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A DEFORMATION ANALYSIS OF FLAT FLEXIBLE GEAR AND ITS EQUATION OF ORIGINAL CURVED SURFACES

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16. Abstract

The equation of the original curved surface of end harmonic gearing is determined by displacement analysis of flat flexible gear. The displacement analysis is also used to calculate the strength and rigidity of the gear. The latter is regarded as a circular plate with two concentrated loads, since its torsional rigidity is much larger than its bending rigidity. Small-deflection theory of thin plates is used to solve for the displacement of any point in the middle plane of the gear. New expressions are given for given for radial and tangential displacements of the middle plane under asymmetrical loading. A digital computer is used to obtain numerical values for the displacements.

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A DEFORMATION ANALYSIS OF FLAT FLEXIBLE GEAR AND ITS EQUATION OF ORIGINAL CURVED SURFACES

SHEN YUNWEN*

ABSTRACT

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Based on the elastic thin plate theory, the displacement of an arbitrary point on the flexible gear affected by a wave generator was investigated, an equation for the original curved surface which is required to study the engaging theory of the end harmonic gearing was derived, analytical solutions as well as calculated results of "u", "v", and "w" were obtained. Thereby, this paper offers the theoretical basis for studying the geometric theory and the strength calculations of the end harmonic gearing transmission.

FORWARD

Although it has been twenty years since the beginning of the development of the harmonic gearing transmission, yet current research and application of the harmonic gearing are limited to radial engagement transmission. Along with the extended application of the harmonic gearing transmission, some equipment require more strictrequirements for the axial dimensions as well as the torsion rigidity of this type of transmission. Consequently, end surface engaging of harmonic gearing transmission is being developed. In foreign countries, some patents have been awarded. However, due to the complexity of the theoretical study, few theoretical papers with systematical analyses have been published. Moreover, some papers even presented dubious results. For example, reference (1) presented an erroneous result with an imperfect analysis. Reference (2) used excessive approximations. Due to the urgent applications of this transmission method, a systematical study on the end harmonic gearing transmission was conducted. This paper not only pointed out the error in reference (1), but also presented an analytical method for the deformation of a flat flexible gear.

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In addition, it also introduced the original curved surface equation which is necessary for the engaging analysis of the end harmonic gearing transmission.

II. MATHEMATICAL MODEL OF DUAL WAVE TRANSMISSION

When we study the flat flexible gear of the end harmonic gearing transmission, we can treat the flat gear as a thin plate of uniform thickness which is held along the circumference of an inside circle with a radius ($r = r_c$).

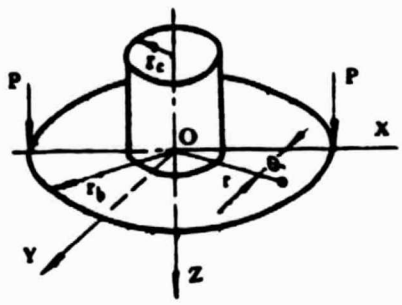

Figure 1. Diagram of the Flat Flexible Gear

As shown in Figure 1, two concentrated loads P are applied on the free boundary of the thin plate which also takes a torsional moment "T" in the middle plane. For more precise analysis, the effect of the bending moment due to engaging which will be distributed according to a certain pattern in the engaging zone should also be considered. For simplification, we temporarily ignore the effect of this engaging force. This effect on the stress distribution will be solved by the proper adjustment of a certain constant (Ref. 3). /176

From the plate and shell theory (Ref. 4), we can see that the stress and the deformation state can be described by the T. Karman's non-linear equations. Let us assume that the middle plane stress function is ϕ, and the displacement of any point on the middle plane along the z-axis is w, then in a polar coordinate ($r \cdot \varphi$), the group of non-linear equations can be written as

$$
\begin{cases}
\frac{D}{\delta}\nabla^2\nabla^2 w = \left[\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2}\right) + \left(\frac{1}{r}\frac{\partial w}{\partial r}\right.\right. \\
\left.\left. + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right)\frac{\partial^2 \Phi}{\partial r^2} - 2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial r}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right)\right] + \frac{q}{\delta}, \\
\nabla^2\nabla^2 \Phi = E\left\{\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right)\right]^2 - \frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}\right.\right. \\
\left.\left. + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\right)\right\}.
\end{cases}
\tag{1}
$$

where, D - cylindrical rigidity of the plate $D = E\delta^3/12(1-\mu^2)$, E and μ, elastic modulus and Poisson ratio respectively; δ - plate thickness, i.e., the wall thickness of the flat flexible gear; q - uniformly distributed load on the plate in the lateral direction; ∇^2 - operator,

$$= \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}.$$

From the engineering point of view, it is not necessary to include the geometric factors and mechanical factors to solve these equations, because these two factors would complicate the problem. This problem can be simplified into a small deflection problem because the effect of the deformation due to the torsion moment "T" can be neglected since the rigidity of the flat flexible gear is high. In addition, the gear is in a very slight bending configuration because the maximum axial deformation W of the flexible gear under the concentratedload is about the same as the working height of the gear-tooth or the gear thickness. Since a circular plate is under a concentrated load "P", the uniform distributed load can be considered as q=0. When we substitute ϕ =0, then equation (1) can be simplified as

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}\right) = 0. \tag{2}$$

Let u, v, and w represent the radial, tangential, and axial displacement respectively, based on the condition that the strain of the middle plane is zero, and according to the

elastic thin plate theory we can easily find the geometric equations as follows

$$\begin{cases} \varepsilon_r = \dfrac{\partial u}{\partial r} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial r}\right)^2 = 0, \\ \varepsilon_\varphi = \dfrac{\partial v}{r\partial\varphi} + \dfrac{u}{r} + \dfrac{1}{2}\left(\dfrac{1}{r}\dfrac{\partial w}{\partial\varphi}\right)^2 = 0, \\ \varkappa_{r\varphi} = \dfrac{1}{r}\dfrac{\partial u}{\partial\varphi} + \dfrac{\partial v}{\partial r} + \left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{1}{r}\dfrac{\partial w}{\partial\varphi}\right) = 0. \end{cases} \tag{3}$$

The physical equations are found to be [4] and [5]

$$\begin{cases} M_r = -D\left[\dfrac{\partial^2 w}{\partial r^2} + \mu\left(\dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial\varphi^2}\right)\right], \\ M_\varphi = -D\left(\dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial\varphi^2} + \mu\dfrac{\partial^2 w}{\partial r^2}\right), \\ M_{r\varphi} = (1-\mu)D\left(\dfrac{1}{r}\dfrac{\partial^2 w}{\partial r\partial\varphi} - \dfrac{1}{r^2}\dfrac{\partial w}{\partial\varphi}\right). \end{cases} \tag{4}$$

and /177

$$\begin{cases} Q_r = -D\dfrac{\partial}{\partial r}\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial\varphi^2}\right), \\ Q_\varphi = -D\dfrac{\partial}{r\partial\varphi}\left(\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial\varphi^2}\right). \end{cases} \tag{5}$$

where, ε_r, ε_φ and $\varepsilon_{r\varphi}$ - radial, tangential, and shear strain respectively; M_r, M_φ and $M_{r\varphi}$ - the bending moment and torsion moment along the directions of r and φ respectively; Q_r and Q_φ - shear forces along the directions of r and φ respectively.

III. AXIAL DISPLACEMENT

The axial displacement ω can be solved with equation (2). The following sections will find the solutions of these differential equations.

1. Boundary Conditions

For a flat flexible gear with an inner holding circle $r = r_c$ the boundary conditions of the inner holding side can be expressed as

$$(w)_{r=r_c} = 0, \quad \left(\frac{\partial w}{\partial r}\right)_{r=r_c} = 0. \tag{6}$$

Since the outside circumference ($r = r_b$) of the flexible gear is free, hence,

$$(M_r)_{r=r_b} = 0; \tag{7}$$

In addition, there are two concentrated loads, 180° apart from the x-axis of the free boundary; therefore the shear force is not continuous. When the positions of the two concentrated forces are treated in such a way that the phase difference is π, then the period will be formed by the accumulation of a 2π - cycle function. In the meantime, if we assume that the effect of the concentrated load is equivalent to a uniformly distributed load q acting on the minute arc $2r_b \cdot \Delta\varphi$, then we can use the method in Reference (6) to expand the concentrated load into a Fourier series, and use relation of $q_b = P/2r \cdot \Delta\varphi$ to determine the corresponding constants. Finally, let us find the limiting value by assuming $\Delta\varphi \to 0$, and accumulate the effect of the two concentrated forces which have a phase difference of π. Then the expanded Fourier series will be

$$P_z = \frac{2P}{\pi r_b}\left(\frac{1}{2} + \sum_{n=2,4,6,} \cos n\varphi\right),$$

and we can also find the other boundary condition of the free boundaries as

$$\left(Q_r - \frac{\partial M_{r\varphi}}{r\partial\varphi}\right)_{r=r_b} = \frac{2P}{\pi r_b}\left(\frac{1}{2} + \sum_{n=2,4,6,} \cos n\varphi\right). \tag{8}$$

2. Solutions of Equation (2)

From Ref. (5), equation (2) can be expressed by a series introduced by A. Clebsch. Since φ is measured on x-axis, we can keep the terms containing only $\cos n\varphi$. Moreover, there are two concentrated forces, hence n is an even number. Then

$$w = R_0(r) + \sum_{n=2,4,6,\ldots} R_n(r)\cos n\varphi. \tag{9}$$

in which the coefficients represented by $R_o(r)$ and $R_n(r)$ ($n=2,4,6,\ldots$) are the functions of the polar radius "r". (They will be abbreviated as R_o, R_n). When equation (9) is substituted into Equation (2), we obtain the following differential equation

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\left(\frac{d^2R_n}{dr^2}+\frac{1}{r}\frac{dR_n}{dr}-\frac{n^2R_n}{r^2}\right)=0. \qquad (10)$$

Let us introduce a non-dimensional variable $\beta = r/r_b$, the we can /178 solve equation (1) as

$$R_0 = A_0 + r_b^2 B_0 \beta^2 + C_0 \ln\beta + r_b^2 D_0 \beta^2 \ln\beta \text{ (当 } n=0 \text{ 时)}, \qquad (11)$$

$$R_n = r_b^n A_n \beta^n + r_b^{-n} B_n \beta^{-n} + r_b^{n+2} C_n \beta^{n+2} + r_b^{-n+2} D_n \beta^{-n+2} \text{ (at } n \geqslant 2 \text{)}. \qquad (12)$$

When we utilized the boundary conditions, i.e., equations (6), (7) and (8), we can obtain the values of the coefficients A_o, B_o, C_o, D_o, A_o, B_o, C_o and D_o.

For the solution R_o, when n = 0, the group of linear equations containing the coefficients A_o, B_o, C_o, and D_o can be derived from the boundary conditions. After the solution is obtained, when we assume

$$a_0 = (\pi D/Pr_b^2)A_0,\ b_0 = (\pi D/P)B_0,\ c_0 = (\pi D/Pr_b^2)C_0,\ d_0 = (\pi D/P)D_0,$$

then "R_o" can be expressed as

$$R_0 = (Pr_b^2/\pi D)(a_0 + b_0\beta^2 + c_0\ln\beta + d_0\beta^2\ln\beta), \qquad (13)$$

in which all the coefficients can be expressed as

$$\begin{cases} a_0 = -b_0\beta_1^2 - c_0\ln\beta_1 - d_0\beta_1^2\ln\beta_1, \\ b_0 = \dfrac{c_0}{2}\dfrac{(1-\mu)}{(1+\mu)} - d_0\dfrac{(3+\mu)}{2(1+\mu)}, \\ c_0 = -d_0\beta_1\dfrac{(2\ln\beta_1+1)(1+\mu)-(3+\mu)}{\beta_1(1-\mu)+\beta_1^{-1}(1+\mu)}, \\ d_0 = -0.25. \end{cases} \qquad (14)$$

where $\beta_1 = r_c/r_b$.

As to the condition $n \geqslant 2$, when we consider

$$(R_0)_{r=r_c} = 0,\quad \left(\frac{\partial R_0}{\partial r}\right)_{r=r_c} = 0,$$

$$\left[\frac{\partial^2 R_0}{\partial r^2} + \mu\left(\frac{1}{r}\frac{\partial R_0}{\partial r}\right)\right]_{r=r_b} = 0,\quad \left[\frac{\partial}{\partial r}\left(\frac{\partial^2 R_0}{\partial r^2} + \frac{1}{r}\frac{\partial R_0}{\partial r}\right)\right]_{r=r_b} = \frac{P}{\pi D},$$

we can obtain the group of linear equations containing the coefficients A_n, B_n, C_n, and D_n, which can be obtained by solving this group of linear equations. Let

$$a_n = (\pi D/Pr_b^{-n+2})A_n,\ b_n = (\pi D/Pr_b^{n+2})B_n,\ c_n = (\pi D/Pr_b^{-n})C_n,\ d_n = (\pi D/Pr_b^{n})D_n.$$

the R_n can be expressed as

$$R_n = (Pr_b^2/\pi D)(a_n\beta^n + b_n\beta^{-n} + c_n\beta^{n+2} + d_n\beta^{-n+2}) \tag{15}$$

where

$$\begin{cases} a_n = -\beta_1^2\left(c_n\dfrac{n+1}{n} + d_n\dfrac{\beta_1^{-2n}}{n}\right), \\ b_n = \beta_1^2\left(c_n\dfrac{\beta_1^{2n}}{n} - d_n\dfrac{n-1}{n}\right), \\ c_n = \dfrac{\left[(n+1)(1-\beta_1^2) - \beta_1^{-2n+2} - \dfrac{3+\mu}{1-\mu}\right]}{n(n+1)(1-\mu)\kappa}, \\ d_n = \dfrac{\left[(n-1)(1-\beta_1^2) + \beta_1^{2n+2} + \dfrac{3+\mu}{1-\mu}\right]}{n(n-1)(1-\mu)\kappa}, \\ \kappa = (1-\beta_1^2)(n^2-1) + \left(\beta_1^{-2n+2} + \dfrac{3+\mu}{1-\mu}\right)\left(\beta^{2n+2} + \dfrac{3+\mu}{1-\mu}\right). \end{cases} \tag{16}$$

Therefore, the axial displacement "w" of any point on the middle plane of the flat flexible gear will be

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$$w = \frac{Pr_b^2}{\pi D}\Big[(a_0 + b_0\beta^2 + c_0\ln\beta + d_0\beta^2\ln\beta) + \sum_{n=2,4,6,\cdots}(a_n\beta^n + b_n\beta^{-n} + c_n\beta^{n+2} + d_n\beta^{-n+2})\cos n\varphi\Big]. \tag{17}$$

When $\beta = 1$ and $\varphi = 0°$, the axial displacement "w" reaches maximum "w_o", then from equation (17), we can use "w_o" to express the equation for an axial displacement as

$$w = \frac{w_0}{(a_0 + b_0) + \sum_{n=2,4,6,\cdots}(a_n + b_n + c_n + d_n)}\Big[(a_0 + b_0\beta^2 + c_0\ln\beta + + d_0\beta^2\ln\beta) + \sum_{n=2,4,6,\cdots}(a_n\beta^n + b_n\beta^{-n} + c_n\beta^{n+2} + d_n\beta^{-n+2})\cos n\varphi\Big] \tag{18}$$

In practical design, w_o for the end harmonic gearing transmission flexible gear is always determined in the design specification. Hence, equation (18) is very simple and easy to use.

IV. RADIAL DISPLACEMENT AND TANGENTIAL DISPLACEMENT

From the first equation of (3), the radial displacement "u" can be expressed as

$$u = -\frac{1}{2}\int\left(\frac{\partial w}{\partial r}\right)^2 dr = -\frac{1}{2}r_0^{-1}\int\left(\frac{\partial w}{\partial \beta}\right)^2 d\beta,$$

when we substitute (18) in this equation, and let $\kappa_0 = (a_0 + b_0) + \sum_{n=2,4,6,\cdots}(a_n + b_n + c_n + d_n)$, then after integration, we obtain

$$u = -(w_0^2/2)\, r_0^{-1}\kappa_0^{-2}\Big[U_0 + 2\sum_{n=2,4,6,\cdots} U_{0n}\cos n\varphi + \sum_{n=2,4,6,\cdots} U_n \cos^2 n\varphi + \sum_{n=2,4,6,\cdots}^{(n\neq m)}\sum_{m=2,4,6,\cdots} U_{nm}\cos n\varphi \cos m\varphi\Big] + C_n(\beta,\varphi). \tag{19}$$

In which

$$U_0 = \frac{4}{3}\beta^3\Big\{b_0^2 + d_0^2\Big[(\ln\beta)^2 - \frac{2}{3}\ln\beta + \frac{17}{36}\Big] + d_0(2b_0 + d_0)\Big(\ln\beta - \frac{1}{3}\Big) + b_0 d_0\Big\} + 2\beta c_0[b_0 + 2d_0\ln\beta - d_0] - c_0^2\beta^{-1}, \tag{20}$$

$$\begin{aligned} U_{0n} = {} & \frac{\beta^{n+1}}{(n+1)}\Big[2nb_0a_n + nd_0a_n + (n+2)c_0c_n + 2nd_0a_n\Big(\ln\beta - \frac{1}{n+1}\Big)\Big] \\ & + \frac{\beta^{-n+1}}{(n-1)}\Big[2nb_0b_n + nd_0b_n + (n-2)c_0d_n + 2nd_0b_n\Big(\ln\beta + \frac{1}{n-1}\Big)\Big] \\ & + \frac{\beta^{n+3}}{(n+3)}\Big[2(n+2)b_0c_n + (n+2)d_0c_n + 2(n+2)d_0c_n\Big(\ln\beta - \frac{1}{n+3}\Big)\Big] \\ & + \frac{\beta^{-n+3}}{(n-3)}\Big[2(n-2)b_0d_n + (n-2)d_0d_n + 2(n-2)d_0d_n\Big(\ln\beta + \frac{1}{n-3}\Big)\Big] \\ & + \frac{\beta^{n-1}}{n-1}nc_0a_n + \frac{\beta^{-n-1}}{n+1}nc_0b_n, \end{aligned} \tag{21}$$

$$
\begin{aligned}
U_n = {} & \frac{n^2 a_n^2}{(2n-1)}\beta^{2n-1} - \frac{n^2 b_n^2}{2n+1}\beta^{-2n-1} + \frac{(n+2)^2 c_n^2}{(2n+3)}\beta^{2n+3} \\
& - \frac{(n-2)^2 d_n^2}{(2n-3)}\beta^{-2n+3} + 2n^2 a_n b_n \beta^{-1} + \frac{2n(n+2)\,a_n c_n}{(2n+1)}\beta^{2n+1} \\
& - \frac{2n(n-2)\,b_n d_n}{(2n-1)}\beta^{-2n+1} - \frac{2(n-2)(n+2)c_n d_n}{3}\beta^3 \\
& - 2n(n-2)a_n d_n \beta - 2n(n+2)b_n c_n \beta_1
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
U_{nm} = {} & \frac{nm a_n a_m}{n+m-1}\beta^{n+m-1} - \frac{nm a_n b_m}{n-m-1}\beta^{n-m-1} \\
& + [n(m+2)a_n c_m + m(n+2)c_n a_m]\frac{\beta^{n+m+1}}{n+m+1} \\
& - \frac{[n(m-2)a_n d_m + m(n+2)c_n b_m]}{n-m+1}\beta^{n-m+1} \\
& - \frac{nm b_n a_m}{m-n-1}\beta^{m-n-1} - \frac{nm b_n b_m}{n+m+1}\beta^{-n-m-1} \\
& - \frac{[n(m+2)b_n c_m + m(n-2)d_n a_m]}{n-m+1} \\
& - \frac{[m(n-2)d_n b_m + n(m-2)b_n d_m]}{n+m-1}\beta^{-n-m+1} \\
& + \frac{(n+2)(m+2)c_n c_m}{n+m+3}\beta^{n+m+3} - \frac{(n+2)(m-2)c_n d_m}{n-m+3}\beta^{n-m+3} \\
& - \frac{(n-2)(m+2)d_n c_m}{m-n+3}\beta^{m-n+3} - \frac{(n-2)(m-2)d_n c_m}{n+m-3}\beta^{-n-m+3}.
\end{aligned}
\tag{23}
$$

The integration constant $C_n(\beta,\varphi)$ can be determined by the boundary conditions. When $\beta = \beta_1$, and $u = 0$, then

$$
C_n(\beta,\varphi) = C_n(\beta_1,\varphi) = \frac{1}{2}\,\omega_0^2 r_b^{-1}\kappa_0^{-2}\Big[U_0 + 2\sum_{n=2,4,6,\cdots} U_{0n}\cos n\varphi
+ \sum_{n=2,4,6,\cdots} U_n \cos^2 n\varphi + \sum_{n=2,4,6,\cdots}^{(n\neq m)}\ \sum_{m=2,4,6,\cdots} U_{nm}\cos n\varphi\cos m\varphi\Big]_{\beta=\beta_1}. \tag{24}
$$

The tangential displacement v can be determined by the second equation of equation (3), thus

$$v = -\int\left[u + \frac{1}{2r}\left(\frac{\partial w}{\partial \varphi}\right)^2\right]d\varphi = -\int\left[u + \frac{1}{2r_0\beta}\left(\frac{\partial w}{\partial \varphi}\right)^2\right]d\varphi. \tag{25}$$

When we substitute equation (19) and the results of the second derivative from the equation (18) with respect to φ, after integration we obtain

$$v = \frac{1}{2}w_0^2\kappa_0^{-2}r_0^{-1}(U_o - \beta^{-1}V_o) + C_v(\beta, \varphi), \tag{26}$$

where

$$U_o = 2\sum_{n=2,4,6,\ldots}\frac{1}{n}(U_{on} - U_{con})\sin n\varphi + \sum_{n=2,4,6,}\frac{1}{n}(U_n - U_{cn})\left(\frac{n\varphi}{2} + \frac{1}{4}\sin 2n\varphi\right)$$
$$+ \sum_{n=2,4,6,}^{(n\neq m)}\sum_{m=2,4,6,}(U_{nm} - U_{cnm})\left[\frac{\sin(m+n)\varphi}{2(m+n)} + \frac{\sin(m-n)\varphi}{2(m-n)}\right], \tag{27}$$

in which U_{con}, U_{cn}, and U_{cum} explain the values of U_{on}, U_n, and U_{nm} when $\beta = \beta_1$. They can be obtained by substituting $\beta = \beta_1$ respectively in the equations (20), (21) and (23).

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$$V_o = -\sum_{n=2,4,6,\ldots}\frac{n}{4}R_n^2\sin 2n\varphi$$
$$+ \sum_{n=2,4,6,}^{(n\neq m)}\sum_{m=2,4,6,\ldots} nmR_nR_m\left[\frac{\sin(m-n)\varphi}{2(m-n)} - \frac{\sin(m+n)\varphi}{2(m+n)}\right]. \tag{28}$$

The integration constant $C_v(\beta,\varphi)$ can be obtained by considering the following two conditions: when $\beta=\beta_1$, v = 0; and when φ = 0° and 90°, v = 0. The physical meaning of the latter is apparent: in order to satisfy the condition that the strain in the middle plane is zero for any value of "r", when φ = 0° and 90° "v" has to be zero. Therefore, we can ignore the terms which do not satisfy the boundary condition in equations (28) and (27), hence,

$$C_v(\beta,\varphi) = C_v(\beta_1, \varphi) = -\frac{1}{2}w_0^2\kappa_0^{-2}r_0^{-1}[U_o - \beta^{-1}V_o]_{\beta=\beta_1}. \tag{29}$$

V. ORIGINAL CURVED SURFACE EQUATION $\tilde{C}_p$

Similar to studying of the engagement theory for the harmonic gearing transmission, we define the middle plane equation after the deformation of the flexible gear with the initial curved surface equation $\tilde{C}_p$. Based on the displacements u, v, and w, we can obtain $\tilde{C}_p$.

Let us assume φ_1 represents the equivalent rotating angle of the vector generated by the tangential displacement of any considered point on the flexible gear, then $\tilde{C}_p$ can be expressed by the first order approximation as

$$\begin{cases} X = (r_b\beta + u)\cos\varphi_1, \\ Y = (r_b\beta + u)\sin\varphi_1, \\ Z = w. \end{cases} \tag{30}$$

in which, $\varphi_1 = \varphi + v/r_b\beta$.

Equation (30) is a basic equation for studying the end harmonic gearing transmission geometry.

VI. CALCULATION RESULTS AND CONCLUSION

For our convenience, we can write Equations (18), (19), and (26) in the following forms

$$\begin{cases} u = -u'(w_n^2/r_b), \\ v = v'(w_0^2/r_b), \\ w = w'w_0. \end{cases} \tag{31}$$

In application, for a flexible gear: $r_c = (0.3 \sim 0.4)r_b$, therefore we conducted a series of calculations assuming $\beta_1 = 0.3$ and 0.4. In the calculations, we select $\beta_1 = 0.3$, and the terms in the series n = 2,4,6,...20. The calculated results u', v', and w', from a FELIX C-256 computer, are shown in Figure 2 (β_1 = 0.3) and Figure 3 ($\beta_1 = 0.4$), in which the solid lines represent the values of w'; dotted lines, v'; and dot-dash lines, u'. From Figure 2 we can see whatever β varies, the maximum values of w' and u' always occur at $\varphi = 0°$, yet the values of "v'" always vary with β. For example, when $\beta = 1.0$, the maximum v' occurs at around $\varphi = 15°$; and $\beta = 0.6$, 25°. Figure 4 shows the curves of the maximum values if u', v', and w' vs β.

When $\beta = 1$, the values of u, v, and w are critical for the engaging analysis of the end harmonic gearing transmission. Hence, for the designer's applications, we calculated the values (Table I) of u', v', and w' under the conditions of $\beta_1 = 0.3$ and 0.4 respectively when $\beta = 1$.

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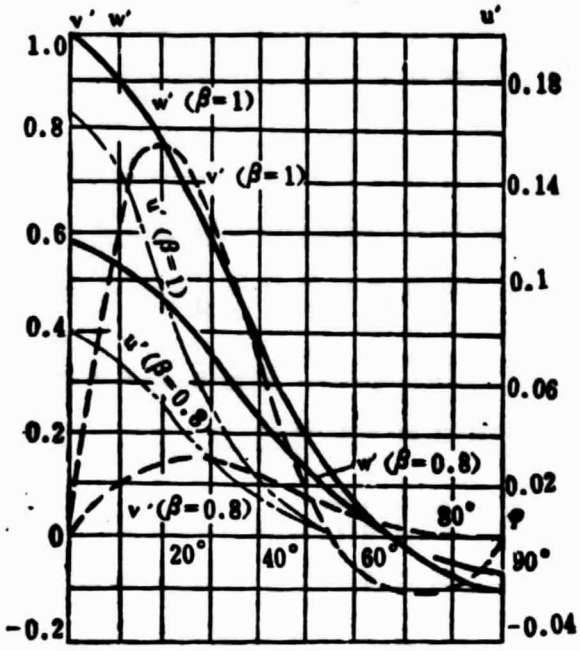

Figure 2 Values of u', v' and w', when $\beta_1 = 0.3$.

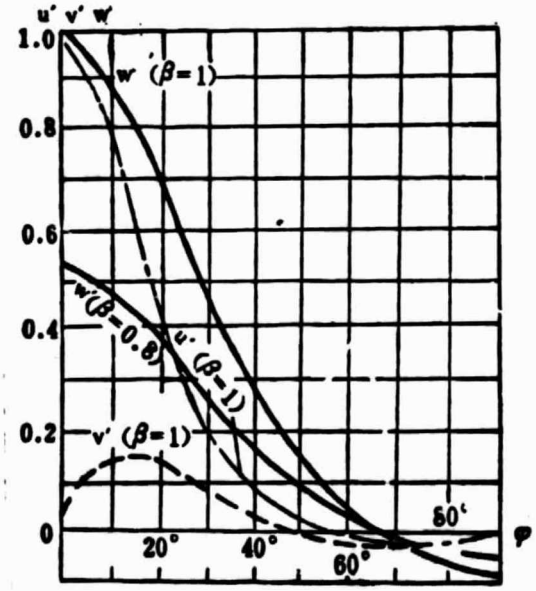

Figure 3 Values of u', v' and w', when $\beta_1 = 0.4$.

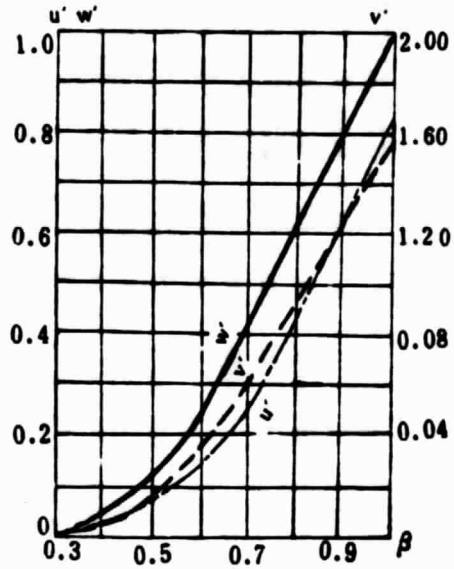

Figure 4 $\beta_1 = 0.3$, u', v' and w' vs β.

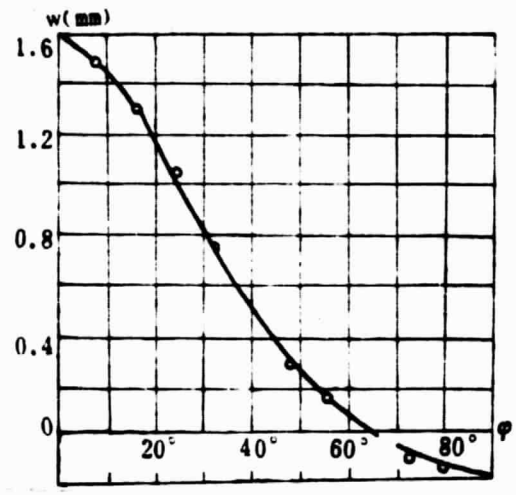

Figure 5 Theoretical Analysis of Actual Measurement ($\beta_1 = 0.37$, $\beta = 1$)

In order to verify the accuracy of this calculation method, we conducted the measurement of a flat flexible gear (with the following dimensions: number of teeth z = 268, gear module m = 0.8mm, tooth width b = 13mm, thickness φ = 2.0mm, r_c = 40mm) under the condition that w_o = 1.6mm. Figure 5 shows the results. In this figure, the solid line represents the calculated values, and the circles represent the measured data (each datum point is

the average of five measurements). The measurement shows calculated value and the measured value exceed no more than 0.015mm. Hence, the error is within the tolerance and the two values are very close. It is concluded that the calculating method recommended in this paper can meet the precision requirement of /183 the end harmonic gearing transmission analysis, and is a feasible method.

Table I. Values of u', v', and w', when $\beta = 1.0$

φ°	$\beta_1 = 0.3$			$\beta_1 = 0.4$		
	u'	v'	w'	u'	v'	w'
0°	0.8321726	0.0000000	1.0000000	0.9775204	0.0000000	1.0000000
5°	0.7965759	0.0764656	0.9798877	0.9260779	0.0868258	0.9749692
10°	0.7017223	0.1314390	0.9235048	0.7916194	0.1416512	0.9054893
15°	0.5757025	0.1561112	0.8410518	0.6198189	0.1541650	0.8059806
20°	0.4468286	0.1556362	0.7449075	0.4541346	0.1364796	0.6937554
25°	0.3325486	0.1406637	0.6453400	0.3179190	0.1071461	0.5827968
30°	0.2385098	0.1192367	0.5481229	0.2150134	0.0777533	0.4803274
35°	0.1636258	0.0947998	0.4548851	0.1396382	0.0513040	0.3873771
40°	0.1052356	0.0684704	0.3653760	0.0849351	0.0271207	0.3021504
45°	0.0615529	0.0417018	0.2798654	0.0464886	0.0051485	0.2235869
50°	0.0314360	0.0172052	0.2001617	0.0217489	-0.0128444	0.1529013
55°	0.0131310	-0.0020920	0.1289173	0.0080536	-0.0247142	0.0926124
60°	0.0038425	-0.0146868	0.6081082	0.0019453	-0.0299428	0.0443598
65°	0.0004946	-0.0209949	0.0180181	0.0001274	-0.0299427	0.0074478
70°	0.0006224	-0.0224488	-0.2250659	0.0004063	-0.0267263	-0.0207643
75°	0.0025784	-0.0203651	-0.0548138	0.0016714	-0.0217060	-0.0430344
80°	0.0051916	-0.0155142	-0.0792365	0.0033314	-0.0154536	-0.0604660
85°	0.0073893	-0.0084348	-0.0948413	0.0047949	-0.0081088	-0.0721371
90°	0.0082658	-0.0000000	-0.1002553	0.0053911	0.0000000	-0.0763083

REFERENCES

[1] N.A. Alfutov, L.G. Alekhin, S.S. Klenikov, V.V. Khrusanov Investigation of the Stressed-Deformational State of the Flexible Wheel of an End-Wave Toothed Transmission, Izvestiya Vuzob, Mashinostroyeiniye, No. 3, 1973, p. 42.

[2] I.A. Shermakov. Determination of the Parameters of the Gearing In an End-Wave Transmission , Izvestiya Vuzob, Mashinostroyeniye, Vol. 1, 1975, p. 58.

[3] Shen Yunwen. "Theoretical Approximation of Strength Calculation for an End-Surface Harmonic Gearing Transmission Flexible Plane Gear", Paper No. 5 submitted to the First National Academic Exchange Meeting on Calculation of Mechanical Parts, 1983.

[4] Vol'mir,A.S. Gibkiye plastinki i obolochki (Flexible plates and shells). Gostekhizdat, 1956, pp 35-36, 171-182.

[5] Timoshenko, S. and Woinowsky-Krieger, S. Theory of Plates and Shells , McGraw-Hill, Book Company, Inc. 1959, pp. 282-293.

[6] Flugge, W. Handbook of Engineering Mechanics , McGraw-Hill Book Company, 1962, Chapter 39 , pp. 24-33.