ANALYSIS OF SPECTRAL OPERATORS IN ONE-DIMENSIONAL DOMAINS

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ABSTRACT
We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to N with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.

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I. INTRODUCTION

This paper presents an investigation of a class of projection operators that arises in the analysis of the approximation of differential equations by spectral methods using Chebyshev decomposition.

Some similar operators have been studied before by Canuto-Quarteroni [1] and Maday-Quarteroni [1], but the existing results are not adequate in many applications. In fact they forbid analysis for the error of the approximation by spectral methods of fourth-order problems and, in several instances, second-order problems (see Canuto-Quarteroni [2]).

We first present some background tools required for our analysis. They consist of Sobolev spaces relative to the weight \( \omega(x) = (1 - x^2)^{-1/2} \) (this weight arises in the relations of orthogonality of Chebyshev polynomials). We recall and complete results proved by Grisvard [1], [2] concerning interpolation theory between these spaces.

Then we present an analysis of projection operators from these spaces into the set of all polynomials of degree lower than \( N \).

Finally we give an application of the results herein proved to a simple test problem.

We shall give other applications in Maday [1] and shall in a future work extend these results to multidimensional domains. Our aim is to apply such results to the analysis of the approximation of Navier-Stokes equations by spectral methods (see Maday-Metivet [1], [2]).

For some different notions about projection operators that arise in spectral methods, see Tadmor [1].
II. PRELIMINARIES: SOME FUNCTION SPACES

Notations and Basic Properties

Let $J$ be an open interval $]a,b[$ of $\mathbb{R}$ ($a<b$); we consider a weight function $\rho(x)$, continuous over $J$, satisfying $\rho(x) \geq \rho_0 > 0$ for any $x \in J$.

Let us set:

\[(2.1) \quad L_\rho^2(J) = \{ \phi : J \to \mathbb{R} \mid \phi \text{ is measurable and } (\phi,\phi)_\rho < +\infty \},\]

equipped with the inner product $(\phi,\psi)_\rho = \int_J \phi(x)\psi(x)\rho(x)\,dx$. For any integer $s \geq 0$ we set:

\[(2.2) \quad \|\phi\|_{s,\rho} = \left( (\phi,\phi)_{s,\rho} \right)_{s,\rho},\]

this space being equipped with the inner product:

\[(2.3) \quad (\phi,\psi)_{s,\rho} = \sum_{k=0}^{s} \frac{d^k \phi}{dx^k} \frac{d^k \psi}{dx^k} \rho.
\]

Clearly, one has the equality:

\[L_\rho^2(I) = H_\rho^0(I).\]

For any real $s \geq 0$, noninteger, $H_\rho^s(J)$ is defined by interpolation between the space $H_\rho^s(J)$ and $H_\rho^{s+1}(J)$, where $\lfloor s \rfloor$ represents the integral part of $s$. 
The method of interpolation can be the complex one, the operator's domain one or the trace one (see Lions-Magenes [1] for more details). Besides we define $H^{s}_{0,\rho}(J)$ as being the closure of $\mathcal{D}(J)$ in $H^{s}_{\rho}(J)$. When $\rho = 1$ these spaces are the usual Sobolev spaces denoted by $H^{s}(J)$ and $H^{s}_{0}(J)$ respectively. For the application to spectral methods we are mostly interested in those spaces when $J = I \equiv [-1,+1]$ and $\rho(x) = \omega(x) \equiv \frac{1}{\sqrt{1 - x^2}}$. Let us recall some results proved in Grisvard [1], [2] valuable for $J = I$, $\rho = \omega$ and for $J = [0,1]$, $\rho = \frac{1}{\sqrt{x}}$.

**THEOREM 2.1 (Grisvard [1]):**

i) For any real $s > 0$, $s \notin \mathbb{N} + \frac{1}{4}$ we have:

\[(2.4) \quad H^{s}_{0,\rho}(J) = \left[ H^{s}_{0,\rho}(J), \overline{H^{s+1}_{0,\rho}(J)} \right]_{s-s}.\]

ii) For any integral $n$ we have:

\[(2.4') \quad \left[ H^{n}_{0,\rho}(J), H^{n+1}_{0,\rho}(J) \right]_{1/4} \subset H^{n+1/4}_{0,\rho}(J).\]

iii) For any real $s > 0$, $s \notin \mathbb{N} + \frac{1}{2}$:

\[H^{s}_{\rho}(J) \subset C^{m}(\overline{J}),\]

the space of continuous mapping defined over $\overline{J}$ whose derivative of order $\leq m$ are continuous over $\overline{J}$, with $m = s - \frac{1}{2}$.

The trace application defined from $C^{\infty}(\overline{J})$ into $\mathbb{R}^{2n}$:

\[ u \mapsto \{u(-1), \frac{du}{dx}(-1), \ldots, \frac{d^n u}{dx^n}(-1), u(1), \frac{du}{dx}(1), \ldots, \frac{d^n u}{dx^n}(1)\} \]

can be extended to a continuous mapping from \( H_{\rho}^{n+\frac{1}{4}+\varepsilon}(I) \) onto \( \mathbb{R}^{2n} \) for any \( \varepsilon > 0 \).

iv) For any real \( \frac{1}{4} < s < \frac{5}{4} \), \( H_{0,\rho}^{S}(J) \) coincide with the subspace of \( \mathcal{H}_{\rho}^{S}(J) \) of functions vanishing at the real boundaries of \( J \).

v) For any real \( s > \frac{1}{2} \), \( \mathcal{H}_{\rho}^{S}(J) \) is an algebra.

**THEOREM 2.2 (Grisvard [2]):**

For any \( 0 < q < s < p \), \( \mathcal{H}_{\rho}^{S}(J) \) satisfies the following double topological imbedding:

\[
[H_{\rho}^{q}(J), H_{\rho}^{p}(J)]_{\theta,1} \subset \mathcal{H}_{\rho}^{S}(J) \subset [H_{\rho}^{q}(J), H_{\rho}^{p}(J)]_{\theta,\infty},
\]

with \( \theta = \frac{s - p}{q - p} \), and the notation holds for the real interpolation (see Lions and Peetre [1]).

The two following results can be found in Canuto-Quarteroni [2] and Maday-Quarteroni [1].

**THEOREM 2.3:**

i) For any real \( s \geq \frac{1}{4} \), \( \mathcal{H}_{\omega}^{S}(I) \subset H_{\omega}^{S-\frac{1}{4}}(I) \).

ii) For any \( 0 \leq r < s \), the imbedding \( \mathcal{H}_{\omega}^{S}(I) \subset H_{\omega}^{r}(I) \) is compact.

In the next section we shall generalize the results (2.4) and (2.4').
Some New Results About Interpolation Between $H^p_{0, \omega}(I)$

This section is devoted to the proof of the following:

**THEOREM 2.4:** For any $0 < q < s < p$ not in $\mathbb{N} + \frac{1}{4}$ we have:

$$[H^q_{0, \omega}(I), H^p_{0, \omega}(I)]_{[s-p]/q-p} = H^s_{0, \omega}(I).$$

This theorem is a consequence of the two following lemma:

**LEMMA 2.1:** For any integer $p \leq n$, we have:

$$u \in H^p_{0, \omega}(I) \Rightarrow \frac{d^p u}{dx^p} \in L^2_{\omega}((0,1), (n-p)+1(I)).$$

**PROOF:** It is an easy matter to check that this result is a consequence of

$$(2.6) \quad u \in H^n_{0, \frac{1}{\sqrt{x}}}(0,1) \Rightarrow \frac{d^p u}{dx^p} \in L^2_{\omega}((0,1), (\frac{1}{\sqrt{x}})^{4(n-p)+1}((0,1),$$

(we shift the difficulties at $\pm 1$ onto $0$). So let $u$ be in $H^n_{0, \frac{1}{\sqrt{x}}}(0,1)$; from Theorem 2.1 (point iv) we have, for any $0 \leq p < n$:

$$\frac{d^p u}{dx^p}(0) = 0,$$

hence

$$\int_0^x \frac{d^{p+1} u}{dx^{p+1}}(t) dt = \frac{d^p u}{dx^p}(x).$$
Besides, from Lemma 6.2.1 of Nečas [1] we have, for any \( \alpha < 1 \) and any \( v \)

such that \( \int_0^1 v^2(x)x^\alpha \, dx < \infty \):

\[
(2.7) \quad \int_0^1 \left( \int_0^x |v(x)| \right)^2 x^{\alpha-2} \, dx \leq \left( \frac{1}{1-\alpha} \right) \int_0^1 |v(x)|^2 x^\alpha \, dx,
\]

taking then \( \alpha = -\frac{1}{2} - 2(n-(p+1)) \) and \( v = \frac{d^{p+1}u}{dx^{p+1}} \) we obtain:

\[
\int_0^1 \left( \frac{d^p u}{dx^p} \right)^2 x^{-\frac{1}{2}} - 2(n-p) \, dx \leq C \int_0^1 \left( \frac{d^{p+1}u}{dx^{p+1}} \right)^2 x^{-\frac{1}{2}} - 2(n-(p+1)) \, dx,
\]

and (2.6) holds by induction over \( p \).

**LEMMA 2.2**: For any integer \( n > 0 \), the mapping \( u \mapsto u \omega^{1/2} \) is an

homeomorphism from \( H_0^n(0,1) \) onto \( H_0^0(0,1) \).

**PROOF**: Here again we prove the result for the weight \( \frac{1}{\sqrt{x}} \), say:

\[
(2.8) \quad u \mapsto ux^{-1/4} \text{ is an homeomorphism from } H_0^n,_{1/\sqrt{x}}(0,1) \text{ onto } H_0^n(0,1).
\]

Let \( \phi \in D(0,1) \), then, for \( 0 \leq m \leq n \):

\[
\frac{d^m}{dx^m} (\phi x^{-1/4}) = \sum_{p=0}^m \binom{m}{p} \frac{d^p \phi}{dx^p} x^{-1/4} - (m-p),
\]

\[
= \sum_{p=0}^m \binom{m}{p} D_m^p \frac{d^p \phi}{dx^p} x^{-1/4} - (m-p),
\]

with \( D_m^p = \left[ -\frac{1}{4} - (m-p+1) \right] D_m^{p+1} \) and \( D_m^m = 1 \). From Lemma 2.1 we then get:
\[ \| \frac{d^m}{dx^m} (x^{1/4}) \|_{0,1} \leq C \| \phi \|_{m,1/\sqrt{x}} \leq C \| \phi \|_{n,1/\sqrt{x}} ; \]

summing up these estimates for \( 0 \leq m \leq n \) we derive:

\[ (2.9) \quad \| \phi x^{1/4} \|_{n,1/\sqrt{x}} \leq C \| \phi \|_{n,1} . \]

Inversely, let us prove that, for any \( \phi \in \mathcal{D}(I) \):

\[ (2.10) \quad \| \phi x^{1/4} \|_{n,1/\sqrt{x}} \leq C \| \phi \|_{n,1} . \]

From Hardy's inequality (Lemma 2.5.1 of Nečas [1]) we derive by induction that, for any \( 0 \leq p \leq m \leq n \):

\[ (2.11) \quad \| \frac{d^p \phi}{dx^p} \|_{0,x^{-2(m-p)}} \leq C \| \phi \|_{m,1}, \]

besides:

\[ \frac{d^m}{dx^m} (x^{1/4}) = \sum_{p=0}^{m} C_p \frac{d^p \phi}{dx^p} \frac{d^m-p}{dx^{m-p}} (x^{1/4}) \]

\[ = \sum_{p=0}^{m} C_p D_m^{-p} \frac{d^p \phi}{dx^p} x^{1/4-(m-p)}, \]

with:

\[ D_m^{-p} = \left[ \frac{1}{4} - (m - p + 1) \right] D_m^{-p+1} \quad \text{and} \quad D_m^{-m} = 1 . \]

Then using (2.11) we get:
and (2.10) is derived by summing up these results for $0 \leq m \leq n$. We can now achieve (2.8) as a consequence of (2.9) and (2.10).

We can now prove the main result of this section.

**PROOF OF THEOREM 2.4:** From (2.4) and Lemma 2.2 we deduce that the mapping $u \mapsto u^{1/2}$ is an homeomorphism from $H^s_{0,\omega}(I)$ onto $H^s_0(I)$ for any $s \geq 0$ not in $\mathbb{N} + 1/4 \cap \mathbb{N} + 1/2$ (see Lions-Magenes [1] for more details about the properties of spaces of interpolation).

Let us recall that, for any $q \leq s \leq p$ not in $\mathbb{N} + 1/2$ we have (see Lions-Magenes [1]):

$$H^s_0(I) = [H^q_0(I), H^p_0(I)]_{s-q}$$

From the previous homeomorphism we deduce that, for any $q \leq s \leq p$ not in $\{ \mathbb{N} + 1/2 \} \cup \{ \mathbb{N} + 1/4 \}$:

$$H^s_{0,\omega}(I) = [H^q_{0,\omega}(I), H^p_{0,\omega}(I)]_{s-q}$$

Let us remark now that the values of $p$, $q$, $s$ in $\mathbb{N} + 1/2$ have only been excluded due to (2.12), these values can now be recovered thanks to the reiteration theorem (Theorem I.6.1 of Lions-Magenes [1]).
III. APPROXIMATION RESULTS OF PROJECTION OPERATOR IN WEIGHTED SOBOLEV SPACES

The previous theorem leads us to define over $\mathcal{H}^{r}_{0,\omega}(I)$ a new scalar product. Indeed, for $p$ not in $\mathbb{N} + 1/8 \cup \mathbb{N} + 1/8$, $\mathcal{H}^{p}_{0,\omega}(I)$ can be seen as the interpolate $1/2$ between $L^{2}_{\omega}(I)$ and $\mathcal{H}^{2p}_{0,\omega}(I)$ and for $p$ in $\mathbb{N} + 1/8$, $\mathcal{H}^{p}_{0,\omega}(I)$ can be seen as the interpolate $1/3$ between $L^{2}_{\omega}(I)$ and $\mathcal{H}^{3p}_{0,\omega}(I)$.

If we consider the domain operator interpolation, this find expression in the existence of a selfadjoint operator $A_{r}$ such that:

* if $r \in \mathbb{N} + 1/8$, the domain $D(A_{r}^{3})$ of the operator $A_{r}^{3}$ in $L^{2}_{\omega}(I)$ is $\mathcal{H}^{3}_{0,\omega}(I)$ if $r \notin \mathbb{N} + 1/8$, the domain of $D(A_{r}^{2})$ of the operator $A_{r}^{2}$ in $L^{2}_{\omega}(I)$ is $\mathcal{H}^{2}_{0,\omega}(I)$.

* The domain $D(A_{r})$ of the operator $A_{r}$ in $L^{2}_{\omega}(I)$ is $\mathcal{H}^{r}_{0,\omega}(I)$ if $r \notin \mathbb{N} + 1/4$ and is included in $\mathcal{H}^{r}_{0,\omega}(I)$ if $r \in \mathbb{N} + 1/4$.

Moreover:

\[(3.1) \quad (u,v) \longmapsto (((u,v)))_{r,\omega} \equiv (A_{r}u, A_{r}v)_{\omega},\]

is a scalar product whose associated norm is equivalent to the one defined in (2.2) if $r \notin \mathbb{N} + 1/4$.

Let us define now $P_{r,N}$ as the projection operator from $\mathcal{H}^{r}_{0,\omega}(I)$ over $S^{r}_{N}$ with respect to the previous scalar product with:

\[S^{r}_{N} = S_{N} \cap \mathcal{H}^{r}_{0,\omega}(I),\]

\[S_{N} = \{ \phi \text{ defined over } I \mid \phi \text{ is a polynomial of degree } \leq N\}.\]
LEMMA 3.1: Let $0 \leq \nu \leq r \leq \sigma$ with $\sigma \notin \mathbb{N} + 1/8$ we have, for any $\phi \in H^\sigma_\omega(I) \cap H^r_{0,\omega}(I)$:

$$\|\phi - P_{r,N} \phi\|_{\nu,\omega} \leq C \nu^{\nu-\sigma} \|\phi\|_{\sigma,\omega}. \tag{3.2}$$

REMARK 3.1: The case $\nu = r = 0$ has been studied in Canuto-Quarteroni [1], the case $0 \leq \nu \leq r = 1$ has been looked at in Maday-Quarteroni [1] (note that the dependence of the constant is then $C(\sigma) = C(\sigma^*)$). Moreover it is proved that no optimal bound was possible for $H^\nu_{0,\omega}(I)$ norms with $\nu > r$. Indeed, for example:

$$\|\phi - P_{0,N} \phi\|_{\nu,\omega} \leq C \nu^{2\nu-\sigma} \|\phi\|_{\sigma,\omega}. \tag{3.4}$$

It is often necessary (see Canuto-Quarteroni [2], Maday-Metivet [2], Maday [1], and (4.14)) to obtain optimal results in higher norms.

PROOF: We shall only consider the case $r \notin \mathbb{N} + 1/8$ for simplicity. The proof is divided in two stages

1) We first prove (3.2) by induction over $r$ in $\mathbb{N}$. So, let us assume that (3.2) is true for $s < r$ in $\mathbb{N}$; let $\phi \in H^r_{0,\omega}(I)$; then $\phi_x \in H^{r-1}_{0,\omega}(I)$ and $P_{r-1,N-1}(\phi_x) \in S^{r-1}_{N-1}$. Moreover if $\phi(-1) = \phi(1) = 0$ we have:

$$\alpha \equiv \int_{-1}^{1} P_{r-1,N-1}(\phi_x)(t)dt = \int_{-1}^{1} [P_{r-1,N-1}(\phi_x) - \phi_x](t)dt.$$  

From the Cauchy-Schwarz inequality we derive:
\[
|a| \leq \left( \int_{-1}^{1} (P_{r-1,N-1}(\phi_x) - \phi_x)^2(t)\omega(t)dt \right)^{1/2} \left( \int_{-1}^{1} (\omega(t))^{-1} dt \right)^{1/2}
\]
\[
\leq C \|P_{r-1,N-1}(\phi_x) - \phi_x\|_{0,\omega};
\]

hence, from the induction hypothesis:

(3.5)  
\[
|a| \leq CN^{1-\sigma} \|\phi_x\|_{\sigma-1,\omega}.
\]

Finally we have:

\[
R_N(x) = \int_{-1}^{1} \left[ P_{r-1,N-1}(\phi_x)(t) - \frac{a(1 - t^2)^{r-1}}{\int_{-1}^{1} (1 - x^2)^{r-1} dx} \right] dt \in S_N^r.
\]

Due to the Poincaré-like inequality, the polynomial satisfies the following:

\[
\|\phi - R_N\|_{r,\omega} \leq \|(\phi - R_N)x\|_{r-1,\omega},
\]

the induction hypothesis, and (3.5) gives us:

\[
\|\phi - R_N\|_{r,\omega} \leq C(N^{(r-1)-(\sigma-1)} + N^{(1-\sigma)}) \|\phi_x\|_{\sigma-1,\omega}
\]
\[
\leq CN^{r-\sigma} \|\phi\|_{\sigma,\omega}.
\]

From the equivalence of the norms \(\|\|_{r,\omega}\) and \(\|\|_{r,\omega}\), and the identity:

\[
\|\|\phi - P_{r,N}\phi\|_{r,\omega} = \inf_{\phi_N \in S_N^r} \|\|\phi - \phi_N\|_{r,\omega},
\]
we obtain for any \( \phi \) in \( H_0^q(I) \cap H_0^r(I) \):

\[
(3.6) \quad \| \phi - \Pr,N \phi \|_{r,\omega} \leq C \| \sigma \|_{r,\omega}.
\]

Besides, since the operator \( \Lambda_r \) is selfadjoint, we have:

\[
\| \phi - \Pr,N \phi \|_{0,\omega} = \inf_{\psi \in L^2_\omega(I)} \frac{(\phi - \Pr,N \phi, \psi)_\omega}{\| \psi \|_{0,\omega}}
= \inf_{\psi \in L^2_\omega(I)} \frac{(\Lambda_r(\phi - \Pr,N \phi), \Lambda_r^{-1} \psi)_\omega}{\| \psi \|_{0,\omega}}.
\]

From (3.1) we then get:

\[
(3.7) \quad \| \psi - \Pr,N \psi \|_{0,\omega} = \inf_{\psi \in L^2_\omega(I)} \frac{((\phi - \Pr,N \phi, \Lambda_r^{-2} \psi))_{r,\omega}}{\| \psi \|_{0,\omega}}.
\]

By definition of \( \Pr,N \) we have, for any \( \psi \) in \( L^2_\omega(I) \):

\[
((\phi - \Pr,N \phi, \Pr,N(\Lambda_r^{-2} \psi)))_{r,\omega} = 0;
\]

hence

\[
\| \phi - \Pr,N \phi \|_{0,\omega} = \inf_{\psi \in L^2_\omega(I)} \frac{((\phi - \Pr,N \phi, (\Lambda_r^{-1} \psi) - \Pr,N(\Lambda_r^{-2} \psi)))_{r,\omega}}{\| \psi \|_{0,\omega}}
\leq \| \phi - \Pr,N \phi \|_{r,\omega} \inf_{\psi \in L^2_\omega(I)} \frac{\| (\Lambda_r^{-2} \psi) - \Pr,N(\Lambda_r^{-2} \psi) \|_{r,\omega}}{\| \psi \|_{0,\omega}}.
\]
Due to (3.6) we then derive:

\[ \| \phi - P_{r,N} \phi \|_{0, \omega} \leq C N^{-r-\sigma} \| \phi \|_{\sigma, \omega} N^{-r} \inf_{\psi \in L^2_{\omega}(I)} \frac{\| A^{-2}_r \psi \|_{0, \omega}}{\| \psi \|_{0, \omega}} 2r, \omega \]

\[ \leq C N^{-r-\sigma} \| \phi \|_{\sigma, \omega} \inf_{\psi \in L^2_{\omega}(I)} \frac{\| A^2_r (A^{-2}_r \psi) \|_{0, \omega}}{\| \psi \|_{0, \omega}} \]

\[ \leq C N^{-r-\sigma} \| \phi \|_{\sigma, \omega}. \]

Now, from the two estimates, valuable for any \( \phi \in H^{\sigma}(I) \cap H^r_{0, \omega}(I) \):

\[ \| \phi - P_{r,N} \phi \|_{r, \omega} \leq C N^{-r-\sigma} \| \phi \|_{\sigma, \omega} \]

\[ \| \phi - P_{r,N} \phi \|_{0, \omega} \leq C N^{-\sigma} \| \phi \|_{\sigma, \omega} \]

we derive that for any \( \theta \in ]0,1[ \):

\[ \| \phi - P_{r,N} \phi \|_{[L^2_{\omega}(I), H^r_{0, \omega}(I)]_{\theta}} \leq C N^{\theta r-\sigma} \| \phi \|_{\sigma, \omega}. \]

Due to (2.4) and (2.4') we deduce that, for any \( 0 \leq \nu \leq r \):

\[ (3.8) \quad \| \phi - P_{r,N} \phi \|_{\nu, \omega} \leq C N^{\nu-\sigma} \| \phi \|_{\sigma, \omega}. \]

ii) Let us now prove (3.3) for nonintegral values of \( r \). Let \( \phi \in \mathcal{D}(I) \), from step (i) we know that, for any \( \sigma \geq r+1 \), \( \sigma \in \mathbb{N} + \frac{1}{4} \).
Due to the interpolation of quotient spaces (see Lions-Magenes [1] Lemma 1.13.2) we have, for any $\theta \in ]0,1[$:

$$
\| \phi \|_{H_0^{r+1}(I)/S_{N}^{r+1}} \leq C N^{-\theta - \sigma} \| \phi \|_{S_{N}^{r+1}}
$$

From (2.4), (2.4') we deduce (we take $\theta = r - \sigma$):

$$
\| \phi \|_{H_0^{r}(I)/S_{N}^{r+1}} \leq C N^{-\sigma} \| \phi \|_{S_{N}^{r+1}}
$$
so that, for any $\sigma \geq \frac{r+1}{2}$ $\sigma \not\in \mathbb{N} + \frac{1}{4}$:

\[
(3.9) \quad \|\phi - P_{r,N}\phi\|_{r,\omega} < C N^{r-\sigma} \|\phi\|_{r,\omega}.
\]

By definition we know that, for any $r \not\in \mathbb{N} + \frac{1}{4}$,

\[
\|\phi - P_{r,N}\phi\|_{r,\omega} < C \|\phi\|_{r,\omega};
\]

hence, (3.9) holds for any $\sigma \geq r$, $\sigma \not\in \mathbb{N} + \frac{1}{4}$ and any $r \in \mathbb{R}^+$. The estimates in lower order norms are obtained following the same lines as in i).

We are now interested in the approximation of the spaces $H^r_\omega(I) \cap H^S_\omega(I)$ for $s \in \mathbb{N}$, $0 \leq s \leq r$ by polynomials therein contained.

For simplicity of exposition we shall consider the case $s = 0$. From point iii) of Theorem 2.4 we can easily exhibit for any $\phi \in H^r_\omega(I)$ a polynomial $\phi_0$ of degree $\leq 2r - 1$ such that: $\phi - \phi_0 \in H^r_0,\omega(I)$ and for any real $p$,

\[
(3.10) \quad \|\phi_0\|_{p,\omega} \leq C \|\phi\|_{r,\omega}.
\]

Due to the previous lemma, we have, for any $0 \leq v \leq r \leq \sigma \not\in \mathbb{N} + \frac{1}{4}$:

\[
\|\phi - \phi_0\|_{v,\omega} \leq C N^{v-\sigma} \|\phi - \phi_0\|_{r,\omega},
\]

and (3.10) then implies:

\[
\|\phi - (\phi_0 + P_{r,N}(\phi - \phi_0))\|_{v,\omega} \leq C N^{v-\sigma} \|\phi\|_{r,\omega}.
\]
For $N$ large enough (more precisely $N \geq 2r - 1$) we then get the existence of an operator $\tilde{P}$ from $H^r(I)$ onto $S_N$ such that:

$$\|\phi - \tilde{P}(\phi)\|_{v,\omega} \leq C N^{v-\sigma} \|\phi\|_{\sigma,\omega}.$$ 

This estimate provides an answer to our question in the case $s = 0$. The same proof can be done to build an operator from $H^r(I) \cap H^s_{0,\omega}(I)$ onto $S_N^s$ satisfying analogous bounds. This leads us to state the main theorem of this paper:

**THEOREM 3.1:** Let $(v, r, \sigma) \in \mathbb{R}, \sigma \notin \mathbb{N} + 1/4, s \in \mathbb{N}, 0 \leq s \leq r,$

$$0 \leq v \leq r \leq \sigma.$$ 

There exists an operator $\Pi^s,0_{r,N}$ from $H^r(I) \cap H^s_{0,\omega}(I)$ onto $S_N^s$ such that, for any $\phi \in H^r(I) \cap H^s_{0,\omega}(I)$ we have:

$$\|\phi - \Pi^s,0_{r,N} \phi\|_{v,\omega} \leq C N^{v-\sigma} \|\phi\|_{\sigma,\omega}.$$ 

**IV. AN APPLICATION**

**Definition of the Problem**

In order to explain how the previous results can be applied, we shall study an approximation of the very simple problem:

Find $\psi$ defined over $I$ such that:
\[
\begin{aligned}
\begin{cases}
\frac{d^4 \psi}{dx^4} = f & \text{over } I, \\
\psi = \frac{d\psi}{dx} = 0 & \text{at } \pm 1.
\end{cases}
\end{aligned}
\] (4.1)

(This problem provides a first step for the analysis of Stokes and Navier-Stokes problems in the \(\psi\)-formulation; see Maday-Metivet [1], [2].) Let us define \(H^{-2}(I)\) as follows:

\[
H^{-2}(I) = \{ f \in \mathcal{D}'(I) | \exists g \in L^2_\omega(I) : f = \frac{d^2 g}{dx^2} \}.
\]

We now want to prove the following:

**Theorem 4.1:** Let \(f \in H^{-2}_\omega(I)\); then there exists one and only one solution \(\psi\) to the problem (4.1) in the space \(H^2_{0,\omega}(I)\).

This theorem is a very simple consequence of Lax Milgram lemma and the two following lemmas.

**Lemma 4.1:** There exist two positive constants \(\delta_1\) and \(\delta_2\) such that, for any \(\phi\) in \(H^2_{0,\omega}(I)\):

\[
\int_I \phi^2 \omega^9 \leq \delta_1 \int_I \left( \frac{d\phi}{dx} \right)^2 \omega^5 \leq \delta_2 \int_I \left( \frac{d^2 \phi}{dx^2} \right)^2 \omega.
\]

This lemma is a corollary of Lemma 2.1.
**Lemma 4.2:** There exist 3 positive constants $\alpha, \beta, \gamma$ such that for any $(\phi, \psi) \in H^2_0(I) \times H^2_0(I)$:

\begin{align*}
(4.2) & \quad (\psi_{xx}, (\psi \omega)_{xx}) \geq \alpha \| \psi \|_{2, \omega}^2,
(4.3) & \quad \| \psi \|_{2, \omega} \leq \beta \| \psi_{xx} \|_{0, \omega},
(4.4) & \quad (\phi_{xx}, (\psi \omega)_{xx}) \omega \leq \gamma \| \phi_{xx} \|_{0, \omega} \| \psi_{xx} \|_{0, \omega}.
\end{align*}

**Proof:**

i) We first note that (4.3) is an easy consequence of the previous lemma, and is an equivalent to the Poincaré inequality.

ii) Next, we get the following equalities, for any $\psi \in D(I)$:

\[
\int_I \psi_{xx} (\psi \omega)_{xx} \, dx = \int_I \psi_{xx}^2 \omega \, dx + 2 \int_I \psi_{xx} \psi_x \omega_x \, dx + \int_I \psi_{xx} \psi_{xx} \omega \, dx
\]

\[
= \int_I \psi_{xx}^2 \omega \, dx + \int_I (\psi_x^2)_{xx} \omega_x \, dx + \int_I (\psi_x^2)_{xx} \, dx
\]

\[
= \int_I \psi_{xx}^2 \omega \, dx - \int_I \psi_x^2 \omega_x \, dx - \int_I \psi_x (\psi_{xx})_x \omega \, dx
\]

\[
= \int_I \psi_{xx}^2 \omega \, dx - 2 \int_I \psi_x^2 \omega_{xx} \, dx + 1/2 \int_I \psi_{xx}^2 \omega_{xxxx} \, dx
\]
let us note that:

\[ \omega_{xx} = (1 + 2x^2)\omega^5, \]

\[ \omega_{xxxx} = (9 + 72x^2 + 24x^4)\omega^9; \]

hence:

(4.5)

\[ \int \psi_{xx}(\psi\omega)_{xx} \, dx = \int \psi_x^2 \, dx - 2\int \psi_x^2 (1 + 2x^2)\omega^5 \, dx + \frac{1}{2} \int \psi^2 (9 + 72x^2 + 24x^4)\omega^9 \, dx. \]

Besides, let us set:

\[ P \equiv \int (\psi_{xx} \omega + 2x\psi_x \omega^3 + (2x^2 + 10^{-2})\omega^5)^2 \omega^{-1} \, dx; \]

\( P \) is \( \geq 0 \) and, an easy calculation gives:

\[ P \equiv \int \psi_{xx}(\psi\omega)_{xx} - 2.10^{-2} \int \psi_x^2 \omega^5 - \int (5.78x^2 + 0.4839)\psi^2 \omega^9, \]

so that

\[ \int \psi_x^2 \omega^5 \leq 50 \int \psi_{xx}(\psi\omega)_{xx}, \]

from which we derive:

\[ \int \psi_x^2 (1 + 2x^2)\omega^5 \leq 150 \int \psi_{xx}(\psi\omega)_{xx}. \]

Using that inequality in (4.5) we obtain \( \alpha > 0 \) such that:
\[ \int_I \psi_{xx}(\psi_\omega)_{xx} \geq \alpha \beta \int_I \psi_{xx}^2 \omega, \]

and (4.2) is an consequence of (4.3).

iii) Finally, let us note that for any \((\phi, \psi)\) in \(H^2(I) \times H^2_0(I)\) we have:

\[ (\phi_{xx}, (\psi_\omega)_{xx})_I = (\phi_{xx}, \psi_{xx})_\omega + 2 \int_I \phi_{xx} \psi_\omega \omega_x + \int_I \phi_{xx} \psi_{xx}. \]

The following inequality is simple:

\[ |(\phi_{xx}, \psi_{xx})_\omega| \leq \|\phi_{xx}\|_0,\omega \|\psi_{xx}\|_0,\omega. \]

Let us examine the second term:

\[ |\int_I \phi_{xx} \psi_\omega \omega_x| = |\int_I \phi_{xx}(\psi_\omega \omega x^{-1})\omega| \leq |\int_I \phi_{xx}^2 \omega|^{1/2} |\int_I \psi_\omega^2 \omega x^{-1}|^{1/2}; \]

since \(\omega_x^2 \omega^{-1} = x^2 \omega^5\), we derive from Lemma 4.1 that:

\[ |\int_I \phi_{xx} \psi_\omega \omega_x| \leq C |\phi_{xx}|_0,\omega \|\psi_{xx}\|_0,\omega. \]

We obtain, in a similar way:

\[ |\int_I \phi_{xx} \psi_{xx}| \leq C |\phi_{xx}|_0,\omega \|\psi_{xx}\|_0,\omega, \]

so that (4.4) is a consequence of (4.6)-(4.9).
PROOF OF THEOREM 4.1: Let \( f = g_{xx} \) be in \( H^{-2}_\omega(I) \).Problem (4.1) is equivalent to the following:

\[
\begin{align*}
\text{Find } \psi \text{ in } V = H^2_{0,\omega}(I) \text{ such that, for any } \phi \in V:
\end{align*}
\]

\[\int_I \psi_{xx}(\phi \omega)_{xx} = \int_I g(\phi \omega)_{xx}.\]

The bilinear form \( a \) defined by: For any \((\chi, \phi)\) in \( V^2 \):

\[a(\phi, \chi) = \int_I \phi_{xx}(\chi \omega)_{xx},\]

is continuous and elliptic over \( V \) (see Lemma 4.2), and Lax-Milgram lemma gives the existence and uniqueness of a solution of (4.10) hence of (4.1).

Approximation of Problem 4.1

We are interested in approximating the solution of (4.1) by a polynomial of degree \( \leq N \). We use a Galerkin method approach known as Spectral Method (see Gottlieb-Orszag [1] for more details); hence from (4.10) we derive an approximate problem:

\[
\begin{align*}
\text{Find } \psi_N \text{ in } V_N = S_N^2 \text{ such that, for any } \phi \in V_N:
\end{align*}
\]

\[\int_I \psi_{Nxx}(\phi \omega)_{xx} = \int_I g(\phi \omega)_{xx}.\]
From Lemma 4.2 we know that problem (4.12) is wellposed in the sense that there exists one and only one solution. Moreover, we derive from (4.10) and (4.12):

\[ a(\psi - \psi_N, \phi) = 0 \quad \text{for any } \phi \text{ in } V_N, \]

so that (remind \( \pi_{2,0}^{2,0} \psi \in V_N\)):

\[ (4.13) \quad a(\psi - \psi_N, \psi - \psi_N) = a(\psi - \psi_N, \phi - \pi_{2,0}^{2,0} \psi). \]

Due to Theorem 3.1 and Lemma 4.2, we then obtain the following

**Theorem 4.2**: There exists one and only one solution \( \psi_N \) to problem (4.12); moreover it verifies, as soon as \( \psi \in H_{\omega}^0(I) \cap H^2_{0,\omega}(I) \):

\[ (4.14) \quad \psi - \psi_N \geq C_N^{2-\sigma} \psi_{\sigma,\omega}. \]

**Remark 4.1**: The previous estimate is an optimal one in the sense that no polynomial of \( \mathcal{S}_N^2 \) is asymptotically nearer from the solution \( \psi \) than the solution of the approximate problem.

**Remark 4.2**: The previous theorem will be extended in a future paper where we shall consider a pseudospectral method (much more efficient from a computational point of view) for approximating a one-dimensional fourth order equation.
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REFERENCES


We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to N with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.