_{i6})}{\Delta_{ij}} \quad i = 1, 2, \ldots, p \tag{3-24}
\]

\[
t_{44} = \frac{a_{22} (a_{63} + a_{44}^2) (f_{i1} + d_{ii} f_{i3}) + (a_{28} + a_{41} + a_{48}) (a_{63} + d_{ii}^2) (f_{i3} + d_{ii} f_{i4})}{\Delta_{ij}} \\
+ \frac{a_{22} a_{45} + a_{28} + a_{41} + a_{48} d_{ii}^2 + d_{ii}^4}{\Delta_{ij}} (f_{i5} + d_{ii} f_{i6}) \tag{3-25}
\]

\[
t_{66} = \frac{a_{22} a_{45} (f_{i1} + d_{ii} f_{i3}) + a_{45} (a_{28} + d_{ii}^2) (f_{i3} + d_{ii} f_{i4})}{\Delta_{ij}} \\
+ \frac{[a_{22} a_{45} + (a_{28} + a_{41} + a_{48}) d_{ii}^2 + d_{ii}^4]}{\Delta_{ij}} (f_{i5} + d_{ii} f_{i6}) \tag{3-26}
\]

\[
t_{11} = \frac{d_{ii} [a_{43} + d_{ii}^2] (a_{63} + d_{ii}^2) + a_{45} d_{ii}^2]}{\Delta_{ij}} (f_{i1} + d_{ii} f_{i3}) + f_{i2} \\
+ \frac{a_{41} d_{ii} [(a_{63} + d_{ii}^2) (f_{i3} + d_{ii} f_{i4}) + a_{43} (f_{i5} + d_{ii} f_{i6})]}{\Delta_{ij}} \tag{3-27}
\]

\[
t_{13} = \frac{a_{22} d_{ii} (a_{63} + d_{ii}^2) (f_{i1} + d_{ii} f_{i3}) + d_{ii} (a_{22} + d_{ii}^2) (a_{63} + d_{ii}^2) (f_{i3} + d_{ii} f_{i4}) + f_{i4}}{\Delta_{ij}} \\
+ \frac{a_{22} d_{ii} (a_{22} + d_{ii}^2)}{\Delta_{ij}} (f_{i5} + d_{ii} f_{i6}) \tag{3-28}
\]

\[
t_{16} = \frac{a_{45} d_{ii} [(a_{28} + d_{ii}^2) (f_{i1} + d_{ii} f_{i2}) + (a_{28} + d_{ii}^2) (f_{i3} + d_{ii} f_{i4})]}{\Delta_{ij}} \\
+ \frac{d_{ii} [(a_{28} + d_{ii}^2) (a_{45} + d_{ii}^2) + a_{41} d_{ii}^2]}{\Delta_{ij}} (f_{i5} + d_{ii} f_{i6}) + f_{i6} \tag{3-29}
\]

The elements, \( t_{11} \) through \( t_{16} \) \( (i = 1, \ldots, p) \), of the \( T \) matrix of the linear observer of order \( p \) corresponding to this model with one or more inaccessible states were found to be affected by the addition of damping at the interface between bodies 1 and 2 as follows.

1. The scalars \( a_{28} \) and \( a_{41} \), were modified to \( a_{28} p_i \) and \( a_{41} p_i \), respectively, in the equations for generating \( t_{12}, t_{14}, t_{16} \) and \( t_{16} \) and in the denominators of the equations for generating \( t_{11} \) and \( t_{13} \) where \( p_i \) was defined in equation (3-19).
2. In the numerator of the equation for generating \( t_{11} \), (3-16), the following changes occurred.
   a. The term, \( \alpha_{23} r_1 d_i^2 (a_{43} + a_{93} + d_i^2) \), was added to the coefficient of \( f_{11} \) where \( r_j \) is defined in equation (3-8).
   b. The scalar, \( \alpha_{41} \), in the coefficient of \( f_{11} \) was changed to \( \alpha_{41} P_1 \).

3. In the numerator of the equation for generating \( t_{18} \), (3-17), the following changes occurred.
   a. The term, \( \alpha_{23} a_{45} r_2 d_i^2 \), was subtracted from the coefficient of \( (f_{11} + d_i f_{12}) \).
   b. The term, \( \alpha_{41} r_1 d_i^2 (a_{43} + d_i^2) \), was added to the coefficient of \( f_{18} \).
   c. Each \( a_{23} \) in the coefficients of \( f_{11} \), \( f_{12} \) and \( f_{18} \) was modified to \( a_{23} p_2 \).
   d. The term, \( \alpha_{41} a_{45} r_2 d_i^2 \), was added to the coefficient of \( (f_{18} + d_i f_{16}) \).

Addition of damping at the interface between bodies 2 and 3 had the following effects.

1. The scalars, \( \alpha_{45} \) and \( \alpha_{65} \), were modified to \( \alpha_{45} p_2 \) and \( \alpha_{65} p_2 \) respectively, in the equations for generating \( t_{11}, t_{12}, t_{14} \) and \( t_{18} \) and the denominators of the equations for generating \( t_{12}, t_{13} \) and \( t_{16} \) where \( p_j \) is defined in equation (3-19).

2. In the numerator of the equation for generating \( t_{18} \), (3-17), the following changes occurred.
   a. The term, \( \alpha_{23} a_{45} r_2 d_i^2 \), was added to the coefficient of \( f_{11} \) where \( r_j \) is defined in equation (3-8).
   b. Each \( a_{45} \) in the coefficients of \( f_{11}, f_{12} \) and \( f_{18} \) was modified to \( a_{45} p_2 \).
   c. The term, \( \alpha_{45} r_2 d_i^2 (a_{23} + d_i^2) \), was added to the coefficient of \( f_{18} \).
   d. The term, \( \alpha_{41} a_{45} r_2 d_i^2 \), was subtracted from the coefficient of \( (f_{18} + d_i f_{16}) \).

3. In the numerator of the equation for \( t_{16} \), (3-17), the following changes occurred.
   a. The term, \( \alpha_{23} a_{45} r_2 d_i^2 \) was added to the coefficient of \( f_{18} \).
   b. The scalar, \( \alpha_{45} \), in the coefficient of \( f_{16} \) was changed to \( \alpha_{45} p_2 \).

Addition of damping at both the interface between bodies 1 and 2 and the interface between bodies 2 and 3 had the following effects.

1. The scalars, \( \alpha_{22}, \alpha_{44} \) and \( \alpha_{66} \), were modified to \( \alpha_{22} p_1 \), \( \alpha_{44} p_1 \), \( \alpha_{66} p_2 \) and \( \alpha_{66} p_2 \) respectively, in the equations for generating \( t_{12}, t_{14} \) and \( t_{16} \) and the denominators of the equations for generating \( t_{13}, t_{14} \) and \( t_{16} \) where \( p_j \) is defined in equation (3-19).

2. In the numerator of the equation for generating \( t_{11} \), (3-16), the following changes occurred.
   a. The term, \( \alpha_{23} r_1 d_i^2 \left[ (a_{45} + a_{65}) p_2 + d_i^2 \right] \), was added to the coefficient of \( f_{11} \), where \( r_j \) is defined in equation (3-8) and \( p_j \) is defined in equation (3-19).
   b. The remaining scalars, \( \alpha_{41}, \alpha_{45} \) and \( \alpha_{65} \), were modified to \( \alpha_{41} p_1 \), \( \alpha_{45} p_2 \) and \( \alpha_{65} p_2 \) respectively with the exception of the \( \alpha_{41} \) common to the coefficients of \( f_{14} \) through \( f_{16} \).

3. In the numerator of the equation for generating \( t_{18} \), (3-17), the following changes occurred.
   a. The term, \( \alpha_{23} a_{45} r_2 (r_2 - r_1) \), was added to the coefficient of \( (f_{11} + d_i f_{12}) \).
   b. The terms, \( \alpha_{41} r_1 d_i^2 (a_{45} p_1 + d_i^2) \) and \( \alpha_{65} r_2 d_i^2 (a_{23} p_1 + d_i^2) \), were added to the coefficient of \( f_{18} \).
   c. The term, \( \alpha_{41} a_{45} r_2 (r_1 - r_2) \), was added to the coefficient of \( (f_{18} + d_i f_{16}) \).
   d. The remaining scalars, \( \alpha_{23} \) and \( \alpha_{65} \), were modified to \( \alpha_{23} p_1 \) and \( \alpha_{65} p_2 \) respectively.

4. In the numerator of the equation for generating \( t_{16} \), (3-17), the following changes occurred.
   a. The term, \( \alpha_{45} r_2 d_i^2 \left[ (a_{22} + a_{41}) p_1 + d_i^2 \right] \), was added to the coefficient of \( f_{16} \).
   b. The remaining scalars, \( \alpha_{22}, \alpha_{41}, \alpha_{45} \) and \( \alpha_{66} \), were modified to \( \alpha_{22} p_1 \), \( \alpha_{41} p_1 \), \( \alpha_{45} p_2 \) and \( \alpha_{66} p_2 \) respectively with the exception of the \( \alpha_{45} \) common to the coefficients of \( f_{11}, f_{12}, f_{13} \) and \( f_{14} \).
2.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

2.4.1 Introduction

Inaccessibility of a scalar state variable in equation set (3-1), (3-2) is reflected by a corresponding null column in the observation matrix, C and, as implied by equation (3-11), in the F matrix for the generation of reduced order observers for the three-body model. The number of inaccessible scalar states can be 1, 2, 3, 4 or 5.

2.4.2 First Order Observers (p = 1)

A first order observer is required when any one of the six scalar state variables of the three body model is inaccessible. The first order form of the linear observer equation is:

\[ \dot{x} = dx + Ew + Gy \]  

(3-30)

The F and T matrices associated with a first order observer for the three body model then reduce to the following row forms.

\[ F = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{bmatrix}^T \]  

(3-31)

\[ T = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{bmatrix}^T \]  

(3-32)

The observer synthesis equations are then given by equation (3-13) through equation (3-23) with \( i = 1 \). Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the \( f_i \) (\( i = 1, 2, 3, 4, 5, 6 \)) = 0.

Example

Suppose that the scalar state representing the angular rate of body 3, \( x_6 \), is inaccessible. Then \( f_6 = 0 \) and the observer synthesis equations reduce to the following forms.

\[ t_2 = \frac{[a_{41}p_1 + d^2][a_{44}p_2 + d^2] + a_{45}p_2d^2]}{\Delta_6} (f_1 + d/f_2) \]

\[ + \frac{a_{41}p_1[(a_{44}p_2 + d^2)/(f_3 + d/f_4) + a_{45}p_2/f_5]}{\Delta_6} \]  

(3-33)

\[ t_4 = \frac{a_{44}p_1(a_{44}p_2 + d^2)(f_1 + d/f_3) + (a_{44}p_1 + d^2)(a_{44}p_2 + d^2)(f_3 + d/f_4)}{\Delta_6} \]

\[ + \frac{a_{44}p_2(a_{44}p_1 + d^2)f_5}{\Delta_6} \]  

(3-34)

\[ t_6 = \frac{a_{44}p_1(a_{44}p_2 + d^2)(f_1 + d/f_3) + (a_{44}p_1 + d^2)a_{44}p_2(f_3 + d/f_4)}{\Delta_6} \]

\[ + \frac{[(a_{44}p_1 + d^2)(a_{44}p_2 + d^2) + a_{44}p_1d^2]}{\Delta_6} f_5 \]  

(3-35)

\[ t_1 = \frac{d\left\{ [(a_{41}p_1 + d^2)(a_{44}p_2 + d^2) + a_{44}p_2d^2] + a_{44}r_1 \left[ (a_{44} + a_{44})p_2 + d^2 \right] \right\}}{\Delta_6} (f_1 + d/f_2) + f_2 \]

\[ + \frac{a_{44}d[(a_{44}p_2 + d^2)(f_3 + d/f_4) + a_{44}p_2/f_5]}{\Delta_6} \]  

(3-36)
\[
t_6 = \frac{a_{4ad}[a_{43d}(r_3 - r_4) + (a_{43d}P_3 + d^3)]}{\Delta'_{1}}(f_1 + d f_3)
\]
\[
+ \frac{d[(a_{32d}P_3 + d^3)(a_{32d}P_3 + d^3) + a_{411}d(a_{43d}P_3 + d^3)]}{\Delta'_{1}}(f_3 + d f_3) + f_4
\]
\[
+ \frac{a_{43d}[a_{411}d(r_1 - r_3) + (a_{32d}P_3 + d^3)]}{\Delta'_{1}}f_5
\]
\[
(3-37)
\]
\[
t_6 = \frac{a_{43d}[a_{32d}P_1 + (a_{32d}P_1 + d^3)]}{\Delta'_{1}}(f_3 + d f_4)
\]
\[
+ \frac{d[a_{44d}d[(a_{28} + a_{41})P_1 + d^3] + [(a_{32d}P_3 + d^3)(a_{44d}P_3 + d^3) + a_{411}P_1 d^3]}{\Delta'_{1}}f_5
\]
\[
(3-38)
\]

where:

\[
r_j = \frac{\dot{z}_j}{k_j} \quad (3-38)
\]
\[
p_j = 1 + r_j d \quad (3-39)
\]
\[
\Delta' = -d^2[(a_{28}a_{48} + a_{28}a_{16} + a_{41}a_{63})P_3 P_3
\]
\[
+ (a_{32d}P_3 + a_{41d}P_1 + a_{44d}P_1 + a_{43d}P_1 + a_{43d}P_2)d^2 + d^4] \quad (3-40)
\]

From equation (2-11) the synthesized scalar state, \(s_6\), is expressed in terms of the observer state variable, \(z\), and the accessible scalar state variables as follows,

\[
\dot{s}_6 = \frac{1}{t_0} [z - \sum_{i=1}^{s} t_i z_i] \quad (3-41)
\]

In this case, it is assumed that:

\[
C = \begin{bmatrix}
    & 0 \\
    I_s & \vdots \\
    & 0
\end{bmatrix} \quad (3-42)
\]

where \(I_s = 5 \times 5\) identity matrix.

From \(F = GC\),

\[
G = [f_1 \ f_2 \ f_3 \ f_4 \ f_5] \quad (3-43)
\]

From \(E = TB\),

\[
E = [t_2 \ t_3 \ t_5] \text{ for } r = 3 \text{ (control torques applied to all three bodies)} \quad (3-44)
\]
\[
E = [t_2 \ t_3 \ 0] \text{ for control applied to bodies 1 and 2} \quad (3-45)
\]
\[
E = [t_2 \ 0 \ 0] \text{ for control applied to body 1} \quad (3-46)
\]
\[
E = [t_2 \ 0 \ t_6] \text{ for control applied to bodies 1 and 3} \quad (3-47)
\]
3.4.3 Observers of Intermediate Order ($p = 2, 3$ or 4)

In the cases in which an intermediate number of the six scalar states of the three-body single-axis model is inaccessible the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by $p$. In each case the number of null columns in the measurement or observation matrix, $C$, and the $F$ matrix also is equal to $p$. The general forms of the $E$, $F$ and $T$ matrices are given in equations (3-10), (3-11) and (3-12) for $p = 2, 3$ or 4 where $p$ represents the number of inaccessible state variables of the model.

Example

Suppose the scalar states, $a_3$ and $a_9$, corresponding to the angular position and rate, respectively, of body 3, are inaccessible. Then $f_{18} = f_{98} = 0$ for $i = 1, 2$ and the observer synthesis equations reduce to the form of equations (3-13) through (3-25) with $f_{18} = f_{98} = 0$. From equation (2-11) the synthesised scalar states, $\hat{a}_8$ and $\hat{a}_9$, are expressed in terms of the observer variables, $a_1$ and $a_2$, and the accessible state variables as follows.

\[
\hat{a}_8 = \frac{(\Delta_3)_{1,1}(a_1 - \sum_{j=1}^{4} t_{1j} x_j) - (\Delta_3)_{2,1}(a_2 - \sum_{j=1}^{4} t_{2j} x_j)}{\Delta_3} \quad (3-48)
\]

\[
\hat{a}_9 = \frac{-(\Delta_3)_{1,2}(a_1 - \sum_{j=1}^{4} t_{1j} x_j) + (\Delta_3)_{2,2}(a_2 - \sum_{j=1}^{4} t_{2j} x_j)}{\Delta_3} \quad (3-49)
\]

where,

\[
\Delta_3 = \begin{vmatrix}
t_{18} & t_{19} \\
t_{28} & t_{29}
\end{vmatrix} = t_{18} t_{29} - t_{19} t_{28} \neq 0 \quad (3-50)
\]

and $(\Delta_3)_{i,j} = \Delta_3$ without the elements of the $i^{th}$ row and $j^{th}$ column.

For $a_8$ and $a_9$ inaccessible, it is assumed that:

\[
C = \begin{bmatrix}
0 & 0 & 0 \\
I_4 & : & : \\
0 & 0 & 0
\end{bmatrix} \quad (3-51)
\]

where $I_4 = 4 \times 4$ identity matrix.

From $F = GC$,

\[
G = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} \\
f_{21} & f_{22} & f_{23} & f_{24}
\end{bmatrix} \quad (3-52)
\]

From $E = TB$,

\[
E = \begin{bmatrix}
t_{12} & t_{14} & t_{16} \\
t_{22} & t_{24} & t_{26}
\end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies)} \quad (3-53)
\]

\[
E = \begin{bmatrix}
t_{12} & t_{14} & 0 \\
t_{22} & t_{24} & 0
\end{bmatrix} \quad \text{for control applied to bodies 1 and 2.} \quad (3-54)
\]

\[
E = \begin{bmatrix}
t_{12} & 0 & t_{16} \\
t_{22} & 0 & t_{26}
\end{bmatrix} \quad \text{for control applied to bodies 1 and 3.} \quad (3-55)
\]
3.4.6 Fifth Order Observers (p = 5)

An observer of at least order five is required when any five of the six scalar state variables of the three body models are inaccessible. The observer synthesis equations are given in equations (3-13) through (3-23) with \( i = 1, 2, \ldots, 5 \). Since a fifth order observer corresponds to five of the six scalar states being inaccessible, \( f_{ij} = 0 \) for five of the six values of the subscript, \( j \).

Example

Suppose that the scalar states, \( x_2, x_3, x_4, x_5 \) and \( x_6 \), representing the angular rate of body 1 and the angular displacements and rates of bodies 2 and 3 are inaccessible. Then \( f_{i5} = f_{i6} = f_{i6} = 0 \) for \( i = 1, 2, \ldots, 5 \) and the observer synthesis equations reduce to the form of equations (3-13) through (3-23) with only \( f_{i1} \neq 0 \). The synthesized scalar state variables, \( \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5 \) and \( \hat{x}_6 \) are expressed in terms of the observer scalar variables, \( x_1, x_2, \ldots, x_6 \), and the accessible state variables, using equation (2-11) as follows:

\[
\sum_{i=1}^{5} (-1)^{i+1} (\Delta_5)_{i,k} (x_i - t_{i1} x_1) = \Delta_6 \quad k = 1, 2, \ldots, 5
\]

\[
\Delta_6 = \begin{bmatrix}
t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\
t_{22} & \vdots & \vdots & \vdots & t_{26} \\
t_{32} & \vdots & \vdots & \vdots & t_{36} \\
t_{42} & \vdots & \vdots & \vdots & t_{46} \\
t_{52} & t_{53} & t_{54} & t_{55} & t_{56}
\end{bmatrix}
\]

\[
= t_{12}(\Delta_4)_{11} - t_{22}(\Delta_4)_{21} + t_{32}(\Delta_4)_{31} - t_{42}(\Delta_4)_{41} + t_{52}(\Delta_4)_{51}
\]

where \( (\Delta_4)_{ij} = \Delta_5 \) without the elements of the \( i^{th} \) row and \( j^{th} \) column.

For only \( x_1 \) accessible, it is assumed that:

\[
C = [ 1 \ 0 \ 0 \ 0 \ 0 \ 0 ]
\]

From \( F = GC \),

\[
G = \begin{bmatrix}
f_{11} \\
f_{21} \\
f_{31} \\
f_{41} \\
f_{51}
\end{bmatrix}
\]

From \( Z = TB \),

\[
E = \begin{bmatrix}
t_{12} & t_{14} & t_{16} \\
\vdots & \vdots & \vdots \\
t_{52} & t_{54} & t_{56}
\end{bmatrix}
\]

for \( r = 3 \) (control torques applied to all three bodies).

3.5 REFERENCES


4.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the four-body single-axis model of a flexible spacecraft with damping shown in Fig. 4-1 may be represented by the following set of equations.

\[\begin{align*}
I_1 \ddot{\theta}_1 &= -c_1 (\dot{\theta}_1 - \dot{\theta}_2) - k_1 (\theta_1 - \theta_2) + q_1 \\
I_2 \ddot{\theta}_2 &= c_1 (\dot{\theta}_1 - \dot{\theta}_2) + k_1 (\theta_1 - \theta_2) + c_2 (\dot{\theta}_2 - \dot{\theta}_3) + k_2 (\theta_2 - \theta_3) + q_2 \\
I_3 \ddot{\theta}_3 &= c_2 (\dot{\theta}_2 - \dot{\theta}_3) + k_2 (\theta_2 - \theta_3) + c_3 (\dot{\theta}_3 - \dot{\theta}_4) + k_3 (\theta_3 - \theta_4) + q_3 \\
I_4 \ddot{\theta}_4 &= -c_3 (\dot{\theta}_3 - \dot{\theta}_4) - k_3 (\theta_3 - \theta_4) + q_4
\end{align*}\]

where:

- \(I_i\) = rotational inertia of body \(i\); \(i = 1, 2, 3, 4\)
- \(\theta_i\) = angular displacement of body \(i\)
- \(\dot{\theta}_i\) = angular rate of body \(i\)
- \(q_i\) = torque applied to body \(i\)
- \(k_j\) = rotational spring coefficient at interface \(j\); \(j = 1, 2, 3\)
- \(c_j\) = rotational damping coefficient at interface \(j\)

4.2 STATE VARIABLE MODEL

The state variable form of the four-body single-axis model of a flexible spacecraft depicted in Fig. 4–1 was written in the following form.

\[\begin{align*}
\dot{x} &= Ax + Bu \\
x_A &= Cx
\end{align*}\]

where:

\[
x = [x_1 \ldots x_8]^T = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2 \ \theta_3 \ \dot{\theta}_3 \ \theta_4 \ \dot{\theta}_4]^T = [x_A^T \ x_i^T]^T = \text{state vector}
\]

\[
x_A = \text{p vector of accessible scalar states}
\]

\[
x_i = \text{m vector of inaccessible scalar states}
\]

\[
u = [u_1 \ldots u_r]^T = \left[\frac{q_1}{I_1} \ldots \frac{q_r}{I_r}\right]^T \quad (r = 1, 2, 3 \text{ or } 4)
\]

\[C = m \times 8 \text{ measurement or observation matrix}
\]

Partitioning of this model by rigid body yields the following forms for its coefficient matrices:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{21} & -a_{22} & a_{22} & a_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{41} & a_{41} & a_{44} & a_{45} & a_{46} & 0 \\
0 & 0 & a_{42} & a_{42} & a_{45} & a_{46} & a_{47} & a_{48} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{61} & a_{62} & a_{64} & a_{66} \\
0 & 0 & 0 & 0 & a_{65} & a_{66} & a_{67} & -a_{68} & -a_{69}
\end{bmatrix}
\]

(4-7)
FIGURE 4-1
FOUR-BODY SINGLE-AXIS MODEL WITH DAMPING AT ALL THREE INTERFACES
The corresponding block diagram appears in Fig. 2-2.

4.3 REDUCED STATE LINEAR OBSERVERS

4.3.1 Introduction

The minimum order of a reduced state linear observer required to reconstruct the $8 - m$ inaccessible scalar state variables of the four body single axis model of a flexible spacecraft represented by equations (4-5) through (4-10) is $p = 8 - m$ where $m = 1, 2, 3, 4, 5, 6$ or 7. All of the reduced state linear observers for this four body model may be written in the form of equations (2-10) and (2-11) under the assumption that the observer coefficient matrix, $D$, is diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form.

$$T = \begin{bmatrix} t_{11} & \cdots & t_{18} \\ \vdots & \ddots & \vdots \\ t_{p,1} & \cdots & t_{p,8} \end{bmatrix} \quad (4-11)$$

From equations (2-12), (4-8) and (4-11).

$$E = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ \vdots & \ddots & \vdots & \vdots \\ t_{p,2} & t_{p,4} & t_{p,6} & t_{p,8} \end{bmatrix} \quad (4-12)$$

$$F = \begin{bmatrix} f_{11} & \cdots & f_{18} \\ \vdots & \ddots & \vdots \\ f_{p,1} & \cdots & f_{p,8} \end{bmatrix} \quad (4-13)$$
The corresponding observer block diagram appears in Fig. 3-4.

### 4.3.2 Observer Synthesis Equations

From Luenberger (4-1), (4-2), (4-3) and Sage (4-4) the equations for synthesizing the reduced state linear observers for the four-body single-axis model represented by equations (4-5) through (4-10) are given by equations (2-13) and (2-14). With coefficient matrices of the form listed in 4.3.1 this set of observer synthesis equations reduces to the following.

\[
    t_{i2} = \frac{(\Delta'_4)_{1,1}(f_{i1} + d_i f_{i2}) - (\Delta'_4)_{2,2}(f_{i8} + d_i f_{i4}) + (\Delta'_4)_{3,3}(f_{i9} + d_i f_{i8})}{\Delta'_4}
    - \frac{(\Delta'_4)_{4,3}(f_{i7} + d_i f_{i9})}{\Delta'_4}
    - \frac{a_{41}p_1 + a_{45}p_2}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i1} + d_i f_{i2})
    + \frac{a_{41}p_1}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i7} + d_i f_{i9})
    \] 

\[
    t_{i4} = \frac{(\Delta'_4)_{1,2}(f_{i1} + d_i f_{i2}) - (\Delta'_4)_{2,3}(f_{i8} + d_i f_{i4}) + (\Delta'_4)_{3,2}(f_{i9} + d_i f_{i8})}{\Delta'_4}
    - \frac{(\Delta'_4)_{4,2}(f_{i7} + d_i f_{i9})}{\Delta'_4}
    - \frac{a_{28}p_1 + a_{45}p_2}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i1} + d_i f_{i2})
    + \frac{a_{28}p_1}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i8} + d_i f_{i4})
    \] 

\[
    t_{i6} = \frac{(\Delta'_4)_{1,3}(f_{i1} + d_i f_{i3}) - (\Delta'_4)_{2,2}(f_{i8} + d_i f_{i4}) + (\Delta'_4)_{3,3}(f_{i9} + d_i f_{i8})}{\Delta'_4}
    - \frac{(\Delta'_4)_{4,3}(f_{i7} + d_i f_{i9})}{\Delta'_4}
    - \frac{a_{45}p_2}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i1} + d_i f_{i3})
    + \frac{a_{45}p_2}{\Delta'_4} \left[ (a_{45}p_3 + d_i^2) (a_{45}p_2 + d_i^2) + a_{47}p_3 d_i^2 \right] (f_{i7} + d_i f_{i9})
    + \frac{a_{45}p_2}{\Delta'_4} (f_{i9} + d_i f_{i8})
    \]
\[ t_6 = -\frac{(\Delta_4)_{04}(f_{11} + d_{ii}/f_{11}) - (\Delta_4)_{04}(f_{10} + d_{ii}/f_{10}) + (\Delta_4)_{04}(f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
+ \frac{(\Delta_4)_{04}(f_{17} + d_{ii}/f_{17})}{\Delta_4} \\
- \frac{a_{48}a_{87}p_{3}p_{0}(a_{28}p_{1} + d_{ii}) (f_{11} + d_{ii}/f_{11}) + (a_{28}p_{1} + d_{ii}) (f_{10} + d_{ii}/f_{10})}{\Delta_4} \\
- \frac{a_{48}a_{87}p_{3}p_{0}(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) + a_{41}d_{ii} (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
- \frac{d_{ii} [a_{41}d_{ii} (a_{44}p_{2} + a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3})] (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
\left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
\Delta_4 \\
\left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
- \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
+ \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
(4-17) \]

\[ t_{11} = \left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) + a_{41}d_{ii} [\left\{ (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
\left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
- \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
+ \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
\right\} \\
(4-18) \]

\[ t_{18} = \left\{ a_{28} \left\{ (a_{44}p_{1} - r_{2} - d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) \right\} \\
\Delta_4 \\
\left\{ a_{44}a_{87}p_{3} (a_{48}a_{87}p_{3} + d_{ii}) (r_{1} - r_{2}) \right\} (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
\left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) + a_{41}d_{ii} [\left\{ (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
\left\{ (a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11}) \right\} \\
\Delta_4 \\
- \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
+ \frac{(a_{28}p_{1} + d_{ii}) (a_{44}p_{2} + d_{ii}) (a_{48}a_{87}p_{3}p_{0} + a_{53}p_{2}p_{3}) (f_{11} + d_{ii}/f_{11})}{\Delta_4} \\
\right\} \\
\Delta_4 \\
\right\} \\
\Delta_4 \\
\right\} \\
\Delta_4 \\
(4-19) \]
\[
\begin{align*}
\Delta_i' &= -\frac{a_{ii}}{\Delta_i} \left[ a_{ii} \phi_i + a_{ir} d_i (r_2 - r_1) + d_i' \right] (f_{i3} + d_{i1} f_{i1}) \\
&+ \frac{d_i \left( a_{ii} + d_i' \right) (a_{ii} + d_i') (a_{ii} + a_{ir} d_i + d_i')}{\Delta_i'} \\
&- \frac{d_i \left[ a_{ii} + a_{ir} d_i + d_i' \right] (a_{ii} + d_i') (a_{ii} + a_{ir} d_i + d_i')}{\Delta_i'} (f_{i3} + d_{i1} f_{i1}) + f_{i3} \\
\end{align*}
\]

\[
\begin{align*}
\Delta_i' &= -\frac{a_{ii}}{\Delta_i} \left[ a_{ii} \phi_i + a_{ir} d_i (r_2 - r_1) + d_i' \right] (f_{i1} + d_{i1} f_{i1}) \\
&+ \frac{d_i \left( a_{ii} + d_i' \right) (a_{ii} + d_i') (a_{ii} + a_{ir} d_i + d_i')}{\Delta_i'} \\
&- \frac{d_i \left[ a_{ii} + a_{ir} d_i + d_i' \right] (a_{ii} + d_i') (a_{ii} + a_{ir} d_i + d_i')}{\Delta_i'} (f_{i1} + d_{i1} f_{i1}) + f_{i1} \\
\end{align*}
\]

where:

\[
\begin{align*}
\Delta_i' &= \left[ \begin{array}{cccc}
-a_{ii} + d_i' & a_{ii} & 0 & 0 \\
0 & a_{ii} & -a_{ii} + d_i' & a_{ii} \\
0 & 0 & -a_{ii} + d_i' & a_{ii} \\
0 & 0 & 0 & -a_{ii} + d_i'
\end{array} \right]
\]

\[
\Delta_i' = d_i' \left[ (a_{ii} + a_{ir} d_i + d_i') (a_{ii} + d_i') (a_{ii} + a_{ir} d_i + d_i') \right]
\]

4.3.3 Comparison of T Matrices For Elimination of Damping at Various Interfaces

Elimination of damping at interface \( j \) of the model corresponds to setting \( r_j = 0 \) and \( p_j = 1 \) in the
equations for generating the elements of the T matrix, equations (4-14) through (4-28). The following damping conditions have been treated for this set of equations.

1. Damping eliminated at the interface between bodies 1 and 2;
2. Damping eliminated at the interface between bodies 2 and 3;
3. Damping eliminated at the interface between bodies 3 and 4;
4. Damping eliminated at the interfaces between bodies 2, 3 and 4;
5. Damping eliminated at the interfaces between bodies 1 and 2, 3 and 4;
6. Damping eliminated at the interfaces between bodies 1, 2 and 3;
7. Damping eliminated from all interfaces.

Example: All interface damping eliminated.

If damping is removed from all three interfaces of the four body model, \( r_j \rightarrow 0, p_j \rightarrow 1, d_{i1} \rightarrow d_{ii}, d_{i2} \rightarrow d_{ii}, d_{i3} \rightarrow d_{ii}, d_{i4} \rightarrow d_{ii} \) and

\[
\Delta_{i4} = a_{45}(a_{i3} + d_{i3}^2) + a_{46}(a_{i3} + d_{i3}^2) + a_{47}(a_{i3} + d_{i3}^2) + a_{48}(a_{i3} + d_{i3}^2)
\]

\[
\Delta_{i4} = \begin{cases} 
\frac{a_{45}}{a_{41}} \left( (a_{i3} + d_{i3}^2)(a_{i5} + d_{i5}^2) + a_{46}(a_{i3} + d_{i3}^2) + a_{47}(a_{i3} + d_{i3}^2) + a_{48}(a_{i3} + d_{i3}^2) \right) & (4-29) \\
\frac{a_{45}}{a_{41}} \left( (a_{i3} + d_{i3}^2)(a_{i5} + d_{i5}^2) + a_{46}(a_{i3} + d_{i3}^2) + a_{47}(a_{i3} + d_{i3}^2) + a_{48}(a_{i3} + d_{i3}^2) \right) & (4-30) \\
\frac{a_{45}}{a_{41}} \left( (a_{i3} + d_{i3}^2)(a_{i5} + d_{i5}^2) + a_{46}(a_{i3} + d_{i3}^2) + a_{47}(a_{i3} + d_{i3}^2) + a_{48}(a_{i3} + d_{i3}^2) \right) & (4-31)
\end{cases}
\]
\[
\begin{align*}
t_{i8} &= -\frac{\phi_{i8}(s_{i5} + di_{i8}) + (a_{28} + d_{i5}^2)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&- \frac{\phi_{i7}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{41}d_{i5}^2}{\Delta_{i4}} \\
&+ \frac{\left(\phi_{i8}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{28}a_{45}d_{i5}^2\right)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&= \frac{\left(d_{i5}(a_{41} + a_{45} + d_{i5}^2)\right)[(a_{41} + d_{i5}^2)(a_{46} + d_{i5}^2) + a_{47}d_{i5}^2]}{\Delta_{i4}} (4-32)
\end{align*}
\]

\[
\begin{align*}
t_{i1} &= -\frac{\phi_{i4}(s_{i5} + di_{i8}) + (a_{28} + d_{i5}^2)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&- \frac{\phi_{i5}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{41}d_{i5}^2}{\Delta_{i4}} \\
&+ \frac{\left(\phi_{i7}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{28}a_{45}d_{i5}^2\right)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&= \frac{\left(d_{i5}(a_{41} + a_{45} + d_{i5}^2)\right)[(a_{41} + d_{i5}^2)(a_{46} + d_{i5}^2) + a_{47}d_{i5}^2]}{\Delta_{i4}} (4-33)
\end{align*}
\]

\[
\begin{align*}
t_{i8} &= -\frac{\phi_{i8}(s_{i5} + di_{i8}) + (a_{28} + d_{i5}^2)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&- \frac{\phi_{i7}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{41}d_{i5}^2}{\Delta_{i4}} \\
&+ \frac{\left(\phi_{i8}(a_{28} + d_{i5}^2)(a_{45} + d_{i5}^2) + a_{28}a_{45}d_{i5}^2\right)(s_{i5} + di_{i8})}{\Delta_{i4}} \\
&= \frac{\left(d_{i5}(a_{41} + a_{45} + d_{i5}^2)\right)[(a_{41} + d_{i5}^2)(a_{46} + d_{i5}^2) + a_{47}d_{i5}^2]}{\Delta_{i4}} (4-34)
\end{align*}
\]
\[ t_{i8} = \frac{-a_{28}d_{i8}(a_{28} + d_{i8})[a_{28}(f_{i1} + d_{i1}f_{i2}) + (a_{28} + d_{i8})(f_{i8} + d_{i8}f_{i4})]}{\Delta_{i8}} + \frac{d_{i8}[a_{28}d_{i8}(a_{28} + d_{i8})(a_{28} + d_{i8}) - a_{41}d_{i8}(a_{28} + d_{i8})](f_{i8} + d_{i8}f_{i4})}{\Delta_{i8}} + f_{i8} \]

\[ t_{i7} = \frac{-a_{07}d_{i7}\left\{ a_{48}[a_{28}(f_{i1} + d_{i1}f_{i2}) + (a_{28} + d_{i8})(f_{i8} + d_{i8}f_{i4})]\right\}}{\Delta_{i8}} + \frac{\left( (a_{28} + d_{i8})(a_{48} + d_{i8}) + a_{41}d_{i8}^2 \right)(f_{i8} + d_{i8}f_{i4})}{\Delta_{i8}} \]

\[ \frac{d_{i7}\left( (a_{28} + d_{i8})(a_{48} + d_{i8})(a_{67} + d_{i8}) \right)}{\Delta_{i8}} + \frac{d_{i7}\left( a_{28}a_{68} + a_{41}a_{68} + a_{41}a_{67} + (a_{41} + a_{68})c_{i7} \right)(f_{i7} + d_{i7}f_{i8})}{\Delta_{i8}} + f_{i8} \] (4-35)

4.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

4.4.1 Introduction

Inaccessibility of a scalar state variable in the model equations (4-5), (4-7) is reflected by a corresponding null column in the C and F matrices as implied in equation (2-14). For the generation of reduced state observers for the four body model the number of inaccessible state variables, \( p \), can be 1, 2, 3, 4, 5, 6 or 7.

4.4.2 First Order Observers (\( p = 1 \))

An observer of order at least one is required when only one of the eight scalar state variables of the four body model is inaccessible. The first order form of the linear observer equation is as follows:

\[ \dot{x} = dx + Bu + Gy \] (4-37)

The \( F \) and \( T \) matrices associated with a first order observer for the four body model then reduce to the following row forms.

\[ F = \begin{bmatrix} f_1 & f_2 & \cdots & f_6 \end{bmatrix} \] (4-38)

\[ T = \begin{bmatrix} t_1 & t_2 & \cdots & t_6 \end{bmatrix} \] (4-39)

The observer synthesis equations are then of the form of equations (4-14) through (4-28) with \( i = 1 \). Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the \( f_i \) (\( i = 1, 2, \ldots, 8 \)) = 0.

Example
Suppose the scalar state representing the angular rate of body 4, \( z_8 \), is inaccessible. Then \( f_8 = 0 \) and the observer synthesis equations reduce to the form of equations (4-14) through (4-28) with \( f_i = 0 \) and \( i = 1 \). From equation (2-11), the synthesised scalar state, \( \hat{z}_8 \), is expressed in terms of the scalar observer variable, \( z \), and the accessible scalar state variables as follows.

\[
\hat{z}_8 = \frac{1}{t_8} [z - \sum_{i=1}^{7} t_i x_i] \tag{4-40}
\]

For \( z_8 \) inaccessible, it is assumed that:

\[
C = \begin{bmatrix}
I_7 & 0 \\
0 & 0
\end{bmatrix}, \tag{4-41}
\]

where \( I_7 = 7 \times 7 \) identity matrix.

From \( F = GC \),

\[
G = [f_1 \ f_2 \ f_3 \ f_4 \ f_5 \ f_6 \ f_7] \tag{4-42}
\]

From \( E = TB \),

\[
E = [t_2 \ t_4 \ t_6 \ t_8] \quad \text{for} \quad r = 4 \quad \text{(control torques on all 4 bodies)} \tag{4-43}
\]

### 4.4.3 Observers of Intermediate Order \((p = 2, 3, 4, 5 \text{ or } 6)\)

For those cases in which an intermediate number of the eight scalar states of the four-body single-axis model is inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by \( p \). In each case the number of null columns in the measurement or observation matrix, \( C \), and the \( F \) matrix also is equal to \( p \). The general forms of the \( E \), \( F \) and \( T \) matrices are given in equations (4-11), (4-12) and (4-13) for \( p = 2, 3, 4, 5 \text{ or } 6 \) where \( p \) represents the number of inaccessible scalar state variables of the model.

**Example**

Suppose the scalar states, \( z_7 \) and \( z_8 \), which represent the angular position and rate of body 4, are inaccessible. Then \( f_7 = f_8 = 0 \) for \( i = 1, 2 \) and the observer synthesis equations reduce to the form of equations (4-14) through (4-28) with the preceding conditions. From equation (2-11), the synthesised scalar states, \( \hat{z}_7 \) and \( \hat{z}_8 \), are expressed in terms of the scalar observer variables, \( z \) and \( z_2 \) and the accessible scalar state variables as follows.

\[
\hat{z}_7 = \frac{\sum_{i=1}^{2} (-1)^{i+1} (\Delta_2)_{i,1} (x_i - \sum_{j=1}^{6} t_{ij} x_j)}{\Delta_2} \tag{4-44}
\]

\[
\hat{z}_8 = \frac{\sum_{i=1}^{2} (-1)^{i+1} (\Delta_2)_{i,2} (x_i - \sum_{j=1}^{6} t_{ij} x_j)}{\Delta_2} \tag{4-45}
\]

For

\[
\Delta_2 = \begin{vmatrix}
 t_{17} & t_{18} \\
 t_{27} & t_{28}
\end{vmatrix} = t_{17}t_{28} - t_{18}t_{27} \neq 0 \tag{4-46}
\]
where \((\Delta t)_{ij} = \Delta t\) without the elements of the \(i^{th}\) row and \(j^{th}\) column.

For \(x_7\) and \(x_8\) inaccessible, it is assumed that:

\[
C = \begin{bmatrix}
0 & 0 \\
L_6 & \vdots & \vdots \\
0 & 0
\end{bmatrix}
\]  

(4-47)

where \(L_6 = 6 \times 6\) identity matrix.

Since \(F = GC\),

\[
G = \begin{bmatrix}
f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\
f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26}
\end{bmatrix}
\]  

(4-48)

From \(E = TB\),

\[
E = \begin{bmatrix}
t_{12} & t_{14} & t_{16} & t_{18} \\
t_{22} & t_{24} & t_{26} & t_{28}
\end{bmatrix}
\]  

for \(r = 4\) (control torques applied to all four bodies)  

(4-49)

\[
E = \begin{bmatrix}
t_{12} & t_{14} & t_{16} & 0 \\
t_{22} & t_{24} & t_{26} & 0
\end{bmatrix}
\]  

for \(r = 3\) (control torques applied to bodies 1, 2 and 3)  

(4-50)

\[
E = \begin{bmatrix}
t_{12} & t_{14} & 0 & 0 \\
t_{22} & t_{24} & 0 & 0
\end{bmatrix}
\]  

for \(r = 2\) (control torques applied to bodies 1 and 2)  

(4-51)

\[
E = \begin{bmatrix}
t_{12} & 0 & 0 & 0 \\
t_{22} & 0 & 0 & 0
\end{bmatrix}
\]  

for \(r = 1\) (control torque applied to body 1)  

(4-52)

### 4.4.4 Seventh Order Observers \((p = 7)\)

When any seven of the eight scalar state variables of the four body model are inaccessible, a linear observer of at least order seven is required. The observer synthesis equations are as presented in equations (4-14) through (4-28) with \(i = 1, 2, \ldots, 7\). Since a seventh order observer corresponds to seven of the scalar states being inaccessible, \(f_{ij} = f_{3j} = \ldots = f_{7j} = 0\) for seven of the eight values of the subscript, \(j\).

**Example**

Suppose only the scalar state variable representing the angular position of body 1, \(x_1\), is accessible. Then the remaining scalar states, \(x_2, x_3, \ldots, x_8\) are inaccessible, \(f_{i2} = f_{i3} = \ldots = f_{i8} = 0\) for \(i = 1, 2, 3, 4, 5, 6\) and 7 and the observer synthesis equations reduce to the form of equations (4-14) through (4-28) with \(f_{12} = f_{13} = \ldots = f_{18} = 0\) and \(i = 1, 2, \ldots, 7\). The synthesized scalar state variables, \(\hat{x}_2\) through \(\hat{x}_8\), are expressed in terms of the observer variables, \(\hat{x}_1\) through \(\hat{x}_7\), and the accessible state variable, \(x_1\), by utilizing equation (2-11) in the following form.

\[
\hat{x}_{k+1} = \frac{\sum_{i=1}^{7}(-1)^{i+1}(\Delta t)_{i,k}(x_i - t_{i1}z_1)}{\Delta t} \quad k = 1, 2, \ldots, 7
\]  

(4-53)

\[
\Delta t = \begin{bmatrix}
t_{12} & \cdots & t_{18} \\
\vdots & \vdots & \vdots \\
t_{72} & \cdots & t_{78}
\end{bmatrix}
\]  

\[
= t_{12}(\Delta t)_{1,1} - t_{22}(\Delta t)_{2,1} + t_{22}(\Delta t)_{3,1} - t_{42}(\Delta t)_{4,1}
\]
\[ + t_{12}(\Delta \gamma)_{6,1} - t_{22}(\Delta \gamma)_{6,1} + t_{72}(\Delta \gamma)_{7,1} \quad (4-54) \]

where \((\Delta \gamma)_{i,j} = \Delta \gamma\), without the elements of the \(i^{th}\) row and the \(j^{th}\) column.

For only \(s_1\) accessible, it is assumed that:

\[ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (4-55) \]

From \(F = GC\),

\[ G = \begin{bmatrix} f_{11} \\ \vdots \\ f_{71} \end{bmatrix} \quad (4-56) \]

From \(E = TB\),

\[ E = \begin{bmatrix} t_{11} & t_{14} & t_{16} & t_{18} \\ \vdots & \vdots & \vdots & \vdots \\ t_{72} & t_{74} & t_{76} & t_{78} \end{bmatrix} \quad \text{for } r = 4 \text{ (control torques applied to all four bodies)} \quad (4-57) \]

4.5 REFERENCES


SECTION 5
DEVELOPMENT OF THE FIVE-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

5.1 ORIGINAL DAMPED MODEL

In earlier work, Guidance Systems Division (5-1), it was shown that one axis of the three-axis five-body approximation of a prototype flexible spacecraft can be decoupled from the other two axes. The four-body single-axis models of a flexible spacecraft developed in the previous section were therefore extended to corresponding five body models to represent the decoupled axis of the three-axis five-body model.

The rotational dynamics of the five-body single-axis model of a flexible spacecraft with damping shown in Fig. 5-1 may be represented by the following set of equations.

\[ I_1 \ddot{\theta}_1 = -c_1 (\dot{\theta}_1 - \dot{\theta}_2) - k_1 (\theta_1 - \theta_2) + q_1 \]  
\[ I_2 \ddot{\theta}_2 = c_1 (\dot{\theta}_1 - \dot{\theta}_2) + h_1 (\theta_1 - \theta_2) + c_2 (\dot{\theta}_2 - \dot{\theta}_3) + k_2 (\theta_3 - \theta_2) + q_2 \]  
\[ I_3 \ddot{\theta}_3 = c_2 (\dot{\theta}_2 - \dot{\theta}_3) + h_2 (\theta_2 - \theta_3) + c_3 (\dot{\theta}_3 - \dot{\theta}_4) + k_3 (\theta_4 - \theta_3) + q_3 \]  
\[ I_4 \ddot{\theta}_4 = c_3 (\dot{\theta}_3 - \dot{\theta}_4) + h_3 (\theta_3 - \theta_4) + c_4 (\dot{\theta}_4 - \dot{\theta}_5) + k_4 (\theta_5 - \theta_4) + q_4 \]  
\[ I_5 \ddot{\theta}_5 = c_4 (\dot{\theta}_4 - \dot{\theta}_5) + h_4 (\theta_4 - \theta_5) + q_5 \]

where:

\[ I_i = \text{rotational inertia of body } i, \quad i = 1, 2, \ldots, 5 \]
\[ \theta_i = \text{angular displacement of body } i \]
\[ \dot{\theta}_i = \text{angular rate of body } i \]
\[ q_i = \text{torque applied to body } i \]
\[ k_j = \text{rotational spring coefficient at interface } j \quad j = 1, 2, 3, 4 \]
\[ c_j = \text{rotational damping coefficient at interface } j \]

5.2 STATE VARIABLE MODEL

The state variable form of the five-body single-axis model of a flexible spacecraft depicted in Fig. 5-1 was written in the following form.

\[ \dot{x} = Ax + Bu \]  
\[ x_A = Cx \]

where:

\[ x = [x_1 \ldots x_{10}]^T = [x_1^T x_2^T]^T = [\theta_1 \dot{\theta}_1 \theta_2 \dot{\theta}_2 \theta_3 \dot{\theta}_3 \theta_4 \dot{\theta}_4 \theta_5 \dot{\theta}_5]^T = \text{state vector} \]
\[ x_A = m \text{ vector of accessible scalar states} \]
\[ x_i = p \text{ vector of inaccessible scalar states} \]
\[ u = [u_1 \ldots u_r]^T = \left[ \frac{q_1}{I_1} \ldots \frac{q_r}{I_r} \right]^T \quad r = 1, 2, \ldots, 5 \]
\[ C = m \times 10 \text{ measurement or observation matrix} \]
FIGURE 8-1
FIVE-BODY SINGLE-AXIS MODEL WITH DAMPING AT ALL FOUR INTERFACES
\[ \mathbf{\Delta} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{28} & -a_{28r1} & a_{28} & a_{28r1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_{41} & a_{41r1} & a_{44} & a_{44} & a_{44r2} & 0 & 0 & 0 \\ 0 & 0 & a_{66} & a_{66r2} & a_{66} & a_{66} & a_{66r2} & 0 \\ 0 & 0 & 0 & 0 & a_{66} & a_{66r2} & a_{66} & a_{66r2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ a_{10,7} & a_{10,7r4} & -a_{10,7} & -a_{10,7r4} \end{bmatrix} \]

\[ a_{28} = \frac{k_1}{I_1} \]
\[ a_{41} = \frac{k_1}{I_2}, \quad a_{45} = \frac{k_2}{I_2}, \quad a_{44} = -(a_{41} + a_{44}), \quad a_{44} = -(a_{41} + a_{44}) \]
\[ a_{66} = \frac{k_2}{I_3}, \quad a_{67} = -(a_{66} + a_{67}), \quad a_{66} = -(a_{66} + a_{67}) \]
\[ a_{88} = \frac{k_3}{I_4}, \quad a_{89} = \frac{k_4}{I_4}, \quad a_{87} = -(a_{88} + a_{89}), \quad a_{88} = -(a_{88} + a_{89}) \]
\[ a_{10,7} = \frac{k_4}{I_5} \]
\[ r_j = \frac{c_i}{k_j}; \quad j = 1, 2, 3, 4 \]

\[ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] for \( r = 5 \)

5.3 REDUCED STATE LINEAR OBSERVERS

5.3.1 Introduction

The minimum order of a reduced state linear observer required to reconstruct the \( 10 - m \) inaccessible scalar state variables of the five-body single-axis model of a flexible spacecraft represented by equations (5-6)
through (5-10) is \( p = 10 - m \) where \( m = 1, 2, 3, 4, 5, 6, 7, 8 \) or 9. All of the reduced state linear observers for this five-body model may be written in the form of equations (2-10) and (2-11) under the assumption that the observer coefficient matrix, \( D \), is diagonal and of dimensions \( p \times p \). The corresponding observer weighting matrix is of the following form

\[
T = \begin{bmatrix}
  t_{11} & \cdots & t_{1,10} \\
  \vdots & & \vdots \\
  t_{p,1} & \cdots & t_{p,10}
\end{bmatrix}
\]  

(5-12)

From equations (2-12), (5-10) and (5-11),

\[
B = \begin{bmatrix}
  t_{12} & t_{14} & t_{16} & t_{18} & t_{110} \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
  t_{p,2} & t_{p,4} & t_{p,6} & t_{p,8} & t_{p,10}
\end{bmatrix}
\]  

(5-13)

\[
F = \begin{bmatrix}
  f_{11} & \cdots & f_{1,10} \\
  \vdots & & \vdots \\
  f_{p,1} & \cdots & f_{p,10}
\end{bmatrix}
\]  

(5-14)

The corresponding block diagram appears in Fig. 2-4.

### 5.3.2 Observer Synthesis Equations

From Luenberger (5-2), (5-3) and (5-4) and Sage (5-5) the equations for synthesizing the reduced state linear observers for the five-body single-axis model represented by equations (5-6) through (5-10) are given by equations (2-13) and (2-14). With coefficient matrices of the form listed in 5.3.1 this set of equations for generating the elements of the \( T \) matrix reduces to the following.

\[
t_{ij} = \left\{ \frac{a_{41} a_{63} P_1 P_2 + a_{47} a_{67} P_1 P_2 + a_{46} a_{66} P_2 P_2 + (a_{41} P_1 + a_{44} P_2 + a_{48} P_2 + a_{47} P_2) d_{ij}^2 + d_{ij}^3}{a_{23} a_{46} P_1 P_3 \Delta_{16}^i} \right\} - \frac{a_{47} a_{67} P_1 P_2}{a_{23} a_{46} P_1 P_3 \Delta_{16}^i} \left( f_{11} + d_{1i} f_{1j} \right)
\]

(5-15)
\[ t_{10} = \frac{(\Delta_{10})_{1,3}}{a_{288} P_1 \Delta_{10}} \left[ a_{288} P_1 (s_i + d_{ii} s_i) + (a_{28} P_1 + d_{ii}) (s_i + d_{ii} s_i) \right] \\
+ \frac{(\Delta_{10})_{1,3} (\Delta_{10})_{1,3}}{a_{288} a_{468} a_{10,7} P_3 P_4 \Delta_{10}^2} (s_i + d_{ii} s_i) \\
+ \frac{(\Delta_{10})_{1,3}}{a_{10,7} P_4 \Delta_{10}} \left[ (s_i + d_{ii} s_i) (s_i + d_{ii} s_i) + a_{10,7} P_4 (s_i + d_{ii} s_i) \right] \]  

\[ t_{10} = \frac{a_{468} a_{47} a_{48} P_3 P_4}{\Delta_{10}} \left[ a_{288} P_1 (s_i + d_{ii} s_i) + (a_{28} P_1 + d_{ii}) (s_i + d_{ii} s_i) \right] \\
+ \frac{a_{468} a_{47} a_{48} P_3 P_4}{a_{288} a_{10,7} P_4 \Delta_{10}^2} (s_i + d_{ii} s_i) \\
+ \left[ \frac{(a_{288} P_1 + a_{468} P_4) (\Delta_{10})_{1,3} - (a_{288} P_1 + d_{ii}) a_{468} a_{48} P_3 P_4}{a_{288} a_{10,7} P_4 \Delta_{10}^2} \right] \left[ (s_i + d_{ii} s_i) (s_i + d_{ii} s_i) + a_{10,7} P_4 (s_i + d_{ii} s_i) \right] \\
+ \left\{ \frac{(a_{288} P_1 + d_{ii}) a_{468} a_{48} P_3 P_4}{a_{288} a_{10,7} P_4 \Delta_{10}^2} \left[ (s_i + d_{ii} s_i) (s_i + d_{ii} s_i) + a_{10,7} P_4 (s_i + d_{ii} s_i) \right] \right\} \right\} \] 

\[ t_{10} = \left\{ \frac{d_{ii} \left[ (a_{41} P_1 + d_{ii}) (a_{468} P_3 + a_{47} P_3 + d_{ii}) + a_{468} P_3 (a_{47} P_3 + d_{ii}) \right]}{a_{239} a_{468} P_1 P_2 \Delta_{10}^2} \right\} \] 

\[ t_{11} = \left\{ \frac{a_{288} P_1 \left[ a_{468} P_3 (d_{ii} + a_{468} P_3 + d_{ii}) \right]}{a_{288} a_{468} P_1 P_2 \Delta_{10}^2} \right\} \] 

\[ t_{11} = \left\{ \frac{a_{288} P_1 \left[ a_{468} P_3 (d_{ii} + a_{468} P_3 + d_{ii}) \right]}{a_{288} a_{468} P_1 P_2 \Delta_{10}^2} \right\} \] 

\[ t_{11} = \left\{ \frac{a_{288} P_1 \left[ a_{468} P_3 (d_{ii} + a_{468} P_3 + d_{ii}) \right]}{a_{288} a_{468} P_1 P_2 \Delta_{10}^2} \right\} \] 

\[ t_{11} = \left\{ \frac{a_{288} P_1 \left[ a_{468} P_3 (d_{ii} + a_{468} P_3 + d_{ii}) \right]}{a_{288} a_{468} P_1 P_2 \Delta_{10}^2} \right\} \]
\[
\begin{align*}
\frac{\{a_{46}(r_1 - r_2) + d_{ii}\} \{a_{47}P_0 + d_{ii}\} + a_{48}P_3 d_{ii}}{a_{48}P_1 P_3 \Delta'_{16}}(\Delta'_{16})_{1,3} \\
\end{align*}
\]

\[
\begin{align*}
t_{16} &= 
\left\{ \frac{a_{46}(r_1 - r_2) + d_{ii}}{\Delta'_{16}} \right\} \{a_{47}P_0 + d_{ii}\} \left( f_{i1} + d_{ii} f_{i3} \right) \\
+ \frac{a_{48}P_3 d_{ii} \left[ a_{22}p_1 + a_{41}r_1 d_{ii} + d_{ii} \right]}{a_{48}P_1 P_3 \Delta'_{16}} \{a_{46}r_2 + d_{ii}\} \\
+ \frac{a_{47}a_{48}P_3 n_{16} \left[ a_{16}P_0 + d_{ii} \right] \left( a_{46}r_2 + d_{ii} \right)}{a_{48}P_1 P_3 \Delta'_{16}} \left( f_{i3} + d_{ii} f_{i4} \right) \\
+ \frac{a_{48}d_{ii} \left[ a_{22}p_1 + a_{41}r_1 (r_1 - r_2) + d_{ii} \right]}{a_{48}P_1 P_3 \Delta'_{16}} \left( f_{i3} + d_{ii} f_{i6} \right) \\
+ a_{43} \left\{ \frac{a_{48}P_3 \Delta'_{16}}{\Delta'_{16}} \right\} \{a_{46}r_2 + d_{ii} d_{ii}\} \left( f_{i3} + d_{ii} f_{i4} - \frac{r_2 \Delta'_{16}}{a_{10},7P_4 \Delta'_{16}} \right) \\
\times \left\{ \frac{a_{10},7P_4 + d_{ii}}{\Delta'_{16}} \right\} \{a_{10},7P_4 \Delta'_{16} \} \\
\end{align*}
\]

(5-21)
\[ t_{16} = \left\{ \frac{d_{ii} - \alpha_{ii} \left( r_2 - r_1 \right)}{\Delta_{ii}^{1,2}} + \frac{a_{ii} \alpha_{ii} \alpha_{ii} \alpha_{ii}}{\Delta_{ii}^{1,2}} \left( \Delta_{ii}^{1,2} \right)_{1,2} + \alpha_{ii} \alpha_{ii} a_{ii} a_{ii} (a_{ii} \alpha_{ii} + d_{ii}) (r_2 - r_1) }{\Delta_{ii}^{1,2}} \right\} \times \left[ a_{ii} \alpha_{ii} a_{ii} a_{ii} a_{ii} \left( f_{i1} + d_{ii} f_{i2} \right) + (a_{ii} \alpha_{ii} + d_{ii}) \left( f_{i1} + d_{ii} f_{i4} \right) \right] \\
- \left[ \frac{(a_{ii} \alpha_{ii} + d_{ii}) a_{ii} \alpha_{ii} a_{ii} a_{ii} (a_{ii} \alpha_{ii} + d_{ii}) (r_2 - r_1) }{a_{ii} \alpha_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii}} \right] \frac{d_{ii} \left( \Delta_{ii}^{1,2} \right)_{1,2}}{\Delta_{ii}^{1,2}} - \frac{d_{ii} \left( \Delta_{ii}^{1,2} \right)_{1,2} (\Delta_{ii}^{1,2})_{6,2} }{a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii}} \left( f_{i6} + d_{ii} f_{i8} \right) + f_{i6} \right] \\
+ \left\{ \frac{a_{ii} a_{ii} a_{ii} a_{ii} (a_{ii} \alpha_{ii} + d_{ii}) (r_2 - r_1) }{a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii}} \right\} \frac{d_{ii} \left( \Delta_{ii}^{1,2} \right)_{5,2}}{\Delta_{ii}^{1,2}} \left( f_{i6} + d_{ii} f_{i8} \right) + f_{i6} \right] \\
\times \left[ \left( a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} a_{ii} \right) (f_{i7} + d_{ii} f_{i8} ) + a_{ii} a_{ii} (f_{i10} + d_{ii} f_{i10}) \right] \right\} \] (5-22)
\[ t_{ij} = \frac{a_{99}d_{ii}}{a_{83}a_{10}a_{72}p_{2}a_{1}a_{10}a_{72}p_{2}a_{1}a_{i}} \left\{ a_{44}a_{75}a_{66}a_{10}a_{72}p_{2}a_{1}a_{2}a_{p_{1}} \left( f_{i1} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i1} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i1} + d_{i1}f_{i1} \right) \right\} + \frac{a_{97}p_{2}a_{10}a_{72}p_{2}a_{2}a_{2}a_{p_{1}} \left( f_{i7} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i7} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \right\} + \frac{\left\{ d_{ii} \left[ a_{65}a_{65}p_{2}a_{1}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} + a_{67}p_{2}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} + \left( a_{23}p_{1} + a_{67}p_{2}a_{1}a_{3}a_{p_{1}} + a_{67}p_{2}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} \right) \left( f_{i7} + d_{i1}f_{i1} \right) \right\} \right\} \left\{ a_{65}a_{65}a_{10}a_{72}p_{2}a_{1}a_{10}a_{72}p_{2}a_{1}a_{i} \right\} \left( \Delta'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i7} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \right\} - \left[ \frac{a_{97}p_{2}a_{10}a_{72}p_{2}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} \left( f_{i7} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i7} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \right\} \right\} \left( \Delta'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) + \frac{\left[ a_{97}p_{2}a_{10}a_{72}p_{2}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} \left( f_{i7} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i7} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \right\} \right\} \left( \Delta'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) + \left[ \frac{a_{97}p_{2}a_{10}a_{72}p_{2}a_{2}a_{10}a_{72}p_{2}a_{2}a_{p_{1}} \left( f_{i7} + d_{i1}f_{i1} \right) + \left( a_{23}p_{1} + d_{ii} \right) \left( f_{i7} + d_{i1}f_{i1} \right) + a_{97}p_{2} \left( t'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \right\} \right\} \left( \Delta'_{ij} \right)_{5,2} \left( f_{i7} + d_{i1}f_{i1} \right) \times \left[ \left( f_{i9} + d_{i1}f_{i1} \right) + f_{i10} \right] + f_{i10} \right\} (5-24) \]

where:

\[ r_j = \frac{c_j}{h_j}; \quad j = 1, 2, 3, 4 \]  
(5-10)

\[ p_j = 1 + r_j d_{ii}; \quad i = 1, 2, \ldots, p \quad j = 1, 2, 3, 4 \]  
(5-25)

\[ d_{ii} = a_{23}p_{1} + d_{ii} \]  
(5-26)

\[ d_{i2} = a_{41}r_{1} + a_{46}r_{2} + d_{ii} \]  
(5-27)

\[ d_{i3} = a_{46}r_{2} + a_{77}r_{3} + d_{ii} \]  
(5-28)

\[ d_{i4} = a_{46}r_{3} + a_{90}r_{4} + d_{ii} \]  
(5-29)

\[ d_{i5} = a_{10}, r_{4} + d_{ii} \]  
(5-30)

\[ \Delta'_{ij} = \begin{array}{cccc}
-a_{23}p_{1} & a_{41}p_{1} & 0 & 0 \\
-a_{23}p_{1} & a_{41}p_{1} & a_{46}p_{2} & 0 \\
0 & a_{46}p_{2} & a_{46}p_{2} & 0 \\
0 & 0 & 0 & a_{46}p_{2}
\end{array} \]  
(5-31)

\[ = -a_{23}p_{1}\left( \Delta'_{ij} \right)_{1,1} + a_{23}p_{1}\left( \Delta'_{ij} \right)_{2,1} \]  
(5-32)

\[ = -a_{23}p_{1}\left( \Delta'_{ij} \right)_{1,1} - a_{23}p_{1}\left( \Delta'_{ij} \right)_{1,2} \]  
(5-33)

5.3.3 Elimination of Damping From Model Interfaces

Elimination of damping at interface \( j \) (\( j = 1, 2, 3, 4 \) for the five body model) corresponds to setting \( r_j = 0 \) and \( p_j = 1 \) in the equations for generating the elements of the \( T \) matrix, equations (5-10) through (5-31).
Example: All Interface Damping Eliminated

If damping is removed from all four interfaces of the five body model, \( r_j = 0 \), \( p_j = 1 \), \( d_{i1} = d_i \), \( d_{i2} = d_i \), \( d_{i3} = d_i \), \( d_{i4} = d_i \), \( d_{i5} = d_i \), and \( \Delta_{i6} = \Delta_{i6} \).

\[
t_{i3} = \left\{ \frac{a_{41}a_{42} + a_{41}a_{47} + a_{44}a_{47} + (a_{41} + a_{45} + a_{48} + a_{47})d_{i1}^2 + d_{i1}^4}{a_{29}a_{45}b_{i6}} \right\}(\Delta_{i6})_{1,3}
\]

\[
- \frac{a_{41}a_{42}(a_{41} + a_{45} + d_{i1}^2)(a_{10,7} + d_{i1})}{\Delta_{i6}} \bigg(f_{i1} + d_{i1}f_{i2} \bigg)
\]

\[
+ a_{41} \left[ \frac{(a_{48} + a_{47} + d_{i1}^2)(\Delta_{i6})_{1,3}}{a_{29}a_{45}b_{i6}} - \frac{a_{41}a_{42}(a_{10,7} + d_{i1})}{\Delta_{i6}} \right] \left(f_{i3} + d_{i1}f_{i4} \right)
\]

\[
+ \frac{a_{41}a_{43}a_{45}}{a_{29}a_{45}b_{i6}} [f_{i5} + d_{i1}f_{i6}]
\]

\[
+ \frac{a_{41}a_{43}a_{45}}{a_{29}a_{45}b_{i6}} \left[(a_{10,7} + d_{i1}) \left(f_{i7} + d_{i1}f_{i8} \right) + a_{10,7} \left(f_{i9} + d_{i1}f_{i10} \right) \right];
\]

\[i = 1, 2, \ldots, p \quad (5-32)\]

\[
t_{i4} = \left[ \frac{(a_{48} + a_{47} + d_{i1}^2)(\Delta_{i6})_{1,3}}{a_{29}a_{45}b_{i6}} - \frac{a_{41}a_{42}(a_{10,7} + d_{i1})}{\Delta_{i6}} \right]
\]

\[
\times \left[ a_{29}(f_{i1} + d_{i1}f_{i2}) + (a_{29} + d_{i1}^2)(f_{i3} + d_{i1}f_{i4}) \right] + \frac{(a_{29} + d_{i1}^2)a_{43}(\Delta_{i6})_{1,3}}{a_{29}a_{45}b_{i6}} [f_{i5} + d_{i1}f_{i6}]
\]

\[
+ \frac{a_{41}a_{43}a_{45}}{a_{29}a_{45}b_{i6}} \left[(a_{10,7} + d_{i1}) \left(f_{i7} + d_{i1}f_{i8} \right) + a_{10,7} \left(f_{i9} + d_{i1}f_{i10} \right) \right]
\]

\[i = 1, 2, \ldots, p \quad (5-33)\]

\[
t_{i6} = \left[ \frac{(\Delta_{i6})_{1,3}}{a_{29}b_{i6}} \right] \left[ a_{29}(f_{i1} + d_{i1}f_{i2}) + (a_{29} + d_{i1}^2)(f_{i3} + d_{i1}f_{i4}) \right]
\]

\[
+ \frac{(\Delta_{i6})_{1,3}(\Delta_{i6})_{5,3}}{a_{29}a_{45}a_{10,7}b_{i6}} [f_{i5} + d_{i1}f_{i6}]
\]

\[
+ \frac{(\Delta_{i6})_{5,3}}{a_{10,7}b_{i6}} \left[(a_{10,7} + d_{i1}) \left(f_{i7} + d_{i1}f_{i8} \right) + a_{10,7} \left(f_{i9} + d_{i1}f_{i10} \right) \right]
\]

\[i = 1, 2, \ldots, p \quad (5-34)\]

\[
t_{i8} = \left[ \frac{a_{48}a_{47}(a_{10,7} + d_{i1}^2)}{a_{29}a_{45}b_{i6}} \right] \left[ a_{29}(f_{i1} + d_{i1}f_{i2}) + (a_{29} + d_{i1}^2)(f_{i3} + d_{i1}f_{i4}) \right]
\]

\[
+ \frac{a_{47}(\Delta_{i6})_{5,3}(a_{10,7} + d_{i1})^2}{a_{29}a_{45}a_{10,7}b_{i6}} [f_{i5} + d_{i1}f_{i6}]
\]

\[
+ \left[ \frac{(a_{29} + a_{47} + d_{i1}^2)(\Delta_{i6})_{5,3}}{a_{29}a_{45}a_{10,7}b_{i6}} - \frac{(a_{29} + d_{i1}^2)a_{48}a_{47}}{a_{29}a_{45}b_{i6}} \right]
\]

\[
\times \left[(a_{10,7} + d_{i1}) \left(f_{i7} + d_{i1}f_{i8} \right) + a_{10,7} \left(f_{i9} + d_{i1}f_{i10} \right) \right]
\]

\[i = 1, 2, \ldots, p \quad (5-35)\]
\[ t_{10} = \frac{a_{46}a_{47}a_{60} \left[ a_{29}(f_{11} + d_{ii}f_{13}) + (a_{28} + d_{ii}^2)(f_{18} + d_{ii}f_{14}) \right]}{\Delta_{15}} \]
\[ + \frac{a_{47}a_{60}(\Delta_{15})_{6,3}}{a_{46}a_{10,7} \Delta_{15}} (f_{16} + d_{ii}f_{16}) \]
\[ + a_{46} \left[ \frac{(a_{48} + a_{67} + d_{ii}^2)(\Delta_{15})_{6,3} - (a_{29} + d_{ii}^2)a_{46}a_{60}}{a_{46}a_{10,7} \Delta_{15}} \right](f_{17} + d_{ii}f_{18}) \]
\[ + \left\{ \frac{a_{46}a_{47}a_{60} + a_{47}a_{60} + (a_{48} + a_{67} + a_{85} + a_{89})d_{ii}^2 + d_{ii}^3}{a_{46}a_{10,7} \Delta_{15}} \right\}(\Delta_{15})_{6,3} \]
\[ - \frac{(a_{29} + d_{ii}^2)a_{46}a_{60}(a_{48} + a_{67} + d_{ii})}{\Delta_{15}} \left\{ f_{19} + d_{ii}f_{110} \right\} \]

\[ t_{i1} = d_{ii} \left\{ \left[ (a_{41} + d_{ii}^2)(a_{48} + a_{67} + d_{ii}^2) + a_{46}(a_{48} + d_{ii}^2) \right] \frac{a_{47}a_{85}(a_{10,7} + d_{ii}^2)(a_{41} + a_{46} + d_{ii}^2)}{a_{29}a_{46} \Delta_{15}} (\Delta_{15})_{1,3} \right\} \]
\[ + \frac{a_{41}}{a_{29}a_{45} \Delta_{15}} \left\{ \frac{a_{47}a_{85}(a_{10,7} + d_{ii}^2)(a_{41} + a_{46} + d_{ii}^2)}{a_{47}a_{85}(a_{10,7} + d_{ii}^2)(a_{10,7} + d_{ii}^2)} (f_{13} + d_{ii}f_{14}) \right\} \]
\[ + a_{46}(\Delta_{15})_{1,3} (f_{15} + d_{ii}f_{16}) \]
\[ + a_{29}a_{46}a_{48}a_{85} \left[ (a_{10,7} + d_{ii}^2)(f_{17} + d_{ii}f_{18}) + a_{10,7}(f_{19} + d_{ii}f_{110}) \right] \right\} + f_{13} \]

\[ t_{i6} = -d_{ii} \left\{ \frac{(a_{48} + a_{67} + d_{ii}^2)(\Delta_{15})_{1,3}}{a_{46} \Delta_{15}} - \frac{a_{46}a_{47}a_{60}(a_{10,7} + d_{ii}^2)}{a_{46} \Delta_{15}} \right\} (f_{11} + d_{ii}f_{12}) \]
\[ - \left\{ \frac{(a_{48} + d_{ii}^2)(a_{48} + d_{ii}^2) + a_{46}(a_{48} + d_{ii}^2)}{a_{29}a_{46} \Delta_{15}} \right\}(\Delta_{15})_{1,3} \]
\[ + \frac{a_{46}a_{47}a_{60}(a_{48} + a_{67} + d_{ii}^2)}{a_{46} \Delta_{15}} \right\} (f_{18} + d_{ii}f_{14}) \]
\[ - \frac{a_{46}(a_{48} + d_{ii}^2)(\Delta_{15})_{1,3}}{a_{29}a_{46} \Delta_{15}} (f_{16} + d_{ii}f_{16}) \]
\[ - \frac{a_{46}a_{47}a_{60}(a_{48} + d_{ii}^2)}{\Delta_{15}} \left[ (a_{10,7} + d_{ii}^2)(f_{17} + d_{ii}f_{18}) + a_{10,7}(f_{19} + d_{ii}f_{110}) \right] \right\} + f_{14} \]
\[ t_{18} = \frac{(\Delta_i)_{18}}{a_{26} a_{16}} \left[ a_{26} (f_{11} + d_{i1} f_{12}) + (a_{28} + d_{i7}) (f_{14} + d_{i4} f_{16}) \right] + f_{16} \]

\[ + \frac{(\Delta_i)_{18}}{a_{26} a_{16}} \left[ a_{10,7} (f_{17} + d_{i7} f_{18}) + a_{10,7} (f_{19} + d_{i9} f_{18}) \right] \] + f_{18} \quad (5-39)

\[ t_{17} = -\frac{a_{48} a_{47} (a_{10,7} + d_{i7})}{a_{26} a_{16}} \left[ a_{26} (f_{11} + d_{i1} f_{12}) + (a_{28} + d_{i7}) (f_{14} + d_{i4} f_{16}) \right] \]

\[ - \frac{a_{27} (a_{10,7} + d_{i7}) (\Delta_i)_{5,3}}{a_{26} a_{16}} \left[ f_{16} + d_{i6} f_{18} \right] \]

\[ - (a_{10,7} + d_{i7}) \left\{ \left[ a_{28} + d_{i7} \right] (\Delta_i)_{5,3} + \left[ a_{28} + d_{i7} \right] a_{48} a_{47} a_{10,7} \right\} \left( f_{19} + d_{i9} f_{18} \right) \] + f_{18} \quad (5-40)

\[ t_{19} = d_{i7} \left[ \frac{a_{26}}{a_{26} a_{16}} \left[ a_{48} a_{47} a_{48} a_{10,7} \left[ a_{28} (f_{11} + d_{i1} f_{12}) \right. \right. \right. \right. \]

\[ + (a_{28} + d_{i7}) (f_{14} + d_{i4} f_{16}) \right] + a_{47} (\Delta_i)_{5,3} (f_{16} + d_{i6} f_{18}) \]

\[ + \left[ (a_{48} + a_{47} + d_{i7}) (\Delta_i)_{5,3} - (a_{48} + d_{i7}) a_{48} a_{47} a_{10,7} \right] \left( f_{19} + d_{i9} f_{18} \right) \] + f_{18} \quad (5-41)

5.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

5.4.1 Introduction

Inaccessibility of a scalar state variable in equations (5-6) and (5-7) is reflected by a corresponding null column in the C and F matrices as implied in equation (2-14). For the generation of reduced state observers for the five body model, the number of inaccessible states can vary between one and nine.

5.4.2 First Order Observers (p = 1)

An observer of order at least one is required when any one of the ten scalar state variables of the five body model is inaccessible. The first order form of the linear observer equation is as follows:

\[ i = ds + Eu + Gy \quad (5-42) \]
The \( F \) and \( T \) matrices associated with a first order observer for the five body model then reduce to the following row forms:

\[
F = \begin{bmatrix} f_1 & f_2 & \cdots & f_{10} \end{bmatrix} \quad (5-43)
\]
\[
T = \begin{bmatrix} t_1 & t_2 & \cdots & t_{10} \end{bmatrix} \quad (5-44)
\]

The observer synthesis equations are then of the form of equations (5-15) through (5-31) with \( i = 1 \). Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the \( f_i \) \((i = 1, 2, \ldots, 10) = 0\).

**Example**

Suppose the scalar state representing the angular rate of body 5, \( z_{10} \), is inaccessible. Then \( f_{10} = 0 \) and the observer synthesis equations reduce to the form of equations (5-15) through (5-31) with the subscript, \( i \), omitted and \( f_{10} = 0 \). From equation (2-11), the synthesized scalar state, \( \dot{z}_{10} \), is expressed in terms of the scalar observer state variable \( z \), and the accessible model scalar state variables as follows.

\[
\dot{z}_{10} = \frac{1}{t_{10}} \left[ z - \sum_{i=1}^{9} t_i z_i \right] \quad (5-45)
\]

For \( z_{10} \) inaccessible, it is assumed that:

\[
C = \begin{bmatrix} 0 & \cdots & 0 \\ I_9 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} \quad (5-46)
\]

where \( I_9 = 9 \times 9 \) identity matrix.

From \( F = GC \),

\[
G = \begin{bmatrix} f_1 & f_2 & \cdots & f_9 \end{bmatrix} \quad (5-47)
\]

From \( Z = TB \),

\[
Z = \begin{bmatrix} t_2 & t_4 & t_6 & t_8 & t_{10} \end{bmatrix} \quad \text{for } r = 5 \text{ (control torques applied to all 5 bodies)} \quad (5-48)
\]

### 5.4.3 Observers of Order Greater Than One \( (1 < p < 10) \)

For those cases in which more than one of the ten scalar states of the five-body single-axis model are inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by \( p \). In each case the number of null columns in the measurement or observation matrix, \( C \), and the \( F \) matrix also is equal to \( p \). The general forms of the \( E, F \) and \( T \) matrices are given in equations (5-12) through (5-14) for \( p = 2, 3, 4, 5, 6, 7, 8 \) or 9.

**Example**

Suppose the scalar states, \( z_9 \) and \( z_{10} \), which represent the angular position and rate of body 5, are inaccessible. Then \( f_{10} = f_{11,10} = 0 \) for \( i = 1, 2 \) and the observer synthesis equations reduce to the form of equations (5-14) through (5-31) with \( i = 1, 2 \) and \( f_{10} = f_{11,10} = 0 \). From equation (2-11) the synthesized scalar states, \( \dot{z}_9 \) and \( \dot{z}_{10} \), are expressed in terms of the scalar observer states, \( x_1 \) and \( x_2 \) and the accessible model scalar state variables as follows.

\[
\dot{z}_9 = \frac{\sum_{i=1}^{9} (-1)^{i+1} (\Delta_x)_{i,1} (x_i - \sum_{j=1}^{9} t_{ij} x_j)}{\Delta_2} \quad (5-49)
\]
\[
\delta_{10} = \frac{\sum_{i=1}^{3} (-1)^{i+1}(\Delta_3)_{i,3} (z_i - \sum_{j=1}^{8} t_{ij} y_j)}{\Delta_3}
\]  

(5-50)

for

\[
\Delta_3 = \begin{bmatrix}
    t_{19} & t_{1,10} \\
    t_{29} & t_{2,10}
\end{bmatrix} = t_{10} t_{210} - t_{1,10} t_{20} \neq 0
\]  

(5-51)

Where \((\Delta_3)_{i,j} = \Delta_3\) without the elements of the \(i^{th}\) row and \(j^{th}\) column.

For \(z_9\) and \(z_{10}\) inaccessible, it is assumed that:

\[
G = \begin{bmatrix}
    0 & 0 \\
    I_8 & \vdots \\
    0 & 0
\end{bmatrix}
\]  

(5-52)

where \(I_8\) is an \(8 \times 8\) identity matrix.

Since \(F = GC\),

\[
G = \begin{bmatrix}
    f_{11} & \cdots & f_{18} \\
    f_{21} & \cdots & f_{28}
\end{bmatrix}
\]  

(5-53)

From \(E = TB\),

\[
E = \begin{bmatrix}
    t_{13} & t_{14} & t_{15} & t_{16} & t_{1,10} \\
    t_{23} & t_{24} & t_{25} & t_{26} & t_{2,10}
\end{bmatrix}
\]  

for \(r = 5\) (control torques applied to all five bodies)  

(5-54)

\[
E = \begin{bmatrix}
    t_{13} & t_{14} & t_{15} & t_{16} & 0 \\
    t_{23} & t_{24} & t_{25} & t_{26} & 0
\end{bmatrix}
\]  

for control torques applied to bodies 1, 2, 3, and 4  

(5-55)

5.5 REFERENCES


SECTION 6
APPLICATION OF MODAL MODELING AND DIRECT MATRIX PRODUCTS

6.1 INTRODUCTION

In the development of reduced state observers for the class of single-axis models of a flexible spacecraft presented in sections 2 through 5 of this report, it was assumed that the state vector coefficient matrix of the observer model was diagonal in order to reduce the amount of computation involved in solving the observer synthesis equation,

\[ TA - DT = F, \]  

for the elements of the T matrix as a function of the elements of the F matrix where

\[ F = GC \]  

Despite this rather arbitrary assumption, the computational effort involved in this solution grew with alarming rapidity as the number of flexibly connected rigid bodies incorporated in the single-axis model was increased. Furthermore, especially for the models incorporating both larger numbers of rigid bodies and damping, the assumption of a diagonal D matrix seemed a rather poor approximation in view of the considerable departure from diagonal form of the state vector coefficient matrices (A matrices) of these models. In view of these problems, Dr. Henry Waites (6-1) of Marshall Space Flight Center suggested that a more fruitful approach to synthesizing observers for this class of single-axis models of a flexible spacecraft treated in the preceding sections of this report would be based upon the following sequence of steps.

1. Recast the state variable forms of each undamped single-axis model into modal form.
2. Add modal damping to each modal model.
3. Recast the observer synthesis equation in terms of direct matrix products.

The advantages cited for this approach include the following.

1. The state vector coefficient matrix, A, in each modal single-axis model appears in 2 x 2 block diagonal form implying that the state vector coefficient matrix, D, of the corresponding observer required to accurately synthesize the inaccessible states would be no less sparse than 2 x 2 block diagonal.
2. The modal model is more amenable to truncation of less significant oscillatory modes.
3. The coefficient of damping associated with each vibrational mode can be specified at the outset of the analysis.

6.2 TRANSFORMATION OF THE TWO-BODY SINGLE-AXIS MODEL TO MODAL FORM

The approach utilised in transforming the undamped two-body single-axis model of Section 2 to damped modal form follows that presented in Thomson (6-2). It consists of the following steps.

1. Write original single-axis model in undamped form.
2. Write undamped single-axis model in terms of inertia and stiffness matrices.
3. Solve extended eigenvalue problem for eigenvalues and corresponding eigenvectors.
4. Normalize the eigenvectors.
5. Construct the modal matrix from the normalized eigenvectors.
6. Transform the model to principal coordinates (modal form) utilizing the modal matrix.
7. Add modal damping to the model in modal form.

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8. Write the modal model with damping in state variable form.

6.2.1 Original Undamped Two-Body Single-Axis Model

The undamped form of equations (2-1) and (2-2) is the following.

\[ I_1 \ddot{\theta}_1 = -k_1 \theta_1 + k_1 \theta_2 = q_1 \]  \hspace{1cm} (6-3)

\[ I_2 \ddot{\theta}_2 = -k_1 \theta_2 + k_1 \theta_1 = q_2 \]  \hspace{1cm} (6-4)

where the coefficients and variables appearing in this set of equations are defined in Fig. 2-1.

6.2.2 Undamped Two-Body Model in Terms of Inertia and Stiffness Matrices

\[ \mathbf{H} + \mathbf{K}x = \mathbf{q} \]  \hspace{1cm} (6-5)

where:

\[ x = [\theta_1 \theta_2]^T = [x_1 \ x_2]^T \]

\[ \mathbf{I} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} = \text{rotational inertia matrix} \]

\[ \mathbf{K} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \text{rotational stiffness matrix} \]

\[ \mathbf{q} = [q_1 \ q_2]^T \]

6.2.3 Determination of Eigenvalues and Eigenvectors

The eigenvalues for equation (6-5) are obtained by solving the extended eigenvalue problem which is equivalent to solving the following equation for \( \lambda \).

\[ \lambda \mathbf{I}x = \mathbf{K}x \]  \hspace{1cm} (6-6)

An equivalent form of the above equation is:

\[ [\lambda \mathbf{I} - \mathbf{K}]x = 0 \]  \hspace{1cm} (6-7)

A non-trivial solution of the extended eigenvalue problem exists if the following holds with the expanded forms of the rotational inertia and stiffness matrices for the two-body model expressed immediately following equation (6-6)

\[ |\lambda \mathbf{I} - \mathbf{K}| = \left| \begin{array}{cc} \lambda I_1 & -k_1 \\ k_1 & \lambda I_2 - k_1 \end{array} \right| \\
= I_1 I_2 \lambda \left( \lambda - k_1 \frac{I_1 + I_2}{I_1 I_2} \right) = 0 \]  \hspace{1cm} (6-8)

The solutions for this extended eigenvalue problem are:

\[ \lambda_1 = 0, \quad \lambda_2 = \frac{k_1}{I_1} + \frac{k_1}{I_2} \]  \hspace{1cm} (6-9)
The eigenvectors corresponding to $\lambda_i$ are obtained by solving equations of the following form for $v_i$ where $i = 1,2$.

$$[\lambda_i I - K]v_i = 0$$  \hspace{1cm} (6-10)

The eigenvectors are normalized by solving the following equation for $c_i$ ($i = 1,2$)

$$v_i^T v_i = 1$$  \hspace{1cm} (6-11)

The eigenvalues, eigenvectors and normalized eigenvector coefficients are displayed in Table 6-1

### 6.3.4 Construction of the Modal Matrix

$$P = [v_1 | v_2] = c_1 \begin{bmatrix} 1 & \left( \frac{I_1}{I_2} \right)^{-1/2} \\ 1 & \left( \frac{I_1}{I_2} \right)^{1/2} \end{bmatrix}$$  \hspace{1cm} (6-12)

### 6.3.5 Transformation to Principal Coordinates

$$y = p^T x$$  \hspace{1cm} (6-13)

$$y_1 = c_1 x_1 + c_1 x_2 = c_1 \theta_1 + c_1 \theta_2$$  \hspace{1cm} (6-14)

$$y_2 = c_1 \left( \frac{I_1}{I_2} \right)^{-1/2} x_1 - c_1 \left( \frac{I_1}{I_2} \right)^{1/2} x_2 = c_1 \left( \frac{I_1}{I_2} \right)^{-1/2} \theta_1 - c_1 \left( \frac{I_1}{I_2} \right)^{1/2} \theta_2$$  \hspace{1cm} (6-15)

$$q' = p^T q$$  \hspace{1cm} (6-16)

$$q_1' = c_1 q_1 + c_1 q_2$$  \hspace{1cm} (6-17)

$$q_2' = c_1 \left( \frac{I_1}{I_2} \right)^{-1/2} q_1 - c_1 \left( \frac{I_1}{I_2} \right)^{1/2} q_2$$  \hspace{1cm} (6-18)

Model in Principal Coordinates (Modal Model)

$$\ddot{\bar{y}}_1 = q_1'$$  \hspace{1cm} (6-19)

$$\ddot{\bar{y}}_2 = -\omega_1^2 \bar{y}_2 + q_2'$$  \hspace{1cm} (6-20)

$$\omega_1 = \left( \frac{k_1}{I_1} + \frac{k_1}{I_2} \right)^{1/2}$$  \hspace{1cm} (6-21)

### 6.3.6 Two Body Modal Model With Damping

Modal damping is added to the modal model described by equations (6-19) and (6-20) by adding a damping term to the equation with which the modal frequency, $\omega_1$, is associated. The two body model with damping in modal form then may be written as follows.

$$\ddot{\bar{y}}_1 = q_1'$$  \hspace{1cm} (6-22)

$$\ddot{\bar{y}}_2 = -2\zeta_1 \omega_1 \bar{y}_2 - \omega_1^2 \bar{y}_2 + q_2'$$  \hspace{1cm} (6-23)

where $\zeta_1$ is the damping ratio associated with $\omega_1$. 

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Table 6-1

Eigenvalues and Eigenvectors for Each Mode of Two Body Model

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Eigenvalue $\lambda_i = \omega_i^2$</th>
<th>Eigenvector $v_i$</th>
<th>Normalized Eigenvector Coefficient, $c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$c_1 \begin{bmatrix} 1 \ 1 \end{bmatrix}$</td>
<td>$\pm (I_1 + I_2)^{-1/2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{k_1}{I_1} + \frac{k_2}{I_2}$</td>
<td>$c_2 \begin{bmatrix} 1 \ -I_1 \ I_2 \end{bmatrix}$</td>
<td>$c_1 \left( \frac{I_2}{I_1} \right)^{-1/2}$</td>
</tr>
</tbody>
</table>

6.2.7 State Variable Form of the Two-Body Modal Model with Damping

The subscript on $y_2$ has been changed to "3" so that the following relationships can be used in constructing the state variable form of the modal model.

\[
\begin{align*}
y_2 &= \dot{y}_1 \\
y_3 &= \dot{y}_2
\end{align*}
\]

(6-24) \hspace{1cm} (6-25)

State Variable Modal Model

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -\omega_1^2 & -2\omega_1\omega_2
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
\]

(6-26)

6.3 Transformation of Three-Body Single-Axis Model to Modal Form

6.3.1 Original Undamped Three-Body Single-Axis Model

The undamped form of equations (3-1) through (3-3) is the following.

\[
\begin{align*}
I_1 \ddot{\theta}_1 &= -k_1 \theta_1 + k_1 \theta_2 + q_1 \\
I_2 \ddot{\theta}_2 &= k_1 \theta_1 + (k_1 + k_2) \theta_2 + k_2 \theta_3 + q_2 \\
I_3 \ddot{\theta}_3 &= k_2 \theta_2 - k_2 \theta_3 + q_3
\end{align*}
\]

(6-27) \hspace{1cm} (6-28) \hspace{1cm} (6-29)

where the coefficients and variables appearing in this set of equations are defined in Fig. 3-1.

6.3.2 Undamped Three-Body Model in Terms of Inertia and Stiffness Matrices

\[
I \ddot{x} + Kx = q
\]

(6-30)
6.3.3 Determination of Eigenvalues and Modal Frequencies

Corresponding to equation (6-30) an extended eigenvalue problem can be defined which consists of solving the following equation for \( \lambda \).

\[
\lambda \mathbf{I} \mathbf{x} = \mathbf{Kx}
\]

(6-31)

This is equivalent to setting the following determinant equal to zero

\[
|\lambda \mathbf{I} - \mathbf{K}| = \lambda I_1 - h_1
\begin{vmatrix}
I_2 - (h_1 + h_2) & k_2
0 & I_3 + k_2
\end{vmatrix}
= I_1 I_2 I_3 \lambda \left( \lambda^2 - \left( \frac{k_1 k_2}{I_1 I_2} + \frac{k_2}{I_3} \right) \lambda + \frac{k_1 k_2}{I_1 I_2} + \frac{k_1 k_2}{I_1 I_3} + \frac{k_1 k_2}{I_2 I_3} \right) = 0
\]

(6-32)

Since each solution for \( \lambda \) corresponds to the square of a modal frequency, equation (6-32) may also be written as follows.

\[
\lambda (\lambda - \omega_1^2)(\lambda - \omega_2^2) = 0
\]

(6-33)

for which the solutions are: \( \lambda_1 = 0 \), \( \lambda_2 = \omega_1^2 \) and \( \lambda_3 = \omega_2^2 \).

The eigenvectors corresponding to \( \lambda_1 \) are obtained by solving equations of the form presented in equation (6-10) for \( \psi_i \) where \( i = 1, 2, 3 \). The resulting pairs of eigenvalues and eigenvectors are displayed in Table 6-2.

Application of the remaining steps in the approach utilized in Section 6.2 yields the following state variable modal form for the three-body single-axis model of a flexible spacecraft.

6.3.4 State Variable Form of Three-Body Modal Model with Damping

\[
\begin{bmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4 \\
\dot{y}_5 \\
\dot{y}_6
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -\omega_1^2 & -2\psi_1 \omega_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\omega_2^2 & -2\psi_2 \omega_2
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4 \\
y_5 \\
y_6
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\psi}_1 \\
\dot{\psi}_2 \\
\dot{\psi}_3
\end{bmatrix}
\]

(6-34)
Table 6-1

Eigenvalues and Eigenvectors for Each Mode

of Three Body Model

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Eigenvalue</th>
<th>Normalized Eigenvector $v_i$</th>
<th>Eigenvector Coefficient, $c_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\lambda_1 = \omega_1^2$</td>
<td>$c_1 \begin{bmatrix} 1 \ 1 \end{bmatrix}$</td>
<td>$\pm (I_1 + I_2 + I_3)^{-1/2}$</td>
</tr>
<tr>
<td>2</td>
<td>$\omega_2^2$</td>
<td>$c_2 \begin{bmatrix} \frac{k_1 - \omega_2^2 I_1}{k_1} \ \frac{k_1 - \omega_2^2 I_2}{k_2} \end{bmatrix}$</td>
<td>$\pm \left[I_1 + \left(\frac{k_1 - r_1 I_1}{k_1}\right)^2 I_2 + \left(\frac{k_1 - r_1 I_2}{k_2}\right)^2 I_3 \right]^{-1/2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\omega_3^2$</td>
<td>$c_3 \begin{bmatrix} \frac{k_1 - \omega_3^2 I_1}{k_1} \ \frac{k_1 - \omega_3^2 I_2}{k_2} \end{bmatrix}$</td>
<td>$\pm \left[I_1 + \left(\frac{k_1 - r_1 I_1}{k_1}\right)^2 I_2 + \left(\frac{k_1 - r_1 I_2}{k_2}\right)^2 I_3 \right]^{-1/2}$</td>
</tr>
</tbody>
</table>

6.4 EXTENSION OF RESULTS TO SINGLE-AXIS MODELS WITH FOUR OR MORE RIGID BODIES

Inspection of the state variable modal form of the three-body single-axis model with modal damping in equation (6-34) and the corresponding modal two-body model in equation (6-26) reveals that the state vector coefficient matrix, $A$, in these modal models could be written in the following $2 \times 2$ block diagonal forms.

Two-Body Model:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \quad (6-35)$$

Three-Body Model:

$$A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \quad (6-36)$$

where $A_{ij}$ and 0 are $2 \times 2$ submatrices.

Furthermore,

$$A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (6-37)$$

and
\[ A_i = \begin{bmatrix} 0 & 1 \\ -\omega_{i-1}^2 & -2\omega_{i-1} \end{bmatrix} \text{ for } i > 1 \]  

(6-38)

It also should be noted that the dimensions of each \( A \) matrix are equal to twice the number of rigid bodies in the model. Application of the approach utilized in transforming the two-body and three-body single-axis models to state variable modal form with damping yields a set of models that extend the patterns for these models. In particular, a single-axis model involving \( r \) rigid bodies can be transformed to a modal state variable model with a state vector coefficient matrix of the following form:

\[
A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n/2,n/2} \end{bmatrix}
\]  

(6-39)

which is a \( 2 \times 2 \) block diagonal matrix of overall dimension \( n \times n \) where \( n = 2r \). The forms of the \( 2 \times 2 \) submatrices along the principal diagonal of this coefficient matrix are given in equations (6-37) and (6-38) for \( i = 1, 2, \ldots, n/2 \). The remaining elements in the \( A \) matrix are zero.

6.5 OBSERVER SYNTHESIS EQUATIONS EXPRESSED IN TERMS OF DIRECT MATRIX PRODUCTS

The observer synthesis equations are expressed in the following form in Section 2 of this report

\[ TA - DT = F = GC \]  

(6-40)

for the state variable form of a single-axis model of a flexible spacecraft, with some scalar states inaccessible,

\[
\dot{x} = Ax + Bu \\
x_a = Cx
\]  

(6-41, 6-42)

where the corresponding reduced state observer is given by:

\[
\dot{s} = Du + Eu + Gx_a \\
s = Tx
\]  

(6-43, 6-44)

The coefficient matrices and vectors appearing in equations (6-40) through (6-44) are defined for a linear model of dimension \( n \) with \( m \) accessible scalar states and \( p \) inaccessible scalar states as follows.

\[
A = n \times n \text{ model state vector coefficient matrix} \\
B = n \times r \text{ model control vector coefficient matrix} \\
C = m \times n \text{ model measurement or observation matrix} \\
D = p \times p \text{ observer state vector coefficient matrix} \\
E = p \times r \text{ observer control vector coefficient matrix} \\
G = p \times m \text{ observer observed vector coefficient matrix} \\
T = p \times n \text{ transformation matrix from model state vector to observer state vector} \\
x = [x^T \ x_a^T]^T = n \text{-vector of model scalar states} \\
x_a = m \text{-vector of accessible scalar states} \\
x_i = p \text{-vector of inaccessible scalar states} 
\]
\[ u. = \text{r-vector of normalized control torques} \]
\[ n. = \text{p-vector of observer scalar states} \]

From these definitions, each of the matrix products appearing in equation (6-40) has the dimensions \( p \times n \). If \( I_p \) is defined as the identity matrix of dimensions \( p \times p \) and \( L_n \) is defined correspondingly, then each of the matrix products, \( L_pTA \) and \( DTL_n \), also has the dimensions \( p \times n \). The observer synthesis equations may now be written in the following form.

\[ L_pTA - DTL_n = GC = F \quad (6-45) \]

However, the definition of a direct matrix product given in Lancaster (6-3) may be used to write the observer synthesis equations in the following equivalent form.

\[ [L_p \otimes A^T - D \otimes L_n] \hat{T} = \hat{F} \quad (6-46) \]

where:

\[ \hat{T} = \begin{bmatrix} T_1^T \\ T_2^T \\ \vdots \\ T_n^T \end{bmatrix} \]

\[ T_i^T = \text{vector comprised of the elements of the } i^{th} \text{ row of the } T \text{ matrix} \]

and \( \hat{F} \) is related to \( F \) in the same way.

From Lancaster (6-3), the direct matrix products appearing in equation (6-40) may be expanded as follows.

\[ L_p \otimes A^T = \begin{bmatrix} A^T & 0 \\ 0 & \ddots & \ddots \\ & 0 & A^T \end{bmatrix}_{np \times np} \quad (6-47) \]

For

\[ D = \begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pp} \end{bmatrix} \quad (6-48) \]

\[ D \otimes L_n = \begin{bmatrix} d_{11}L_n & \cdots & d_{1p}L_n \\ \vdots & \ddots & \vdots \\ d_{p1}L_n & \cdots & d_{pp}L_n \end{bmatrix}_{np \times np} \quad (6-49) \]

Solving for \( \hat{T} \) yields

\[ \hat{T} = [L_p \otimes A^T - D \otimes L_n]^{-1} \hat{G} \hat{C} \quad (6-50) \]

In general, this solution would require inversion of a matrix of dimension \( np \times np \).
6.6 OBSERVER SYNTHESIS EQUATIONS FOR SINGLE-AXIS MODELS IN MODAL FORM

If $A$ is the state vector coefficient matrix of a single-axis model in modal form, it was shown in Subsection 4.4 that it assumes the form given in equation (6-39). The transpose of each $2 \times 2$ block diagonal matrix also is block diagonal and of the following form.

$$A^T = \begin{bmatrix} A_{11}^T & A_{12}^T & \cdots & 0 \\ 0 & A_{22}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{n/2,n/2}^T \end{bmatrix}$$  \hspace{1cm} (6-51)

where each submatrix, $A_{ii}$, is $2 \times 2$ and the remaining elements in $A^T$ are zero.

For an observer of even order, $p$, of a state variable single-axis model in modal form with an $A$ matrix of the $2 \times 2$ block diagonal form appearing in equation (6-39) the observer state vector coefficient matrix is of the following $2 \times 2$ block diagonal form.

$$D = \begin{bmatrix} D_{11} & D_{22} & \cdots & 0 \\ 0 & D_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n/2,n/2} \end{bmatrix}$$  \hspace{1cm} (6-52)

where $D_{ii} = 2 \times 2$ submatrix on the principal diagonal.

$$D \otimes I_n = \begin{bmatrix} D_{11} \otimes I_n & D_{22} \otimes I_n & \cdots & 0 \\ 0 & D_{22} \otimes I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n/2,n/2} \otimes I_n \end{bmatrix}$$  \hspace{1cm} (6-53)

where:

$$D_{11} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$  \hspace{1cm} (6-54)

$$D_{22} = \begin{bmatrix} d_{22} & d_{24} \\ d_{42} & d_{44} \end{bmatrix}$$  \hspace{1cm} (6-55)

$$D_{i_1,i_2} = \begin{bmatrix} d_{p-2,g-2} & d_{p-2,g-1} \\ d_{p-1,g-2} & d_{p-1,g-1} \end{bmatrix}$$  \hspace{1cm} (6-56)

$$D_{i_1,i_2} = \begin{bmatrix} d_{p-1,g-2} & d_{p-1,g-1} \\ d_{p,g-2} & d_{p,g-1} \end{bmatrix}$$  \hspace{1cm} (6-57)
The equation for generating the elements of the $T$ matrix may now be written in the following form:

$$
\begin{bmatrix}
\Delta_{T_{ij}} - d_{1,1}I_2 & -d_{1,2}I_2 \\
-d_{2,1}I_2 & \Delta_{T_{ij}} - d_{2,2}I_2 \\
\vdots & \vdots \\
-d_{p-1,1}I_2 & \Delta_{T_{ij}} - d_{p-1,2}I_2 \\
-d_{p,1}I_2 & \Delta_{T_{ij}} - d_{p,2}I_2 \\
\end{bmatrix}
\begin{bmatrix}
T_{1,1} \\
T_{2,1} \\
\vdots \\
T_{p-1,1} \\
T_{p,1} \\
\end{bmatrix}
= 
\begin{bmatrix}
F_{1,1} \\
F_{2,1} \\
\vdots \\
F_{p-1,1} \\
F_{p,1} \\
\end{bmatrix}
$$

where:

$$
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
T_{ij} = [t_{i-1,j-1}, t_{i,j}]^T \quad i = 1, 2, \ldots, p
$$

$$
F_{ij} = [f_{i-1,j-1}, f_{i,j}]^T
$$

It should be noted that all of the $2 \times 2$ submatrices appearing in the coefficient matrices of equation set (6-58) are commutative under multiplication because this property is useful in the solution for the elements of the $T$ matrix for $p > 1$. Since all diagonal matrices commute, it is necessary only to show that the matrices, $A_{T_{ij}} - d_{ik}I_2$ and $A_{T_{ij}} - d_{ik}I_2$, commute for $j = 1, 2, \ldots, n/2$.

$$
[A_{T_{ij}} - d_{ik}I_2][A_{T_T} - d_{ik}I_2] = \begin{bmatrix} -d_{ik} & 0 \\ 1 & -d_{ik} \end{bmatrix} \begin{bmatrix} -d_{ik} & 0 \\ 1 & -d_{ik} \end{bmatrix} = 
$$

$$
= \begin{bmatrix} -d_{ik}d_{ro} & 0 \\ -(d_{ik}d_{ro} + d_{ik}d_{ro}) & d_{ik}d_{ro} \end{bmatrix} = [A_{T_{ij}} - d_{ro}I_2][A_{T_T} - d_{ro}I_2]
$$

$$
\begin{bmatrix} \Delta_{T_{ij}} - d_{ik}I_2 \end{bmatrix} = \begin{bmatrix} -d_{ik} & -\omega_{j-1}^2 \\ 1 & -d_{ik} + 2\omega_{j-1}\omega_{j-1} \end{bmatrix} = \begin{bmatrix} -d_{ik} & -\omega_{j-1}^2 \\ 1 & -d_{ik} + 2\omega_{j-1}\omega_{j-1} \end{bmatrix}
$$

Hence, the $2 \times 2$ submatrices, $A_{T_{ij}} - d_{ik}I_2$ and $A_{T_{ij}} - d_{ik}I_2$, are commutative under matrix multiplication for $j = 1, 2, \ldots, n/2$.

For an even $p$ the solution for the elements of the $T$ matrix now involves $np/4$ inversions of the coefficient matrices of dimensions $4 \times 4$ partitioned into $2 \times 2$ submatrices. Since the $2 \times 2$ submatrices are commutative under multiplication, each of the $np/4$ vector-matrix equations of the set can be solved in terms of each $T_{ij}$ which has the effect of reducing dimensions of the matrix inversion involved by a factor of two. The definitions of $T_{ij}$ and $F_{ij}$ in equations (6-59) and (6-60) in terms of the individual elements of the $T$ and $F$ matrices would then be invoked to complete the solution.

For the case in which $p$ is an odd integer, the corresponding $D$ matrix is $2 \times 2$ block diagonal except at one location along its principal diagonal where there occurs a degenerate "block" in the form of a single non-zero scalar element. With the assumption that the individual scalar state variables can be reordered, this isolated principal diagonal element can be placed at the lower right hand corner so that the $D$ matrix assumes the following form.

$$
D = 
\begin{bmatrix}
D_{11} & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
D_{p,1} & \ldots & \ldots & \ldots \\
D_{p,p} & \ldots & \ldots & \ldots \\
\end{bmatrix}
$$

(6-63)
where \( D_{ii} \) are \( 2 \times 2 \) submatrices defined in the same way as for \( p \) even for \( i = 1, 2, \ldots, (p-1)/2 \) and \( d_{p,p} \) is a scalar on the principal diagonal of the \( D \) matrix.

\[
D \otimes I_n = \begin{bmatrix}
D_{11} \otimes I_n & D_{21} \otimes I_n & \cdots & D_{2n-1} \otimes I_n \\
D_{21} \otimes I_n & \ddots & & \vdots \\
\vdots & & \ddots & \\
D_{2n-1} \otimes I_n & \cdots & & D_{p,p} \otimes I_n 
\end{bmatrix}
\]  
(6-64)

By the same procedure as utilized for even \( p \) the equations for generating the elements of the \( T \) matrix when \( p \) is odd may be expressed in the following form with \( F_{ij} \) and \( T_{ij} \) defined in equations (6-59) and (6-60).

\[
\begin{bmatrix}
A_{11}^T - d_{11} I_2 & A_{12}^T - d_{12} I_2 \\
A_{21}^T - d_{21} I_2 & A_{22}^T - d_{22} I_2 \\
\vdots & \vdots \\
A_{p-1,1}^T - d_{p-1,1} I_2 & A_{p-1,2}^T - d_{p-1,2} I_2 \\
A_{p,1}^T - d_{p,1} I_2 & A_{p,2}^T - d_{p,2} I_2 \\
\end{bmatrix}
\begin{bmatrix}
T_{1,j} \\
T_{2,j} \\
\vdots \\
T_{p-1,j} \\
T_{p,j}
\end{bmatrix}
= \begin{bmatrix}
F_{1,j} \\
F_{2,j} \\
\vdots \\
F_{p-1,j} \\
F_{p,j}
\end{bmatrix}
\]  
(6-65)

From equation set (6-65), it is evident that when \( p \) is an odd integer, \( n/2 \) of the equations in the set reduce to the form,

\[
[A_{jj}^T - d_{p,p} I_2] T_{p,j} = F_{p,j} 
\]  
(6-66)

where it has been assumed that the state variable model can be rearranged, if necessary, so that this vector-matrix equation appears last in each of the \( \frac{n}{2} \) sets of equations.

From equation (6-37),

\[
[A_{11}^T - d_{44} I_2]^{-1} = \begin{bmatrix}
-d_{44} & 0 \\
-1 & -d_{44}
\end{bmatrix}
\]  
(6-67)

From equation (6-38),

\[
[A_{jj}^T - d_{44} I_2]^{-1} = \begin{bmatrix}
-(2\xi_jI - \omega_j - 1 + d_{44}) & \omega_j^2 - 1 \\
-1 & -d_{44}
\end{bmatrix}
\]  
(6-68)

Hence, for the case in which the order of the observer, \( p \), is an odd integer the solution for the elements of the \( T \) matrix in terms of the elements of the \( F \) matrix reduces to \( n(p-1)/4 \) inversions of coefficient matrices of dimension \( 4 \times 4 \) partitioned into \( 2 \times 2 \) submatrices and \( n/2 \) inversions of coefficient matrices of dimensions \( 2 \times 2 \). Since all of the \( 2 \times 2 \) matrices of equation set (6-56) commute under multiplication, the dimensions of the \( n(p-1)/4 \) coefficient matrices to be inverted are in effect reduced by a factor of two by first solving for the \( T_{i,j} \)'s in terms of the \( F_{i,j} \)'s and then applying equations (6-59) and (6-60).

### 6.6.1 First Order Observers (\( p = 1 \))

Since \( p \) is an odd integer, equation (6-65) applies and reduces to the following form.

\[
[A_{jj}^T - d_{11} I_2] T_{1,j} = F_{1,j} 
\]  
(6-69)

where \( F_{1,j} \) and \( T_{1,j} \) are defined in equations (6-59) and (6-60) and the \( F \) matrix, which has a single row, may be written as follows:

\[
F = [F_{11} \quad F_{12} \quad \cdots \quad F_{1,n/2}]^T
\]  
(6-70)
The corresponding single row \( T \) matrix may be written in the same form as the \( F \) matrix. Solving equation (6-59) for \( T_{ij} \) yields the following:

\[
T_{ij} = [A_{ij}^T - d_{11}I_2]^{-1} F_{ij}
\]  
(6-71)

which, in view of equations (6-59) and (6-60), becomes:

\[
\begin{bmatrix}
  t_{1,2j-1} \\
  t_{1,2j}
\end{bmatrix} = \begin{bmatrix}
  A_{ij}^T - d_{11}I_2 \\
  d_{12}I_2
\end{bmatrix}^{-1} \begin{bmatrix}
  f_{i,2j-1} \\
  f_{i,2j}
\end{bmatrix}
\]

(6-72)

where \([A_{ij}^T - d_{11}I_2]^{-1}\) is given in equation (6-67) and \([A_{ij}^T - d_{11}I_2]^{-1}\) is given in equation set (6-68) for \( j = 2, 3, \ldots, n/2 \).

6.6.2 Second Order Observers \((p = 2)\)

Since \( p \) is even, equation (6-58) applies. For \( p = 2 \) it reduces to:

\[
\begin{bmatrix}
  A_{ij}^T - d_{11}I_2 \\
  -d_{12}I_2
\end{bmatrix} \begin{bmatrix}
  T_{ij} \\
  T_{2j}
\end{bmatrix} = \begin{bmatrix}
  F_{ij} \\
  F_{2j}
\end{bmatrix}; \quad j = 1, 2, \ldots, \frac{n}{2}
\]

(6-73)

where \( T_{ij} \) and \( F_{ij} \) are defined in equations (6-59) and (6-60).

\[
\Delta_{2j} = \begin{bmatrix}
  A_{ij}^T - d_{11}I_2 \\
  -d_{12}I_2
\end{bmatrix}
\]

(6-74)

Since \( A_{ij}^T - d_{4j}I_2 \) commutes with \( A_{ij}^T - d_{4j}I_2 \) under matrix multiplication

\[
|\Delta_{2j}| = \|[A_{ij}^T - d_{11}I_2][A_{ij}^T - d_{22}I_2] - d_{12}d_{21}I_3|
\]

(6-75)

Then

\[
T_{1j} = \frac{A_{ij}^T - d_{22}I_2}{|\Delta_{2j}|} F_{1j} + \frac{d_{12}I_2}{|\Delta_{2j}|} F_{2j} \quad j = 1, 2, \ldots, \frac{n}{2}
\]

(6-76)

\[
T_{2j} = \frac{d_{21}I_2}{|\Delta_{2j}|} F_{1j} + \frac{A_{ij}^T - d_{11}I_2}{|\Delta_{2j}|} F_{2j}
\]

(6-77)

Solutions for the individual elements of the \( T \) matrix in terms of the elements of the \( F \) matrix are then obtained by application of the definitions of \( T_{ij} \) and \( F_{ij} \) in equations (6-59) and (6-60).

\[
\begin{bmatrix}
  t_{1,2j-1} \\
  t_{1,2j}
\end{bmatrix} = \begin{bmatrix}
  A_{ij}^T - d_{22}I_2 \\
  d_{12}I_2
\end{bmatrix}^{-1} \begin{bmatrix}
  f_{i,2j-1} \\
  f_{i,2j}
\end{bmatrix}
\]

(6-78)

\[
\begin{bmatrix}
  t_{2,2j-1} \\
  t_{2,2j}
\end{bmatrix} = \begin{bmatrix}
  d_{12}I_2 \\
  |\Delta_{2j}|
\end{bmatrix} \begin{bmatrix}
  f_{i,2j-1} \\
  f_{i,2j}
\end{bmatrix} + \begin{bmatrix}
  A_{ij}^T - d_{11}I_2 \\
  |\Delta_{2j}|
\end{bmatrix} \begin{bmatrix}
  f_{i,2j-1} \\
  f_{i,2j}
\end{bmatrix}
\]

(6-79)

6.6.3 Observers of Higher Order

For \( p = 3 \) equation (6-55) reduces to the following.

\[
\begin{bmatrix}
  A_{ij}^T - d_{11}I_2 \\
  -d_{12}I_2
\end{bmatrix} \begin{bmatrix}
  T_{ij} \\
  T_{2j}
\end{bmatrix} = \begin{bmatrix}
  T_{1j} \\
  T_{2j}
\end{bmatrix}
\]

(6-73)
The observer synthesis equations for $p = 3$ differ from those for $p = 2$ by the addition of equation set (6-80) which is of the same form as the observer synthesis equations for $p = 1$ with the subscript, 3, substituted for the subscript, 1. Therefore, the solutions for the elements of the first two rows of the $T$ matrix for $p = 3$ are identical with those for the two rows of the matrix for $p = 2$, equations sets (6-76) and (6-77). While the solutions for the elements of the third row are of the same form as those for the $T$ matrix for $p = 1$, equation set (6-71) with the numerical subscript, 3, substituted for the subscript, 1.

For $p = 4$ equation set (6-58) reduces to one equation set identical with the one for $p = 2$, and another equation set of the same form with each numerical subscript incremented by one. Hence, the solutions for the elements of the first two rows of the $T$ matrix for $p = 4$ are identical with those for the two rows of the $T$ matrix for $p = 2$, equation set (6-76) and (6-77). The solutions for the elements of the third and fourth rows are given by the same equations with each of the numerical subscripts incremented by one.

The solutions for the elements of the $T$ matrix for larger values of $p$ follow the same pattern. Thus, they can be constructed directly by using the solutions for $p = 1$ and $p = 2$ as "building blocks" as was demonstrated for $p = 3$ and $p = 4$.

### 6.7 SOLUTION FOR SYNTHESIZED STATE VARIABLES

In Subsection 6.4 it was shown that the single-axis modal models of a flexible spacecraft treated in this report can be written in the state variable form in terms of the modal state vector as follows.

\[ \dot{y} = Ay + Bu \]  
\[ y_A = Cy \]

where

- $A = n \times n$ state vector coefficient matrix
- $B = n \times r$ control vector coefficient matrix
- $C = m \times n$ observation or measurement matrix
- $y = \text{modal state vector of dimension } n$
- $y_A = \text{vector of accessible modal state variables of dimension } m$
- $u = \text{control vector of dimension } r$

The block diagram corresponding to this model is the same as Fig. 2-2 except that the vectors, $x$ and $x_A$, are replaced by $y$ and $y_A$, respectively. The $2 \times 2$ block diagonal form of the $A$ matrix is shown in equation (6-39).

If the number of inaccessible modal model scalar states is given by $p = n - r$ ($1 < p < n$), then the corresponding reduced modal state linear observer model is the following.

\[ \dot{s} = Ds + Eu + G y_A \]  
\[ s = Ty \]

where

- $D = p \times p$ observer state vector coefficient matrix
- $E = p \times r$ observer control vector coefficient matrix
- $G = p \times m$ observer observed vector coefficient matrix
- $T = p \times n$ observer weighting matrix
The block diagram for this observer is the same as that shown in Fig. 2-4 except that the vector, $y$, is substituted for the vector $x$.

After the observer synthesis equation given by equation (6-40) or one of its equivalent forms such as equation set (6-68) for even $p$ and equation set (6-85) for odd $p$ has been solved for the elements of the $T$ matrix, $t_{ij}$, equation (6-84) can be solved to express the synthesized inaccessible modal model scalar states in terms of the accessible modal model scalar states. This last step generally will require the inversion of a $p \times p$ matrix. A block diagram of the modal model of a flexible spacecraft and its reduced state linear observer appears in Fig. 6-1.

Example: Solution for two synthesized modal states in the two body model.

Suppose that for the state variable modal two-body model the modal scalar states, $y_2$ and $y_4$ are inaccessible. Then $p = 2$, $n = 4$ and the $T$ matrix is thus:

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix}$$

(6-85)

corresponding to:

$$y = [y_1\ y_2\ y_3\ y_4]^T$$

(6-86)

$$x = [x_1\ x_2]^T$$

(6-87)

If the remaining modal model scalar states, $y_1$ and $y_3$ and all of the elements of the $T$ matrix are known then equation (6-84) can be solved to express the synthesized inaccessible modal model scalar states $\hat{y}_2$ and $\hat{y}_4$ as follows.

$$\hat{y}_2 = \frac{(\Delta_2)_{1,1}}{\Delta_2} (x_1 - t_{11}y_1 - t_{12}y_2) - \frac{(\Delta_2)_{2,1}}{\Delta_2} (x_2 - t_{21}y_1 - t_{22}y_2)$$

(6-88)

$$\hat{y}_4 = \frac{(\Delta_2)_{1,2}}{\Delta_2} (x_1 - t_{11}y_1 - t_{12}y_2) - \frac{(\Delta_2)_{2,2}}{\Delta_2} (x_2 - t_{21}y_1 - t_{22}y_2)$$

(6-89)

where

$$\Delta_2 = \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} = t_{11}t_{22} - t_{12}t_{21} \neq 0$$

(6-90)

$$(\Delta_2)_{i,j} = \Delta_2 \text{ without the elements of the } i^{th} \text{ row and the } j^{th} \text{ column.}$$

6.5 REFERENCES

\[ u = \text{vector of scalar inputs to vehicle modal model} \]
\[ y_A = \text{vector of accessible states of modal model} \]
\[ z = \text{vector of scalar states of observer} \]
\[ T = \text{observer weighting matrix} \]
\[ \hat{y}_I = \text{vector of reconstructed scalar states of model} \]
\[ \hat{y} = \begin{bmatrix} y_A \\ \hat{y}_I \end{bmatrix} \quad \text{reconstructed vector of all scalar state variables of vehicle model} \]

**FIGURE 6-1**

BLOCK DIAGRAM OF MODAL SPACECRAFT MODEL AND ITS REDUCED STATE LINEAR OBSERVER
SECTION 7

CONCLUSIONS AND RECOMMENDATIONS

During the period covered by this report, the class of single-axis state variable models with some inaccessible states was extended three ways.

1. The patterns involved in the prior development of the state variable forms of the two-body, three-body, and four-body single-axis models of a flexible spacecraft were extended to produce a five-body model that could represent the single axis that was found to be decoupled from the remaining axes of a five-body three axis model treated in earlier work.

2. A rotational damping coefficient was added to each flexible interconnection between the rigid bodies comprising each model.

3. Each undamped single axis model was transformed to a modal model with one or more inaccessible model state variables.

For each combination of single axis state variable model and inaccessible scalar state(s) a reduced state linear observer was generated to reconstruct those scalar states that were inaccessible. This was done because the application of linear quadratic regulator (LQR) and closely related time domain approaches to attitude control utilise all or nearly all of the scalar states of the model of the spacecraft to be controlled.

7.1 CONCLUSIONS

The following conclusions were drawn mainly from the development of the damped two-, three-, four- and five-body single-axis models with inaccessible scalar state variables of a prototype flexible spacecraft and the generation of the corresponding linear observers of minimum order required to reconstruct these inaccessible scalar states.

7.1.1 Observers Generated for Single-Axis Models Based on Angular Displacement and Rate State Variables

1. Since, of the four coefficient matrices appearing in the observer synthesis equation, A, D, F and T, only the state vector coefficient matrix in the single-axis model, A, is known a priori, the following approach was used to generate the elements of the coefficient matrix, T, for the transformation from the state vector of the model, x, to the state vector of the observer, s.

   a. The elements of F can be determined by utilizing the known values for the elements of C, the observation matrix in the single-axis model and the assumed values of the elements of G in conjunction with the equation, F = GC.

   b. Assuming that D is diagonal simplifies significantly the solution of the equations for determining the elements of T.

2. The minimum order required for a reduced state linear observer to reconstruct p inaccessible scalar states of a single-axis state variable model with a total of n scalar states is p where p = 1, 2, ..., n - 1. Therefore, the number of elements in the T matrix to be determined equals np and solving for the p inaccessible synthesized scalar state variables requires the inversion of a p x p coefficient matrix.

3. The rigid-body flexible-joint single-axis models of a flexible spacecraft treated in this report are in a more general form when damping is added to each joint connecting the rigid bodies. Therefore it is far easier to develop the observer synthesis equations for the damped models than to begin with the equations for the undamped models and generalize them to account for the effects of added damping.

4. If n - 1 of the n scalar state variables of the single-axis model are accessible, a reduced state observer of order at least one (p = 1) is required to synthesize the inaccessible state variable. The number of elements in the T matrix to be determined equals n and solving for the one inaccessible scalar state variable does not require the inversion of a matrix.
As the number of accessible scalar state variables decreases, the number of inaccessible scalar states, \( p \), the number of elements in the T matrix to be determined, \( np \) and the dimensions of the coefficient matrix to be inverted in solving for the inaccessible synthesized scalar state variables, \( p \times p \), increase, which increases the number of computations required.

At least one of the \( n \) state variables of the single axis model must be accessible in order for the inaccessible state variables to be synthesized by a reduced state observer.

### 7.1.2 Observers Generated for Single-Axis Models Based on Modal State Variables

1. The state vector coefficient matrix, \( A \), in each modal single-axis model appears in \( 2 \times 2 \) block diagonal form implying that the state vector coefficient matrix, \( D \), in the corresponding reduced state observer is \( 2 \times 2 \) block diagonal.

2. When the observer synthesis equation is expressed in terms of direct matrix products, solution for the elements of the T matrix generally requires inversion of an \( np \times np \) coefficient matrix.

3. When the number of inaccessible states of the model, \( p \), is even, use of \( A \) and \( D \) matrices in \( 2 \times 2 \) block diagonal form in the observer synthesis equation reduces the solution for the elements of the T matrix to the inversion of \( \frac{np}{4} \times 4 \) matrices partitioned into \( 2 \times 2 \) submatrices all of which commute under multiplication.

4. When the number of inaccessible states of the model is odd, use of \( A \) and \( D \) matrices in \( 2 \times 2 \) block diagonal form in the observer synthesis equation reduces the solution for the elements of the T matrix to the inversion of \( \frac{np+1}{4} \times 4 \) matrices partitioned into \( 2 \times 2 \) commutative submatrices and \( \frac{np-1}{2} \times 2 \times 2 \) matrices.

5. The modal matrix operates on only the angular displacement state variables and thus each modal state variable generally is a weighted linear sum of all of the angular displacements.
   a. Reduced state linear observers predicated upon a modal single axis model generally require that at least one of the modal state variables be accessible which is equivalent to requiring that all of the angular displacement state variables of the original state variable model be accessible.
   b. Reduced state observers based on the modal model can be used to synthesize one or more inaccessible angular rate state variables of the original state model.
   c. If no modal state variable is accessible or, equivalently, if any one of the angular displacement state variables for the original state model is inaccessible, reduced state observers predicated upon the modal model cannot be used to synthesize any state variables.

6. Even if all of the necessary conditions required for synthesis of state variables by a reduced state observer predicated upon a modal single axis model are satisfied, two significant disadvantages of this approach are the following:
   a. Modal state variables that are weighted sums of angular displacements and rate state variables are difficult to interpret physically.
   b. Transformation from the modal state variables to the angular displacement and rate state variables may be very complicated.

### 7.2 RECOMMENDATIONS

The following directions are suggested for future study in the application of attitude control to state variable models of flexible spacecraft for which one or more scalar states are inaccessible.

1. The modular control techniques developed for the attitude control of models of flexible spacecraft for which all scalar state variables are accessible should be modified for application to series of single axis models and their associated reduced state linear observers developed in the work treated in this report.

2. Selected combinations of single axis model and its associated linear observer and modular attitude control system should be simulated on a digital computer to support investigation of effects of changes in the following single-axis model and observer characteristics.
a. Ratios between the masses (rotational inertias) of bodies comprising the single-axis model.

b. Magnitudes of spring and damping coefficients at the interfaces between the rigid bodies of the single-axis model.

3. The generation of reduced state observers to reconstruct inaccessible scalar states of a model of a flexible spacecraft should be extended to the three-axis five-body model of a prototype flexible spacecraft developed earlier.

4. The application of modular techniques to the attitude control of selected combinations of a single-axis model and its corresponding reduced state linear observer should be extended to the combinations of the single-axis and two-axis five body models representing the prototype flexible spacecraft and the corresponding reduced state observers.

5. The combination of single-axis and two-axis five body models and their linear observers and modular attitude control systems should be simulated on a digital computer.

6. Coefficients representing the sensitivity of the scalar states to parameters of the combination of single-axis and two-axis five body models and their linear observers and modular attitude control systems should be developed.

7. Since, for a model with $n$ scalar state variables, the number of elements of the $T$ matrix to be determined, $np$, and the dimensions of the coefficient matrices to be inverted in solving for the synthesized inaccessible variables, $p \times p$, increase as the number of inaccessible variables, $p$, increases, it would be desirable to determine whether there is a value for the ratio, $\frac{p}{n}$, at which a full state observer would be more readily implemented than a reduced state observer.

8. In view of the especially convenient forms of the modal state variable single-axis models of the flexible spacecraft and of the corresponding observer synthesis equation it appears worthwhile to investigate ways to mitigate the requirement that all rotational displacement state variables in the original model be accessible.