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ALGORITHMS FOR THE EULER AND NAVIER-STOKES EQUATIONS
FOR SUPERCOMPUTERS

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ALGORITHMS FOR THE EULER AND NAVIER-STOKES EQUATIONS FOR SUPERCOMPUTERS

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Abstract

We consider the steady state Euler and Navier-Stokes equations for both compressible and incompressible flow. Methods are found for accelerating the convergence to a steady state. This acceleration is based on preconditioning the system so that it is no longer time consistent. In order that the acceleration technique be scheme independent this preconditioning is done at the differential equation level. Applications are presented for very slow flows and also for the incompressible equations.

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1. INTRODUCTION

With the introduction of the latest class of supercomputers, e.g., the Cray XMP and Cyber 205, it has begun to be feasible to solve the Euler and Navier-Stokes equations for three-dimensional configurations. The major added difficulty in solving the Navier-Stokes equations is in the need to resolve the boundary layers. This is especially difficult for turbulent flow. Most codes rely on algebraic turbulence models, but even these require an extremely fine mesh to resolve the sublayers. The use of one or two equation turbulence models requires even finer meshes [15]. Hence, a Navier-Stokes code about a wing-body configuration requires a mesh that the new computers can just meet both in terms of speed and memory. Even with the new generation of supercomputers, it is not feasible to routinely run three-dimensional codes. It is therefore necessary to introduce new algorithms that will reduce the storage requirements and the running time compared with present schemes. Since several sophisticated schemes already exist, it would be advantageous if the new algorithms could be incorporated within the presently existing codes.

In this paper we will only consider steady state problems. This will enable us to change the time-dependent equations in any way that does not change the steady state. Thus, the approach that we use can be classified as a pseudo-unsteady approach to the steady state [19]. In addition we shall only consider conservation equations as this gives us greater flexibility in the problems that can be solved.

2. PRECONDITIONING

We now consider two-dimensional equations in conservation form

$$f_x + g_y = 0 \quad \text{in} \quad D \tag{1}$$

with appropriate boundary conditions. We consider schemes that are pseudo-time dependent. This approach allows the same code to treat true time-dependent problems by removing the pseudo-time elements. In addition the pseudo-time changes can all be done locally. The present analysis is based on constant coefficient equations. However, both the Euler and Navier-Stokes equations are nonlinear equations. Hence, the preconditioners that will be developed will, in practice, vary at each mesh point. It will also be necessary to blend different regions together, which will not be discussed in this paper. As a result when we consider subsonic flow there is no need for the flow to be subsonic everywhere. Hence, even when discussing very slow flow we wish the equations to be in conservation form since there may be shocks in other regions of the domain. Similarly, when we consider supersonic flow one cannot march in space as there may be regions of subsonic flow.

According to our philosophy of having the applications as general as possible, the analysis will be done at the differential equation level. Hence, the results are scheme independent and apply to both explicit and implicit methods. Though we are interested in the steady state, we shall use a time-like approach. Hence, we consider the system

$$w_t + f_x + g_y = 0 \tag{2}$$

where (x,y) represent general curvilinear coordinates. Since we are only interested in the steady state, we replace (2) with the system

$$E^{-1} w_t + f_x + g_y = 0. \tag{3}$$

The minimum requirements on E are that E be nonsingular and that (3) be a well-posed problem with boundary conditions that are consistent with those imposed on (1). It is straightforward to solve (3) with an explicit method. Using an implicit method only the diagonal block of the matrix to be inverted is changed compared with (2). We first consider the case that (2) is a hyperbolic system. Though the code solves (2) we will only consider the constant coefficient problem. Thus (2) is replaced by

$$w_t + Aw_x + Bw_y = 0 \tag{4}$$

while the preconditioned system (3) is replaced by

$$E^{-1} w_t + Aw_x + Bw_y = 0 \tag{5}$$

where A and B are the Jacobians of f and g with respect to w respectively. Also A, B and E are frozen at constant values. Let $w = Tv$ then (4) becomes

$$Tv_t + ATv_x + BTv_y = 0.$$

Multiplying this equation by S we find that (4) is equivalent to

$$STv_t + A_o v_x + B_o v_y = 0 \tag{6}$$

with $A_o = SAT$ and $B_o = SBT$. We will choose S and T to be nonsingular and such that A_o and B_o are "nice" matrices. We stress that there is no need for the transformation from (4) to (6) to be an equivalence transformation. Since (4) can be transformed into (6) it is sufficient to analyze (6). We now precondition the system (6) and consider

$$E_o^{-1} v_t + A_o v_x + B_o v_y = 0 \tag{7}$$

with an appropriate E_o. Returning to the original w variables we find that (7) can be transformed into (5) and hence (3) with $E^{-1} = S^{-1} E_o^{-1} T^{-1}$ or $E = TE_o S$. Using an explicit scheme we wish to find the matrix E in (3) while for an implicit scheme we wish to construct E^{-1}. We thus wish

<u>Objective No. 1:</u>

Choose E_o so that

1. E_o is invertible;
2. (7) is well-posed with appropriate boundary conditions;
3. (7) approaches the steady state as rapidly as possible.

We really wish to analyze (3) rather than (7) but we ignore nonlinear effects in this paper. The first property is straightforward. Implementation of the second and third properties will be discussed in the coming sections.

3. BOUNDARY CONDITIONS

Though the question of well-posedness is not the objective of this paper, nevertheless, we wish to point out several difficulties. By well-posed we mean that the solution exists, is unique, and depends continuously on the data. In discussing appropriate boundary conditions, we must distinguish between three systems. First is the transformed steady state system,

$$A_o\, v_x + B_o\, v_y = 0. \tag{8}$$

Next there is the transformed time-dependent system (6) and finally there is the preconditioned system given by (7). We assume that the matrices A_o and B_o are symmetric and that ST and E_o are symmetric positive definite. Then both (6) and (7) form a symmetric hyperbolic system as considered by Friedrichs [7]; hence both are well-posed for appropriate boundary data. If the boundary data are dissipative for (6) in L^2 with weight ST, then the same data will be dissipative and hence well-posed for (7) in L^2 with weight E_o. For more general boundary data it is not clear that data which make (6) well-posed will also make (7) well-posed.

Furthermore, it is not known if data that make (6) well-posed will also make (8) well-posed when a steady state is achieved. Thus, for example, one must rule out the possibility that the Helmholtz equation can be the steady state solution of a hyperbolic system. Even though the Helmholtz equation is well-posed in the sense of Lopatinski, this is not enough to yield uniqueness.

When the system (6) is strictly hyperbolic then one only needs analyze solutions to (6) of the form

$$v(x,y,t) = e^{\omega t} f(x,y).$$

Since we assume that a steady state is reached we must have that $\mathrm{Re}\ \omega < 0$. Hence, all the eigenvalues of (8) are in the left half plane. This is enough to ensure well-posedness in the sense of Lopatinski [17]. When the steady state equation is elliptic this guarantees regularity but not uniqueness. To show well-posedness in the sense of Hadamard one must also get uniform bounds on how close to the imaginary axis the eigenvalues can be. In particular (8) may have a zero eigenvalue so that there are solutions to (8) that cannot be achieved by a time-dependent process in addition to the solutions that are steady states of (6). Hence, we conclude that steady state solutions to (6) or (7) are solutions to (8), but we have no guarantee, even for constant coefficients, that these are the only solutions to (8) or that (8) is well-posed in the sense of Hadamard under the same boundary conditions.

We also wish to point out that if one begins with the steady state equations (8) then there are many possible boundary conditions that yield solutions, but not all of them are physically relevant boundary conditions. One way of choosing the relevant boundary conditions is to demand that the solution be the limit of an appropriate time-dependent problem. An alternative approach is to demand that the solution to (8) be the smooth limit of an appropriate viscous problem. As an example we consider the simple steady state

$$u_x = f \qquad 0 \leq x \leq 1. \tag{9}$$

The differential equation (9) is well-posed if we impose

$$u(0) = \text{given} \tag{10a}$$

or if we impose

$$u(1) = \text{given}. \tag{10b}$$

To decide which boundary condition is physically relevant we must determine, physically, whether (9) is the limit of

$$u_t + u_x = f \tag{11a}$$

or

$$-u_t + u_x = f \tag{11b}$$

as t goes to infinity. Equivalently, we can choose (11a) and decide whether (9) is the limit as the time goes to plus infinity or backwards to minus infinity. Since a hyperbolic equation is reversible in time both possibilities are legitimate. For a nonlinear problem reversing time will reverse the entropy inequality.

An alternative method to choose between the boundary conditions (10a) and (10b) is to claim that (9) is the smooth limit of a viscous system. Hence, (9) is the limit of either

$$u_x = \varepsilon u_{xx} + f \qquad \varepsilon > 0 \tag{12a}$$

or

$$u_x = -\varepsilon u_{xx} + f \qquad \varepsilon > 0. \tag{12b}$$

Equation (12a) will have a boundary layer near $x = 1$. By eliminating the boundary condition at $x = 1$ for $\varepsilon = 0$ the boundary layer does not appear

in the limiting solution. Equivalently we can eliminate the boundary layer for (12a) by specifying a Neumann type boundary condition rather than a Dirichlet condition. In the limit of small ε the Dirichlet condition at $x = 0$ remains while the boundary condition at $x = 1$ disappears. Of course, the roles of the two boundaries are interchanged when we choose (12b) instead of (12a).

In this case everything is obvious. A physically more relevant case is to consider flow through a nozzle. If the flow is subsonic then one should specify two conditions at inflow and one boundary condition at outflow. However, the steady state is unique if one specifies the total mass, the total enthalpy, and the entropy. It makes no difference where these quantities are specified [26]. Thus, for example, one could specify two of these quantities at outflow and only one at inflow. Nevertheless, the physically appropriate conditions are to specify two at inflow and one at outflow. This follows from the time-dependent Euler equations or the steady Navier-Stokes equations.

We hence conclude that the matrix E_0 in (7) must be chosen as positive definite whenever A_0 and B_0 are symmetric and ST is positive definite. This guarantees that we do not change the direction of the characteristics, so information flows in the same direction as before. Therefore, the number of boundary conditions is not changed.

4. ACCELERATION TO A STEADY STATE

We wish to choose E_0 in (7) so that we reach a steady state as fast as possible. When the equation is parabolic we can choose the free parameters so as to maximize the rate of decay to the steady state. This was first done by

Garabedian [8] in analyzing SOR. For a hyperbolic equation with constant coefficients, energy is conserved except for boundary effects. Hence, the only way to introduce dissipation is through the artificial surfaces [1]. We shall therefore ignore dissipative mechanisms. Instead we consider explicit schemes, and then we reach a steady state faster by choosing a larger time step within the stability limits. The methods to be developed are also effective for implicit methods using space factorization such as A.D.I. type methods. In order to compare different preconditionings we must normalize the time. Consider

$$u_t + u_x = f. \tag{13}$$

Let $\tau = at$; then (13) becomes

$$au_\tau + u_x = f. \tag{14}$$

When a is less than one, then we reach a steady state faster in terms of absolute quantities. However, we do not achieve the steady state faster in terms of physical time scales. For example, using a typical explicit scheme one requires that $\Delta\tau/\Delta x \leq a$. Thus, the smaller a is the less time it takes to reach a steady state, but at the same time the time steps are correspondingly smaller. The number of time iterations to reach a steady state is independent of a.

We therefore conclude that we cannot compare the absolute time step allowed by different preconditioners. Instead we must scale all speeds by a given reference speed. Hence, we rephrase the third condition of Objective No. 1 as

Objective No. 2:

Choose E_o so that we minimize the ratio of the fastest speed to the slowest speed of (7). Equivalently, choose E_o, positive definite, to minimize the condition number of

$$\kappa[E_o(\omega_1 A_o + \omega_2 B_o)] \quad \text{with} \quad \omega_1^2 + \omega_2^2 = 1. \tag{15}$$

We now consider the question of minimizing (15) when (6) has different time scales. Kreiss [14] has developed a normal form for symmetric hyperbolic systems with three equations. The two-dimensional Euler equations has four equations. However, since the entropy equation essentially decouples from the other three equations the two-dimensional Euler equations are included in the theory. Tadmor [21] has extended the normal form to systems with more equations. Browning and Kreiss [3] have also analyzed nonlinear equations. In this study we wish to do the opposite of what Kreiss did. Instead of treating the initial conditions to filter the fast waves, we wish to precondition the equations so that there is only one time scale. We shall choose E_o so as to equilibrate the time scales for the Euler equations. The normal form of Kreiss demonstrates that once we have accomplished this for the Euler equations, we have done the general two-dimensional symmetric hyperbolic system with three equations.

It also follows from [14] that this approach will work only if the two time scales separate uniformly in the Fourier variables (ω_1, ω_2). The simplest case where the time scales are uniform in the Fourier variables is one-dimensional flow, since there is only one Fourier variable,

$$u_t + A_o\, u_x = f. \tag{16}$$

In this case (15) becomes: find E_o, positive definite, so that $\kappa(E_o\, A_o^{-1})$ is minimum. The obvious choice is $E_o = |A^{-1}|$ where the absolute value of a matrix is found by going to diagonal form in an equivalence transformation, taking absolute values and then transforming back. Thus, the optimal preconditioned form for (16) is

$$|A_o|\, u_t + A_o\, u_x = f. \tag{17}$$

All the speeds of (17) are ± 1 and so the condition number is equal to 1. In two space dimensions this recipe doesn't work since $E_o = |\omega_1\, A_o + \omega_2\, B_o|$ implies that E_o is a pseudodifferential operator. Furthermore, since neither ω_1 nor ω_2 is small, in general, there are no obvious expansions. One possibility is to minimize this quantity in a root mean square sense over all ω_1 and ω_2.

Another possibility is to minimize the condition number in physical space rather than in Fourier space. If we replace the derivatives in space by central differences on a uniform periodic mesh, then we wish to choose a $(4n) \times (4n)$ matrix so as to minimize the condition number of

$$\begin{bmatrix} E_o & & & & & & \\ & \bigcirc & & & & & \\ & & E_o & & & & \\ & & & \ddots & & & \\ & & & & E_o & & \\ & & & & & \bigcirc & \\ & & & & & & E_o \end{bmatrix} \qquad \begin{bmatrix} 0 & A_o & 0 & \cdots & B_o & 0 & \cdots & 0 \\ -A_o & 0 & A_o & & & B_o & & \vdots \\ 0 & -A_o & \ddots & \ddots & & & \ddots & 0 \\ \vdots & & & \ddots & & & & B_o \\ \vdots & & \ddots & \ddots & & & & 0 \\ 0 & & & & & & & \vdots \\ -B_o & 0 & & \ddots & & \ddots & & \vdots \\ 0 & -B_o & & & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & & & 0 \\ \vdots & & & \ddots & & & & A_o \\ 0 & \cdots & 0 & -B_o & 0 & \cdots & 0 & -A_o & 0 \end{bmatrix}$$

This is similar to the preconditioning that appears in the use of the conjugate gradient method [6]. However, now A_o and B_o are themselves matrices. Furthermore, the matrix to be conditioned is not symmetric but antisymmetric. Hence, this approach is not very useful for general A_o and B_o. We therefore abandon the attempt to find a general solution to Objective No. 2. Instead we shall consider specific cases for the Euler equations.

5. LOW SPEED FLOWS

When $u^2 + v^2 \ll c^2$ standard explicit schemes are inefficient. The time step is governed by $1/c$ while most important phenomena move at the convective speed. Implicit methods, especially A.D.I. type methods, also slow

down due to the presence of different time scales. One possibility is to use a semi-implicit method, but this is hard to implement in a conservative manner. If one is interested in time accuracy then one also needs to filter the high frequency content and then use an implicit method on the incompressible portion [10]. We shall instead precondition the Euler equations to remove the dependence on the sound speed, c. Viviand [25], and Briley, McDonald, and Shamroth [2] have considered similar problems for the reduced isoenergetic equations. We shall also discuss this case in a later section. We now consider the full Euler equations so that we can easily extend the results to both the compressible and incompressible Navier-Stokes equations. The conservative Euler equations in curvilinear coordinates (x,y) can be symmetrized by an equivalence transform with $S = T^{-1}$, [22], [23]. We then recover (6) with $ST = I$ and

$$A_o = \begin{bmatrix} q & Y_y c & -X_y c & 0 \\ Y_y c & q & 0 & 0 \\ -X_y c & 0 & q & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

$$B_o = \begin{bmatrix} r & -Y_x c & X_x c & 0 \\ -Y_x c & r & 0 & 0 \\ X_x c & 0 & r & 0 \\ 0 & 0 & 0 & r \end{bmatrix} \tag{18}$$

$$T_o = \begin{bmatrix} \rho/c & 0 & 0 & -1/c \\ u/c & \rho & 0 & -u/c \\ v/c & 0 & \rho & -v/c \\ h/c & u & v & (u^2 + v^2)/2c \end{bmatrix}$$

where (X,Y) are the Cartesian coordinates and q and r are the contravariant components of velocity given by

$$q = Y_y\, u - X_y\, v \qquad r = X_x\, v - Y_x\, u. \tag{19}$$

We then choose the preconditioner E_o in (7) as

$$E_o = \begin{bmatrix} z^2/c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

where $z^2 = \max(\varepsilon^2, u^2 + v^2)$ is introduced so that E_o is nonsingular at stagnation points. Typically ε is chosen as $.001c$ so that $z^2/c^2 > .001$. Transforming back to (3) we find that [24]

$$E = I + dQ \qquad\qquad E^{-1} = I + eQ$$

$$d = (\gamma - 1)(z^2/c^2 - 1)/c \qquad e = hd$$

where h is the enthalpy, $h = c^2/(\gamma - 1) + s^2$, $s^2 = (u^2 + v^2)/2$ and

$$Q = \begin{bmatrix} s^2 & -u & -v & 1 \\ us^2 & -u^2 & -uv & u \\ vs^2 & -uv & -v & v \\ hs^2 & -uh & -vh & h \end{bmatrix}. \tag{21}$$

We note that the lower three rows of Q are obtained by multiplying the first row of Q by u, v, and h respectively. Hence, Q times a vector can be computed using six multiplies.

Let M be the Mach number defined by $M^2 = z^2/c^2$. Then the largest eigenvalue of $D = A\omega_1 + B\omega_2$ is given by

$$2\lambda = |w| \quad (1 + M^2) + \sqrt{w^2(1 - M^2) + 4(a^2 + b^2)z^2} \tag{22}$$

where

$$w = q\omega_1 + r\omega_2, \qquad a = Y_y\ \omega_1 - Y_x\ \omega_2, \qquad b = X_x\ \omega_2 - X_y\ \omega_1.$$

Hence, near a stagnation point $M = O(\varepsilon)$ and $\lambda = O(\varepsilon)$. It follows that at low speeds $\Delta t/\Delta x = K/\max\left(\sqrt{u^2 + v^2}, \varepsilon\right)$ and so Δt is independent of c. Briley et al. [2] present results for the Navier-Stokes equations with turbulence using an implicit method. They show the advantages for a similar preconditioning for the isoenergetic equations.

6. ISOENERGETIC EQUATIONS

The steady state Euler equations have the property that the total specific enthalpy, $h = (E + p)/\rho$ is constant along streamlines. Hence, when the flow comes from a common reservoir the total enthalpy is constant throughout the entire field. Thus, various authors have replaced the energy equation in the inviscid equations by the algebraic condition that $h = h_o$. This system is no longer time consistent but gives the correct solution in the steady state.

In two space dimensions the isoenergetic equations form a 3×3 hyperbolic system. The theory of these equations was first discussed in [9]. Since there are several errors in that paper we shall derive the pertinent results. The nondimensional isentropic equations can be written in the form (4), [9], with

$$A = \begin{bmatrix} u & \rho & 0 \\ c^2/\gamma p & (1 - 2R)u & -2Rv \\ 0 & 0 & u \end{bmatrix} \quad B = \begin{bmatrix} v & 0 & \rho \\ 0 & v & 0 \\ c^2/\gamma p & -2Ru & (1 - 2R)v \end{bmatrix} \tag{23}$$

where $R = (\gamma - 1)/2\gamma$ and $c^2 = \gamma p/\rho$.

Note, that the definition given for c differs slightly from that given in [9]. We now define

$$a_{\pm} = Ru \pm \sqrt{R^2 u^2 + c^2/\gamma} \; .$$

It is easily seen that a_+ is always positive while a_- is always negative. Using the technique described in [9] we let

$$T = \begin{bmatrix} \dfrac{c^2}{\gamma(\gamma - 1)} \sqrt{\dfrac{-a_-}{a_+ - a_-}} & \dfrac{c^2}{\gamma(\gamma - 1)} \sqrt{\dfrac{a_+}{a_+ - a_-}} & 0 \\ 0 & 0 & c^2/\gamma(\gamma - 1) \\ \dfrac{-\rho c^2}{\gamma(\gamma - 1)a_+} \sqrt{\dfrac{-a_-}{a_+ - a_-}} & \dfrac{-\rho c^2}{\gamma(\gamma - 1)a_-} \sqrt{\dfrac{a_+}{a_+ - a_-}} & 2vR \end{bmatrix} .$$

It can then be verified that

$$T^{-1}\,AT = \begin{bmatrix} u - a_+ & 0 & 0 \\ 0 & u - a_- & 0 \\ 0 & 0 & u \end{bmatrix}$$

and

$$T^{-1}\,BT = vI - \frac{2}{a_+ - a_-}\begin{bmatrix} Rva_+ & Rvc/\delta\gamma & \sqrt{\frac{a_+(a_+ - a_-)}{\gamma}}\,\frac{c}{2} \\ Rvc/\delta\gamma & -Rva_- & \sqrt{\frac{-a_-(a_+ - a_-)}{\gamma}}\,\frac{c}{2} \\ \sqrt{\frac{a_+(a_+ - a_-)}{\gamma}}\,\frac{c}{2} & \sqrt{\frac{-a_-(a_+ - a_-)}{\gamma}}\,\frac{c}{2} & 0 \end{bmatrix} \tag{24}$$

and so A and B can be simultaneously symmetrized. This property is also necessary if we wish to construct any entropy function [16]. Since the isoenergetic equations are a symmetric hyperbolic system, we can use energy methods to determine well-posed boundary conditions as well as the normal mode approach used in [9]. Furthermore, we define a state as supersonic if numbers ω_1 and ω_2 exist such that $\omega_1 A + \omega_2 B$ is positive definite. It can then be shown that the isoenergetic equations are supersonic if and only if $u^2 + v^2 > c^2$.

Since the isoenergetic equations are symmetrizable we can use the theory developed in the previous sections. If we choose E_o as

$$E_o = \begin{bmatrix} z^2/c^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad z \text{ as in (20)}, \tag{25}$$

then the condition number of $E^{-1}(\omega_1 A_o + \omega_2 B_2)$ is independent of c. This is similar to the preconditioning previously considered and also similar to that considered in [2].

7. INCOMPRESSIBLE FLOW

We next consider the steady state, inviscid, incompressible fluid dynamic equations. Klainerman and Majda [12] have proven that these equations are the asymptotic reduced equations of the Euler equations. Hence, one method of solving the incompressible equations is to numerically solve the homentropic Euler equations or Navier-Stokes equations, e.g., [20] with a small Mach number and then use the preconditioning of Section 5 to remove the stiffness of the equations. In this section we shall consider ways to directly integrate the incompressible equations. With both approaches the introduction of viscous terms does not introduce any fundamental difficulties especially with a high Reynolds number. Since we are interested in a pseudo-time approach, we consider the artificial density algorithm [5].

In conservation form the time-dependent equations are

$$u_x + v_y = 0 \tag{26a}$$

$$u_t + (u^2 + p)_x + (uv)_y = 0 \tag{26b}$$

$$v_t + (uv)_x + (v^2 + p)_y = 0. \tag{26c}$$

Using the artificial density approach [18] we replace (26a)

$$p_t/c + u_x + v_y = 0. \tag{26a'}$$

It is easy to verify that the resultant system is hyperbolic but not symmetrizable. Instead we replace (26) by

$$\begin{aligned} &p_t/c + u_x + v_y = 0 \\ &aup_t/c + u_t + (u^2 + p)_x + (uv)_y = 0 \\ &avp_t/c + v_t + v_t + (uv)_x + (v^2 + p)_y = 0 \end{aligned} \tag{27}$$

with a to be defined. Equivalently,

$$E^{-1} w_t + Aw_x + Bw_y = 0, \qquad w = (p,u,v)$$

and so (27) can be considered as a preconditioning of the system (26). c is an artificial sound speed which need not be constant. We shall later discuss how to choose c.

When $a = 1$ in (27) then this system is equivalent to a symmetric hyperbolic system. In this case the eigenvalues of $E(A\omega_1 + B\omega_2)$ are

$$q \; ; \; \left(q \pm \sqrt{q^2 + 4(\omega_1^2 + \omega_2^2)c^2}\right)/2, \qquad q = u\omega_1 + v\omega_2. \tag{28}$$

It thus follows that this system is always subsonic independent of the value we choose for c. This is to be expected as we do not wish an incompressible fluid to behave like a supersonic flow with shocks even in the nonphysical

time-dependent phase. When a differs from 1 the system is no longer symmetrizable though still hyperbolic. In this case the eigenvalues of (27) are

$$q \; ; \quad \left((2 - a)q \pm \sqrt{(2 - a)^2 q + 4(\omega_1^2 + \omega_2^2)c^2}\right)/2. \tag{29}$$

P. Roe (private communication) has noted that for $a = 2$ the eigenvalues have the simple form

$$q \; ; \; \pm c. \tag{30}$$

Hence, in this case the speed of the sound waves is independent of the convective speed, and hence the sound waves spread isotropically even in the presence of a flow. We shall later see that this allows a more optimal selection for the artificial speed of sound c. We therefore rewrite (27) with $a = 2$ in nonconservative form

$$\begin{aligned} &p_t/c^2 + u_x + v_y = 0 \\ &up_t/c^2 + u_t + uu_x + vu_y + p_x = 0 \\ &vp_t/c^2 + v_t + uv_x + vv_y + p_y = 0 \end{aligned} \tag{31}$$

or equivalently

$$\begin{aligned} &p_t + c^2(u_x + v_y) = 0 \\ &u_t + vu_y - uv_y + p_x = 0 \\ &v_t + uv_x - vu_x + p_y = 0. \end{aligned} \tag{32}$$

The eigenvalues of this new artificial density equation are given by (30). The improvement in the sound speed is achieved at the expense of the loss of symmetry. It is not clear that this loss of symmetry is of any importance since all the coefficients that appear are well-behaved and the system is strictly hyperbolic. The original pseudo-density equations in nonconservative form are

$$
\begin{aligned}
&p_t/c^2 + u_x + v_y = 0 \\
&u_t + uu_x + vu_y + p_x = 0 \qquad (33) \\
&v_t + uv_x + vv_y + p_y = 0.
\end{aligned}
$$

This is equivalent to (27) with $a = 1$, and so is symmetric.

The question of how to choose the artificial sound speed c remains. As we have stressed, for inviscid flow we wish to reduce the ratio of the largest eigenvalue to the smallest eigenvalue. For the system (31), (32), or (27) with $a = 2$, the eigenvalues are given by (30). Hence, we would like to choose $c = q = \omega_1 u + \omega_2 v$. This choice would give us a condition number of one. However, we cannot allow c to depend on the Fourier variables ω_1 and ω_2. Hence, an alternate choice is to set $c^2 = u^2 + v^2$.

For the original equations (33) or else, (27) with $a = 1$, we wish to minimize both

$$\left(1 + \sqrt{1 + 4c^2/q^2}\right)/\left(1 - \sqrt{1 + 4c^2/q^2}\right) \quad \text{and} \quad 1 + \sqrt{1 + 4c^2/q^2}\ .$$

If we choose c small we enlarge the first ratio while if we choose c small we increase the second ratio. It is easy to calculate that the minimum of the maximum of both ratios is reached when $c^2 = 3q^2/4$. In that case the condition number is three. Hence, if we could choose this value for c the original pseudo-density system (33) would be three times slower than the new version given by (31) or (32). As before, this choice for c is not legitimate since it depends on the Fourier variables (ω_1,ω_2). As before, an alternative is to choose $c^2 = 3(u^2 + v^2)/4$. In this analysis we have only considered the effect of the inviscid time step on c. In [4] the effect of the viscous terms is considered.

When the incompressible Navier-Stokes equations are considered, the pseudo-density approach can be easily modified to include these terms. When the Reynolds number is sufficiently large, for a given mesh, the time step is only governed by the inviscid part and the previous analysis is valid. For lower cell Reynolds number one can treat the viscous terms implicitly. Since the coefficients are constant for the viscous portion, a backward Euler method even in several space dimensions is feasible.

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16. Abstract

We consider the steady state Euler and Navier-Stokes equations for both compressible and incompressible flow. Methods are found for accelerating the convergence to a steady state. This acceleration is based on preconditioning the system so that it is no longer time consistent. In order that the acceleration technique be scheme independent this preconditioning is done at the differential equation level. Applications are presented for very slow flows and also for the incompressible equations.

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