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National Aeronautics
and Space Administration

Scientific and Technical
Information Branch

NASA TECHNICAL PAPER

AN INTEGRAL REPRESENTATION OF THE GENERALIZED EULER-MASCHERONI CONSTANTS

The generalized Euler-Mascheroni constants are defined by

$$\gamma = \lim_{N\to\infty}\left[\sum_{k=1}^{N}\frac{\ln^n k}{k} - \frac{\ln^{n+1}N}{n+1}\right] \quad , \quad n = 0, 1, 2, \ldots \tag{1}$$

and are coefficients of the Laurent expansion of the Riemann Zeta function $\zeta(z)$ about $z = 1$

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{(-1)^n\,\gamma_n\,(z-1)^n}{n!} \quad , \quad \mathrm{Re}(z) \geqslant 0 \quad . \tag{2}$$

These were first defined by Stieltjes in 1885, discussed by Stieltjes and Hermite [1], and have been periodically reinvented over the years [2,3,4]. Computation of these numbers has attracted several writers, among them: Jensen [5], n=1 to 9; Gram [6], n=1 to 16; Liang and Todd [7], n=1 to 20; and Ainsworth and Howell [8], n=1 to 31. Reference 7 uses Euler-Maclaurin summation with an extended precision software package but gives no error bounds. Reference 8 extends the list by another 11 and does almost as well in terms of accuracy as Reference 7 using standard precision and some mathematical analysis and does include error bounds.

However it seems that 35 is about the limit that one can reach using Euler-Maclaurin summation of the series as given in equation (1) so we shall express this series as an integral which is easy to numerically evaluate. This integral will allow us to compute the first 2000 Euler-Mascheroni constants.

Consider the integral

$$I = \frac{1}{2i}\int_C \cot \pi z \ln^n z \frac{dz}{z} \tag{3}$$

taken about the contour C in Figure 1a (Fig. 1b depicts the magnitude of the integrand for n=4) where $0 < y < R$, $1 < x < N + 1/2$. Further, we take

$$\frac{\cot \pi z}{2i} = \begin{cases} -1/2 - \dfrac{1}{e^{-2\pi iz} - 1} & \mathrm{Im}(z) > 0 \\[2ex] 1/2 + \dfrac{1}{e^{2\pi iz} - 1} & \mathrm{Im}(z) < 0 \end{cases} \quad . \tag{4}$$

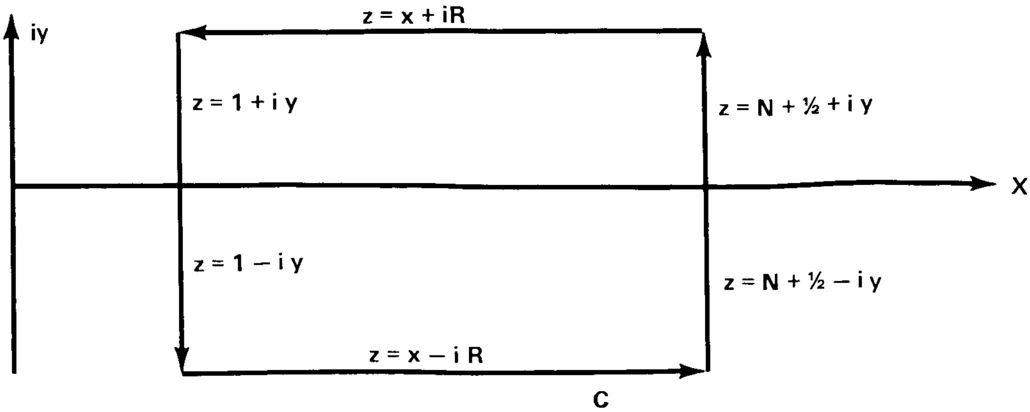

Figure 1a. Contour of integration.

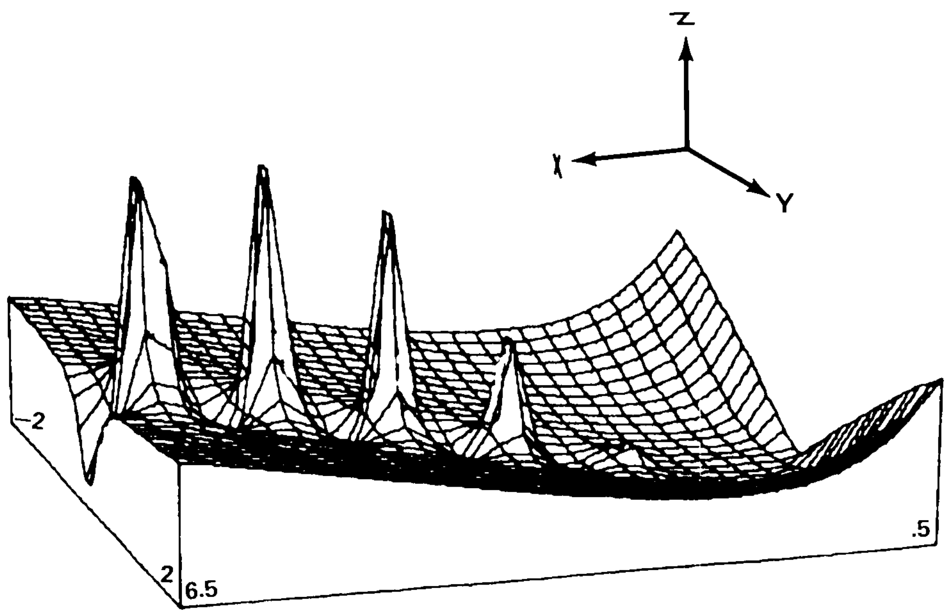

Figure 1b. Magnitude of the integrand in equation (3) for n = 4.

Therefore,

$$I = \int_R^0 \left[-\frac{1}{2} - \frac{1}{e^{-2\pi i(1+iy)} - 1} \right] \frac{\ln^n (1+iy)}{1+iy} \, idy + \int_0^R \left[\frac{1}{2} + \frac{1}{e^{2\pi i(1-iy)} - 1} \right] \frac{\ln^n (1-iy)}{1-iy} (-idy)$$

$$+ \int_1^{N+1/2} \left[\frac{1}{2} + \frac{1}{e^{2\pi i(x-iR)} - 1} \right] \frac{\ln^n (x-iR)}{x-iR} \, dx$$

$$+ \int_R^0 \left[\frac{1}{2} + \frac{1}{e^{2\pi i(N+1/2-iy)} - 1} \right] \frac{\ln^n (N+1/2-iy)}{N+1/2-iy} (-idy)$$

$$+ \int_0^R \left[-\frac{1}{2} - \frac{1}{e^{-2\pi i(N+1/2+iy)} - 1} \right] \frac{\ln^n (N + 1/2 + iy)}{N + 1/2 + iy} \, idy$$

$$+ \int_{N+1/2}^1 \left[-\frac{1}{2} - \frac{1}{e^{-2\pi i(x+iR)} - 1} \right] \frac{\ln^n (x + iR)}{x + iR} \, dx = \sum_{k=1}^{N} \frac{\ln^n k}{k} \quad .$$

Now consider the first part of I_2, I_3, and I_4 (the second, third, and fourth integrals). We have:

$$\frac{1}{2} \int_0^R \frac{\ln^n(1-iy)}{1-iy} (-idy) + \frac{1}{2} \int_1^{N+1/2} \frac{\ln^n(x-iR)}{x-iR} \, dx + \frac{1}{2} \int_R^0 \frac{\ln^n(N+1/2-iy)}{N+1/2-iy} (-idy)$$

$$= \frac{1}{2} \frac{1}{n+1} [\ln^{n+1}(1-iR) - \ln^{n+1}(1) + \ln^{n+1}(N+1/2-iR) - \ln^{n+1}(1-iR) + \ln^{n+1}(N+1/2)$$

$$- \ln^{n+1}(N+1/2-iR)] = \frac{\ln^{n+1}(N+1/2)}{2(n+1)} \quad .$$

The first part of I_1, I_5, and I_6 yield the same. The $-1/2$ factor changes the signs but the reversal of the limits makes it also

$$\frac{\ln^{n+1}(N+1/2)}{2(n+1)} \quad ,$$

and the sum

$$\frac{\ln^{n+1}(N+1/2)}{n+1}$$

may be replaced by

$$\frac{\ln^{n+1}(N+1)}{n+1}$$

when we let N approach infinity.

We now have, upon slight rearrangement,

$$\int_0^R \frac{\ln^n(1+iy)}{(e^{2\pi y}-1)\,(1+iy)}\, idy + \int_0^R \frac{\ln^n(1-iy)\,(-idy)}{(e^{2\pi y}-1)\,(1-iy)} + \int_0^R \frac{\ln^n(N+1/2-iy)}{N+1/2-iy} \cdot \frac{(-idy)}{e^{2\pi y}+1}$$

$$+ \int_0^R \frac{\ln^n(N+1/2+iy)}{N+1/2+iy} \cdot \frac{idy}{e^{2\pi y}+1} + \int_1^{N+1/2} \frac{1}{e^{2\pi ix}\, e^{2\pi R}-1} \cdot \frac{\ln^n(x-iR)}{x-iR}\, dx$$

$$+ \int_1^{N+1/2} \frac{\ln^n(x+iR)}{e^{-2\pi ix}\, e^{2\pi R}-1}\, \frac{dx}{(x+iR)} = \sum_{k=1}^{N} \frac{\ln^n k}{k} - \frac{\ln^{n+1}(N+1)}{n+1}$$

Because of the $e^{2\pi R}$ factor in the denominator, the last two integrals obviously tend to zero as R approaches infinity. Also, the first two integrals are conjugates so that

$$\sum_{k=1}^{N} \frac{\ln^n k}{k} - \frac{\ln^{n+1}(N+1)}{n+1} = 2\ \mathrm{Re} \int_0^N \frac{\ln^n(1-iy)\,(-idy)}{(e^{2\pi y}-1)\,(1-iy)} + R_N$$

where

$$R_N = 2\ \mathrm{Re} \int_0^\infty \frac{\ln^n(N+1/2-iy)\,(-idy)}{(e^{2\pi y}+1)\,(N+1/2-iy)}$$

which tends to zero as N approaches infinity. Therefore,

$$\lim_{N\to\infty} \left[\sum_{k=1}^{N} \frac{\ln^n k}{k} - \frac{\ln^{n+1}(N+1)}{n+1} \right] = 2 \operatorname{Re} \int_0^\infty \frac{\ln^n(1-iy)\,(-idy)}{(e^{2\pi y}-1)\,(1-iy)} \quad ,$$

or

$$\gamma_n = 2 \operatorname{Re} \int_0^\infty \frac{(y-i)\ln^n(1-iy)}{(1+y^2)\,(e^{2\pi y}-1)}\, dy \quad , \quad n = 1, 2, 3, \ldots \quad .$$

As a check, we make the substitution

$$\gamma_n = \int_0^\infty \frac{1}{(1+y^2)\,(e^{2\pi y}-1)} \left[(y-i)\ln^n(1-iy) + (y+i)\ln^n(1+iy)\right] dy$$

in the series portion of equation (2) to obtain:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n (z-1)^n}{n!} = \frac{1}{2} + \int_0^\infty \frac{1}{(1+y^2)\,(e^{2\pi y}-1)} \left[\frac{y-i}{(1-iy)^{z-1}} + \frac{y+i}{(1+iy)^{z-1}}\right] dy \quad ,$$

where we have made use of the fact that

$$\gamma_0 = \frac{1}{2} + 2 \int_0^\infty \frac{y}{(1+y^2)\,(e^{2\pi y}-1)}\, dy \quad .$$

Simplifying gives:

$$\sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n (z-1)^n}{n!} = \frac{1}{2} + \int_0^\infty \frac{i}{(e^{2\pi y}-1)} \left[\frac{1}{(1+iy)^z} - \frac{1}{(1-iy)^z}\right] dy = \frac{1}{2} - \int_0^\infty \left\{ \frac{i}{(1+y^2)^{z/2}\,(e^{2\pi y}-1)} \right.$$

$$\left. \left[e^{iz\tan^{-1} y} - e^{-iz\tan^{-1} y} \right] \right\} dy = \frac{1}{2} + 2 \int_0^\infty \frac{\sin(z \tan^{-1} y)}{(1+y^2)^{z/2}\,(e^{2\pi y}-1)}\, dy \quad .$$

Thus, we obtain for the Riemann zeta function

$$\zeta(z) = \frac{1}{2} + \frac{1}{z-1} + 2\int_0^\infty \frac{\sin(z \tan^{-1} y)}{(1+y^2)^{z/2}\,(e^{2\pi y}-1)}\,dy$$

in agreement with the formula derived by the Danish mathematician J. L. W. V. Jensen in 1893 [9].

COMPUTATION

Using the new representation

$$\gamma_n = 2\,\mathrm{Re}\int_0^\infty \frac{(y-i)\ln^n(1-iy)}{(1+y^2)\,(e^{2\pi y}-1)}\,dy \quad , \quad n = 1, 2, 3, \ldots ,$$

we write

$$\gamma_n = 2\,\mathrm{Re}\int_0^\infty \frac{(y-i)}{(1+y^2)\,(1-e^{-2\pi y})}\; e^{n \ln\ln(1-iy) - 2\pi y}\,dy \quad .$$

Setting

$$\ln\ln(1-iy) = \frac{1}{2}\ln\left[\frac{1}{4}[\ln(1+y^2)]^2 + (\tan^{-1}y)^2\right] - i\tan^{-1}\left(\frac{2\tan^{-1}y}{\ln(1+y^2)}\right) ,$$

we obtain

$$\gamma_n = 2\,\mathrm{Re}\int_0^\infty \frac{(y-i)}{(1+y^2)\,(1-e^{-2\pi y})}\left\{\cos\left[n\tan^{-1}\left(\frac{2\tan^{-1}y}{\ln(1+y^2)}\right)\right] - i\sin\left[n\tan^{-1}\left(\frac{2\tan^{-1}y}{\ln(1+y^2)}\right)\right]\right\} e^{\frac{n}{2}\ln\left[\frac{1}{4}[\ln(1+y^2)]^2 + (\tan^{-1}y)^2\right] - 2\pi y}\,dy$$

or,

$$\gamma_n = 2 \int_0^{\infty} \frac{1}{(1+y^2)(1-e^{-2\pi y})} \left\{ y \cos\left[n \tan^{-1}\left(\frac{2 \tan^{-1} y}{\ln(1+y^2)}\right)\right] - \sin\left[n \tan^{-1}\left(\frac{2 \tan^{-1} y}{\ln(1+y^2)}\right)\right]\right\} e^{\frac{n}{2} \ln\left[\frac{1}{4}[\ln(1+y^2)]^2 + (\tan^{-1} y)^2\right] - 2\pi y} dy \ .$$

Now, for $n > 150$, the integrand becomes too large for computational purposes. We thus introduce a scale factor β^n which can be chosen to keep the integrand well behaved. Thus, we obtain

$$\gamma_n = 2\,\beta^n \int_0^{\infty} \frac{1}{(1+y^2)(1-e^{-2\pi y})} \left\{ y \cos\left[n \tan^{-1}\left(\frac{2 \tan^{-1} y}{\ln(1+y^2)}\right)\right] - \sin\left[n \tan^{-1}\left(\frac{2 \tan^{-1} y}{\ln(1+y^2)}\right)\right]\right\} e^{\frac{n}{2} \ln\left[\frac{1}{4}[\ln(1+y^2)]^2 + (\tan^{-1} y)^2\right] - 2\pi y - n\ln\beta} dy \ .$$

After the integral is evaluated, it can be easily multiplied by the scale factor to give the desired result.

The Gauss numerical integration formula was used and if the value was unchanged by taking $n=N$ and $n=2N$, we assumed the integral to be correct as far as the agreement extended. The integrand is much too complex to use the standard remainder terms and the bounds that we were able to find in Reference 8 were very much too conservative. A further check is that the results were in agreement with those obtained when rectangular integration was used.

Table 1 gives γ_n, $n = 1\ (1)\ 200$, and assorted values to $n = 2000$.

We now take n in γ_n to be a non-negative continuous variable and obtain the graph (Fig. 2) plus a listing of the first 13 zeros (Table 2). Conceivably the integral of γ_n or some function of its roots might lead to some esoteric information regarding the Riemann zeta function.

TABLE 1. GENERALIZED EULER-MASCHERONI CONSTANTS

n	γ_n	n	γ_n	n	γ_n
0	5.772156649E-1	40	2.487215594E-1	80	2.516344101E10
1	-7.281584548E-2	41	-7.195748469E-1	81	1.510585108E11
2	-9.690363193E-3	42	-2.638794927E0	82	4.379044312E11
3	2.053834420E-3	43	-5.264930312E0	83	9.317068469E11
4	2.325370065E-3	44	-7.188745890E0	84	1.472099819E12
5	7.933238173E-4	45	-5.072344590E0	85	1.259044968E12
6	-2.387693454E-4	46	6.609915609E0	86	-1.958810225E12
7	-5.272895671E-4	47	3.403977498E1	87	-1.295154550E13
8	-3.521233538E-4	48	7.868247976E1	88	-3.929715388E13
9	-3.439477443E-5	49	1.258443876E2	89	-8.753044740E13
10	2.053328149E-4	50	1.268236027E2	90	-1.471610494E14
11	2.701844395E-4	51	-1.919691183E1	91	-1.477977001E14
12	1.672729121E-4	52	-4.631889230E2	92	1.294632141E14
13	-2.746380660E-5	53	-1.340659144E3	93	1.188562976E15
14	-2.092092621E-4	54	-2.572454741E3	94	3.920684627E15
15	-2.834686553E-4	55	-3.457141209E3	95	9.341670850E15
16	-1.996968583E-4	56	-2.055275816E3	96	1.707524745E16
17	2.627703711E-5	57	5.372282214E3	97	2.071798354E16
18	3.073684081E-4	58	2.401938938E4	98	-2.854307850E15
19	5.036054530E-4	59	5.742431929E4	99	-1.125848108E17
20	4.663435615E-4	60	9.854325459E4	100	-4.253401572E17
21	1.044377698E-4	61	1.116709578E5	101	-1.107785955E18
22	-5.415995822E-4	62	5.333665226E3	102	-2.232843819E18
23	-1.243962090E-3	63	-3.909726873E5	103	-3.249016082E18
24	-1.588511279E-3	64	-1.303180713E6	104	-1.590464731E18
25	-1.074591953E-3	65	-2.845076553E6	105	1.017821085E19
26	6.568035186E-4	66	-4.540526610E6	106	4.868854920E19
27	3.477836914E-3	67	-4.341905139E6	107	1.424392927E20
28	6.400068532E-3	68	2.871566946E6	108	3.200736262E20
29	7.371151770E-3	69	2.660490855E7	109	5.487745762E20
30	3.557728856E-3	70	7.932166312E7	110	5.449976660E20
31	-7.513325998E-3	71	1.662151340E8	111	-6.729019783E20
32	-2.570372911E-2	72	2.551532583E8	112	-5.611293844E21
33	-4.510673411E-2	73	2.126556317E8	113	-1.930664135E22
34	-5.112692802E-2	74	-2.987670894E8	114	-4.902622623E22
35	-2.037304360E-2	75	-1.919487428E9	115	-9.759599993E22
36	7.248215882E-2	76	-5.515574258E9	116	-1.373492481E23
37	2.360263823E-1	77	-1.148345099E10	117	-3.817890627E22
38	4.289634464E-1	78	-1.757015228E10	118	5.896847847E23
39	5.179218427E-1	79	-1.396102146E10	119	2.656742888E24

TABLE 1. (Concluded)

n	γ_n	n	γ_n	n	γ_n
120	7.818403277E24	155	-2.740591577E38	190	9.870435858E52
121	1.796461571E25	156	-4.790923159E38	191	2.007116958E53
122	3.181276594E25	157	-4.247747471E38	192	2.676425145E53
123	3.336371622E25	158	1.143679930E39	193	-1.391163860E53
124	-3.782268183E25	159	7.970742318E39	194	-2.601448677E54
125	-3.475516117E26	160	2.910661837E40	195	-1.167447626E55
126	-1.257375875E27	161	8.179814602E40	196	-3.817302910E55
127	-3.362216797E27	162	1.863819327E41	197	-1.019713344E56
128	-7.131871952E27	163	3.236126153E41	198	-2.217662282E56
129	-1.115323781E28	164	2.703985771E41	199	-3.442070818E56
130	-6.637629576E27	165	-8.811044847E41	200	-6.974649720E55
131	3.605865029E28	166	-5.871467633E42	250	3.059212855E79
132	1.936502639E29	167	-2.142378188E43	300	-5.556728220E102
133	6.244838068E29	168	-6.061386654E43	350	-1.785446107E127
134	1.564173210E30	169	-1.395436964E44	400	-1.761642187E152
				500	-1.165505188E204
135	3.101716787E30	170	-2.457784157E44	750	-2.1645871E341
136	4.226701286E30	171	-2.124640956E44	1000	-1.5709538E486
137	-7.995222356E27	172	6.605630678E44	1500	-3.1170139E791
138	-2.523130101E31	173	4.538504341E45	2000	2.68024593E1109
139	-1.103550302E32	174	1.687141829E46		
140	-3.323756372E32	175	4.863746403E46		
141	-7.953050911E32	176	1.144982704E47		
142	-1.490838047E33	177	2.086041054E47		
143	-1.747042301E33	178	2.021973061E47		
144	1.370337533E33	179	-4.769958441E47		
145	1.679138192E34	180	-3.664699738E48		
146	6.623981071E34	181	-1.414931352E49		
147	1.914935612E35	182	-4.203887079E49		
148	4.445803239E35	183	-1.022891247E50		
149	7.981621278E35	184	-1.958109259E50		
150	8.028853731E35	185	-2.218374141E50		
151	-1.396229273E36	186	3.112089130E50		
152	-1.134586466E37	187	3.061460410E51		
153	-4.241080464E37	188	1.255253421E52		
154	-1.199627658E38	189	3.887946022E52		

TABLE 2. ROOTS ρ_i LESS THAN 60 OF γ, WHERE $|\gamma_{\rho_i}| < 10^{-9}$.

2.623384858
5.684688574
9.114834513
12.86713391
16.90281884
21.18896
25.6981707
30.40797324
35.29985207
40.35837636
45.57051042
50.9251026
56.41250687

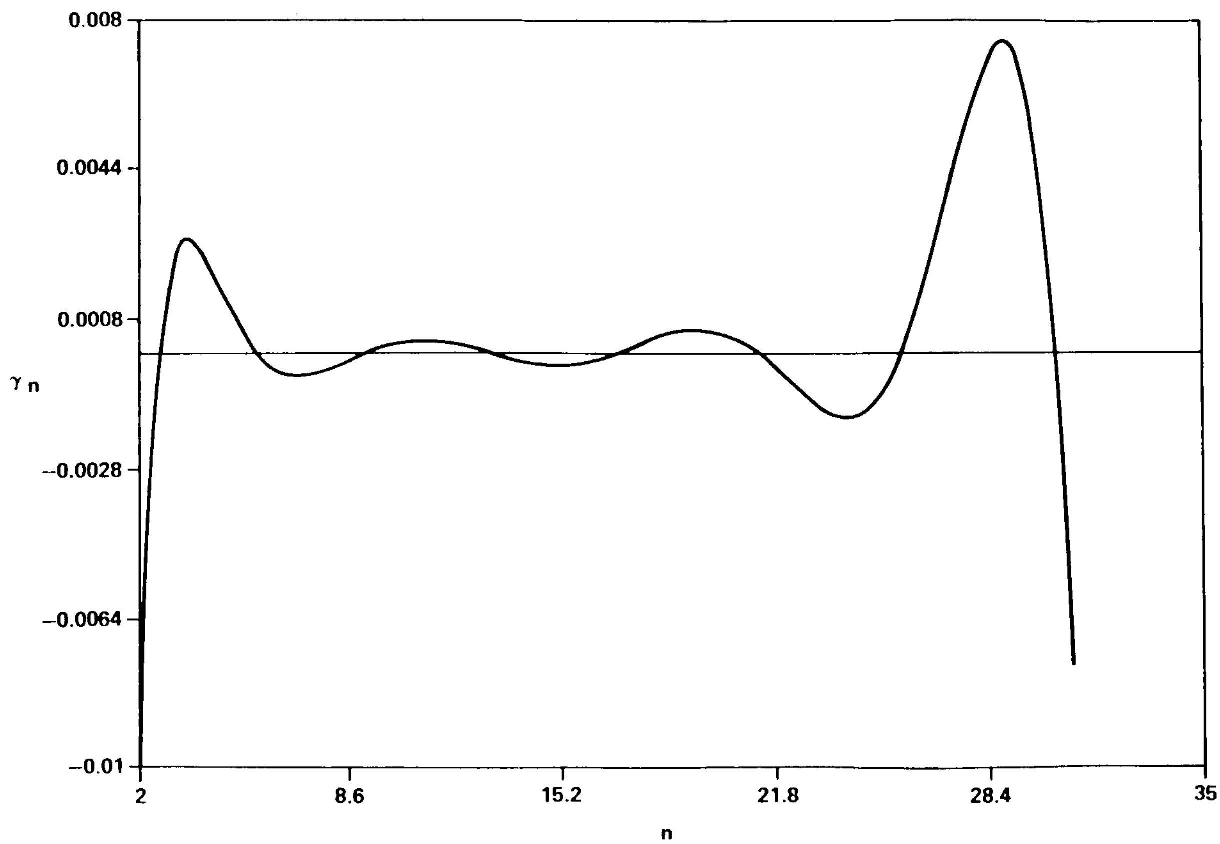

Figure 2. The Euler-Mascheroni function.

REFERENCES

1. Mitrovic, Dragisa: Math Reviews, Vol. 29, No. 2232, 1965.

2. Kluyver, J. C.: On Certain Series of Mr. Hardy. Quart. J. of Pure and Applied Math., Vol. 50, 1927, pp. 185-192.

3. Briggs, W. E., and Chowla, S.: The Power Series Coefficients of $\zeta(s)$. Amer. Math. Monthly, Vol. 62, 1955, pp. 323-325.

4. Wilton, J. R.: A Note on the Coefficients in the Expansion of $\zeta(s,z)$ in Powers of s - 1. Quart. J. of Pure and Applied Math., Vol. 50, 1927, pp. 329-332.

5. Jensen, J. L. W. V.: Sur la fonction $\zeta(s)$ de Riemann. Comptes Rendus (Paris), Vol. 104, 1887, pp. 1156-1159.

6. Gram, J. P.: Note sur le calcul de la fonction $\zeta(s)$ de Riemann. Oversig. K. Danske Vidensk. (Selskab Forhandlingar. 305-308, 1895).

7. Liang, J. J. Y., and Todd, John: The Stieltjes Constants. J. of Research of the National Bureau of Standards; Vol. 76B, Nos. 3 and 4, 1972, pp. 161-178.

8. Ainsworth, O. R., and Howell, L. W.: The Generalized Euler-Mascheroni Constants, NASA TP 2264, 1984.

9. Jensen, J. L. W. V.: Opgaver Til Losning. 34; Nyt Tidsskrift for Mathematik, BD. 4B, pp. 54, 1893.

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16. ABSTRACT

The limit series for the Euler-Mascheroni constants is represented as an integral. Using this new representation, we compute the first 200 values and assorted others up to 2000. The first 13 roots of γ_n, where n is a positive continuous variable, are also given.

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