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MATHEMATICAL PHYSICS APPROACHES TO LIGHTNING DISCHARGE PROBLEMS

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INTRODUCTION

In this report mathematical physics arguments useful for lightning discharge and generation problems are pursued. The idea of using a more mathematical approach than heretofore is based on the prospects for clarification of the physical phenomena involved which in many cases is obscured by the multiplicity of factors at work.

The first section treats a soliton Ansatz for the lightning stroke including a charge generation term which is the ultimate source for the phenomena. In this way the number of functions required to completely specify electric and magnetic fields, charge and current densities is reduced to a minimum. For purposes of simplification it is supposed that the ionization channel radius is independent of time.

The second section establishes dynamical, electrical and thermal equations for a partially ionized plasma including the effects of pressure, magnetic field, electric field, gravitation, viscosity and temperature. From these equations is then derived the Non-Stationary Generalized Ohm's Law essential for describing field/current density relationships in the ionization channel of the lightning stroke. Arguments are then given for the essential participation of ionic generation processes in the "exponentially" increasing current density and charge density which develop during the stroke.

The third section deals with the discharge initiation problem and argues that the ionization rate drives both the convective current and electric displacement current to increase "exponentially" but that because of relative saturation of the former compared to the latter the convective current is unable to "relieve" the electric field which eventually increases to breakdown unleashing the lightning stroke. In this section the non-linear term of the Non-Stationary Ohm's Law is retained without approximation so that the temporal development of the lightning discharge may be precisely formulated.

The fourth section deals with the statistical distributions of charge in the thundercloud preceding a lightning discharge. Defining centers of positive and negative charge, centers of generation and recombination as well as centers of current efflux and influx for the cloud it becomes possible to statistically characterize the development of the cloud dipole moment and the relative velocity between the centers of charge as functions of time.

The fifth section contains some physical comments on the stability of the pre-lightning charge distributions and the use of Boltzmann relaxational equations to determine them. Also the argument for aircraft providing a lowered impedance path for the stroke is given subject to the additional effect of field enhancement factors such as aircraft curvatures.

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MATHEMATICAL PHYSICS APPROACHES TO
LIGHTNING DISCHARGE PROBLEMS

LIST OF SYMBOLS

q charge per unit volume (net signed charge in a unit of volume)

q_+ absolute (non-negative) quantity of positive charge per unit volume

q_- absolute (non-negative) quantity of negative charge per unit volume

$\bar{J}$ net signed charge crossing unit area per unit time: electrical current density

$\bar{J}_+$ absolute (non-negative) quantity of positive charge crossing unit area per unit time: electrical current density of (absolute) positive charge

$\bar{J}_-$ absolute (non-negative) quantity of negative charge crossing unit area per unit time: electrical current density of (absolute) negative charge

note: The directions of the current densities are given by the vectorial average of the velocities of the individual charges crossing unit area in unit time; with this convention the above definitions of the magnitudes of the current densities are completed vectorially

G net signed charge generated per unit volume per unit time (includes ion production and recombination)

G_+ absolute (non-negative) quantity of positive charge generated per unit volume per unit time

G_- absolute (non-negative) quantity of negative charge generated per unit volume per unit time

c light speed

t elapsed time

τ distance unit of elapsed time = product of light speed and elapsed time

∇ del operator with scalar components representing partial differentiations in the coordinate directions

μ magnetic permeability of medium (plasma)

ε electric permittivity of medium (plasma)

$\bar{E}$ electric intensity field vector

$\bar{H}$	magnetic intensity field vector
e	charge (absolute) of electron
Z	valence of ion, i.e. non-negative integer multiple of e which is absolute ionic charge: for electron or singly-charged + ion Z=1
D	diffusion coefficient (one third of square of step length per mean free time in a random walk)
$\bar{w}$	drift velocity in a random walk
$\bar{u}_r$	unit vector orthogonal to axis of cylindrical coordinate system
$\bar{u}_\theta$	unit vector in direction of circumferential (θ) increase in cylindrical coordinate system
$\bar{u}_z$	unit vector in axial direction of cylindrical coordinate system
∂_r	operator of partial differentiation in direction of u_r
∂_θ	operator of partial differentiation in direction of u_θ
∂_z	operator of partial differentiation in direction of u_z
H_r	radial component of magnetic field intensity
H_θ	circumferential component of magnetic field intensity
H_z	axial component of magnetic field intensity
E_r	radial component of electric field intensity
E_θ	circumferential component of electric field intensity
E_z	axial component of electric field intensity
ξ	fraction of light speed at which soliton (pulse) propagates
e(r)	radial dependence of axial component of electric field intensity
h(r)	radial dependence of circumferential component of magnetic field intensity
k	reciprocal channel size constant (of ionization channel)
$K_o(kr)$	modified Bessel function of second kind
$I_o(kr)$	modified Bessel function of first kind
J_r	radial component of current density vector

J_θ circumferential component of current density vector

J_z axial component of current density vector

n absolute number of particles per unit volume

n_+ absolute number of positively charged particles per unit volume

n_- absolute number of negatively charged particles per unit volume

n_o absolute number of uncharged particles per unit volume

Z_+ valence of positively charged particles (averaged for aggregate)

Z_- valence of negatively charged particles (averaged for aggregate)

ρ mass density: total mass of particles in a unit volume

$\bar{J}_m$ mass current density = momentum density: total vectorial momentum of all particles in a unit volume

G_o number of neutral particles generated per unit volume per unit time (including recombination and ionization)

D_t Eulerian derivative operator for differentiation with respect to time while following moving portion of plasma $D_t = \partial_t + \bar{v}.\nabla$

B magnetic induction field vector

$\bar{v}_+$ vectorial average velocity of positively charged particles

$\bar{v}_-$ vectorial average velocity of negatively charged particles

$\bar{v}_o$ vectorial average velocity of uncharged particles (averaged over small portion of plasma)

m_+ mass of positive ions (averaged over small portion of plasma)

m_- mass of negative ions (averaged over small portion of plasma)

m_o mass of uncharged particles (averaged over small portion of plasma)

$\bar{T}_+$ vectorial momentum transfer to positive ions per unit volume per unit time

$\bar{T}_-$ vectorial momentum transfer to negative ions per unit volume per unit time

$\bar{T}_o$ vectorial momentum transfer to neutral particles per unit volume per unit time (momentum transfer excludes contributions to pressure and viscosity)

p_+	partial pressure due to positive ions
p_-	partial pressure due to negative ions (including electrons)
p_o	partial pressure due to neutral particles
η_+	viscosity for positive ions
η_-	viscosity for negative ions
η_o	viscosity for neutral particles
ϕ	gravitational potential
η_A	arithmetic mean viscosity
$\bar{v}$	viscous average velocity of small portion of plasma
η	resistivity of plasma
α	reciprocal valence weighted mass of charged particles $= 1/(Z_+m_-+Z_-m_+)$
m_G	geometric mean mass of charged particles $= (m_+m_-)^{1/2}$
m_A	valence weighted arithmetic mean mass of charged particles
$\langle Z \rangle$	average valence of + and - ions
$\langle Z^2 \rangle$	average squared valence of + and - ions
m_H	harmonic mean mass of charged particles weighted for valence and concentration
n_A	valence weighted concentration average for charged particles
Z_G	geometric mean valence of charged particles
Z_δ	mass weighted valence difference parameter
Δ	Laplacian operator
Γ	electrical collisional frequency
$\bar{\Omega}_+$	vectorial cyclotron frequency for positive ions
$\bar{\Omega}_-$	vectorial cyclotron frequency for negative ions
T	absolute temperature (Kelvin)
$\bar{J}_\theta$	thermal current density

κ	thermal conductivity
S	power per unit volume radiated
c_+	specific heat at constant pressure per unit mass for + ions
c_-	specific heat at constant pressure per unit mass for - ions
c_o	specific heat at constant pressure per unit mass for neutral particles
S_+	power radiated per unit volume by + ions
S_-	power radiated per unit volume by - ions
S_o	power radiated per unit volume by neutral particles
T_+	temperature of + ions
T_-	temperature of - ions
T_o	temperature of neutral particles
κ_+	thermal conductivity for + ions
κ_-	thermal conductivity for - ions
κ_o	thermal conductivity for neutral particles
D_+	diffusion coefficient for + ions
D_-	diffusion coefficient for - ions
D_o	diffusion coefficient for neutral particles
λ	mean free path length (τ = mean free time)
v	mean free speed
γ_+	collisional frequency for + ions
γ_-	collisional frequency for - ions
γ_o	collisional frequency
c_p	specific heat per unit mass averaged for +, - and neutrals
$\langle m \rangle$	arithmetic mean mass of particles
$\langle D \rangle$	average diffusion coefficient
$\langle \gamma \rangle$	average collisional frequency

k	Boltzmann constant
σ	conductivity
V	volume of region
S	surface of region
$\langle\rangle_V$	volume average
$\langle\rangle_S$	surface average
$\bar{n}$	unit normal vector for surface
$\alpha(\tau)$	time dependence of charge density
$\beta(\tau)$	time dependence of electrical current density
$\gamma(\tau)$	time dependence of electric field
$\bar{E}_+$	vectorial electric field intensity generated by + charges
$\bar{E}_-$	vectorial electric field intensity generated by - charges
Q_+	total absolute positive charge
Q_-	total absolute negative charge
$\bar{R}_+$	center of positive charge
$\bar{R}_-$	center of negative charge
V_{++}	portion of total volume over which net production of + ions occurs
V_{+-}	portion of total volume over which net recombination of + ions occurs
$\bar{R}^G_{++}$	center of generation for + charges
$\bar{R}^G_{+-}$	center of recombination for + charges
$\bar{R}^G_{-+}$	center of generation for - charges
$\bar{R}^G_{--}$	center of recombination for - charges
S_{++}	portion of closed surface over which current due to + charges is efflux

S_{+-} portion of closed surface over which current due to + charges is influx

S_{-+} portion of closed surface over which current due to - charges is efflux

S_{--} portion of closed surface over which current due to - charges is influx

$\bar{R}^{I}_{++}$ center of efflux of current on surface S due to + charges

$\bar{R}^{I}_{+-}$ center of influx of current on surface S due to + charges

$\bar{R}^{I}_{-+}$ center of efflux of current on surface S due to - charges

$\bar{R}^{I}_{--}$ center of influx of current on surface S due to - charges

N_{+} total number of + charges generated per unit time in V

N_{-} total number of - charges generated per unit time in V

A total absolute quantity of positive or negative charge (equal)

$\bar{v}_{rel}$ relative vectorial velocity of centers of positive and negative charge

$\bar{M}$ vectorial dipole moment of thundercloud

B_t Boltzmannian derivative operator for differentiation with respect to time while following states of small portion of plasma in phase space

$$B_t = \partial_t + \bar{v} \cdot \nabla_{\bar{R}} + \bar{a} \cdot \nabla_{\bar{v}}$$

$d\tau_{\bar{R}}$ differential volume element in locational space

$d\tau_{\bar{v}}$ differential volume element in velocity space

note: occasionally the same symbols have been used for different variables but the difference in meaning should be clear from the difference in context and in location on the list of symbols: one should especially note the difference between τ mean free time and distance unit of elapsed time; k reciprocal channel size constant and the Boltzmann constant; viscosity and resistivity η

1. Soliton Ansatz for the Lightning Stroke with Charge Generation Term

The basic objective of this section is to derive a mathematical framework in terms of which observational and experimental data can be used to identify and interpret physical features of the propagation of fields, charges and currents during a typical lightning stroke. By characterizing the field and current density components in terms of a small number of scalar functions it should become easier to relate input data to the actual development of charge, currents and fields.

The starting point is the set of Maxwell equations modified by the inclusion of a charge generation term G describing the number of charges (of valence Z) generated per unit volume per unit time. The absolute charge density q is taken to be positive corresponding to the absolute value of charge per unit volume. Then in principle each of the following equations after (1.2) is doubled with a separate version for positive and negative charges and for the currents and fields associated with them. The actual signed charge density q and current density J are related to these unsiged quantities by

$$q = q_+ - q_- \tag{1.1}$$

$$\bar{J} = \bar{J}_+ - \bar{J}_- \tag{1.2}$$

but for conciseness the generics q, $\bar{J}$ will be used for typical q_+, $\bar{J}_+$ or q_-, $\bar{J}_-$ corresponding to G_+, G_-. Generally the inclusion of a G term means that charge conservation will be violated for + and - species separately but it will be expected that overall conservation holds for the net charge given by (1.1).

Turning now to the Maxwell equations one has using $\tau = ct$ and Gaussian units

$$\nabla \times \bar{E} = -\mu\partial_\tau \bar{H} \tag{1.3}$$

$$\bar{J} = c(\nabla \times \bar{H} - \varepsilon\partial_\tau \bar{E}) \tag{1.4}$$

$$q = \varepsilon\nabla \cdot \bar{E} + \frac{Ze}{c}\int_0^\tau G d\tau \tag{1.5}$$

$$\nabla \cdot \bar{H} = 0 \tag{1.6}$$

Non-Conservation of charge then follows by adding the partial time derivative of (1.5) to the divergence of (1.4)

$$\partial_t q + \nabla \cdot \bar{J} = ZeG \tag{1.7}$$

At the outset it may be noted that the assumption

$$\bar{J} = -D\nabla q + \bar{w}q \tag{1.8}$$

converts (1.7) to a Diffusion Equation with drift

$$\partial_t q = D\Delta q + \bar{w}\cdot\nabla q + q\nabla\cdot\bar{w} + ZeG \tag{1.9}$$

In the case of a lightning stroke it is to be expected that the $D\Delta q$ term which acts as a randomizing element in (1.9) will be small compared with the drift term $w\cdot\nabla q$ producing an often small random deviation in direction of the stroke (whose direction is principally determined by $\bar{w}$). This point which is of passing interest will not be further pursued here.

The modified Maxwell Equations (1.3), (1.4), (1.5), (1.6) are to be solved in cylindrical coordinates with a soliton Ansatz for the case of no θ dependence. Components in the θ direction will, however, be retained. The expressions for curl and divergence in (r, θ, z) cylindrical coordinates under these conditions are recalled to be

$$\nabla \times \bar{H} = -\bar{u}_r\partial_z H_\theta + \bar{u}_\theta(\partial_z H_r - \partial_r H_z) + \bar{u}_z \frac{\partial_r(rH_\theta)}{r} \tag{1.10}$$

$$\nabla \cdot \bar{H} = \frac{1}{r}\partial_r(rH_r) + \partial_z H_z \tag{1.11}$$

where $\bar{u}_r$, $\bar{u}_\theta$, $\bar{u}_z$ are the unit vectors in the radial, circumferential and axial directions.

By virtue of (1.11), (1.6) may be written

$$\partial_r(rH_r) = \partial_z(-rH_z) \tag{1.12}$$

So that the magnetic intensity components may be expressed in terms of a function $\Psi(r,z,\tau)$ by

$$rH_r = \partial_z\Psi \tag{1.13}$$

$$-rH_z = \partial_r \Psi \tag{1.14}$$

If $\Psi(r,z,\tau) = rh(r)F(z-\xi\tau)$ one has

$$H_r = h(r)F'(z - \xi\tau) \tag{1.15}$$

$$H_z = \frac{-[rh(r)]'}{r}F(z - \xi\tau) \tag{1.16}$$

This suggests an Ansatz for H_θ of the form

$$H_\theta = h(r)H(z - \xi\tau) \tag{1.17}$$

and a corresponding Ansatz for E_z of the form

$$E_z = e(r)E'(z - \xi\tau) \tag{1.18}$$

The cylindrical components of the Faraday Induction Law (1.3) are

$$-\partial_z E_\theta = -\mu\partial_\tau H_r \tag{1.19}$$

$$\partial_z E_r - \partial_r E_z = -\mu\partial_\tau H_\theta \tag{1.20}$$

$$\frac{1}{r}\partial_r(rE_\theta) = -\mu\partial_\tau H_z \tag{1.21}$$

which upon substitution from (1.15), (1.16), (1.17), (1.18) become

$$\partial_z E_\theta = -\mu\xi h(r)F''(z - \xi\tau) \tag{1.22}$$

$$\partial_z E_r - e'(r)E'(z - \xi\tau) = \mu\xi h(r)H'(z - \xi\tau) \tag{1.23}$$

$$\frac{1}{r}\partial_r(rE_\theta) = -\mu\xi \frac{[rh(r)]'}{r}F'(z - \xi\tau) \tag{1.24}$$

Clearly, (1.22) is satisfied by

$$E_\theta = -\mu\xi h(r)F'(z - \xi\tau) \tag{1.25}$$

in which case (1.24) is automatically satisfied. According to (1.23)

$$\partial_z E_r = e'(r)E'(z - \xi\tau) + \mu\xi h(r)H'(z - \xi\tau) \tag{1.26}$$

which is satisfied by

$$E_r = e'(r)E(z - \xi\tau) + \mu\xi h(r)H(z - \xi\tau) \tag{1.27}$$

Collecting the components of electric and magnetic intensities one then has

$$E_z = e(r)E'(z - \xi\tau) \tag{1.28}$$

$$E_\theta = -\mu\xi h(r)F'(z - \xi\tau) \tag{1.29}$$

$$E_r = e'(r)E(z - \xi\tau) + \mu\xi h(r)H(z - \xi\tau) \tag{1.30}$$

$$-H_z = \frac{[rh(r)]'}{r}F(z - \xi\tau) \tag{1.31}$$

$$H_\theta = h(r)H(z-\xi\tau) \tag{1.32}$$

$$H_r = h(r)F'(z -\xi\tau) \tag{1.33}$$

For the generation term G in (1.5) and (1.7) one takes

$$G(r,z,\tau) = \xi g(r)\ \Gamma'\ (z-\xi\tau) \tag{1.34}$$

where Γ is an as yet unspecified soliton function (not the Gamma function). In terms of Γ the integral of G can be written

$$\int_0^\tau G d\tau = [\Gamma(z) - \Gamma(z-\xi\tau)]g(r) \tag{1.35}$$

It should be noted that (1.5) assumes no generation before time $t=0$. Using (1.35) in (1.5) one has

$$\frac{q}{\varepsilon} = \partial_z E_z + \frac{1}{r}\,\partial_r(rE_r) + \frac{Ze}{\varepsilon c}\,g(r)\,[\Gamma(z) - \Gamma(z-\xi\tau)] \tag{1.36}$$

or substituting from (1.28), (1.30)

$$\frac{q}{\varepsilon} = e(r)E''(z-\xi\tau) + \frac{[re'(r)]'}{r}\,E(z-\xi\tau) + \mu\xi\,\frac{[rh(r)]'}{r}\,H(z-\xi\tau)$$

$$+ \frac{Ze}{\varepsilon c}\,g(r)\,[\Gamma(z) - \Gamma(z-\xi\tau)] \tag{1.37}$$

At time $t=0$ this becomes

$$\frac{q}{\varepsilon} = e(r)E''(z) + \frac{[re'(r)]'}{r}\,E(z) + \mu\xi\frac{[rh(r)]'}{r}\,H(z) \tag{1.38}$$

For zero initial charge density $q=0$

$$e(r)E''(z) + \frac{[re'(r)]'}{r}\,E(z) + \mu\xi\frac{[rh(r)]'}{r}\,H(z) = 0 \tag{1.39}$$

and it must be satisfied with r independent of z. This can be accomplished by

$$k^2 e(r) = \frac{(re'(r)]'}{r} = \mu\xi\frac{[rh(r)]'}{r} \tag{1.40}$$

and

$$E''(z) = k^2 E(z) + k^2 H(z) = 0 \tag{1.41}$$

The first equation of (1.40) implies

$$e''(r) + \frac{e'(r)}{r} - k^2 e(r) = 0 \tag{1.42}$$

which is satisfied by the modified Bessel functions $I_o(kr)$ and $K_o(kr)$ of which $K_o(kr)$ exhibits exponential decay as $r \to \infty$ while $I_o(kr)$ becomes exponentially infinite as $r \to \infty$. Thus $K_o(kr)$ is the better solution and one may take

$$e(r) = A\, K_o(kr) \tag{1.43}$$

The second equation of (1.40) is satisfied by the choice

$$\mu\xi h(r) = e'(r) = AkK_o'(kr) \tag{1.44}$$

while (1.41) is satisfied by

$$-k^2H(z) = E''(z) + k^2E(z) \tag{1.45}$$

Substituting these results into (1.28), (1.29), (1.30), (1.31), (1.32), (1.33) one has

$$E_z = AK_o(kr)E'(z-\xi\tau) \tag{1.46}$$

$$E_\theta = -AkK_o'(kr)F'(z-\xi\tau) \tag{1.47}$$

$$E_r = \frac{-A}{k}\, K_o'(kr)E''(z-\xi\tau) \tag{1.48}$$

$$H_z - \frac{-Ak^2}{\mu\xi}K_o(kr)F(z-\xi\tau) \tag{1.49}$$

$$H_\theta = \frac{-A}{\mu\xi k}K_o'(kr)\,[k^2E(z-\xi\tau) + E''(z-\xi\tau)] \tag{1.50}$$

$$H_r = \frac{Ak}{\mu\xi}K_o'(kr)F'(z-\xi\tau) \tag{1.51}$$

The components of the Ampere Circuital Law (1.4) are

$$J_z/c = \frac{1}{r}\partial_r(rH_\theta) - \varepsilon\partial_\tau E_z \tag{1.52}$$

$$J_\theta/c = (\partial_z H_r - \partial_r H_z) - \varepsilon\partial_\tau E_\theta \tag{1.53}$$

$$J_r/c = -\partial_z H_\theta - \varepsilon\partial_\tau E_r \tag{1.54}$$

so that substitution from (1.28), (1.29), (1.30), (1.31), (1.32), (1.33) yields

$$J_z/c = \frac{[rh(r)]'}{r} H(z - \xi\tau) + \varepsilon\xi e(r)E''(z - \xi\tau) \tag{1.55}$$

$$J_\theta/c = h(r)F''(z - \xi\tau) + \left[\frac{[rh(r)]'}{r}\right]' F(z - \xi\tau)$$

$$- \varepsilon\mu\xi^2 h(r)F''(z - \xi\tau) \tag{1.56}$$

$$J_r/c = -h(r)H'(z - \xi\tau) + \varepsilon\xi[e'(r)E'(z - \xi\tau) + \mu\xi h(r)H'(z - \xi\tau)] \tag{1.57}$$

for the current densities.

Substitution from (1.43), (1.44) and (1.45) then yields

$$J_z/c = \frac{-AK_o(kr)}{\mu\xi} [k^2E(z-\xi\tau) + (1-\varepsilon\mu\xi^2)E''(z-\xi\tau)] \tag{1.58}$$

$$J_\theta/c = \frac{AK_o'(kr)}{\mu\xi} [k^2F(z-\xi\tau) + (1-\varepsilon\mu\xi^2)F''(z-\xi\tau)] \tag{1.59}$$

$$J_r/c = \frac{AkK_o'(kr)}{\mu\xi} [E'(z-\xi\tau) + \frac{1}{k^2}(1-\varepsilon\mu\xi^2)E'''(z-\xi\tau)] \tag{1.60}$$

while (1.39) applied to (1.37) results in

$$\frac{q}{\varepsilon} = \frac{Ze}{\varepsilon c} [\Gamma(z) - \Gamma(z-\xi\tau)]g(r) \tag{1.61}$$

which for $g(r) = e(r) = AK_o(kr)$ is

$$\frac{q}{\varepsilon} = \frac{AZe}{\varepsilon c} K_o(kr)[\Gamma(z) - \Gamma(z-\xi\tau)] \tag{1.62}$$

The solution might be further simplified by taking $F=0$ if experimental justification excluding circumferential currents could be found. This would leave only $E(z-\xi\tau)$ and $\Gamma(z-\xi\tau)$ in the solution considerably simplifying the correlation of field data with the propagation properties of the lightning stroke. Return strokes are easily handled in the Ansatz by replacing ξ by $-\xi$. The pulse speed ξ is known to be a proper fraction of light speed. Another result $F=E'$ is obtained if the circumferential component of the Poynting vector is set equal to zero.

It seems clear physically that a constant pulse shape can hardly be maintained indefinitely as it propagates along its ionization channel. However, it does not follow that in the times considered the pulse shape will change very much and in any case it is primarily important at this stage to discern the way in which the various vectorial components depend upon the E and Γ functions. In this way it is hoped that measurements will eventually clarify the entire phenomenological sequence of events of the stroke.

2. Plasma Dynamics and the Non-Stationary Generalized Ohm's Law

The purpose of this section is to systematically construct the dynamical, electrical, and thermal equations describing the behavior of a typical portion of plasma containing positive and negative ions (including electrons) and neutral particles <u>entirely free of mathematical approximations</u>. This is to be effected by means of suitably defined averages. The electrical equations are then used to establish (2.54) the Non-Stationary Ohm's Law relating electrical field E to current density J.

The formulation will include effects of partial pressures due to + and - ions and neutral particles, external electromagnetic fields, a gravitational potential field and viscosity terms.

There will naturally still remain physical approximations and statistical assumptions concerning the extent to which such averages can reasonably represent the parameters entering into the formulation.

The restriction here to an isotropic model is not serious since the pressures may be readily generalized to stresses as in the transition from an ideal fluid to an anisotropic viscous fluid. The tensor extensions possible in the model are generally easily handled and will be ignored here.

First relevant averaging parameters are defined and then equations for mass, charge, momentum, and thermal transport and generation are established.

2a. Definitions and Formulations

The notation adopted associates a + subscript with positive ions, a - subscript with negative ions (including electrons) and a zero subscript with neutrals. The ions are supposed to be of valences Z_+ and Z_-.

We define

$$n = \underbrace{n_+}_{\text{ions}^+} + \underbrace{n_-}_{\text{ions}^-} + \underbrace{n_o}_{\text{neutrals}} = \text{number density} \tag{2.1}$$

$$q = (n_+Z_+ - n_-Z_-)e = \text{charge density} \tag{2.2}$$

$$\rho = n_+m_+ + n_-m_- + n_om_o = \text{mass density} \tag{2.3}$$

$$\bar{J}_m = n_+m_+\bar{v}_+ \; n_-m_-\bar{v}_- + n_om_o\bar{v}_o = \rho\bar{v} \tag{2.4}$$

mass current density (defines v)
(also equal to momentum density)

$$\bar{J} = Z_+en_+\bar{v}_+ - Z_-en_-\bar{v}_- = \text{electrical current density} \tag{2.5}$$

so that

$$\partial_t\rho = m_+\partial_tn_+ + m_-\partial_tn_- + m_o\partial_tn_o \tag{2.6}$$

$$\nabla \cdot \bar{J}_m = m_+\nabla\cdot(n_+\bar{v}_+) + m_-\nabla\cdot(n_-\bar{v}_-) + m_o\nabla\cdot(n_o\bar{v}_o) \tag{2.7}$$

$$\partial_tq = Z_+e\partial_tn_+ - Z_-e\partial_tn_- \tag{2.8}$$

$$\nabla \cdot \bar{J} = Z_+e\nabla\cdot(n_+\bar{v}_+) - Z_-e\nabla\cdot(n_-\bar{v}_-) \tag{2.9}$$

We assume no net mass generation and no net charge generation

$$\underbrace{\partial_t\rho + \nabla \cdot \bar{J}_m = 0}_{\text{No net mass generation}} = \underbrace{\partial_tq + \nabla \cdot \bar{J}}_{\text{No net charge generation}} \tag{2.10}$$

With G_+, G_-, G_o the generation rate for + ions, - ions and neutrals (the latter is initially negative and finally positive in a transient ionization-recombination sequence) one has

$$\partial_tn_+ + \nabla \cdot (n_+\bar{v}_+) = G_+ \quad \text{generation rate of + ions} \tag{2.11}$$

$$\partial_tn_- + \nabla \cdot (n_-\bar{v}_-) = G_- \quad \text{generation rate of - ions (incl. electrons)} \tag{2.12}$$

$$\partial_tn_o + \nabla \cdot (n_o\bar{v}_o) = G_o \quad \text{generation rate of neutrals} \tag{2.13}$$

so that the ionic generation rates are related to the neutral rate as shown below (2.16) (2.17)

$$m_+G_+ + m_-G_- + m_oG_o = 0 \quad \text{no net mass generation} \tag{2.14}$$

$$Z_+G_+ - Z_-G_- = 0 \quad \text{no net charge generation} \tag{2.15}$$

Consequently,

$$G_+ = \frac{-m_o Z_-}{m_-Z_+ + m_+Z_-} \qquad G_o = \text{number of positive ions generated per unit volume per unit time} \tag{2.16}$$

$$G_- = \frac{-m_oZ_+}{m_-Z_+ + m_+Z_-} \qquad G_o = \text{number of negative ions (or electrons) generated per unit volume per unit time} \tag{2.17}$$

The electromagnetic fields are regarded as externally imposed, it being supposed that the particle motions are sufficiently random as to produce effective cancellation of the fields generated by individual particles except when their motions become so highly correlated as to produce well defined resultant fields.

The Eulerian derivative $D_t = \partial_t + \bar{v} \cdot \nabla$ of the mass current density $\bar{J}_m$ (mass/area time = momentum/vol.) and of the electrical current density $\bar{J}$ are given by (2.18) and (2.22)

$$D_t \bar{J}_m = m_+D_t(n_+\bar{v}_+) + m_-D_t(n_-\bar{v}_-) + m_oD_t(n_o\bar{v}_o) \tag{2.18}$$

$$\bar{J} \times \bar{B} = \bar{J}_+ \times \bar{B} - \bar{J}_- \times \bar{B} \tag{2.19}$$

$$\bar{J}_+ = en_+Z_+\bar{v}_+ \tag{2.20}$$

$$\bar{J}_- = en_-Z_-\bar{v}_- \tag{2.21}$$

$$D_t\bar{J} = eZ_+D_t(n_+\bar{v}_+) - eZ_-D_t(n_-\bar{v}_-) \tag{2.22}$$

By solving for $\bar{v}_+$ and $\bar{v}_-$ in the expressions for the two current densities $\bar{J}$ and J_m these quantities may be expressed in terms of them and of the velocity of neutral particles $\bar{v}_o$ as shown in (2.26) and (2.28)

$$m_+ \qquad \left\{ \begin{array}{l} en_+Z_+\bar{v}_+ - en_-Z_-\bar{v}_- = \bar{J} \\ \\ n_+m_+\bar{v}_+ + n_-m_-\bar{v}_- = \bar{J}_m - n_om_o\bar{v}_o \end{array} \right\} \qquad \begin{array}{l} m_- \\ \\ eZ_- \end{array} \qquad \begin{array}{r} (2.23) \\ \\ (2.24) \end{array}$$

$$en_-(Z_+m_- + Z_-m_+)\bar{v}_- = eZ_+\bar{J}_m - m_+\bar{J} - en_om_oZ_+\bar{v}_o \tag{2.25}$$

$$\bar{v}_- = \frac{eZ_+\bar{J}_m - m_+\bar{J} - en_om_oZ_+\bar{v}_o}{en_-(Z_+m_- + Z_-m_+)} \tag{2.26}$$

$$en_+(Z_+m_- + Z_-m_+)\bar{v}_+ = eZ_-\bar{J}_m + m_-\bar{J} - en_om_oZ_-\bar{v}_o \tag{2.27}$$

$$\bar{v}_+ = \frac{eZ_-\bar{J}_m + m_-\bar{J} - en_om_oZ_-\bar{v}_o}{en_+(Z_+m_- + Z_-m_+)} \tag{2.28}$$

$$m_-n_-\bar{v}_- = \frac{[eZ_+\bar{J}_m - m_+\bar{J} - en_om_oZ_+\bar{v}_o\ ]m_-}{e(Z_+m_- + Z_-m_+)} \tag{2.29}$$

$$m_+n_+\bar{v}_+ = \frac{[eZ_-\bar{J}_m + m_-\bar{J} - en_om_oZ_-\bar{v}_o\ ]m_+}{e(Z_+m_- + Z_-m_+)} \tag{2.30}$$

The momentum densities for the 3 species satisfy the equations

$$D_t(n_+m_+\bar{v}_+) = en_+Z_+\bar{E} + en_+Z_+\left(\frac{\bar{v}_+}{c}\right) \times \bar{B} - \nabla p_+ - n_+m_+\nabla\phi + \bar{T}_+$$
$$+ \eta_+\Delta\bar{v}_+ + \frac{\eta_+}{3}\nabla(\nabla \cdot \bar{v}_+) \tag{2.31}$$

$$D_t(n_-m_-\bar{v}_-) = -en_-Z_-\bar{E} - en_-Z_-\left(\frac{\bar{v}_-}{c}\right) \times \bar{B} - \nabla p_- - n_-m_-\nabla\phi + \bar{T}_-$$
$$+ \eta_-\Delta\bar{v}_- + \frac{\eta_-}{3}\nabla(\nabla \cdot \bar{v}_-) \tag{2.32}$$

$$D_t(n_om_o\bar{v}_o) = \eta_o\Delta\bar{v}_o + \frac{\eta_o}{3}\nabla(\nabla \cdot \bar{v}_o) - \nabla p_o - n_om_o\nabla\phi + \bar{T}_o \tag{2.33}$$

The terms $\bar{T}_+$, $\bar{T}_-$, $\bar{T}_o$ represent momentum transfer rates per unit volume to + ions, - ions and neutrals due to collisions. ϕ is gravitational potential and η_+, η_- and η_o are viscosities. Addition yields

$$D_t\bar{J}_m = D_t(\rho\bar{v}) = (\bar{J} \times \bar{B})/c - \nabla p - \rho\nabla\phi + 3\eta_A\Delta < \bar{v} > + \eta_A(\nabla \cdot <\bar{v}>) \qquad (2.34)$$

under the assumption that the net tranfer of momentum for the whole system is zero. Thus $\bar{T}_+ + \bar{T}_- + \bar{T}_o = \bar{0}$. <v> is a "viscous average" velocity defined by

$\eta_+\bar{v}_+ + \eta_-\bar{v}_- + \eta_o\bar{v}_o = 3\eta_A<\bar{v}>$ with η_A arithmetic mean viscosity.

The electrical current densities for the ions satisfy

$$D_t(en_+Z_+\bar{v}_+) = \frac{e^2n_+Z_+^2}{m_+}\bar{E} + \frac{e^2n_+Z_+^2}{m_+c}(\bar{v}_+ \times \bar{B}) - \frac{eZ_+}{m_+}\nabla p_+ - en_+Z_+\nabla\phi$$
$$+ \frac{eZ_+}{m_+}\eta_+\Delta\bar{v}_+ + \frac{eZ_+}{3m_+}\eta_+\nabla(\nabla \cdot \bar{v}_+) + \frac{eZ_+T_+}{m_+} \qquad (2.35)$$

$$D_t(en_-Z_-\bar{v}_-) = \frac{-e^2n_-Z_-^2}{m_-}\bar{E} - \frac{e^2n_-Z_-^2}{m_-c}(\bar{v}_-x_-\bar{B}) - \frac{eZ_-}{m_-} - \nabla p_- - en_-Z_-\nabla\phi$$
$$+ \frac{eZ_-}{m_-}\eta_-\Delta\bar{v}_- + \frac{eZ_-}{3m_-}\eta_-\nabla(\nabla \cdot \bar{v}_-) + \frac{eZ_-}{m_-}\bar{T}_- \qquad (2.36)$$

These may be subtracted to yield

$$D_t\bar{J} = e^2\left(\frac{n_+Z_+^2}{m_+} + \frac{n_-Z_-^2}{m_-}\right)\bar{E} + \frac{\alpha e^2Z_+Z_-}{c}\left(\frac{Z_+}{m_+} + \frac{Z_-}{m_-}\right)\bar{J}_m \times \bar{B}$$
$$+ \frac{\alpha e}{c}\left(\frac{Z_+^2m_-}{m_+} - \frac{Z_-^2m_+}{m_-}\right)\bar{J} \times \bar{B} - \frac{\alpha e^2Z_+Z_-n_om_o}{c}\left(\frac{Z_+}{m_+} + \frac{Z_-}{m_-}\right)\bar{v}_o \times \bar{B}$$
$$+ \frac{eZ_-}{m_-}\nabla p_- - \frac{eZ_+}{m_+}\nabla p_+ + e(n_-Z_- - n_+Z_+)\nabla\phi + e\left(\frac{Z_+}{m_+}\bar{T}_+ - \frac{Z_-}{m_-}\bar{T}_-\right)$$
$$+ e\Delta\left[\frac{Z_+n_+\bar{v}_+}{m_+} - \frac{Z_-n_-\bar{v}_-}{m_-}\right] + \frac{e}{3}\nabla\left(\nabla \cdot \left[\frac{Z_+n_+\bar{v}_+}{m_+} - \frac{Z_-n_-\bar{v}_-}{m_-}\right]\right) \qquad (2.37)$$

where α is defined by $\alpha(Z_+m_- + Z_-m_+) = 1$. A resistivity η may be defined in terms of the momentum transfer terms for ions thus

$$- e \left(\frac{Z_+}{m_+} \bar{T}_+ - \frac{Z_-}{m_-} \bar{T}_-\right) = e^2\left(\frac{n_+Z_+^2}{m_+} + \frac{n_-Z_-^2}{m_-}\right) \eta\bar{J} \tag{2.38}$$

η = resistivity (not to be confused with viscosities η_+, η_-, η_o)

By introducing the following means, the electrical current density equation can be made more concise

$$m_G^2 = m_-m_+ \tag{2.39}$$

$$m_A = \left(\frac{Z_+m_+ + Z_-m_-}{Z_+ + Z_-}\right) \tag{2.40}$$

$$\langle Z\rangle = \left(Z_+ + Z_-\right)/2 \tag{2.41}$$

$$\langle Z^2\rangle = \left(Z_+^2 + Z_-^2\right)/2 \tag{2.42}$$

$$\frac{1}{m_H} = \left(\frac{n_+Z_+^2}{m_+} + \frac{n_-Z_-^2}{m_-}\right) /(n_+Z_+^2 + n_-Z_-^2) \tag{2.43}$$

$$n_A = (n_+Z_+^2 + n_-Z_-^2)/(Z_+^2 + Z_-^2) \tag{2.44}$$

$$\frac{2 < Z^2 > n_A}{m_H} = \frac{n_+Z_+^2}{m_+} + \frac{n_-Z_-^2}{m_-} \tag{2.45}$$

$$Z_G^2 = Z_-Z_+ \tag{2.46}$$

$$\alpha = \frac{1}{(Z_+m_- + Z_-m_+)} \tag{2.47}$$

$$Z_\delta = \frac{Z_+^2 m_-}{m_+} - \frac{Z_-^2 m_+}{m_-} \tag{2.48}$$

Δ = Laplacian operator, ∇ = del operator

In these terms, one has

$$D_t \bar{J} = \frac{2e^2 < Z^2 > n_A}{m_H} (\bar{E} - \eta \bar{J}) + \frac{e^2 Z_G^2 \rho}{m_G^2 c} (\bar{v} \times \bar{B}) + \frac{\alpha e Z_\delta^2}{c} \bar{J} \times \bar{B}$$

$$- \frac{e^2 Z_G^2 n_o m_o}{m_G^2 c} (\bar{v}_o \times \bar{B}) + e\left(\frac{Z_-}{m_-} \nabla p_- - \frac{Z_+}{m_+} \nabla p_+\right) + e(n_- Z_- - n_+ Z_+)\nabla\phi$$

$$+ e\Delta \left[\frac{Z_+ n_+ \bar{v}_+}{m_+} - \frac{Z_- n_- \bar{v}_-}{m_-}\right] + \frac{e}{3} \nabla\left(\nabla \cdot \left[\frac{Z_+ n_+ \bar{v}_+}{m_+} - \frac{Z_- n_- \bar{v}_-}{m_-}\right]\right) \tag{2.49}$$

Defining the conductivity σ by

$$\sigma = \frac{1}{\eta} \tag{2.50}$$

and the electrical collisional frequency Γ by

$$\Gamma = \frac{2e^2 < Z^2 > n_A}{m_H} \eta \tag{2.51}$$

and introducing the vectorial cyclotron frequencies

$$\bar{\Omega}_+ = - \frac{Z_+ e \bar{B}}{m_+ c} \tag{2.52}$$

$$\bar{\Omega}_- = \frac{Z_- e \bar{B}}{m_- c} \tag{2.53}$$

The second and third terms may be replaced so that

$$D_t\bar{J} = \Gamma(\sigma\bar{E} - \bar{J}) + \frac{eZ_-m_+n_+}{m_-}(\bar{\Omega}_+ \times \bar{v}_+) - \frac{eZ_+m_-n_-}{m_+}(\bar{\Omega}_- \times \bar{v}_-)$$

$$+ \frac{\alpha Z_\delta^2}{2}\left[\frac{m_-}{Z_-}(\bar{J} \times \bar{\Omega}_-) - \frac{m_+}{Z_+}(\bar{J} \times \bar{\Omega}_+)\right] + e\left(\frac{Z_-}{m_-}\nabla p_- - \frac{Z_+}{m_+}\nabla p_+\right)$$

$$+ e(n_-Z_- - n_+Z_+)\nabla\phi + e\Delta\left(\frac{Z_+n_+\bar{v}_+}{m_+} - \frac{Z_-n_-\bar{v}_-}{m_-}\right)$$

$$+ \frac{e}{3}\nabla\left[\nabla \cdot \left(\frac{Z_+n_+\bar{v}_+}{m_+} - \frac{Z_-n_-\bar{v}_-}{m_-}\right)\right] \tag{2.54}$$

The simplest non-stationary case is with $B = 0$ and negligible viscosity, pressure and gravitational gradients.

Then (2.54) becomes

$$D_t\bar{J} = \Gamma(\sigma\bar{E} - \bar{J}) \tag{2.55}$$

as the generalized Ohm's Law which reduces to $\bar{E} = \eta\bar{J}$ or $\bar{J} = \sigma\bar{E}$ for

$$D_t\bar{J} = \bar{0} = \bar{B} = \nabla\phi \quad \text{and} \quad \frac{Z_-}{m_-}\nabla p_- = \frac{Z_+}{m_+}\nabla p_+ \quad \text{with} \quad n_+ = n_- = n_o = 0 \tag{2.56}$$

(2.55) will be considered in more detail at the end of this section.

In terms of g the vectorial acceleration due to gravity (2.54) can also be written

$$\bar{J} = \sigma\bar{E} - \frac{m_H\sigma}{2e^2 < Z^2 > n_A}D_t\bar{J} + \frac{m_HZ_G^2\rho\sigma}{2 < Z^2 > n_Am_G^2c}(\bar{v} \times \bar{B})$$

$$+ \frac{\alpha m_HZ_\delta^2\sigma}{2ce<Z^2>n_A}\bar{J} \times \bar{B} - \frac{Z_G^2n_om_om_H\sigma}{2 < Z^2 > n_Am_G^2c}(\bar{v}_o \times \bar{B})$$

$$+ \frac{m_H\sigma}{2e < Z^2 > n_A}\left(\frac{Z_-}{m_-}\nabla p_- - \frac{Z_+}{m_+}\nabla p_-\right) + \frac{m_H\sigma}{2e < Z^2 > n_A}(n_-Z_- - n_+Z_+)\bar{g}$$

$$+ \frac{m_H \sigma}{2e < Z^2 > n_A} \Delta \quad \frac{Z_+ n_+ \bar{v}_+}{m_+} - \frac{Z_- n_- \bar{v}_-}{m_-}$$

$$- \frac{m_H \sigma}{6e < Z^2 > n_A} \nabla \quad \nabla \cdot \quad \frac{Z_+ n_+ \bar{v}_+}{m_+} - \frac{Z_- n_- \bar{v}_-}{m_-} \tag{2.57}$$

The heat transport equations consider Joule heating, viscous heating and radiation. Thus in general one has (c_p = specific heat/mass at constant pressure)

$$\rho c_p \partial_t T + \nabla \cdot \bar{J}_\theta = \bar{E} \cdot \bar{J}/c + n_+ \bar{v}_+ \cdot \Delta \bar{v}_+ + \frac{n_+ (\bar{v}_+ \cdot \nabla)(\nabla \cdot \bar{v}_+)}{3}$$

$$+ n_- \bar{v}_- \cdot \Delta \bar{v}_- + \frac{n_- (\bar{v}_- \cdot \nabla)(\nabla \cdot \bar{v}_-)}{3}$$

$$+ n_o \bar{v}_o \cdot \Delta \bar{v}_o + \frac{n_o (\bar{v}_o \cdot \nabla)(\nabla \cdot \bar{v}_o)}{3}$$

$$+ S \tag{2.58}$$

S is the energy radiated per unit volume per unit time

where J_θ, the thermal current density is taken to be proportional to the temperature gradient

$$\bar{J}_\theta = - \kappa \nabla T \tag{2.59}$$

It is understood that J is obtained from the generalized Ohm's Law. Hence, one may write for the 3 species of particle

$$n_+ m_+ c_+ \partial_t T_+ - \nabla \cdot (\kappa_+ \nabla T_+) = e n_+ Z_+ (\bar{E} \cdot \bar{v}_+)/c + \bar{n}_+ v_+ \cdot \Delta \bar{v}_+ + \frac{n_+}{3} (\bar{v}_+ \cdot \nabla)(\nabla \cdot \bar{v}_+) + S_+ \tag{2.60}$$

$$n_- m_- c_- \partial_t T_- - \nabla \cdot (\kappa_- \nabla T_-) = -e n_- Z_- (\bar{E} \cdot \bar{v}_-)/c \; n_- \bar{v}_- \cdot \Delta \bar{v}_- + \frac{n_-}{3} (\bar{v}_- \cdot \nabla)(\nabla \cdot \bar{v}_-) + S_- \tag{2.61}$$

$$n_o m_o c_o \partial_t T_o - \nabla \cdot (\kappa_o \nabla T_o) = \eta_o \bar{v}_o \cdot \Delta \bar{v}_o + \frac{\eta_o}{3} (\bar{v}_o \cdot \nabla)(\nabla \cdot \bar{v}_o) + S_o \tag{2.62}$$

where T_+, T_- and T_o are the temperatures of + ions, - ions and neutral particles.

A plasma temperature may be defined by

$$\frac{n_+ m_+ c_+ T_+ + n_- m_- c_- T_- + n_o m_o c_o T_o}{n_+ m_+ c_+ + n_- m_- c_- + n_o m_o c_o} = T \tag{2.63}$$

Then a specific heat at constant pressure c_y may be defined by

$$n_+ m_+ c_+ + n_- m_- c_- + n_o m_o c_o = \rho c_p \tag{2.64}$$

where

$$n_+ m_+ + n_- m_- + n_o m_o = \rho \tag{2.65}$$

Hence

$$n_+ m_+ c_+ T_+ + n_- m_- c_- T_- + n_o m_o c_o T_o = \rho c_p T \tag{2.66}$$

Diffusion coefficients may be defined by

$$n_+ m_+ c_+ D_+ = \kappa_+, \quad n_- m_- c_- D_- = \kappa_-, \quad n_o m_o c_o D_o = \kappa_o \tag{2.67}$$

in terms of the thermal conductivities κ_+, κ_-, κ_o for the 3 species.

Since diffusion coefficients in general are obtained from a random walk model with mean free time τ and mean free path λ one has

$$D = \frac{\lambda v}{3} \qquad \text{where} \quad v = \frac{\lambda}{\tau} \tag{2.68}$$

Hence

$$D = \frac{v^2 \tau}{3} \tag{2.69}$$

and since

$$\frac{mv^2}{2} = \frac{3}{2} kT \tag{2.70}$$

one obtains

$$D = \frac{kT\tau}{m} = \frac{kT}{m\gamma} \tag{2.71}$$

with collisional frequency γ. Substituting this result for each species into the expression for $\rho c_p kT$ one has

$$\gamma_+ n_+ m_+^2 c_+ D_+ + \gamma_- n_- m_-^2 c_- D_- + \gamma_o n_o m_o^2 c_o D_o = \rho c_p kT \tag{2.72}$$

if $\langle m \rangle$ denotes the arithmetic mean mass with

$$3 \langle m \rangle = m_+ + m_- + m_o \tag{2.73}$$

one can define a statistical average collisional frequency $\langle \gamma \rangle$ by

$$\langle \gamma \rangle = \frac{\gamma_+ n_+ m_+^2 c_+ + \gamma_- n_- m_-^2 c_- + \gamma_o n_o m_o^2 c_o}{\langle m \rangle (n_+ m_+ c_+ + n_- m_- c_- + n_o m_o c_o)} \tag{2.74}$$

or

$$\gamma_+ n_+ m_+^2 c_+ + \gamma_- n_- m_-^2 c_- + \gamma_o n_o m_o^2 c_o = \langle \gamma \rangle \langle m \rangle \rho c_p \tag{2.75}$$

so that an average diffusion coefficient $\langle D \rangle$ becomes

$$\langle D \rangle = \frac{\gamma_+ n_+ m_+^2 c_+ D_+ + \gamma_- n_- m_-^2 c_- D_- + \gamma_o n_o m_o^2 c_o D_o}{\langle \gamma \rangle \langle m \rangle \rho c_p} \tag{2.76}$$

Hence the expression for $\rho c_p kT$ yields

$$\langle D \rangle = \frac{kT}{\langle m \rangle \langle \gamma \rangle} \tag{2.77}$$

Returning now to consideration of the Non-Stationary Ohm's Law (2.55) and approximating the Eulerian derivative $D_t\bar{J}$ by $\partial_t\bar{J}$ one has

$$\partial_t\bar{J} = \Gamma(\sigma\bar{E} - \bar{J}) \tag{2.78}$$

Then (1.7) can be partially differentiated with respect to time to yield

$$\partial_t^2 q + \nabla\cdot(\partial_t\bar{J}) = Ze\partial_t G \tag{2.79}$$

into which (2.78) may be substituted to yield

$$\partial_t^2 q + \Gamma\sigma\nabla\cdot\bar{E} - \Gamma\nabla\cdot\bar{J} = Ze\partial_t G \tag{2.80}$$

Substitution from (1.5) and (1.7) then yields

$$\partial_t^2 q + \Gamma\partial_t q + \frac{\Gamma\sigma}{\varepsilon} q = \Gamma ZeG + Ze\partial_t G + \frac{\Gamma\sigma Ze}{\varepsilon c}\int_0^t G dt \tag{2.81}$$

in which the charge generation terms act as forcing functions in the above inhomogeneous equation. For $G=0$ one obtains the corresponding homogeneous equation

$$\partial_t^2 q_H + \Gamma\partial_t q_H + \frac{\Gamma\sigma}{\varepsilon} q_H = 0 \tag{2.82}$$

which is satisfied by

$$q_H = q_1 e^{\lambda t} \tag{2.83}$$

for

$$\lambda^2 + \Gamma\lambda + \frac{\Gamma\sigma}{\varepsilon} = 0 \tag{2.84}$$

which is seen to have no roots with negative real part since

$$2\lambda = -\Gamma \pm \sqrt{\Gamma^2 - \frac{4\Gamma\sigma}{\varepsilon}} \tag{2.85}$$

corresponding to exponential decay or oscillation under an exponentially decaying envelope in (2.83) depending upon whether

$$\Gamma \geq \frac{4\sigma}{\varepsilon} \tag{2.87}$$

or

$$\Gamma < \frac{4\sigma}{\varepsilon} \tag{2.88}$$

In any case the argument indicates that decay of charge is inevitable unless $G \neq 0$. Thus nonvanishing charge generation is a <u>sine qua non</u> for an "exponentially" increasing charge density.

In the next section examples of increasing charge will be derived without neglecting the non-linear term in (2.55).

3. Lightning Stroke Initiation Problem

The purpose of the section is to attempt to characterize the mathematical features of the initiation problem in sufficient detail to clarify the physical theory of the phenomenon and provide a framework for the design of experiments and observations of the development of lightning discharges.

The starting point is the set of Maxwell equations modified by inclusion of a charge generation term G which is a function of position $\bar{R}$ and time $\tau = ct$ signifying the number of charges (of valence Z) generated per unit volume per unit time.

$$\nabla \times \bar{E} = -\mu\partial \bar{H}_\tau \tag{3.1}$$

$$\bar{J} = c(\nabla \times \bar{H} - \varepsilon\partial \bar{E}_\tau) \tag{3.2}$$

$$q = \varepsilon\nabla\cdot\bar{E} + \frac{Ze}{c}\int_0^\tau G d\tau \tag{3.3}$$

$$\nabla\cdot\bar{H} = 0 \tag{3.4}$$

Elimination of $\bar{E}$ between the time derivative of (3.3) and the divergence of (3.2) leads to

$$\partial_t q + \nabla\cdot\bar{J} = ZeG \tag{3.5}$$

In this q is meant to be the absolute quantity of a single-signed charge per unit volume while J is the corresponding current density and G refers only to the absolute number of correspondingly charged particles generated per unit volume per unit time. In the same way E and H fields are generated by these (single-signed) charges. Hence, (3.5) typefies one of the two equations

$$\partial_t q_+ + \nabla\cdot\bar{J}_+ = ZeG_+ \tag{3.6}$$

$$\partial_t q_- + \nabla\cdot\bar{J}_- = ZeG_- \tag{3.7}$$

with the actual (signed) charge density q and actual current density $\bar{J}$ given by

$$q = q_+ - q_- \tag{3.8}$$

$$\bar{J} = \bar{J}_+ - \bar{J}_- \tag{3.9}$$

but it will be more concise to simply deal with (3.5) recalling that with this specification both q_+ and q_- are positive.

Integrating (3.5) over a simply-connected volume V bounded by a surface S the Gauss Divergence Theorem yields

$$\langle\partial_t q\rangle_V + \frac{S}{V}\langle\bar{n}\cdot\bar{J}\rangle_S = Ze\,\langle G\rangle_V \tag{3.10}$$

where $\langle\ \rangle_V$ and $\langle\ \rangle_S$ indicate volume and surface averages respectively. One notes that for $\langle G\rangle_V = 0$

$$\langle\partial_t q\rangle_V = \frac{-S}{V}\langle\bar{n}\cdot\bar{J}\rangle_S \tag{3.11}$$

so that it is not possible to have an increase in the absolute quantity of charge in a region together with an efflux of current unless $\langle G\rangle_V$ is positive. The normal vector n is taken to point outward (away from V) on S.

To the above equations one also adds the Non-Stationary Ohm's Law previously derived.

$$D_t\bar{J} = \Gamma(\sigma\bar{E} - \bar{J}) \tag{3.12}$$

which may also be written

$$q\partial_t\bar{J} + (\bar{J}\cdot\nabla)\bar{J} = q\Gamma\sigma\bar{E} - q\Gamma\bar{J} \tag{3.13}$$

If a solution of this partial differential equation be sought which is separable in q, $\bar{J}$ and $\bar{E}$ one has

$$q(\bar{R}, \tau) = \hat{q}(\bar{R})\alpha(\tau) \tag{3.14}$$

$$\bar{J}(\bar{R}, \tau) = \hat{J}(\bar{R})\beta(\tau) \tag{3.15}$$

$$\bar{E}(\bar{R}, \tau) = \hat{E}(\bar{R})\gamma(\tau) \tag{3.16}$$

(3.13) becomes

$$c\hat{q}\hat{J}\alpha\beta' + \beta^2(\hat{J}\cdot\nabla)\hat{J} = \hat{q}\hat{E}\Gamma\sigma\alpha\gamma - \hat{q}\hat{J}\Gamma\alpha\beta \tag{3.17}$$

or

$$\hat{q}\hat{J}(\Gamma\beta+c\beta')\alpha + \beta^2(\hat{J}\cdot\nabla)\hat{J} = \hat{q}\hat{E}\Gamma\sigma\alpha\gamma \tag{3.18}$$

which can be satisfied separately in time and position provided

$$(\beta + \frac{c}{\Gamma}\beta')\alpha = \beta^2 = \alpha\gamma \tag{3.19}$$

and (absorbing constant factors in $\hat{q}$, $\hat{J}$, $\hat{E}$)

$$\Gamma\hat{q}\hat{J} + (\hat{J}\cdot\nabla)\hat{J} = \hat{q}\hat{E}\Gamma\sigma \tag{3.20}$$

In other words

$$\gamma = \beta + \frac{c}{\Gamma}\beta' \tag{3.21}$$

$$\alpha = \frac{\Gamma\beta^2}{\Gamma\beta + c\beta'} \tag{3.22}$$

For a temporally increasing current density $\beta' > 0$ and (3.21) indicates that the field will increase faster than the current density. Also

$$\gamma' = \beta' + \frac{c}{\Gamma}\beta'' \tag{3.23}$$

For

$$\beta = (e^{\lambda\tau} - 1)^k = W^k \tag{3.24}$$

with $W' = \lambda(W + 1)$

$$\beta' = kW^{k-1}W' = \lambda k(W^k + W^{k-1}) \tag{3.25}$$

$$\beta'' = \lambda k(kW^{k-1} + (k-1)W^{k-2})\lambda(W+1)$$

or

$$\beta'' = \lambda^2 k[kW^k + (2k-1)W^{k-1} + (k-1)W^{k-2}] \tag{3.26}$$

whence

$$\gamma = \left(1 + \frac{\lambda ck}{\Gamma}\right) W^k + \frac{\lambda ck}{\Gamma} W^{k-1} \tag{3.27}$$

$$\gamma' = \lambda k \left[(1 + \frac{\lambda ck}{\Gamma})W^k + \left[1 + \frac{\lambda c(2k-1)}{\Gamma}\right] W^{k-1} + \frac{\lambda c(k-1)}{\Gamma} W^{k-2}\right] \tag{3.28}$$

and by (3.22)

$$\alpha = W^{k+1} / \left[\left(1 + \frac{\lambda ck}{\Gamma}\right) W + \frac{\lambda ck}{\Gamma}\right] \tag{3.29}$$

all of which are initially zero.

For times $\tau \gg 1/\lambda$ $(t \gg 1/\lambda c)$ one has the asymptotic formulae for the temporal dependence of current density, electric field, charge and displacement current

$$\beta \sim e^{\lambda k\tau} \tag{3.30}$$

$$\gamma \sim \left(1 + \frac{\lambda ck}{\Gamma}\right) e^{\lambda k\tau} \tag{3.31}$$

$$\alpha \sim e^{\lambda k \tau} / \left(1 + \frac{\lambda k \tau}{\Gamma}\right) \tag{3.32}$$

$$\gamma^1 \sim \lambda k \left(1 + \frac{\lambda c k}{\Gamma}\right) e^{\lambda k \tau} \tag{3.33}$$

so that λk can be chosen sufficiently large that the ratio of displacement to convective current is large.

In the presence of "exponential" forcing from the ionic generation term G both convective current and displacement current will increase but eventually the convective current will be "relatively saturated" with respect to the displacement current which thus provides the physical mechanism for increase of the field to breakdown. It should be clearly understood that the displacement current referred to here is monotonically increasing not oscillatory as in most electromagnetic applications.

After breakdown occurs, it leads to the development of an ionized channel with enhanced conductivity so that it then becomes possible for larger convective current to flow which ultimately "relieves" the high $\bar{E}$ field and the lightning stroke proper is over. The disturbance in charge balance can, however, lead to return strokes and further effects. The constants Γ and λ must be determined experimentally.

Naturally the increase in q, $\bar{J}$ and $\bar{E}$ will not occur everywhere but only in locations subject to (3.20). Applying the Gauss Divergence Theorem to (3.20) one has

$$\Gamma < \hat{q}\hat{J} >_V + \frac{S}{V} < (\hat{J}\cdot\bar{n})\hat{J} >_S = \Gamma < \sigma\hat{q}\hat{E} >_V \tag{3.34}$$

or

$$\frac{S}{V} < (\hat{J}\cdot\bar{n})\hat{J} >_S = \Gamma < \sigma(\hat{q}\hat{E} - \hat{J}) >_V \tag{3.35}$$

which suggests that the larger the discrepancy from the static Ohm's Law the less likely the discharge region is to be spherical since the sphere has the smallest surface S for a given volume.

It should be particularly noted in the arguments of this section that the results (3.24), (3.29), (3.27) and (3.28) for the current density, charge, field and displacement current take full account exactly of the non-linear $(\bar{J}\cdot\nabla)\bar{J}$ term in (3.13) the Non-Stationary Ohm's Law appropriate for this case.

4. The Separate Charge Distributions in a Thundercloud

In this section the statistical properties of the separate + and - charge distributions in a thundercloud will be investigated and relations involving the cloud dipole moment will be established.

Writing (1.5) for the separate charge species one has

$$q_+ = \frac{Z_+ e}{c} \int_0^\tau G_+ d\tau + \varepsilon \nabla \cdot \bar{E}_+ \tag{4.1}$$

$$q_- = \frac{Z_- e}{c} \int_0^\tau G_- d\tau - \varepsilon \nabla \cdot \bar{E}_- \tag{4.2}$$

for the positive and negative <u>absolute</u> charge densities.
Using the Gauss Divergence Theorem this leads to total charges

$$Q_+ = \frac{Z_+ e}{c} \int_0^\tau \int G_+ dV d\tau + \varepsilon \oint \bar{n} \cdot \bar{E}_+ dS \tag{4.3}$$

$$Q_- = \frac{Z_- e}{c} \int_0^\tau \int G_- dV d\tau - \varepsilon \oint \bar{n} \cdot \bar{E}_- dS \tag{4.4}$$

In terms of these (positive) total quantities of positive and negative charge one can define

$$\bar{R}_+ = \frac{1}{Q_+} \quad \bar{R} q_+ dV \tag{4.5}$$

as the <u>center of positive charge</u> and

$$\bar{R}_- = \frac{1}{Q_-} \int \bar{R} q_- dV \tag{4.6}$$

as the <u>center of negative charge</u>.

The rates of change with time of the total charges of the two species is found from (4.3) and (4.4) to be

$$d_t Q_+ = Z_+ e \int G_+ dV + \varepsilon \oint \bar{n} \cdot \partial_t \bar{E}_+ dS \tag{4.7}$$

$$d_t Q_- = Z_- e \int G_- dV - \varepsilon \oint \bar{n} \cdot \partial_t \bar{E}_- dS \tag{4.8}$$

The velocities of the + and - charge centers can be found from (4.5) and (4.6)

$$\dot{\bar{R}}_+ = d_t \bar{R}_+ = \frac{1}{Q_+} \int \bar{R} \partial_t q_+ dV - \bar{R}_+ d_t (\ln Q_+) \tag{4.9}$$

$$\dot{\bar{R}}_- = d_t \bar{R}_- = \frac{1}{Q_-} \int \bar{R} \partial_t q_- dV - \bar{R}_- d_t (\ln Q_-) \tag{4.10}$$

Substituting from the modified continuity equations

$$\partial_t q_+ = Z_+ e G_+ - \nabla \cdot \bar{J}_+ \tag{4.11}$$

$$\partial_t q_- = Z_- e G_- + \nabla \cdot \bar{J}_- \tag{4.12}$$

(4.9) and (4.10) become

$$\dot{\bar{R}}_+ = \frac{Z_+ e}{Q_-} \int \bar{R} G_+ dV - \frac{1}{Q_+} \int \bar{R} (\nabla \cdot \bar{J}_+) dV - \bar{R}_+ d_t (\ln Q_+) \tag{4.13}$$

$$\dot{\bar{R}}_- = \frac{Z_- e}{Q_-} \int \bar{R} G_- dV + \frac{1}{Q_-} \int \bar{R}(\nabla \cdot \bar{J}_-) dV - \bar{R}_- d_t(\ln Q_-) \tag{4.14}$$

so that if V_{++} is the portion of total volume V over which net production of + ions is occurring and V_{+-} is the portion of total volume V over which recombination of + ions is occurring one has $V = V_{++} + V_{+-}$ and one may define

$$\bar{R}^G_{++} = \frac{1}{V_{++}} \int_{V_{++}} \bar{R} |G_+| dV \tag{4.15}$$

$$\bar{R}^G_{+-} = \frac{1}{V_{+-}} \int_{V_{+-}} \bar{R} |G_+| dV \tag{4.16}$$

as the <u>center of generation</u> for + charges and the <u>center of recombination</u> for + charges. Similarly

$$\bar{R}^G_{-+} = \frac{1}{V_{-+}} \int_{V_{-+}} \bar{R} |G_-| dV \tag{4.17}$$

$$\bar{R}^G_{--} = \frac{1}{V_{--}} \int_{V_{--}} \bar{R} |G_-| dV \tag{4.18}$$

are the <u>center of generation</u> for - charges and the <u>center of recombination</u> for - charges. The terms in (4.13) and (4.14) containing the divergences of current densities can be written in terms of surface integrals by the Gauss Divergence Theorem. Thus

$$\int \bar{R}(\nabla \cdot \bar{J}) dV = \oint \bar{R}(\bar{n} \cdot \bar{J}) dS \tag{4.19}$$

so if the portion of S in (4.13) corresponding to current efflux is S_{++} and the portion corresponding to current influx is S_{+-} one can define

$$\bar{R}^I_{++} = \frac{1}{S_{++}} \int_{S_{++}} \bar{R} |(\bar{n} \cdot \bar{J}_+)| dS \tag{4.20}$$

$$\bar{R}^I_{+-} = \frac{1}{S_{+-}} \int_{S_{+-}} \bar{R}\,|(\bar{n}\cdot\bar{J}_+)|\,dS \tag{4.21}$$

as the centers of current efflux and current influx due to positive charge. Similarly in (4.14)

$$\bar{R}^I_{-+} = \frac{1}{S_{-+}} \int_{S_{-+}} \bar{R}\,|(\bar{n}\cdot\bar{J}_-)|\,dS \tag{4.22}$$

$$\bar{R}^I_{--} = \frac{1}{S_{--}} \int_{S_{--}} \bar{R}\,|(\bar{n}\cdot\bar{J}_-)|\,dS \tag{4.23}$$

In terms of (4.15), (4.16), (4.20), (4.21), (4.22), (4.23) one has from (4.13) and (4.14)

$$\dot{\bar{R}}_+ = \frac{Z_+ e}{Q_+}\left(V_{++}\bar{R}^G_{++} - V_{+-}\bar{R}^G_{+-}\right) - \frac{\left(S_{++}\bar{R}^I_{++} - S_{+-}\bar{R}^I_{+-}\right)}{Q_+} - \bar{R}_+ d_t(\ln Q_+) \tag{4.24}$$

$$\dot{\bar{R}}_- = \frac{Z_- e}{Q_-}\left(V_{-+}\bar{R}^G_{-+} - V_{--}\bar{R}^G_{--}\right) + \frac{\left(S_{-+}\bar{R}^I_{-+} - S_{--}\bar{R}^I_{--}\right)}{Q_-} - \bar{R}_- d_t(\ln Q_-) \tag{4.25}$$

Returning now to (4.7) and (4.8) one notes that according to (1.4)

$$\varepsilon \oint \bar{n}\cdot\partial_\tau \bar{E}\, dS = \oint \bar{n}\cdot\nabla \times \bar{H}\, dS - \frac{1}{c}\oint \bar{n}\cdot\bar{J}\, dS$$

and

$$\oint \bar{n}\cdot\nabla \times \bar{H}\, dS = 0 \tag{4.26}$$

over a closed surface so

$$\varepsilon \oint \bar{n}\cdot\partial_t \bar{E}_+ dS = -I_+ \tag{4.27}$$

$$\varepsilon \oint \bar{n}\cdot\partial_t \bar{E}_- dS = -I_- \tag{4.28}$$

where I_+ and I_- are the currents flowing out of V due to + and − charges respectively. Thus (4.7) and (4.8) become

$$\dot{Q}_+ = d_t Q_+ = Z_+ e\dot{N}_+ - I_+ \tag{4.29}$$

$$\dot{Q}_- = d_t Q_- = Z_- e\dot{N}_- + I_- \tag{4.30}$$

where $\dot{N}_+$ and $\dot{N}_-$ are the total number of charges + and − generated in V per unit time.

In case of net charge zero $Q_+ = Q_- = Q$ and the <u>relative velocity</u> of the + and − charge centers is given by

$$\bar{V}_{rel} = \dot{\bar{R}}_+ - \dot{\bar{R}}_- \tag{4.31}$$

so

$$Q\bar{V}_{rel} = \bar{W}^G - \bar{W}^I - Q(\bar{R}_+ - \bar{R}_-)d_t(\ell n\, Q) \tag{4.32}$$

with

$$\bar{W}^G = Z_+ e(V_{++}\bar{R}^G_{++} - V_{+-}\bar{R}^G_{+-}) - Z_- e\,(V_{-+}\bar{R}^G_{-+} - V_{--}\bar{R}^G_{--}) \tag{4.33}$$

$$\bar{W}^I = (S_{++}\bar{R}^I_{++} - S_{+-}\bar{R}^I_{+-}) + (S_{-+}\bar{R}^I_{-+} - S_{--}\bar{R}^I_{--}) \tag{4.34}$$

The dipole moment $\bar{M}$ of the cloud is

$$\bar{M} = Q(\bar{R}_+ - \bar{R}_-) \tag{4.35}$$

and its time rate of change is

$$\dot{\bar{M}} = Q\bar{V}_{rel} + \bar{M}\, d_t(\ell n\, Q) \tag{4.36}$$

The condition for constant separation of the charge centers is $\bar{V}_{rel} = \bar{0}$ hence (4.34) implies

$$\bar{M}\, d_t(\ell n\, Q) = \bar{W}^G - \bar{W}^I \tag{4.37}$$

Thus the averaging arguments to establish the statistical structure of the cloud charge distributions is established.

5. Physical Arguments Bearing on Lightning Discharge Problems

According to Earnshaw's Theorem electric charges cannot be maintained in equilibrium by electrical forces alone so it is important to recognize that static models have a limited use in explaining lightning and the development of thunderclouds capable of generating it. Actually one can start with static models in unstable equilibrium and augment the electric and gravitational forces by aerodynamical flows giving at least temporary stability to the structures while the charge concentrations develop in the cloud. The proper tool for the temporal development of these + and - charge distributions is the Boltzmann equation

$$B_t p = \partial_t p + \bar{v}\cdot\nabla_{\bar{R}} p + \bar{a}\cdot\nabla_{\bar{v}} p = \delta_t p \tag{5.1}$$

where

$$p(\bar{R}, \bar{v}, t)\, d\tau_{\bar{R}} d\tau_{\bar{v}} \tag{5.2}$$

is the probability that a charge is located within $d\tau_R$ of R and with a velocity within $d\tau_v$ of v and $\delta_t p$ is the so-called "collisional derivative" which is not actually a derivative at all but an integral giving the rate of change of p due to collisions. Under some conditions which are probably adequate for the problem of explaining the physical basis of charge separation in the cloud in mathematical detail the $\delta_t p$ may be replaced by γp to form the <u>Relaxational Boltzmann equation</u>[1]

$$B_t p = \gamma p \tag{5.3}$$

In the <u>stationary</u>, <u>collisionless</u> case (5.3) is easily solved in terms of the canomical ensemble for which

$$p = p_o e^{-E/kT} \tag{5.4}$$

The problem of charge separation is clearly <u>neither stationary nor collisionless</u> so (5.3) must considered as it is and its solution is obviously somewhat more complex than (5.4). Since (5.2) must satisfy the normalization condition

$$\iint p\, d\tau_{\bar{R}}\, d\tau_{\bar{v}} = 1 \tag{5.5}$$

---

[1]For negative γ, 5.3 corresponds to decay to a null state; if it is desired to decay to a final state p_o, then p should be replaced by $p-p_o$.

for all time it is clear that γ cannot be taken to be constant since this will lead to a probability density with an exponential factor $e^{\gamma t}$ in the solution so the (5.5) cannot be satisfied. Thus (5.3) must be studied in generality to rigorously establish the properties of the separately signed charge distributions and it is intended to pursue this direction in a subsequent publication. Here we merely note that it follows from (5.3) that the locations of maximum probability density correspond to the positions of null resultant force for each charge species and the physical explanation of the charge separation process must be found in the fact that the null force altitudes are different for positively and negatively charged particles. The three types of force term most relevant for the altitude segregation process are the electric field force, the gravitational force and the average force due to collisions. The electric field force is altitude dependent and oppositely directed for the two charge species. The gravitational force is altitude dependent via the altitude variation of the acceleration due to gravity and will reflect the mass difference of the two charge species. The collisional drag force will depend on the altitude variation of temperature and concentration which will also play a role in the thermal biasing of fracture processes in riming. Although the null force condition for the two charge species may be fulfilled at separate altitudes it cannot be expected that such a separation of charges will remain stable for any appreciable time without the influence of a vorticial flow which would tend to keep the separated charge configuration within its low pressure region.

The average fair weather electric field of the Earth is directed downward and decreases monotonically with increasing altitude. Typical values are 120 volts/meter at the surface, 25 volts/meter at 3 km. and 6 volts/meter at 10 km. Below 3 km. the field is sufficiently strong to accelerate electrons upward out of the layer of atmosphere closest to the Earth. The acceleration of positive ions is less because of the mass difference between electrons and positive ions and the net result is to establish an excess of positive charge in this region. However the most important features of the situation are not the static ones. There is a continuous upward transport of electrons and a continuous downward transport of positive ions to the negatively charged Earth under fair weather conditions. This is the observed Sky-Earth current. Far above it is to be expected that there will be an escape of electrons into the thermosphere analogous to the escape of hydrogen since the electric fields are presumably negligible and the electron gas is even less dense than hydrogen. This is of little relevance to events in the troposphere except as a consistency condition for the general upward flow of electrons.

With the advent of pre-storm conditions there is a disruption of the normal adiabatic lapse rate so that the lower troposphere becomes considerably cooler than it was under fair weather conditions. This is due to enhanced mixing between layers at different altitudes. In this way the lower troposphere attains a temperature intermediate between the fair weather surface temperature and the tropopause temperature. This quasi-isothermal layer interferes with the vertical transport of heat leading to the development of thermal vortices with updrafts near the surface and it also interferes with the transport of electrons upward and the transport of positive ions downward so that each of these species tends to collect at characteristic altitudes with the positive ions usually concentrating at higher altitudes because they are subject to much higher drag in the thermal updrafts. There is a very large

size difference between the electrons and the positive ions. Again it should be emphasized that this charge separation process is dynamical rather than static and that it depends upon the aerodynamic flow to remain stable for any appreciable length of time. After the separation of charges becomes accomplished one can expect that the local effects of the charge maxima will result in electrons being repelled upwards and downwards near the negative charge center at altitudes above and below it while positive ions will be likewise repelled upwards and downwards near the positive charge center above and below it.

It should be clear that continuous transport of electrons away from Earth and continuous transport of positive ions to Earth under fair weather conditions cannot be maintained indefinitely if the Earth is to retain its negative charge. In order for this to be preserved one expects that the net result of the lightning ground strokes in thunderstorms will be to return a net negative charge to the Earth so that a consistent fair weather electric field may be maintained. With regard to individual strokes this could be accomplished by transporting negative charge to Earth or positive charge from Earth.

Under storm conditions charge separations of 2 to 6 kms. occur with charge concentrations of the order of tens of coulombs and these generate breakdown fields (3 megavolts/meter in dry air; 1 megavolt/meter in clouds). Since breakdown is observed to occur it is a simple matter to compute

$$E = k\, Q^2/r^2 = 10^6 \text{ volts/meter} \tag{5.6}$$

with $k = 9 \times 10^9$ so

$$Q/r \sim .01 \tag{5.7}$$

and to have breakdown there must be roughly 10 coulombs of charge of each sign for each kilometer of separation to generate such a field.

The fair weather surface field of about 120 volts/meter corresponds to a surface charge density of about a nanocoulomb/meter^2. This yields an Earth total charge of about half a megacoulomb over the total area of the Earth of 5×10^{14} meters^2. With an observed air conductivity of 2×10^{-14}/meter ohm a Sky-Earth current density of about 2.4 picoamperes/meter^2 is obtained. Hence the total current over the whole Earth is 1200 amperes according to the stationary form of Ohm's Law. This must then be compensated during lightning strikes in thunderstorms by much larger current acting for much shorter times.

6. Summary and Conclusions

The basic purpose of the soliton Ansatz used in the first section is to reduce the number of functions in terms of which the 3-component electric field intensity, convective current density, magnetic field intensity and the scalar

electrical charge density and ionic generation may be described. The intention is that the functions left unspecified in that section be determined by experimental data. Once they are specified the full functional dependence of all components would be determined. Also the relationship of the input data to the full characterization of the vectorial fields would be precisely specified and the directional properties of the lightning strike fields clarified.

In order to understand the processes underlying the initiation of lightning discharges it is essential to have a physically reasonable, mathematically manageable formulation of the relationship between the electric intensity field and the electrical current density. The purpose of the second section is to give a systematic plasma dynamic derivation of the nonlinear, nonstationary generalization of Ohm's Law appropriate for the plasma of the ionization channel. The simplest case (2.55) suitable for the lightning discharge problem involves an electrical collisional frequency in addition to the conductivity characteristic of the stationary form of Ohm's Law which results when the left side of (2.55) vanishes. The modifications of (2.55) the generalized Ohm's Law which would ensue due to the influences of magnetic field, gravitation, pressure and viscosity would follow from (2.54) but are not required in the present discussion.

One finds that the electrical charge density naturally decays (with or without oscillation) unless there is an ionic generation term which produces inhomogeneities in the differential equation for the charge density. These act as forcing terms and no matter how small they may be initially, they make possible the development of "exponentially increasing" charge and current densities.

In section 3 a separation of variables argument is pursued to determine the relationships between the time-dependences of the charge density, current density and electric field without dropping the nonlinear term of the generalized Ohm's Law. It is found that there is a relative saturation of the convective current which allows the electric field to increase monotonically with time corresponding to a non-oscillatory displacement current. This continues until dielectric breakdown occurs which is accompanied by the development of an ionization channel with a greater conductivity than the pre-stroke medium. When this channel is developed a larger convective current can flow thereby relieving the high field which led to breakdown. After the channel has been developed the stochastic nature of the charge exchange process can lead to a succession of return and direct strokes.

If one considers a crude model for field propagation along the ionization channel with the rather drastic assumptions that appreciable values of the fields are restricted to a cylindrical portion of the channel of radius r and length h and that all losses from this portion are radiative, one obtains for the power loss P

$$P = S_r 2\pi rh + S_z \pi r^2 \tag{6.1}$$

where S_r, S_z are the radial and axial components of the Poynting vector. If the power were then taken to be produced by Joule heating one would have

$$E_z J_z = 2(S_r/r) + (S_z/h) \tag{6.2}$$

For $S_r \approx S_z$ and $h >> r$ this would yield

$$r = (2\ S_r/E_z J_z) \tag{6.3}$$

for the radius of the ionization channel. S_r could be obtained from the luminosity of the lightning flash and using $J_z = I/\pi r^2$ (6.3) becomes

$$r = (2\ S_r/\pi E_B I)^{1/3} \tag{6.4}$$

where E_z has been replaced by the breakdown field E_B and I is the current.

In the fourth section the statistical descriptions of the two charge species is given quantitatively in terms of the corresponding absolute densities of positive and negative charge which separate out in a thundercloud with the development of the electrical dipole moment of the cloud. In the fifth section physical arguments bearing on the charge separation processes are reviewed.

It is believed by the author that considerable progress has been made in making the quantitative arguments of the first, third, and fourth sections with the work of the third section on discharge initiation being founded on the plasma-theoretic derivation of the generalized Ohm's Law in the second section. There is every indication that the concepts of section 5 can be supported by precise treatments of the Boltzmann transport equation in a subsequent publication. Finally a comment about triggering will be made. In addition to the field enhancement factors associated with excessive curvatures of portions of an aircraft one can expect triggering to occur because the aircraft themselves lower the impedance path for a lightning stroke. For metal-bodied aircraft this is because they decrease the free air path of the stroke and for composite-bodied aircraft because they raise the permittivity of the stroke locus.

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15. Supplementary Notes

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16. Abstract

Mathematical physics arguments useful for lightning discharge and generation problems are pursued. This approach is based on the prospects for clarification of the physical phenomena involved which in many cases is obscured by the multiplicity of factors at work. The first section treats a soliton Ansatz for the lightning stroke including a charge generation term which is the ultimate source for the phenomena. The second section establishes equations for a partially ionized plasma including the effects of pressure, magnetic field, electric field, gravitation, viscosity and temperature. From these equations is then derived the Non-Stationary Generalized Ohm's Law essential for describing field/current density relationships in the ionization channel of the lightning stroke. Arguments are then given for the essential participation of ionic generation processes in the "exponentially" increasing current density and charge density which develop during the stroke. The third section deals with the discharge initiation problem and argues that the ionization rate drives both the convective current and electric displacement current to increase "exponentially" but that because of relative saturation of the former compared to the latter the convective current is unable to "relieve" the electric field which eventually increases to breakdown unleashing the lightning stroke. The fourth section deals with the statistical distributions of charge in the thundercloud preceding a lightning discharge. The last section contains some physical comments on the stability of the pre-lightning charge distributions and the use of Boltzmann relaxational equations to determine them along with a discussion of the covered impedance path provided by the aircraft.

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