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APPROXIMATION OF THE NATURAL MODES OF VIBRATION
FOR FLEXIBLE BEAMS WITH TIP BODIES

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SPLINE-BASED RAYLEIGH-RITZ METHODS FOR THE APPROXIMATION OF THE NATURAL MODES
OF VIBRATION FOR FLEXIBLE BEAMS WITH TIP BODIES*

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ABSTRACT

Rayleigh-Ritz methods for the approximation of the natural modes for a class of vibration problems involving flexible beams with tip bodies using subspaces of piecewise polynomial spline functions are developed. An abstract operator theoretic formulation of the eigenvalue problem is derived and spectral properties investigated. The existing theory for spline-based Rayleigh-Ritz methods applied to elliptic differential operators and the approximation properties of interpolatory splines are used to argue convergence and establish rates of convergence. An example and numerical results are discussed.

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1. Introduction

Recently there has been a surge of interest in the use of continuum models to study the dynamics, stability, and control of vibrating structures. This trend has been especially apparent with regard to large flexible spacecraft. Often an important tool in these studies is the structure's natural vibrational frequencies and corresponding mode shapes.

A class of structures which is playing an ever increasing role in these studies is the one which consists of flexible beams with tip bodies. We have investigated the use of piecewise polynomial spline-based Rayleigh-Ritz-Galerkin schemes for the approximation of the natural modes for the transverse vibration of structures of this type.

It is often the case in engineering practice that in using the Rayleigh-Ritz method to determine the natural modes of a complex structure the approximating subspaces of trial functions are chosen as the span of a finite number of mode shapes for a related, but more easily analyzed, structure. In our investigation, which was motivated by one such instance of this (see [18]), we observed that the spline-based schemes were computationally attractive and yielded significant increases in accuracy and stability. In addition these benefits were achieved for relatively low orders of approximation.

Although we treat a relatively specific structure, it is not difficult to see how our results would extend to a more general class of problems. Mathematically, we have proven few new results, but rather have modified and applied several existing ones which have appeared throughout the literature. The system which describes the transverse vibration of a flexible beam with tip body is a hybrid of ordinary and partial differential equations. In section 2 we call upon the well known product space theory for functional differential equations to develop an appropriate abstract operator theoretic formulation for the eigenvalue problem and to establish relevant

spectral results. In section 3, the Rayleigh-Ritz method and the associated convergence theory for the approximation of eigenvalues and eigenfunctions of elliptic differential operators as developed in [2], [3], and [14] are discussed in the context of the present problem. In section 4 we outline the application of these results to spline-based methods in the spirit of the treatment in [15]. In section 5 an example and numerical results are discussed.

Notation is standard throughout. The Sobolev space of functions ϕ defined on the interval (a,b) for which $D^{k-1}\phi$ is absolutely continuous with $D^k\phi \in L_2$ is denoted by $H^k(a,b)$. The usual Sobolev inner products and norms are denoted by $\langle \cdot, \cdot \rangle_k$ and $|\cdot|_k$ respectively. The spectrum, point spectrum, and continuous spectrum of a linear operator T are denoted by $\Sigma(T)$, $\Sigma_p(T)$, and $\Sigma_c(T)$. For $\lambda \in P(T)$, the resolvent set of T , the resolvent of T at λ is denoted by $R_\lambda(T)$.

2. Formulation of the eigenvalue problem and spectral results

We consider a long, slender, flexible beam of length ℓ having spatially varying linear mass density ρ and flexural stiffness EI which is clamped at one end and free at the other with a rigidly attached tip body. The mass properties of the tip body are assumed to be: mass m centered at a distance c from the tip of the beam and directed along the longitudinal axis of the beam and moment of inertia J about its center of mass (see Fig. 2.1).

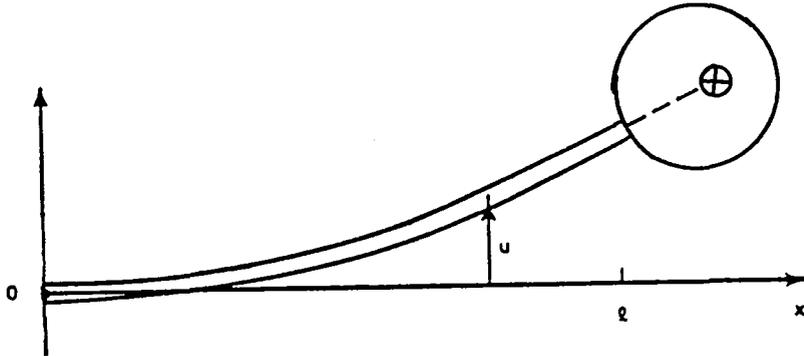


Figure 2.1

Letting $u(t,x)$ denote the vertical displacement of the beam at time $t > 0$ at position $x \in [0,\ell]$ and assuming small deformations ($|u(t,x)| \ll \ell$), the Euler-Bernoulli theory for the transverse vibration of a flexible beam yields the equation (see [4], [18])

$$(2.1) \quad \rho(x)D_t^2 u(t,x) + D_x^2 EI(x)D_x^2 u(t,x) - D_x \sigma(x)D_x u(t,x) = 0$$

where σ denotes the internal tension which results from externally applied, temporally invariant, axially directed loading. Elementary Newtonian mechanics can be used to derive the boundary conditions which describe the motion of the end of the beam with the tip body. Translational and rotational equilibrium respectively yield the two boundary conditions at $x = \ell$ given by (see [4],[18])

$$(2.2) \quad mD_t^2 u(t, \ell) + mcD_t^2 D_x u(t, \ell) - D_x EI(\ell) D_x^2 u(t, \ell) + \sigma(\ell) D_x u(t, \ell) = 0$$

and

$$(2.3) \quad mcD_t^2 u(t, \ell) + (J+mc^2) D_t^2 D_x u(t, \ell) + EI(\ell) D_x^2 u(t, \ell) + c\sigma(\ell) D_x u(t, \ell) = 0.$$

At the clamped end, $x = 0$, we have zero displacement and zero slope as given by the geometric boundary conditions

$$(2.4) \quad u(t, 0) = 0$$

and

$$(2.5) \quad D_x u(t, 0) = 0$$

respectively.

Remark If the center of mass of the tip body were not directed along the longitudinal axis of the beam, that is, the tip body had a non-zero mass center offset, the resulting set of equations would be nonhomogeneous. Consequently a classical eigenvalue problem for the natural modes of vibration would not result. We do note, however, that by employing a suitable transformation of the parameters c and J the homogeneous part of the resulting equations can be put in the form of (2.1)-(2.5) (See [18]).

The natural frequencies and mode shapes for the system described above are determined by assuming a solution to (2.1)-(2.5) of the form $u(t, x) = e^{i\omega t} \phi(x)$. The eigenvalue problem on ϕ with eigenvalues $\lambda = \omega^2$

$$(2.6) \quad D^2 EI(x) D^2 \phi(x) - D\sigma(x) D\phi(x) = \lambda \rho(x) \phi(x)$$

$$(2.7) \quad -DEI(\ell)D^2\phi(\ell) + \sigma(\ell)D\phi(\ell) = \lambda(m\phi(\ell) + mcD\phi(\ell))$$

$$(2.8) \quad EI(\ell)D^2\phi(\ell) + c\sigma(\ell)D\phi(\ell) = \lambda(mc\phi(\ell) + (J+mc^2)D\phi(\ell))$$

$$(2.9) \quad \phi(0) = 0$$

$$(2.10) \quad D\phi(0) = 0$$

results. In order to study the eigenvalue problem (2.6)-(2.10), we develop an appropriate abstract operator theoretic formulation.

Define the Hilbert space $H = \mathbb{R}^2 \times H^0(0, \ell)$ with inner product

$$\langle (\eta, \phi), (\xi, \psi) \rangle_H = \eta^T \xi + \langle \phi, \psi \rangle_0.$$

We make the standing assumptions

$$(H1) \quad \rho \in C(0, \ell), \rho(x) > 0 \quad x \in [0, \ell]$$

$$(H2) \quad EI \in C^2(0, \ell), EI(x) > 0 \quad x \in [0, \ell]$$

$$(H3) \quad \sigma \in C^1(0, \ell)$$

and of course that $m, J, c > 0$. The smoothness requirements in (H1), (H2), and (H3) above can in fact be relaxed. This will be discussed further at the end of the section.

Define the operators $M : H \rightarrow H$, $A_0 : \text{Dom}(A) \subset H \rightarrow H$ and $B_0 : \text{Dom}(B) \subset H \rightarrow H$ by

$$M(\eta, \phi) = (M_0\eta, \rho\phi),$$

$$M_0 = \begin{bmatrix} m & mc \\ mc & J+mc^2 \end{bmatrix},$$

$$\text{Dom}(A) = \{(\eta, \phi) \in H : \phi \in H^4(0, \ell), \phi(0) = D\phi(0) = 0, \eta = (\phi(\ell), D\phi(\ell))^T\},$$

$$A_0 \hat{\phi} = ((-DEI(\ell)D^2\phi(\ell), EI(\ell)D^2\phi(\ell))^T, D^2EID^2\phi), \hat{\phi} = ((\phi(\ell), D\phi(\ell))^T, \phi),$$

$$\text{Dom}(B) = \{(\eta, \phi) \in H : \phi \in H^2(0, \ell), \phi(0) = D\phi(0) = 0, \eta = (\phi(\ell), D\phi(\ell))^T\},$$

$$B_0 \hat{\phi} = ((\sigma(\ell)D\phi(\ell), c\sigma(\ell)D\phi(\ell))^T, -D\sigma D\phi).$$

Defining $L_0 : \text{Dom}(L) \subset H \rightarrow H$ by $\text{Dom}(L) = \text{Dom}(A)$, $L_0 = A_0 + B_0$ we consider the abstract formulation of (2.6)-(2.10) given by

$$(2.11) \quad L_0 \hat{\phi} = \lambda M \hat{\phi} \quad \hat{\phi} \in \text{Dom}(L).$$

Since M is invertible, the abstract generalized eigenvalue problem (2.11) is equivalent to

$$(2.12) \quad L \hat{\phi} = \lambda \hat{\phi} \quad \hat{\phi} \in \text{Dom}(L)$$

where

$$L \equiv M^{-1}L_0 = M^{-1}A_0 + M^{-1}B_0 \equiv A + B,$$

with

$$M^{-1}(\eta, \phi) = (M_0^{-1}\eta, \frac{1}{\rho} \phi),$$

$$M_0^{-1} = \frac{1}{Jm} \begin{bmatrix} J+mc^2 & -mc \\ -mc & m \end{bmatrix}.$$

We investigate the spectral properties of the operator L by first characterizing the spectrum of A and then treating B as an A-bounded perturbation. Defining the $\langle \cdot, \cdot \rangle_H$ - equivalent inner product on H, $\langle \cdot, \cdot \rangle_M$ by

$$\langle (\eta, \phi), (\xi, \psi) \rangle_M = \langle M(\eta, \phi), (\xi, \psi) \rangle_H,$$

it is not difficult to argue that A is densely defined, self adjoint, and positive with respect to the $\langle \cdot, \cdot \rangle_M$ inner product. Consequently A is closed and its spectrum, $\Sigma(A)$, is real with $\Sigma(A) \subset (0, \infty)$ and $\Sigma(A) = \Sigma_p(A) \cup \Sigma_c(A)$.

For each $\lambda \in \mathbb{C}$ let ϕ_λ^1 and ϕ_λ^2 denote two linearly independent solutions to

$$\frac{1}{\rho} D^2 E I D^2 \phi - \lambda \phi = 0$$

$$\phi(0) = 0$$

$$D\phi(0) = 0$$

and define $\Delta_\lambda \in \mathbb{C}^{2 \times 2}$ by

$$(2.13) \quad \Delta_\lambda = M_0^{-1} \begin{bmatrix} -DEI(\ell)D^2\phi_\lambda^1(\ell) & -DEI(\ell)D^2\phi_\lambda^2(\ell) \\ EI(\ell)D^2\phi_\lambda^1(\ell) & EI(\ell)D^2\phi_\lambda^2(\ell) \end{bmatrix}$$

$$- \lambda \begin{bmatrix} \phi_\lambda^1(\ell) & \phi_\lambda^2(\ell) \\ D\phi_\lambda^1(\ell) & D\phi_\lambda^2(\ell) \end{bmatrix}.$$

Using arguments similar to those which can be found in [21], it can be demonstrated

that $A - \lambda I$ is injective/surjective if and only if Δ_λ is injective/surjective. Consequently, since Δ_λ is finite dimensional we have the following theorem.

Theorem 2.1 The spectrum of A is discrete. That is $\Sigma(A) = \Sigma_p(A)$ with $\lambda \in \Sigma(A)$ if and only if $\det \Delta_\lambda = 0$.

Moreover, A has compact resolvent.

Theorem 2.2 For each $\lambda \in P(A)$, $R_\lambda(A)$ is a compact linear operator.

Pf

Let $\lambda \in P(A)$ (that is $\det \Delta_\lambda \neq 0$) and suppose that λ is not an eigenvalue of the clamped-clamped beam vibration problem

$$(2.14) \quad \frac{1}{\rho} D^2 E I D^2 \phi = \lambda \phi$$

$$(2.15) \quad \phi(0) = D\phi(0) = \phi(\ell) = D\phi(\ell) = 0.$$

Furthermore, let $G(x,y; \lambda)$, $0 < x, y < \ell$ denote the Green's function corresponding to (2.14), (2.15), and λ (see [5]). The operator $R_\lambda(A)$ can then be written as the sum of two bounded operators,

$$R_\lambda(A) = T_1 + T_2,$$

where T_2 is given by

$$T_2(\xi, \psi) = ((0,0)^T, \int_0^\ell G(\cdot, \theta; \lambda) \psi(\theta) d\theta)$$

and T_1 has range in the finite dimensional space

$$\{(\eta, \phi) \in H : \eta = (\phi(\ell), D\phi(\ell))^T, \phi = c_1 \phi_\lambda^1 + c_2 \phi_\lambda^1\}.$$

Now T_2 is essentially an integral operator on $H^0(0, \ell)$ and as such is compact (see [7]). The finite rank operator T_1 is compact from which we may conclude that $R_\lambda(A)$ is compact being the sum of two compact operators.

Having now shown that $R_\lambda(A)$ is compact for at least one $\lambda \in P(A)$, we may conclude that A is discrete and hence that $R_\lambda(A)$ is compact for all $\lambda \in P(A)$ (see [7], Vol III, p. 2291).

A straightforward application of the spectral theorem for compact self adjoint operators in a Hilbert space (see [20]) yields the following well known result.

Theorem 2.3 $\Sigma(A) = \{\lambda_k\}_{k=1}^\infty$ with $0 < \lambda_1 < \lambda_2 < \lambda_3 \dots < +\infty$ and $\lambda_k \rightarrow +\infty$ as $k \rightarrow \infty$. Each λ_k is of multiplicity 1 or 2 and the corresponding set of eigenvectors, $\{\hat{\phi}_k\}_{k=1}^\infty \subset \text{Dom}(A)$, is complete and orthonormal with respect to the $\langle \cdot, \cdot \rangle_M$ inner product. We have

$$\hat{\phi} = \sum_{k=1}^{\infty} \langle \hat{\phi}, \hat{\phi}_k \rangle_M \hat{\phi}_k, \quad \hat{\phi} \in \text{Dom}(A)$$

and if $\lambda \in P(A)$

$$R_\lambda(A)(\eta, \phi) = \sum_{k=1}^{\infty} \frac{\langle (\eta, \phi), \hat{\phi}_k \rangle_M}{\lambda_k - \lambda} \hat{\phi}_k,$$

$(\eta, \phi) \in H$.

The operator B is densely defined with $\text{Dom}(A) \subset \text{Dom}(B)$ and symmetric ($B \subset B^*$) with respect to the $\langle \cdot, \cdot \rangle_M$ inner product. Furthermore, hypotheses (H1) - (H3) and the

Sobolev interpolation inequalities for intermediate derivatives (see [1]) can be used to argue that B is A-bounded with A-bound 0 (see [10]). That is, for all $\epsilon > 0$ sufficiently small there exist constants α and β , independent of ϵ , for which

$$(2.16) \quad |B\hat{\phi}|_M < \alpha\epsilon^{-1}|\hat{\phi}|_M + \beta\epsilon|A\hat{\phi}|_M, \quad \hat{\phi} \in \text{Dom}(A).$$

It follows from (2.16) that $L = A + B$ is self adjoint (see [10]) and therefore closed. Since A is self adjoint and positive it is m-accretive and as such satisfies

$$|R_{-\lambda}(A)|_M < \frac{1}{\text{Re } \lambda}$$

and

$$|AR_{-\lambda}(A)|_M < 1$$

for all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > 0$. These estimates imply that

$$\alpha\epsilon^{-1}|R_{-\lambda}(A)|_M + \beta\epsilon|AR_{-\lambda}(A)|_M < \frac{\alpha\epsilon^{-1}}{\text{Re } \lambda} + \beta\epsilon < 1$$

for ϵ chosen sufficiently small and all $\lambda \in \mathbb{C}$ with $\text{Re } \lambda > 0$ sufficiently large. We may conclude therefore (see [10], page 214, Theorem 3.17) that there exists a constant γ such that $\{\lambda \in \mathbb{C} : \text{Re } \lambda < \gamma\} \subset P(L)$ with $R_\lambda(L)$, $\lambda \in P(L)$, compact. It is not difficult to argue that

$$(2.17) \quad \gamma > -4\alpha\beta.$$

As was the case with A, the spectral properties of L are easily characterized using the spectral theorem for compact self adjoint operators on a Hilbert space. Indeed, the conclusions of Theorem 2.3 are valid with A replaced by L and with the

exception of the fact that the spectrum of L is bounded below by $-4\alpha\beta$ rather than 0.

The smoothness assumptions in (H1) - (H3) can be relaxed to $\rho, EI, \sigma \in L_\infty(0, \ell)$ with EI bounded away from 0 and σ defined at ℓ by turning to a weak formulation. Define the space

$$V = \{(\eta, \phi) \in H : \phi \in H^2(0, \ell), \phi(0) = D\phi(0) = 0, \eta = (\phi(\ell), D\phi(\ell))^T\}$$

with inner product

$$\langle \hat{\phi}_1, \hat{\phi}_2 \rangle_V = \langle D^2\phi_1, D^2\phi_2 \rangle_0$$

where $\hat{\phi}_i = ((\phi_i(\ell), D\phi_i(\ell))^T, \phi_i)$, $i = 1, 2$. The usual dense embeddings $V \subset H \subset V'$ hold with the injection $V \hookrightarrow H$ compact. Define the bilinear forms on $V \times V$

$$a(\hat{\phi}, \hat{\psi}) = \langle EID^2\phi, D^2\psi \rangle_0$$

$$b(\hat{\phi}, \hat{\psi}) = c\sigma(\ell)D\phi(\ell)D\psi(\ell) + \langle \sigma D\phi, D\psi \rangle_0$$

and

$$l(\hat{\phi}, \hat{\psi}) = a(\hat{\phi}, \hat{\psi}) + b(\hat{\phi}, \hat{\psi}).$$

The form $a(\cdot, \cdot)$ is V -elliptic (see [17]) while the form $l(\cdot, \cdot)$ is V - H -elliptic.

Indeed recalling (2.17) we have

$$l(\hat{\phi}, \hat{\phi}) + 4\alpha\beta|\hat{\phi}|_M^2 > \delta|\hat{\phi}|_V^2$$

for $\hat{\phi} \in V$. The forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, and $l(\cdot, \cdot)$ are related to the operators A, B ,

and L defined earlier in the usual manner. Consider the form $a(\cdot, \cdot)$. Using the Riesz Theorem define the operator $A_1 : \text{Dom}(A_1) \subset H \rightarrow H$ by

$$\text{Dom}(A_1) = \{\hat{\phi} \in V : \text{The map } \hat{\psi} \rightarrow a(\hat{\phi}, \hat{\psi}) \text{ is continuous on } V \text{ with respect to the } |\cdot| \text{ norm on } H\},$$

$$a(\hat{\phi}, \hat{\psi}) = \langle A_1 \hat{\phi}, \hat{\psi} \rangle_M, \quad \hat{\phi} \in \text{Dom}(A_1), \quad \hat{\psi} \in V.$$

It can be shown that

$$\text{Dom}(A_1) = \{\hat{\phi} \in V : \text{EID}^2 \phi \in H^2(0, \ell)\}$$

and if EI is sufficiently smooth, that $\text{Dom}(A_1) = \text{Dom}(A)$ and $A_1 = A$. In addition if the $\langle \cdot, \cdot \rangle_M$ inner product is interpreted as the duality pairing between V and V' then A_1 can be extended to an operator $\bar{A}_1 \in L(V, V')$. Similar correspondences exist between the form $b(\cdot, \cdot)$ and the operator B and the form $l(\cdot, \cdot)$ and the operator L .

The weak form of the eigenvalue problem (2.12) is given by

$$(2.18) \quad l(\hat{\phi}, \hat{\psi}) = \lambda \langle \hat{\phi}, \hat{\psi} \rangle_M \quad \hat{\psi} \in V.$$

Standard results (see [17], page 78, Corollary 7D) yield the existence of a set of orthonormal (with respect to the $\langle \cdot, \cdot \rangle_M$ inner product) eigenvectors $\{\hat{\phi}_k\}_{k=1}^{\infty} \subset \text{Dom}(A_1) \subset V$ which are complete in H and a corresponding set of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ which satisfy $-4\alpha\beta < \lambda_1 < \lambda_2 < \dots < \infty$ with $\lambda_k \rightarrow +\infty$ as $k \rightarrow \infty$.

3. Approximation and Convergence - The Rayleigh-Ritz - Galerkin Method

Since the addition of a constant multiple of the identity to an operator simply translates the spectrum by the same constant value, we assume throughout, without loss of generality so far as approximation is concerned, that L is positive. For (2.17) implies that the operator $L_1 \equiv L + 4\alpha\beta I$ satisfies

$$\langle L_1 \hat{\phi}, \hat{\phi} \rangle_M = \langle L \hat{\phi}, \hat{\phi} \rangle_M + 4\alpha\beta \langle \hat{\phi}, \hat{\phi} \rangle_M - 4\alpha\beta \langle \hat{\phi}, \hat{\phi} \rangle_M + 4\alpha\beta \langle \hat{\phi}, \hat{\phi} \rangle_M = 0.$$

The Rayleigh-Ritz method for the approximation of the eigenvalues and eigenvectors of a positive self adjoint operator is based upon their characterization as the extrema and critical values (stationary points) of the Rayleigh quotient. For $\hat{\phi} \in V$, $\hat{\phi} \neq 0$ define

$$R[\hat{\phi}] = l(\hat{\phi}, \hat{\phi}) / \langle \hat{\phi}, \hat{\phi} \rangle_M.$$

We note that for $\hat{\phi} \in \text{Dom}(L)$ and $\hat{\psi} \in V$ we have $l(\hat{\phi}, \hat{\psi}) = \langle L \hat{\phi}, \hat{\psi} \rangle_M$ and that V is the closure of $\text{Dom}(L)$ in V with respect to norm $|\hat{\phi}|_L = (l(\hat{\phi}, \hat{\phi}))^{\frac{1}{2}}$ on V . The eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and corresponding eigenvectors $\{\phi_k\}_{k=1}^{\infty}$ can then be characterized by (see [9], [22])

$$(3.1) \quad \lambda_k = \min\{R[\hat{\phi}] : \hat{\phi} \in V, \hat{\phi} \neq 0, \langle \hat{\phi}, \hat{\phi}_j \rangle_M = 0, j = 1, 2, \dots, k-1\} = R[\hat{\phi}_k].$$

There exist other characterizations of the spectrum equivalent to (3.1) which are often more useful for computational and theoretical purposes (see [14]).

For each $N = 1, 2, \dots$ let V^N denote a finite dimensional subspace of H with $V^N \subset V$, $\dim V^N = k^N$ and $V^N = \text{span} \{\hat{\psi}_k^N\}_{k=1}^{k^N}$. The classical Rayleigh-Ritz method consists of determining the extrema and critical values of $R[\cdot]$ over V^N . Requiring that the first variation of $R[\cdot]$ vanish over V^N results in the matrix generalized eigenvalue problem

$$(3.2) \quad L_{V^N}^N = \lambda M_{V^N}^N$$

where

$$[L^N]_{ij} = 1(\hat{\psi}_i^N, \hat{\psi}_j^N) \quad i, j = 1, 2, \dots, k^N$$

and

$$[M^N]_{ij} = \langle \hat{\psi}_i^N, \hat{\psi}_j^N \rangle_M \quad i, j = 1, 2, \dots, k^N.$$

The matrices L^N and M^N are real, symmetric, and positive definite. Let

$$C^N = (M^N)^{-\frac{1}{2}} L^N (M^N)^{-\frac{1}{2}}.$$

Then the generalized eigenvalue problem (3.2) is equivalent to the standard eigenvalue problem

$$C_{w^N}^N = \lambda w^N$$

where $w^N = (M^N)^{\frac{1}{2}} v^N$. The matrix C^N is real, symmetric, and positive definite. It has k^N positive eigenvalues $\{\lambda_k^N\}_{k=1}^{k^N}$, $0 < \lambda_1^N < \lambda_2^N < \dots < \lambda_{k^N}^N$ and k^N corresponding

eigenvectors $\{w_k^N\}_{k=1}^{k^N}$ which can be chosen to be orthonormal,

$$(w_i^N)^T w_j^N = \delta_{ij} \quad i, j = 1, 2, \dots, k^N.$$

The $\{\lambda_k^N\}_{k=1}^{k^N}$ are known as the Reyleigh-Ritz approximate eigenvalues and $\{\phi_k^N\}_{k=1}^{k^N}$,

$\hat{\phi}_j^N = \sum_{i=1}^{k^N} (v_j^N)_i \hat{\psi}_i^N$ the corresponding Rayleigh-Ritz approximate eigenvectors of L. The

$\{\hat{\phi}_k^N\}_{k=1}^{k^N}$ are orthonormal in H with respect to the $\langle \cdot, \cdot \rangle_M$ inner product,

$$\langle \hat{\phi}_i^N, \hat{\phi}_j^N \rangle_M = (v_i^N)^T M v_j^N = (w_i^N)^T w_j^N = \delta_{ij}$$

$i, j = 1, 2, \dots, k^N.$

The weak formulation (2.18) is the basis for the Galerkin method. It leads to the finite dimensional eigenvalue problem which consists of finding $\hat{\phi}^N \in V^N$ and λ^N that satisfy

$$(3.3) \quad l(\hat{\phi}^N, \hat{\psi}^N) = \lambda^N \langle \hat{\phi}^N, \hat{\psi}^N \rangle_M, \quad \hat{\psi}^N \in V^N.$$

For the problem considered here, the Galerkin equations (3.3) and the Rayleigh quotient characterization (3.1) both lead to the same matrix generalized eigenvalue problem.

The convergence theory for the Rayleigh-Ritz method for the approximation of the spectrum of self adjoint elliptic differential operators is well documented in the literature. Convergence results for the eigenvalues and eigenvectors of Sturm-Liouville systems can be found in [2]. These ideas were then extended to apply to more general problems and improved upon in [3] and [14]. Although the operators L_0 and M are not precisely of the form of those considered, it is not difficult, however, to argue that the formulation we have employed renders the existing convergence theory directly applicable.

Let $P^N : V \rightarrow V^N$ denote the orthogonal projection of V onto V^N with respect to the $\langle \cdot, \cdot \rangle_L$ inner product on V defined by $\langle \hat{\phi}, \hat{\psi} \rangle_L = l(\hat{\phi}, \hat{\psi})$. Assume that the subspaces V^N have the property that

$$\lim_{N \rightarrow \infty} |P^N \hat{\phi} - \hat{\phi}|_M = 0 \quad \hat{\phi} \in V.$$

Theorem 3.1 (Eigenvalue Error Estimate). Let k be a fixed positive integer with $k < k^N$. Then the sequence $\{\lambda_k^N\}_{N=1}^\infty$ converges to λ_k from above, there exists a positive number \bar{N} such that for all $N > \bar{N}$

$$\lambda_k < \lambda_k^N < \lambda_k + \frac{\sum_{j=1}^k |P^N \hat{\phi}_j - \hat{\phi}_j|_L^2}{(1 - (\sum_{j=1}^k |P^N \hat{\phi}_j - \hat{\phi}_j|_M^2)^{\frac{1}{2}})^2}$$

and hence

$$\lambda_k^N < \lambda_k + \gamma^N \sum_{j=1}^k |P^N \hat{\phi}_j - \hat{\phi}_j|_L^2$$

with $\gamma^N \rightarrow 1^+$ as $N \rightarrow \infty$.

Theorem 3.2 (Eigenvector Error Estimates). Suppose λ_k is of multiplicity $m_k + 1$, $m_k = 0$ or 1 and $k + m_k < k^N$. Then:

- (i) If $m_k = 0$ (λ_k is simple) there exist positive numbers γ and \bar{N} depending only upon k for which

$$|\hat{\phi}_k^N - \hat{\phi}_k|_L < \gamma |P^N \hat{\phi}_k - \hat{\phi}_k|_L$$

for all $N > \bar{N}$.

(ii) If $m_k = 1$ (λ_k is of multiplicity 2), let $\hat{\phi}$ be any element in $\text{span}\{\hat{\phi}_k, \hat{\phi}_{k+1}\}$ and let $\hat{\phi}^N$ be the orthogonal projection of $\hat{\phi}$ onto $\text{span}\{\hat{\phi}_k^N, \hat{\phi}_{k+1}^N\}$ with respect to the $\langle \cdot, \cdot \rangle_M$ innerproduct. Then, there exist positive numbers γ and \bar{N} depending only upon k for which

$$|\hat{\phi}^N - \hat{\phi}|_L < \gamma |\hat{\phi}|_M^{\frac{1}{2}} \sum_{j=k}^{k+1} |P^N \hat{\phi}_j - \hat{\phi}_j|_L$$

for all $N > \bar{N}$.

In the next section we describe a particular class of schemes which are based upon choosing the V^N as subspaces of spline functions. We use Theorems 3.1 and 3.2 together with the approximation properties of splines to establish convergence and estimate rates of convergence.

4 Spline Approximations

For each $N = 1, 2 \dots$ let π^N denote a partition of $[0, \ell]$, $\pi^N: 0 = x_0 < x_1 < x_2 \dots < x_N = \ell$ and let $h^N = \max_{i=1, 2, \dots, N} (x_i - x_{i-1})$ and $g^N = \min_{i=1, \dots, n} (x_i - x_{i-1})$. For $m = 1, 2 \dots$ let $S(m, \pi^N, z)$ denote the order $2m$ (degree $2m - 1$) polynomial spline spaces corresponding to the partition π^N and the node incidence vector $z \in R^{N-1}$ (see [19]). Recall that $S(m, \pi^N, z) \subset H^{2m-\mu}(0, \ell)$ where $1 < \mu \equiv \max_{i=1, 2, \dots, N-1} z_i < m$ and $\dim S(m, \pi^N, z) = 2m + n(N, z)$ where $n(N, z) \equiv \sum_{i=1}^N z_i$.

Of particular interest in practice are the following two special cases:

(1) $z = (m, m, \dots, m)$. In this case $S(m, \pi^N, z) = H_m^N(\pi^N)$, the Hermite spaces, $\dim H_m^N(\pi^N) = m(N+1)$ and $H_m^N(\pi^N) \subset H^m(0, \ell)$.

(2) $z = (1, 1, \dots, 1)$. In this case $S(m, \pi^N, z) = S_m^N(\pi^N)$, the standard polynomial spline spaces, $\dim S_m^N(\pi^N) = N - 1 + 2m$ and $S_m^N(\pi^N) \subset H^{2m-1}(0, \ell)$.

For $m > 2$ let $S^0(m, \pi^N, z) = \{s \in S(m, \pi^N, z) : s(0) = Ds(0) = 0\}$ and define

$$V_m^N(z) = \{\hat{\phi} = ((\phi(\ell), D\phi(\ell)))^T, \phi : \phi \in S^0(m, \pi^N, z)\}.$$

Then $k_m^N(z) = \dim V_m^N(z) = 2(m-1) + n(N, z)$ and since $m > 2$, $V_m^N(z) \subset V$. For $k > 2$ define

$$V_k = \{(\eta, \phi) \in H : \phi \in H^k(0, \ell), \phi(0) = D\phi(0) = 0, \eta = (\phi(\ell), D\phi(\ell))^T\}.$$

Then $V_k \subset V$, $V_{k+1} \subset V_k$, $k = 2, 3, \dots$ and $V_2 = V$.

In order to simplify notation, we suppress showing explicit dependence on z with the understanding that the spaces V_m^N and the interpolation and projection operators defined below do in fact depend upon the choice of the node incidence vector.

Define the interpolation operators $I_{m,k}^N : H^k(0, \ell) \rightarrow S(m, \pi^N, z)$ by letting $I_{m,k}^N \phi$ denote the unique element in $S(m, \pi^N, z)$ (see [19]) which satisfies

$$D^j(\phi - I_{m,k}^N \phi)(x_i) = 0 \quad \begin{array}{l} 0 < j < \min(k-1, z_i-1) \\ 0 < i < N \end{array}$$

$$D^j I_{m,k}^N \phi(x_i) = 0 \quad \begin{array}{l} \min(k-1, z_i-1) < j < z_i-1 \\ 0 < i < N \end{array}$$

where $z_0 = z_N \equiv m$. For $m > 2$ define $\mathcal{I}_{m,k}^N : V_k \rightarrow V_m^N$ by

$$\mathcal{I}_{m,k}^N \hat{\phi} = ((\phi_m^N(\ell), D\phi_m^N(\ell))^T, \phi_m^N)$$

where $\hat{\phi} = ((\phi(\ell), D\phi(\ell))^T, \phi) \in V_k$ and $\phi_m^N = I_{m,k}^N \phi$.

If the sequence of partitions $\{\pi_{N=1}^{\infty}\}$ satisfies the uniformity condition $\frac{h^N}{g^N} \leq \tau$ for some $\tau > 1$ independent of N and $\lim_{N \rightarrow \infty} h^N = 0$ we have the following error estimates.

Theorem 4.1 For $m = 2, 3, \dots$, $k = 2, 3, \dots$ $2m$ and $\hat{\phi} = ((\phi(\ell), D\phi(\ell))^T, \phi) \in V_k$

$$(i) \quad |\mathcal{I}_{m,k}^N \hat{\phi} - \hat{\phi}|_V \leq \gamma (h^N)^{k-2} |D^k \phi|_0$$

and

$$(ii) \quad |\mathcal{I}_{m,2}^N \hat{\phi} - \hat{\phi}|_V \rightarrow 0 \text{ as } N \rightarrow \infty, \hat{\phi} \in V_2 = V$$

where γ is a constant independent of N .

Pf

Statement (i) is a direct consequence of standard spline interpolation error estimates (see [19]). To argue the validity of (ii) let $\hat{\phi} \in V$. Then for $\psi \in H^3(0, \ell)$

$$|\mathcal{I}_{m,2}^N \hat{\phi} - \hat{\phi}|_V \leq |\phi_m^N - \phi|_2 \leq |\phi_m^N - \psi_m^N|_2 + |\psi_m^N - \psi|_2 + |\psi - \phi|_2$$

where $\psi_m^N = I_{m,2}^N \psi$. Now (see [19])

$$|\phi_m^N - \psi_m^N|_2 \leq \gamma |D^2(I_{m,2}^N(\phi - \psi))|_0 \leq \gamma |D^2(\phi - \psi)|_0 \leq \gamma |\phi - \psi|_2$$

and therefore

$$(4.1) \quad |\mathcal{I}_{m,2}^N \hat{\phi} - \hat{\phi}|_V \leq (1 + \gamma) |\phi - \psi|_2 + |\psi_m^N - \psi|_2.$$

Corresponding to z and $I_{m,2}^N = I_{m,2}^N(z)$ there exists a \tilde{z} and $\tilde{I}_{m,3}^N = \tilde{I}_{m,3}^N(\tilde{z})$ such that $I_{m,2}^N \psi = \tilde{I}_{m,3}^N \psi$. Therefore

$$(4.2) \quad |\psi^N - \psi|_2 = |I_{m,2}^N \psi - \psi|_2 = |\tilde{I}_{m,3}^N \psi - \psi|_2 \leq \gamma_1 h^N |D^3 \psi|_0.$$

The estimates (4.1) and (4.2) together with the density of $H^3(0, \ell)$ in $H^2(0, \ell)$ and the fact that $\lim_{N \rightarrow \infty} h^N = 0$ yield the desired result.

Let $P_m^N : V \rightarrow V_m^N$ denote the orthogonal projection of V onto V_m^N computed with respect to the $\langle \cdot, \cdot \rangle_L$ inner product. Arguments similar to those found in [15] together with the interpolation error estimates given in Theorem 4.1 can be used to verify

Theorem 4.2 If $\hat{\phi} = ((\phi(\ell), D\phi(\ell))^T, \phi) \in V_k$ with $2 < k < 2m$ then

$$(i) \quad |P_m^N \hat{\phi} - \hat{\phi}|_M \leq \gamma (h^N)^k |D^k \phi|_0,$$

$$(ii) \quad |P_m^N \hat{\phi} - \hat{\phi}|_L \leq \gamma (h^N)^{k-2} |D^k \phi|_0 \quad \text{and} \quad |P_m^N \hat{\phi} - \hat{\phi}|_L \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \quad \text{if} \quad k=2,$$

$$(iii) \quad |D^j(\phi_m^N - \phi)|_0 \leq \gamma (h^N)^{k-j} |D^k \phi|_0, \quad j=0, 1, 2 \quad \text{and}$$

$$|D^2(\phi_m^N - \phi)|_0 \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty \quad \text{if} \quad k=2,$$

$$(iv) \quad |D^j(\phi_m^N - \phi)|_\infty \leq \gamma (h^N)^{k-j-\frac{1}{2}} |D^k \phi|_0, \quad j=0, 1,$$

where $P_m^N \hat{\phi} = ((\phi_m^N(\ell), D\phi_m^N(\ell))^T, \phi_m^N)$ and γ is a constant which is independent of $\hat{\phi}$ and N .

Theorems 3.1, 3.2 and 4.2 yield the following eigenvalue and eigenvector error estimates (see [15]).

Theorem 4.3 Suppose λ_i is simple and $\hat{\phi}_i \in V_k$. Let $\lambda_i^{N,m}$ and $\hat{\phi}_i^{N,m}$ denote respectively the i^{th} Rayleigh-Ritz approximate eigenvalue and eigenvector for L computed over V_m^N where N and m have been chosen so that $k_m^N = \dim V_m^N > i$. Then

(i) $0 < \lambda_i^{N,m} - \lambda_i < \gamma(h^N)^{2(r-2)}$ and $\lambda_i^{N,m} - \lambda_i \rightarrow 0^+$ as $N \rightarrow \infty$ if $r=2$,

(ii) $|\hat{\phi}_i^{N,m} - \hat{\phi}_i|_L < \gamma(h^N)^{r-2} |D^r \phi_i|_0$ and $|\hat{\phi}_i^{N,m} - \hat{\phi}_i|_L \rightarrow 0$ as $N \rightarrow \infty$ if $r=2$,

(iii) $|\hat{\phi}_i^{N,m} - \hat{\phi}_i|_H < \gamma(h^N)^r |D^r \phi_i|_0$,

(iv) $|D^j(\phi_i^{N,m} - \phi_i)|_0 < \gamma(h^N)^{r-j} |D^r \phi_i|_0$, $j=0,1,2$ and

$|D^2(\phi_i^{N,m} - \phi_i)|_0 \rightarrow 0$ as $N \rightarrow \infty$ if $r=2$,

(v) $|D^j(\phi_i^{N,m} - \phi_i)|_\infty < \gamma(h^N)^{r-j-\frac{1}{2}} |D^r \phi_i|_0$, $j=0,1$

where $\hat{\phi}_i = ((\phi_i(\ell), D\phi_i(\ell))^T, \phi_i)$, $\hat{\phi}_i^{N,m} = ((\phi_i^{N,m}(\ell), D\phi_i^{N,m}(\ell))^T, \phi_i^{N,m})$, γ is a constant which is independent of N and $r=\min(k, 2m)$. If λ_i is of multiplicity 2, we have (i) - (v) above holding with $\hat{\phi}_i$ replaced by $\hat{\phi}$ and $\hat{\phi}_i^{N,m}$ by $\hat{\phi}^{N,m}$ where $\hat{\phi}$ is any element in $\text{span}\{\hat{\phi}_i, \hat{\phi}_{i+1}\}$ and $\hat{\phi}^{N,m}$ is the orthogonal projection of $\hat{\phi}$ onto $\text{span}\{\hat{\phi}_i^{N,m}, \hat{\phi}_{i+1}^{N,m}\}$ computed with respect to the $\langle \cdot, \cdot \rangle_M$ inner product.

Remark When V_m^N is defined using $S_m(\pi^N)$ with π^N a uniform partition the L_∞ error estimates given in the previous two theorems can be improved. Orders of convergence in Theorem 4.2 (iv) and 4.3 (v) can be increased by $\frac{1}{2}$.

5. An Example

We consider the approximation of the natural frequencies for the transverse vibration of a uniform, cantilevered beam with tip body subject to an axially directed base thrust. More specifically, we assume that the linear mass density ρ and flexural stiffness EI are spatially invariant. Furthermore we assume that the beam is acted upon by a constant, axially directed load $P_0 > 0$ applied to its base inducing an acceleration in the positive x -direction (see fig 2.1). It can be derived (see [18]) that the internal tension σ is given by

$$\sigma(x) = -P_0 \left(\frac{m + \int_x^\ell \rho}{m + \int_0^\ell \rho} \right) = -P_0 \left(\frac{m + (\ell - x)\rho}{m + \ell\rho} \right).$$

We note that the formulation described above leads naturally to a classical buckling problem. Indeed, for P_0 sufficiently large the beam will buckle. The critical buckling loads, P_0^{cr} are those values of P_0 for which steady state solutions to (2.1) - (2.5) exist; that is those values of P_0 for which $\lambda = 0$ is a solution to the eigenvalue problem (2.6) - (2.10). For the problem stated here, the critical buckling loads can be computed directly as the roots of a transcendental equation involving Bessel functions of the first kind (see [18]).

The B-spline representation for polynomial spline spaces leads to computationally efficient algorithms for the evaluation of the spline functions, their derivatives, and inner products (see [16]). We considered a scheme using the standard polynomial spline spaces $S_m(\pi^N)$ defined over the uniform partition π^N with $h^N = \ell/N$. In this case the relevant formulas become especially simple.

For $k > 0$ let $(x-y)_+^k = (x-y)^k (x-y)_+^0$ where

$$(x-y)_+^0 = \begin{cases} 0 & x < y \\ 1 & x > y . \end{cases}$$

Define the fundamental B-spline of order $2m$ (degree $2m-1$) with knots $\{0,1,2,\dots,2m\}$ and support $[0, 2m]$ by

$$B_m(x) = \delta_y^{2m} (x-y)_+^{2m-1} \Big|_{y=0} = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (x-j)_+^{2m-1}$$

where δ_y is the forward difference operator on y ; $\delta_y f(x,y) = f(x,y+1) - f(x,y)$. If we let

$$B_i^{N,m}(x) = B_m\left(\frac{N}{m}(x - (i-m)h^N)\right), \quad i=1-m, \dots, N+m-1$$

then $S_m(\pi^N) = \text{span}\{B_i^{N,m}\}_{i=1-m}^{N+m-1}$. The basis splines $B_i^{N,m}$ for $S_m(\pi^N)$ are scaled-

translates of the fundamental B-spline, B_m , and as such have the desirable property

that their supports, $((i-m)h^N, (i+m)h^N)$ are local. A basis for

$S_m^0(\pi^N) = \{s \in S_m(\pi^N) : s(0) = Ds(0) = 0\}$ is easily constructed from the $B_i^{N,m}$ by

taking linear combinations of basis elements which do not satisfy the boundary

conditions at zero to form basis elements which do. We obtain

$S_m^0(\pi^N) = \text{span}\{\beta_i^{N,m}\}_{i=1-m+2}^{N+m-1}$. For example, in the case $m=2$ one possible choice for

the $\beta_i^{N,m}$ is given by

$$\beta_1^{N,2} = B_0^{N,2} - 2B_1^{N,2} - 2B_{-1}^{N,2}$$

$$\beta_j^{N,2} = B_j^{N,2}, \quad j = 2, 3, \dots, N+1.$$

Defining $\hat{\beta}_j^{N,m} = ((\beta_j^{N,m}(\ell), D\beta_j^{N,m}(\ell))^T, \beta_j^{N,m}) \in V$, $j = 1-m+2, \dots, N+m-1$, $m > 2$, we have $V_m^N = \text{span} \{ \hat{\beta}_j^{N,m} \}_{j=1-m+2}^{N+m-1}$. The matrices L^N and M^N which appear in the generalized eigenvalue problem (3.2) are given by

$$(5.1) \quad [L^N]_{ij} = EI \int_0^\ell D^2 \beta_i^{N,m} D^2 \beta_j^{N,m} - \frac{P_0}{m+\ell\rho} \int_0^\ell (m+(\ell-x)\rho) D\beta_i^{N,m} D\beta_j^{N,m} \\ - \frac{-mc P_0}{m+\ell\rho} D\beta_i^{N,m}(\ell) D\beta_j^{N,m}(\ell)$$

and

$$(5.2) \quad [M^N]_{ij} = [\beta_i^{N,m}(\ell), D\beta_i^{N,m}(\ell)] \begin{bmatrix} m & mc \\ mc & J+mc^2 \end{bmatrix} \begin{bmatrix} \beta_j^{N,m}(\ell) \\ D\beta_j^{N,m}(\ell) \end{bmatrix} + \rho \int_0^\ell \beta_i^{N,m} \beta_j^{N,m}$$

$i, j = 1-m+2, \dots, N+m-1$.

Since the B-splines are polynomials of degree $2m-1$ on each subinterval of π^N , the integrals in the above expressions can be computed exactly using a composite Gaussian Quadrature formula. The efficient evaluation of the B-splines and their derivatives is facilitated by the fact that the fundamental B-spline satisfies the recurrence relations

$$B_m(x) = x^2 B_{m-1}(x) + (2x(2m-x) - 2m) B_{m-1}(x-1) + (2m-x)^2 B_{m-1}(x-2)$$

$$DB_m(x) = (2m-1)[xB_{m-1}(x) + 2(m-x)B_{m-1}(x-1) - (2m-x)B_{m-1}(x-2)]$$

$$D^2 B_m(x) = (2m-1)(2m-2)[B_{m-1}(x) - 2B_{m-1}(x-1) + B_{m-1}(x-2)].$$

The local support property of the B-splines necessarily implies that the matrices L^N and M^N are banded. This can be exploited when (3.2) is solved (see [6]).

We chose $\ell = 1.0$, $\rho = 2.0$, $EI = 1.0$, $m = 4.0$, $c = .1$, $J = .52$ and $P_0 = 2.5$ and solved the generalized eigenvalue problem (3.2) with the matrices L^N and M^N given by (5.1) and (5.2) respectively for $m = 2$ (cubic splines), 3 (quintic splines) and 4 (septic splines) and various values of N . The eigenvalue problem (3.2) was solved using the IMSL routine EIGZS. This routine finds the Cholesky decomposition $R^N(R^N)^T$ of M^N and then uses the QR algorithm to compute the eigenvalues of the transformed matrix $(R^N)^{-1}L^N(R^N)^{-T}$. Our results are given in Tables 5.2, 5.3, and 5.4. Recall that $\dim(S_m(\pi^N)) = N+2m-3$.

We also computed approximate eigenvalues using the Rayleigh-Ritz method over two modal subspaces. The spaces were taken to be the span of the first N natural mode shapes for the corresponding (1) clamped-free beam ($m = c = J = P_0 = 0$) and (2) clamped-free beam with tip body ($P_0 = 0$). These results are given in Tables 5.5 and 5.6 respectively.

As an independent basis for comparison, we also computed approximate eigenvalues by using an iterative root finding technique to compute the zeros of the frequency equation corresponding to (2.6) - (2.10). A numerical integrator was used to find two appropriate linearly independent solutions to the initial value problem (2.6), (2.9), (2.10) for a given value of λ in each iteration. The frequency equation arises from the setting to zero of the determinant of a matrix which is similar in form to Λ_λ given in (2.13) (see [18]). The first five eigenvalues computed in this manner are given in Table 5.1 below.

i	λ_i
1	.0365798
2	9.597101
3	271.624402
4	1934.262208
5	7352.906500

Table 5.1

i	$N=2$	$N=4$	$N=8$	$N=16$	$N=32$
1	.0367460	.0365892	.0365804	.0365799	.0365799
2	9.611071	9.597834	9.597144	9.597104	9.597107
3	279.748161	272.659779	271.677636	271.627567	271.624598
4		2008.349824	1937.362657	1934.430532	1934.272355
5		7808.026186	7406.496727	7355.451937	7353.054424

Table 5.2 - Spline Scheme - $m = 2$

i	$N=2$	$N=4$	$N=8$	$N=16$	$N=32$
1	.0365798	.0365798	.0385798	.0365798	.0365798
2	9.597110	9.597101	9.597101	9.597101	9.597101
3	271.637063	271.626382	271.624408	271.624403	271.624403
4	2024.194063	1937.055444	1934.264769	1934.262236	1934.262227
5	7730.251147	7358.613802	7353.153706	7352.907163	7352.906727

Table 5.3 - Spline Scheme - $m = 3$

$\downarrow i \ N \rightarrow$	2	4	8	16
1	.0365798	.0365798	.0365798	.0365798
2	9.597101	9.597101	9.597101	9.597101
3	271.624474	271.624406	271.624403	271.624392
4	1935.739405	1934.300633	1934.262216	1934.262210
5	7362.669220	7353.063985	7352.907860	7352.906573

Table 5.4 - Spline Scheme - $m = 4$

$\downarrow i \ N \rightarrow$	2	4	8	16
1	.0438998	.0391454	.0375638	.0370661
2	20.580764	13.553588	11.326133	10.581254
3		445.668737	340.718106	310.220838
4		3401.113918	2441.500175	2211.540908
5			9373.564654	8425.737638

Table 5.5 - Clamped Free Modes

$\downarrow i \ N \rightarrow$	2	4	8	16
1	.0382750	.0365966	.0365801	.0365608
2	9.609008	9.597279	9.597107	9.571124
3		271.642143	271.624818	270.693532
4		1934.288529	1934.263452	1927.899003
5			7352.912644	7328.632345

Table 5.6 - Clamped Free with Tip Body Modes

From the tables above it is immediately clear that the spline schemes exhibit rapid convergence with no apparent deterioration of accuracy for large N . Rapid convergence was also observed in higher frequencies ($i > 5$) than those shown in the tables. The cantilever mode based scheme converged very slowly. The cantilever with tip body modes performed better. However, there is an apparent stability problem when N becomes large. This is most likely a consequence of the high frequencies involved (which must be determined as the roots of the transcendental equation $\det \Delta_\lambda = 0$) and the fact that the stiffness matrices L^N are full. The cantilever with tip body modes performed optimally when $N = 12$. Rapid eigenvalue convergence was also observed with Hermite spline-based schemes. However these schemes did exhibit some stability problems when N was large. Their overall performance was inferior to the $S_m(\pi^N)$ - based methods.

Finally we note that if damping is introduced either into the beam or tip body dynamics it is likely that the resulting stiffness operator would not be self adjoint. Consequently the theory presented here would no longer be directly applicable. However, the results in [8] and [13] or [11] and [12] may well apply depending upon whether or not the associated resolvent operators were compact. We have not as yet considered the damped problem.

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