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THREE-DIMENSIONAL MASS CONSERVING ELEMENTS FOR COMPRESSIBLE FLOWS

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Abstract

A variety of finite element schemes has been used in the numerical approximation of compressible flows particularly in underwater acoustics. In many instances instabilities have been generated due to the lack of mass conservation. In this paper we develop new two- and three-dimensional elements which avoid these problems.

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INTRODUCTION

It has been known for several years that many finite element formulations of fluid flows have potentially serious instabilities (see, e.g., [1]-[8]). A great deal of attention has been given to the incompressible case, and a variety of techniques for dealing with these instabilities has been developed. These include reduced integration [9], artificial diffusion [10], penalty methods [11], and the development of special stable elements [12]-[14], to cite only a few references in a very large literature. The special elements have proven to be of great practical value since they typically can be implemented by using standard elements in a Galerkin formulation, and then applying various filtering techniques to the pressures computed from the latter.

The compressible case is similar in the sense that elements which are unstable for incompressible flows are also unstable in the compressible case. This has been analyzed mathematically and shown to be equivalent to the failure of the Babuska-Brezzi angle condition between the finite spaces used in the formulation [3]-[4]. On the other hand, the compressible case is dissimilar in that the angle condition is not sufficient for stability. Put differently, there are elements which perform quite satisfactorily for incompressible flows but which exhibit a number of instabilities in the compressible case. These are usually of the secondary nature; i.e., they do not destroy the entire calculation, but they do lead to slower rates of convergence than normal and, hence, less accurate and less efficient approximations.

The key to these has been shown to be related to mass conservation [1], [4], [7]. In the compressible case it is necessary to have mass conservation (i.e., a divergence free velocity field) only in an appropriate averaged sense. For example, if linear elements are used to represent the fluid velocity $\underset{\sim}{u}_h$, typically one has

$$\int_T \operatorname{div} \underset{\sim}{u}_h = 0 \tag{1.1}$$

for each subdivision T defining the grid. This is different from

$$\operatorname{div} \underset{\sim}{u}_h = 0 \tag{1.2}$$

holding at each point, the latter being equivalent to exact mass conservation.

In the steady compressible case, the relevant vector field is the mass flow $\underset{\sim}{u}_h$, and the assertion is that without (1.2) serious errors will occur in the approximation that do not exist in the incompressible case.

Fortunately, there are a number of mass conserving elements that have been developed in the two-dimensional case. The most widely used are the second-order accurate union jack or criss-cross element [3], and the first-order linear element developed by Thomas [14]. The goal of this paper is to develop analogous elements for three spatial dimensions. The analog of the Thomas element is given in Section 2, and the analog of the criss-cross element is given in Section 4. In Section 3 we give an apparently new element, which like the criss-cross element, is second-order accurate, but (like the Thomas element) does not have restrictive grid regularity conditions.

To describe these elements we consider the specific case of steady potential flow. We let ϕ denote the potential and ρ the density. The governing equations in the flow region Ω are:

$$\mathrm{div}\ \rho\ \overrightarrow{\mathrm{grad}}\ \phi = 0 \qquad \text{in} \quad \Omega\ , \tag{1.3}$$

$$\phi = \phi_\Gamma \qquad \text{on} \quad \Gamma_D\ , \tag{1.4}$$

$$\overrightarrow{\mathrm{grad}}\ \phi\cdot\underline{n} = v_n \qquad \text{on} \quad \Gamma_N\ . \tag{1.5}$$

On one part of the boundary of Ω, namely Γ_D, the potential ϕ_Γ is given while on the other, Γ_N, the normal velocity v_n is given. The density ρ and velocity $\overrightarrow{\mathrm{grad}}\ \phi$ are related through Bernoulli's equation.

Galerkin formulations typically work in terms of the mass flow

$$\underline{u} = \rho\ \overrightarrow{\mathrm{grad}}\ \phi, \tag{1.6}$$

since jump conditions across a shock are equivalent to the continuity of $\underline{u}$, and thus are an intrinsic part of the formulation. Entropy conditions ruling out expansion shocks, on the other hand, are introduced through modifications of the density ρ. The formulation consists of a finite element space V_h in which the mass flow is represented and a finite space S_h for the potential. The parameter $h > 0$ denotes an generic mesh spacing. One seeks a $\underline{u}_h$ in V_h and a ϕ_h in S_h such that

$$\int_\Omega \frac{\underline{u}_h\cdot\underline{v}^h}{\rho} + \int_\Omega \phi_h\ \mathrm{div}\ \underline{v}^h = \int_{\Gamma_D} \phi_\Gamma\ \underline{v}^h\cdot\underline{n} \tag{1.7}$$

$$\int_\Omega \mathrm{div}\ \underline{u}_h\ \psi^h = 0. \tag{1.8}$$

The approximate mass flow $\underline{u}_h$ is required to satisfy

$$\underline{u}_h \cdot \underline{n} = u_n \qquad \text{on} \quad \Gamma_N, \tag{1.9}$$

where u_n is the given normal mass flow, and (1.7) holds for all $\underline{v}^h$ in V_h whose normal components are zero on Γ_N. The equation (1.8), on the other hand, holds for all ψ^h in S_h, the boundary condition (1.4) being natural in this formulation. Once a basis for V_h and S_h has been selected, (1.7)-(1.8) reduce to a system of nonlinear equations, the nonlinearity coming from the functional dependence of the density ρ on the mass flow $\underline{u}_h$.

In the next three sections we shall display pairs V_h, S_h which satisfy exact mass conservations (1.2). In each of these cases this property is a direct consequence of the following inclusion property [3]:

$$S_h = \mathrm{div}[V_h]. \tag{1.10}$$

That is, each element ψ^h is a divergence $\mathrm{div}\,\underline{v}^h$ of some $\underline{v}^h$ in V_h, and conversely. Indeed, suppose (1.10) were true for the pair S_h, V_h. Then if $\underline{u}_h$ is the mass flow arising from (1.7)-(1.8) we have

$$\mathrm{div}\,\underline{u}_h \in S_h.$$

Thus letting $\psi^h = \mathrm{div}\,\underline{u}_h$ in (1.8) we get

$$\int_\Omega (\mathrm{div}\,\underline{u}_h)^2 = 0,$$

and hence (1.2) holds.

The inclusion property (1.10) is the primary vehicle in the paper for constructing mass conserving elements. The next two sections S_h is taken as a suitable space of piecewise constant functions, and V_h is constructed so that (1.10) holds. In the last section the converse is used. Here V_h is taken as a suitable space of piecewise linear functions, and S_h is defined by (1.10).

Each of the spaces introduced in the next sections also satisfies the Babuska-Brezzi angle condition. The proofs will be omitted since they are very close to the two-dimensional proofs and quite technical in nature.

2. A FIRST-ORDER ELEMENT

We suppose for simplicity that Ω is a polynomial region in $\mathbb{R}^3$, and we subdivide it into tetrahedral elements, $T_1, \cdots, T_n$. The space S_h consists of all piecewise constant functions on this grid. Thus S_h has n degrees of freedom and a local basis $\psi_1, \cdots, \psi_n$ is defined in the standard manner, i.e.,

$$\psi_j(T_\ell) = \begin{cases} 1 & \text{if } j = \ell \\ 0 & \text{if } j = \ell \ . \end{cases} \tag{2.1}$$

To define V_h we use the reference tetrahedron T_{ref} shown in Figure 2.1.

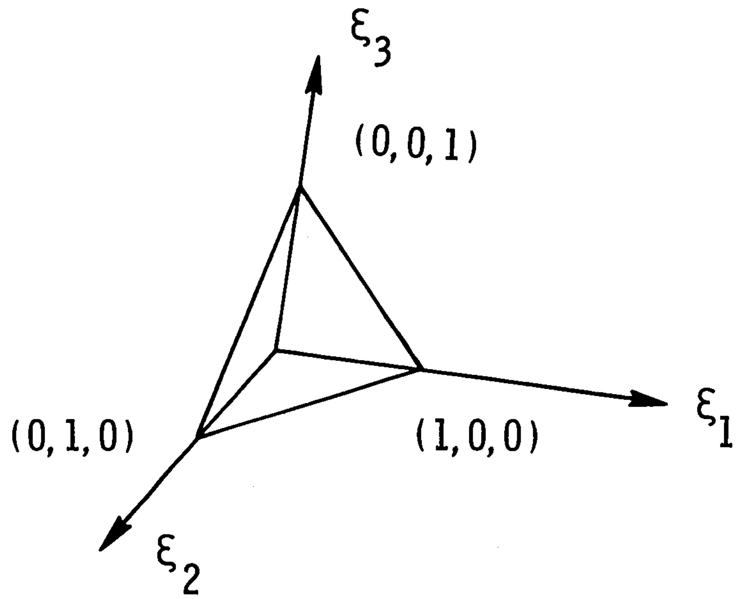

Figure 2.1: T_{ref}

In particular, we consider incomplete linear functions in T_{ref} having the form

$$\underline{v} = \underline{\alpha} + \underline{\beta}\ \underline{\xi}\ . \tag{2.2}$$

The four degrees of freedom $\underline{\alpha}$, β are uniquely determined by specifying the normal component $\underline{v}\cdot\underline{\nu}$ at the centroid $\underline{m}$ of each of the four faces of T_{ref}. The space V_h will consist of images of these functions.

More precisely, let T_j be a tetrahedron in the grid, and consider the affine mapping

$$\sigma_j : T_{ref} \longrightarrow T_j. \tag{2.3}$$

Then functions $\underline{v}_h$ in V_h have the form in T_j:

$$\underline{v}_h(\sigma_j(\xi)) = \underline{\alpha} + \beta\underline{\xi}. \tag{2.4}$$

The parameters α,β are determined by specifying $\underline{v}\cdot\nu$ at the centroid of each face.

Suppose $\{F_1,\cdots,F_N\}$ is the set of all faces and let $\underline{m}_j$ be the centroid of F_j with $\underline{\nu}_j$ being a unit normal to F_j. Then V_h has N degrees of freedom, and any $\underline{v}_h$ in V_h is uniquely determined by specifying

$$\underline{v}_h\cdot\underline{\nu}_j(\underline{m}_j) \qquad (j=1,\cdots,N). \tag{2.5}$$

A basis $\underline{\phi}_1,\cdots,\underline{\phi}_N$ for V_h is obtained by requiring

$$\phi_j\cdot\underline{\nu}_\ell(\underline{m}_\ell) = \begin{cases} \dfrac{1}{|F_j|} & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell \end{cases} \tag{2.6}$$

where $|F_j|$ is the area of F_j. Observe that since $\underline{\phi}_j\cdot\underline{\nu}_\ell$ is linear on F_j

$$\int_{S_\ell} \underline{\phi}_j\cdot\underline{\nu}_\ell = \begin{cases} 1 & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell. \end{cases} \tag{2.7}$$

Moreover, ϕ_j is identically zero in any T_ℓ which does not contain F_j as a face; i.e., $\underline{\phi}_1,\cdots,\underline{\phi}_N$ is a locally defined basis.

Functions $\underline{v}_h$ in V_h are not necessarily continuous, but they are square integrable, and have a square integrable divergence. In fact, for each tetrahedron T

$$\text{div } \underline{v}_h = \frac{1}{|T|}\int_T \text{div } \underline{v}_h = \frac{1}{|T|}\int_{\partial T} \underline{v}_h\cdot\underline{\nu} \qquad \text{in } T, \tag{2.8}$$

where $|T|$ denotes the volume of T and $\underline{\nu}$ is the outer normal. This is the only regularity required on the variational principle (1.7)-(1.8). These elements, however, cannot be used for the incompressible case since continuity is required there.

We assert that the inclusion property (1.10) holds for this pair of spaces. Since $\underline{v}_h$ is linear, div $\underline{v}_h$ is a piecewise constant and hence is in S_h. We must show that every function ψ_h has this form. To see this we first select the function $\underline{v}$ satisfying

$$\operatorname{div} \underline{v} = \psi_h \qquad \text{in } \Omega. \tag{2.9}$$

Using the basis $\underline{\phi}_j$ let

$$\underline{v}_h = \sum_{j=1}^{N} \left(\int_{F_j} \underline{v}\cdot\underline{\nu}_j\right)\underline{\phi}_j . \tag{2.10}$$

Observe that for any T_ℓ

$$\operatorname{div} \underline{v}_h = \frac{1}{|T_\ell|}\int_{T_\ell} \operatorname{div} \underline{v}_h = \frac{1}{|T_\ell|}\int_{\partial T_\ell} \underline{v}_h\cdot\underline{\nu} . \tag{2.11}$$

But using (2.7) and (2.10) we have

$$\int_{\partial T_\ell} \underline{v}_h\cdot\underline{\nu} = \int_{\partial T_\ell} \underline{v}\cdot\underline{\nu} = \int_{T_\ell} \operatorname{div} \underline{v} . \tag{2.12}$$

In light of (2.9) we therefore have

$$\operatorname{div} \underline{v}_h = \frac{1}{|T_\ell|}\int_{T_\ell} \operatorname{div} \underline{v}_h = \frac{1}{|T_\ell|}\int_{T_\ell} \psi_h = \psi_h \tag{2.13}$$

in T_ℓ. Since ψ_h is an arbitrary function in S_h, it follows that (1.10) must hold.

Incidentally, the Babuska-Brezzi angle condition requires that, in addition to (2.13), $\underline{v}_h$ must satisfy

$$\|\underline{v}_h\|_0 + \|\text{div } \underline{v}_h\|_0 \leq C\|\psi_h\|_0 \tag{2.14}$$

where

$$\|\underline{v}\|_0 = \{\int_\Omega |\underline{v}|^2\}^{1/2}, \tag{2.15}$$

and $0 < C < \infty$ is uniformly bounded independent of the grid. This can be verified directly from (2.10) once explicit formulas are obtained for the basis $\underline{\phi}_1,\cdots,\underline{\phi}_N$. This calculation is similar to the one in [14], and it does require that the grid be quasi-regular in the sense that the ratios of the maximum side length to the diameter of each T be uniformly bounded.

Functions in V_h are capable of only first-order accuracy. The reason for this can be seen from (2.2). The first component v_1 of $\underline{v}$ does not contain terms involving ξ_2 and ξ_3. Hence, $\underline{v}_h \in \upsilon_h$ is not a complete piecewise linear function.

Since S_h contains only piecewise constant functions, it too is capable of only first-order accuracy. Thus, if $\underline{u}_h$ and ϕ_h are the approximations obtained from (1.7)-(1.8), then

$$\|\underline{u} - \underline{u}_h\|_0 \leq Ch\|\text{grad } \underline{u}\|_0 \tag{2.16}$$

$$\|\phi - \phi_h\|_0 \leq Ch\|\text{grad } \phi\|_0, \tag{2.17}$$

where h is the maximum diameter of tetrahedrons T_ℓ.

3. A SECOND-ORDER ELEMENT

In many applications it is important to have second-order approximations to the mass flow and velocities. To achieve this we use the same grid as in Section 2, but with a larger space V_h. The idea is to increase the size of

$$\eta = \{\underline{v}_h : \operatorname{div} \underline{v}_h = 0\}, \tag{3.1}$$

so that the inclusion property (1.10) is not affected.

Indeed, we retain the same notation introduced in Section 2, and as before, let S_h denote the space of piecewise constant functions. To define V_h we let $F_1,\cdots,F_N$ denote the faces of the tetrahedrons with $\underline{m}_1,\cdots,\underline{m}_N$ being the associated centroids. With each face F_j we select a normal $\underline{\nu}_j$ along with two independent tangential directions $\underline{\tau}_j^{(1)}$, $\underline{\tau}_j^{(2)}$. A function $\underline{v}_h$ in V_h, if and only if,

(i) $\underline{v}_h$ is a linear polynomial in each T_j.

(ii) $\underline{v}_h$ is continuous at each centroid $\underline{m}_j$.

In the previous section, functions $\underline{v}_h$ were required only to have a continuous normal component at the centroids $\underline{m}_j(j=1,\cdots,N)$. Here all three components of $\underline{v}_h$ are continuous. Moreover, the representation of $\underline{v}_h$ in each T_j is in terms of a complete linear polynomial.

As in the last section, $\underline{v}_h$ in V_h are not necessarily continuous everywhere in Ω. There are, however, square integrable with square integrable divergences. Moreover,

$$\operatorname{div} \underline{v}_h = \frac{1}{|T|} \int_{\partial T} \underline{v}_h \cdot \underline{\nu} \tag{3.2}$$

holds for each tetrahedron T.

Observe that V_h has $3N$ degrees of freedom, and a locally defined basis $\underline{\phi}_1^{(i)},\cdots,\underline{\phi}_N^{(i)} (i=0,1,2)$ can be selected so that for $j=1,\cdots,N$ we

$$\underline{\phi}_j^{(0)}\cdot\underline{\nu}_\ell(\underline{m}_\ell) = \begin{cases} |F_\ell| & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell \end{cases} \tag{3.3}$$

$$\underline{\phi}_j^{(0)}\cdot\underline{\tau}_\ell^{(t)}(\underline{m}_\ell) = 0 \qquad \ell=1,\cdots,N,\ t=1,2, \tag{3.4}$$

and for $i=1,2$

$$\underline{\phi}_j^{(i)}\cdot\underline{\nu}_\ell(\underline{m}_\ell) = 0 \qquad \text{all } \ell \tag{3.5}$$

$$\underline{\phi}_j^{(i)}\cdot\underline{\tau}_\ell^{(t)}(\underline{m}_\ell) = \begin{cases} |F_\ell| & \text{if } i = t \text{ and } j = \ell \\ 0 & \text{otherwise .} \end{cases} \tag{3.6}$$

Note that since $\underline{\phi}_j^{(i)}$ is a linear function on the faces F_ℓ we have

$$\int_{S_\ell} \underline{\phi}_j^{(0)}\cdot\underline{\nu}_\ell = \begin{cases} 1 & \text{if } j = \ell \\ 0 & \text{if } j \neq \ell \end{cases} \tag{3.7}$$

$$\int_{S_\ell} \underline{\phi}_j^{(0)}\cdot\underline{\tau}_\ell^{(t)} = 0 \qquad \text{all } \ell,t \tag{3.8}$$

and for $i=1,2$

$$\int_{S_\ell} \phi_j^{(i)}\cdot\underline{\nu}_\ell = 0 \qquad \text{all } \ell \tag{3.9}$$

$$\int_{S_\ell} \underline{\phi}_j^{(i)} \cdot \underline{\tau}_\ell^{(t)} = \begin{cases} 1 & \text{if } j = \ell \text{ and } i = t \\ 0 & \text{otherwise .} \end{cases} \tag{3.10}$$

Thus any $\underline{v}_h$ in V_h can be written

$$\underline{v}_h = \sum_{j=1}^{N} \left(\int_{S_j} \underline{v}_h \cdot \underline{\nu}_j\right) \phi_j^{(0)} + \sum_{j=1}^{2} \sum_{j=1}^{N} \left(\int_{S_j} \underline{n}_h \cdot \underline{\tau}_j^{(i)}\right) \underline{\phi}_j^{(i)} . \tag{3.11}$$

It follows from (3.2) and (3.9) that the second sum in (3.11) is divergence free. It therefore does not play a role in the inclusion property (1.10), but it does contain the extra terms (missing in the space defined in Section 2) giving second-order accuracy.

To verify (1.10) for this case we select a function $\underline{v}$ satisfying (2.9), and let

$$\underline{v}_h = \sum_{j=1}^{N} \left(\int_{S_j} \underline{v} \cdot \underline{\nu}_j\right) \underline{\phi}_j^{(0)} , \tag{3.12}$$

(i.e., the divergence free part is omitted). The equations (2.11)-(2.13) remain valid, and hence as before we have

$$\operatorname{div} \underline{v}_h = \psi_h \quad \text{in } T. \tag{3.13}$$

Assuming the grid is quasi-regular in the sense cited in Section 2, one can directly verify the angle condition introduced in [3]. A consequence is that second-order accuracy, i.e.,

$$\| \underline{u} - \underline{u}_h \|_0 \le Ch^2 \| D^2 u \|_0 , \tag{3.14}$$

is obtained for the mass flow. Since S_h consists only of piecewise constant elements, first-order accuracy (i.e., (2.17)) is the best one we can obtain to the potential ϕ. However, the results in [3] indicate that ϕ_h is a second-order accurate approximation to the average

$$\frac{1}{|T|}\int_T \phi$$

in each tetrahedron T.

4. THE THREE-DIMENSIONAL CRISS-CROSS ELEMENT

Let Ω be a rectangular domain in $\mathbb{R}^3$, divided into $m_1 m_2 m_3$ cubes $B_{\underset{\sim}{\alpha}}$ where $\underset{\sim}{\alpha} = (i,j,k)(0 \le i < m_1,\ 0 \le j < m_2,\ 0 \le k < m_3)$, by a uniform grid of mesh spacing h. We denote the corners of the cubes by ω_{ijk}, then centers by $\underline{\omega}_{i+1/2,j+1/2,k+1/2}$ and the center of each face by $\underline{\omega}_{i+1/2,j+1/2,k}$, $\underline{\omega}_{i,j+1/2,k+1/2}$ or $\underline{\omega}_{i+1/2,j,k+1/2}$.

Each cube is partitioned into 24 congruent tetrahedrons by considering divisions along the four diagonals of each cube, the two diagonals of each face of the cube and the three lines joining the centers of opposite faces of the cube [Figure 4.1]. We number the faces of each cube from 1 to 6 as shown in Figure 4.2. Each of these six faces of the cube will contain the bases of four tetrahedrons. We can now identify any tetrahedron in the grid by $T^{r,s}_{\underset{\sim}{\alpha}}$ where $\underset{\sim}{\alpha} = (i,j,k)$ gives the cube in which the tetrahedron is contained, $1 \le r \le 6$ specifies the face on which the base is located, and $1 \le s \le 4$ specifies the tetrahedron completely.

Let $\hat{V}_h$ be the space of all $\mathbb{R}^3$ valued continuous functions over Ω which are piecewise linear for each tetrahedron. We take our approximate space V_h to be that subspace of $\hat{V}_h$ in which the nodes at the centers of the faces of each cube are condensed out. This is done by specifying the value at such a node to be the average of the values at the four vertices of the face. More precisely, for $v_h = \sum_{\ell=1}^{3} v^\ell \underline{e}_\ell{}^{(1)} \in \upsilon_h$,

$$v^\ell(\underline{\omega}_{i+1/2,j+1/2,k}) = v^\ell_{i+1/2,j+1/2,k} = 1/4 \sum_{q=0}^{1} \sum_{p=0}^{1} v^\ell_{i+p,j+q,k}, \quad \ell = 1,2,3. \tag{4.1}$$

The number $v^\ell(\underline{\omega}_{i+1/2,j,k+1/2})$ and $v^\ell(\underline{\omega}_{i,j+1/2,k+1/2})$ will be similar averages.

The dimensions of V^h is equal to 3N, where N, the number of unconstrained nodes is given by $(m_1 + 1)(m_2 + 1)(m_3 + 1) + m_1 m_2 m_3$. Let η be the set of all nodes (constrained or unconstrained). For every node $\underline{\omega}_{\underset{\sim}{\alpha}}$, define a function $\phi_{\underset{\sim}{\alpha}} \in \hat{V}_h$ by

$$\begin{aligned} \phi_{\underset{\sim}{\alpha}}(\underline{\omega}_{\underset{\sim}{\beta}}) = \delta_{\underset{\sim}{\alpha}\underset{\sim}{\beta}} &= 1 \qquad \text{if } \underset{\sim}{\alpha} = \underset{\sim}{\beta} \\ &= 0 \qquad \text{if } \underset{\sim}{\alpha} \neq \underset{\sim}{\beta}. \end{aligned}$$

Then any function $\hat{\underline{v}}^h$ in $\hat{V}_h$ can be represented as $\underline{v}^h = \sum_{\ell=1}^{3} \left(\sum_{\alpha \in \eta} v^\ell_{\underset{\sim}{\alpha}} \phi_{\underset{\sim}{\alpha}} \right) \underline{e}_\ell$. For a function in V_h, the coefficients for constrained nodes are replaced by suitable averages.

---

(1) $\underline{e}_1, \underline{e}_2, \underline{e}_3$ for an orthonormal basis for $\mathbb{R}^3$.

Let $\hat{S}_h$ be the set of functions that are constant over each tetrahedron. $\phi_{\underset{\sim}{\alpha}}^{r,s}$ will denote the value of ϕ in $T_{\underset{\sim}{\alpha}}^{r,s}$. Define S_h to be the set of $\phi \in \hat{S}_h$ such that for any $\underset{\sim}{\alpha}$,

$$\text{(i)} \qquad \phi_{\underset{\sim}{\alpha}}^{r,1} + \phi_{\underset{\sim}{\alpha}}^{r,3} = \phi_{\underset{\sim}{\alpha}}^{r,2} + \phi_{\underset{\sim}{\alpha}}^{r,4} \qquad (r = 1,\cdots,6) \tag{4.2}$$

$$\text{(ii)} \qquad \sum_{S=1}^{4} \left(\phi_{\underset{\sim}{\alpha}}^{r,s} + \phi_{\underset{\sim}{\alpha}}^{r+3,s}\right) = C_{\underset{\sim}{\alpha}}, \text{ a constant} \qquad (r = 1,2,3) \tag{4.3}$$

With the choice of S_h and V_h, we have $S_h = \text{div}[V_h]$ as proved in the appendix.

Moreover, our proof also shows the Babuska-Brezzi condition is satisfied so that for every $\phi \in S_h$, we can find a $\underline{w} \in V_h$ satisfying

$$\text{div } \underline{w} = \phi \quad \text{in} \quad \Omega$$

$$\|\underline{w}\|_0 \leq C\|\phi\|_{-1}$$

where $0 < C < \infty$ is bounded independently of the grid.

The second-order accurate space of functions that are linear over each $B_{\underset{\sim}{\alpha}}$ is easily seen to be a subspace of V_h. Similarly, the space of functions constant over each $B_{\underset{\sim}{\alpha}}$ is a subset of S_h.

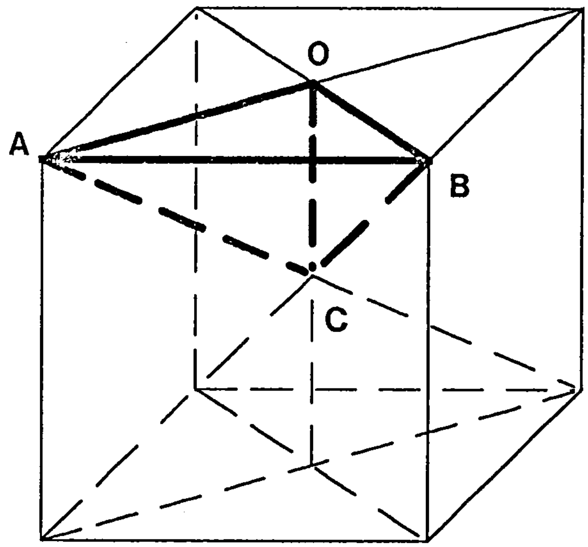

Figure 4.1

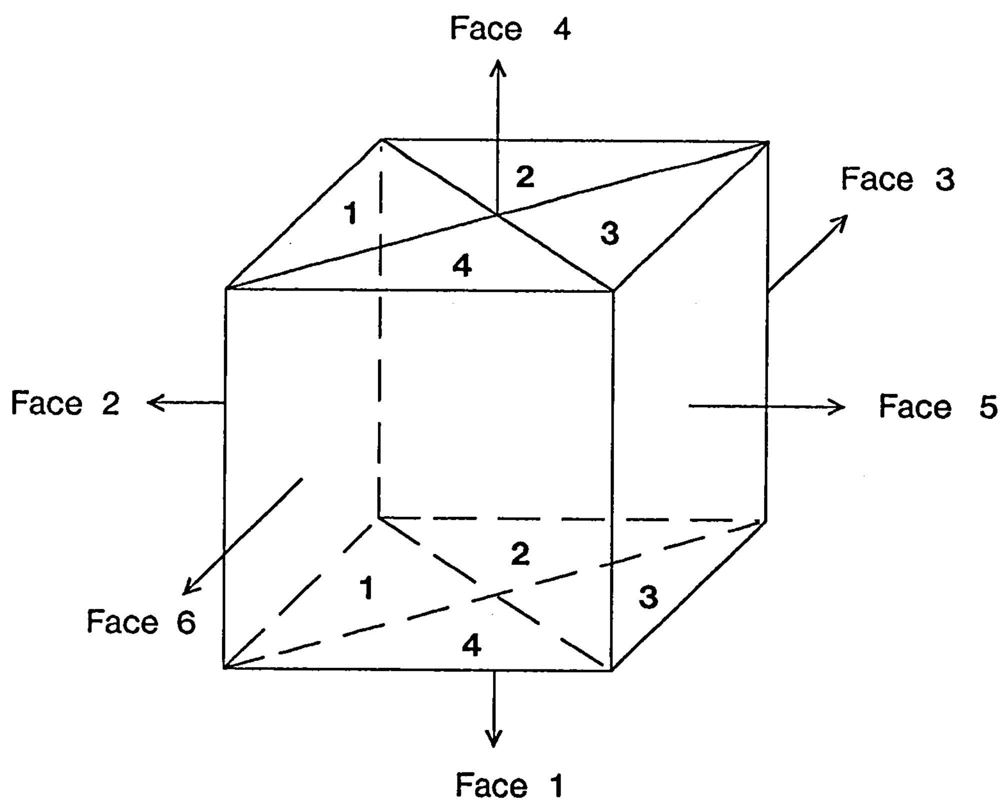

Figure 4.2

APPENDIX

We now prove the following theorem:

Theorem: <u>For the space</u> V_h <u>and</u> S_h <u>mentioned in Section 4, we have</u> $S_h = \mathrm{div}[V_h]$.

<u>Moreover, for any</u> $\phi \in S_h$, <u>we can find a</u> $\underline{w} \in V_h$ <u>satisfying</u>

$$\begin{aligned} \mathrm{div}\ \underline{w} &= \phi \quad \underline{\text{in}} \quad \Omega \\ \|\underline{w}\|_0 &\le C\|\phi\|_{-1}. \end{aligned} \tag{A.1}$$

<u>Proof:</u> Let $\phi = \mathrm{div}\ \vec{v}$, where $\vec{v} \in V_h$. Referring to Figure A.1 and denoting $\phi_{\underset{\sim}{\alpha}}^{1,s}$ by ϕ_s, we have

$$\phi_1 = \frac{v_A^1 - v_0^1}{h} + \frac{2v_B^2 - v_A^2 - v_0^2}{h} + \frac{2v_0^3 - 2v_B^3}{h} .$$

Similarly,

$$\phi_2 = \frac{2v_B^1 - v_0^1 - v_E^1}{h} + \frac{v_E^2 - v_0^2}{h} + \frac{2v_C^3 - 2v_B^3}{h}$$

$$\phi_3 = \frac{v_D^1 - v_E^1}{h} + \frac{v_D^2 + v_E^2 - 2v_B^2}{h} + \frac{2v_C^3 - 2v_B^3}{h}$$

$$\phi_4 = \frac{v_D^1 + v_A^1 - 2v_B^1}{h} + \frac{v_D^2 - v_A^2}{h} + \frac{2v_C^3 - 2v_B^3}{h} .$$

From this, we see that

$$\phi_1 + \phi_3 = \phi_2 + \phi_4 = \frac{v_A^1 + v_D^1 - v_0^1 - v_E^1}{h} + \frac{v_D^2 + v_E^2 - v_0^2 - v_A^2}{h} + 4\,\frac{(v_C^3 - v_B^3)}{h} \,. \tag{A.2}$$

This proves (4.2) for the function ϕ in the case $r = 1$. The other cases can be verified similarly. We now prove (4.3) for ϕ. We use ϕ_s and ϕ_s' to denote $\phi_{\underset{\sim}{\alpha}}^{1,S}$ respectively. By calculations similar to the ones above, we obtain

$$\frac{1}{8}\sum_{s=1} (\phi_s + \phi_s') = \frac{(v_A^1 + v_A^1\ 1 + v_D^1 + v_D^1\ 1) - (v_0^1 + v_0^1\ 1 + v_E^1 + v_E^1\ 1)}{4h}$$

$$+ \frac{(v_D^2 + v_D^2\ 1 + v_E^2 + v_E^2\ 1) - (v_A^2 + v_A^2\ 1 + v_0^2 + v_0^2\ 1)}{4h}$$

$$+ \frac{(v_A^2\ 1 + v_0^3\ 1 + v_E^3\ 1 + v_D^3\ 1) - (v_A^3 + v_0^3 + v_E^3 + v_D^3)}{4h}$$

$$= D_x\, v^1\, D_y\, v^2 + D_3\, v^3 = \mathrm{div}_h\, v\Big|_{\underset{\sim}{\alpha}} \,. \tag{A.3}$$

We have used averaging expressions for v_B^3 and v_B^3 in the above. By the symmetry of the expression for $\mathrm{div}_h\, v$, it follows that the same result must be obtained for $r = 2$ and $r = 3$, i.e., (4.3) holds. Hence, $\phi \in S_h$, i.e., $\mathrm{div}[V_h] \subset S_h$. Now let $\phi \in S_h$. We will find $\underline{v}^h \in V_h$ satisfying (A.3), which we rewrite as

$$\mathrm{div}_h\, \underline{v}\Big|_{i,j,k} = \overline{\phi}\Big|_{i,j,k} \,. \tag{A.4}$$

Here, the left-hand side stands for the expression

$$\frac{1}{4h}\sum_{q=0}^{1}\sum_{p=0}^{1}\{(v^1_{i+1,j+p,k+q}-v^1_{i,j+p,k+q})+(v^2_{i+1,j+1,k+p}-v^2_{i+q,j,k+p})$$

$$+(v^3_{i+p,j+q,k+1}-v^3_{i+p,j+q,k})\}$$

$$=D_x\,v^1\Big|_{i,j,k}+D_y\,v^2\Big|_{i,j,k}+D_z\,v^3\Big|_{i,j,k}. \tag{A.5}$$

The right-hand side equals

$$\frac{1}{h^3}\int_{B_{i,j,k}}\operatorname{div}\underline{v}.$$

Using (A.2), we can obtain the value of v^3 at the central node as

$$v^3_{i+1/2,j+1/2,k+1/2}=\frac{h}{4}\,[\phi^{1,2}_{i,j,k}+\phi^{1,4}_{i,j,k}]-\frac{1}{4}\sum_{p=0}^{1}\{(v^1_{i+1,j+p,k}-v^1_{i,j+p,k})$$

$$+(v^2_{i+p,j+1,k}-v^2_{i+p,j,k})+(v^3_{i+p,j,k}+v^3_{i+p,j+1,k})\}. \tag{A.6}$$

Similar formulas hold for the other components of $\underline{v}$ at the central node of $B_{\underset{\sim}{\alpha}}$. We shall use (A.4) to determine $\underline{v}$ at the corner nodes and then define $\underline{v}$ at central nodes by (A.6). To solve (A.4), we introduce a discrete potential $\{\theta_{i+1/2,j+1/2,k+1/2}\}$, constant over each box $B_{i,j,k}$ and satisfying:

$$v^1_{i,j,k} = \frac{1}{4h}\sum_{q=0}^{1}\sum_{p=0}^{1}\left(\theta_{i+1/2,j-1/2+p,k-1/2+q} - \theta_{i-1/2,j-1/2+p,k-1/2+q}\right)$$

$$= (G^h_x\,\theta)_{i,j,k} \tag{A.7}$$

$$v^2_{i,j,k} = \frac{1}{4h}\sum_{q=0}^{1}\sum_{p=0}^{1}\left(\theta_{i-1/2+q,j+1/2,k-1/2+p} - \theta_{i-1/2+q,j-1/2,k-1/2+p}\right)$$

$$= (G^h_y\,\theta)_{i,j,k} \tag{A.8}$$

$$v^3_{i,j,k} = \frac{1}{4h}\sum_{q=0}^{1}\sum_{p=0}^{1}\left(\theta_{i-1/2+p,j-1/2+q,k+1/2} - \theta_{i-1/2+p,j-1/2+q,k-1/2}\right)$$

$$= (G^h_z\,\theta)_{i,j,k}. \tag{A.9}$$

Then (A.4) implies that θ satisfies the following discrete Poisson equation

$$\Delta_h\,\theta = (D_x\,G^h_x\,\theta + D_y\,G^h_y\,\theta + D_z\,G^h_z\,\theta)\Big|_{i,j,k} = \bar{\phi}\Big|_{i,j,k}$$

where Δ_h is the discrete Laplacian shown in Figure 2. (Δ_h is the same as the "one-point quadrature stencil" described in [15].)

The equation above only involves θ at the centers of boxes. Since all boundary conditions are natural, we can extend the grid to cover Ω and let $\theta_{i+1/2,j+1/2,k+1/2}$ be zero for points outside Ω. We now show that such a θ exists.

Let

$$\Delta_h\,\psi = f \quad \text{in} \quad \Omega \quad \text{and} \quad \psi = 0 \quad \text{outside} \quad \Omega. \tag{A.10}$$

[Here f is constant over each box $B_{i,j,k}$.] Then, multiplying both side by ψ and performing a discrete integration by parts, we obtain

$$-\Big\{\sum_{(i,j,k)\in\Omega}(G_x^h\,\theta)^2_{i,j,k} + (G_y^h\,\theta)^2_{i,j,k} + (G_z^h\,\theta)^2_{i,j,k}\Big\} = \sum (f\theta)_{i+1/2\,,j+1/2\,,k+1/2}$$

Put $f \equiv 0$. Then, since $\theta = 0$ outside Ω, by starting from a corner of Ω and working through Ω, we see that $\theta \equiv 0$ in Ω. Hence, the Fredholm Alternative, (A.10) has a unique solution for each f. We therefore obtain a solution for (A.4). Defining v^1, v^2, v^3 by (A.7)-(A.9) we get the standard estimate

$$h^2 \sum_{\ell=1}^{3} \sum_{i,j,k} |^2 \leq C\|\phi\|^2_{-2} \tag{A.11}$$

for some absolute constant $0 < C < \infty$. In addition, defining v at the centers of boxes by (A.6), we obtain

$$h^2 \sum_{\ell=1}^{3} \sum_{(i+1/2\,,j+1/2\,,k+1/2\,)\in\Omega} |v^{\ell}_{i+1/2\,,j+1/2\,,k+1/2}|^2 \leq C^2\{\|\phi\|^2_{-1} + h^2\|\phi\|^2_0\}.$$

Letting $\vec{v}$ be the continuous piecewise linear function whose value at each node is defined as above, we get

$$\sum_{i=1}^{3} \|v^{\ell}\|^2_0 \leq C\{\|\phi\|^2_{-1} + h^2\|\phi\|^2_0\} \quad \text{and} \quad \operatorname{div}\vec{v} = \phi.$$

Finally, using the inverse inequality for the uniform grid on Ω, we obtain for $\phi \in S_h$,

$$\|\phi\|_0 \leq Ch^{-1}\|\phi\|-1.$$

This proves the theorem.

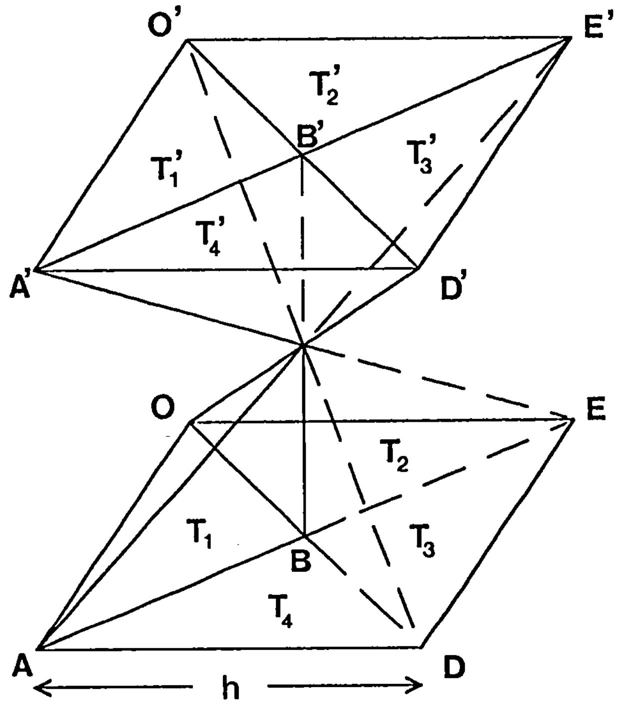

Figure A.1

$$\Delta_h = \frac{1}{16\,h^2}$$

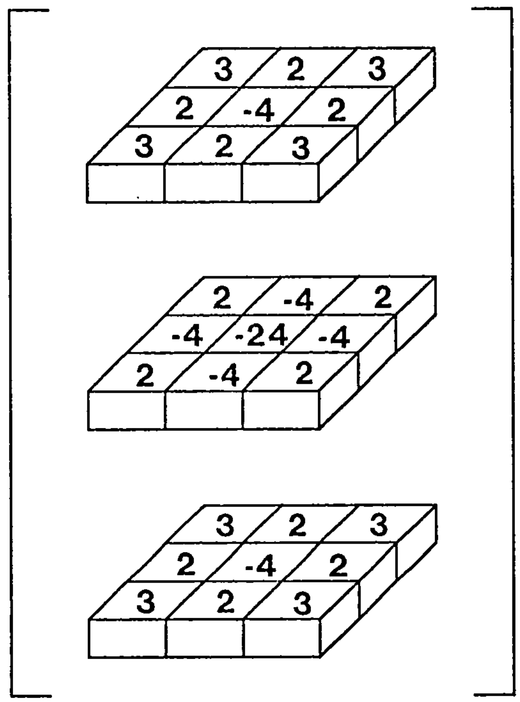

Figure A.2

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16. Abstract

A variety of finite element schemes has been used in the numerical approximation of compressible flows particularly in underwater acoustics. In many instances instabilities have been generated due to the lack of mass conservation. In this paper we develop new two- and three-dimensional elements which avoid these problems.

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