ANALYSIS OF SPECTRAL OPERATORS IN ONE-DIMENSIONAL DOMAINS

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ABSTRACT

We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to \( N \) with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.

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I. INTRODUCTION

This paper presents an investigation of a class of projection operators that arises in the analysis of the approximation of differential equations by spectral methods using Chebyshev decomposition.

Some similar operators have been studied before by Canuto-Quarteroni [1] and Maday-Quarteroni [1], but the existing results are not adequate in many applications. In fact they forbid analysis for the error of the approximation by spectral methods of fourth-order problems and, in several instances, second-order problems (see Canuto-Quarteroni [2]).

We first present some background tools required for our analysis. They consist of Sobolev spaces relative to the weight \( \omega(x) = (1 - x^2)^{-1/2} \) (this weight arises in the relations of orthogonality of Chebyshev polynomials). We recall and complete results proved by Grisvard [1], [2] concerning interpolation theory between these spaces.

Then we present an analysis of projection operators from these spaces into the set of all polynomials of degree lower than \( N \).

Finally we give an application of the results herein proved to a simple test problem.

We shall give other applications in Maday [1] and shall in a future work extend these results to multidimensional domains. Our aim is to apply such results to the analysis of the approximation of Navier-Stokes equations by spectral methods (see Maday-Metivet [1], [2]).

For some different notions about projection operators that arise in spectral methods, see Tadmor [1].
II. PRELIMINARIES: SOME FUNCTION SPACES

Notations and Basic Properties

Let \( J \) be an open interval \([a,b[\) of \( \mathbb{R} \) \((a<b)\); we consider a weight function \( \rho(x) \), continuous over \( J \), satisfying \( \rho(x) \geq \rho_0 > 0 \) for any \( x \in J \).

Let us set:

\[
L^2_{\rho}(J) = \{ \phi : J + \mathbb{R} | \phi \text{ is measurable and } (\phi, \phi)_\rho < +\infty \},
\]
equipped with the inner product \((\phi, \psi)_\rho = \int_J \phi(x)\psi(x)\rho(x)dx\). For any integer \( s \geq 0 \) we set:

\[
H^s_{\rho}(J) = \{ \phi \in L^2_{\rho}(J) | \|\phi\|_s,\rho < \infty \},
\]
where:

\[
\|\phi\|_{s,\rho} = ((\phi, \phi))_{s,\rho},
\]
this space being equipped with the inner product:

\[
((\phi, \psi))_{s,\rho} = \sum_{k=0}^{s} \frac{d^k \phi}{dx^k} , \frac{d^k \psi}{dx^k}_\rho.
\]

Clearly, one has the equality:

\[
L^2_{\rho}(I) = H^0_{\rho}(I).
\]

For any real \( s \geq 0 \), noninteger, \( H^s_{\rho}(J) \) is defined by interpolation between the space \( H^s_{\rho}(J) \) and \( \bar{H}^{s+1}_{\rho}(J) \), where \( \bar{s} \) represents the integral part of \( s \).
The method of interpolation can be the complex one, the operator's domain one or the trace one (see Lions-Magenes [1] for more details). Besides we define $H^{s}_{0,p}(J)$ as being the closure of $\mathcal{D}(J)$ in $H^{s}_{p}(J)$. When $p = 1$, these spaces are the usual Sobolev spaces denoted by $H^{s}(J)$ and $H^{s}_{0}(J)$ respectively. For the application to spectral methods we are mostly interested in those spaces when $J = I \equiv ]-1,+1[$ and $\rho(x) = \omega(x) \equiv \frac{1}{\sqrt{1 - x^2}}$. Let us recall some results proved in Grisvard [1], [2] valuable for $J = I$, $\rho = \omega$ and for $J = ]0,1[$, $\rho = \frac{1}{\sqrt{x}}$.

**THEOREM 2.1 (Grisvard [1]):**

i) For any real $s > 0$, $s \notin \mathbb{N} + \frac{1}{4}$ we have:

$$H^{s}_{0,p}(J) = [H^{s}_{0,p}(J), H^{s+1}_{0,p}(J)]_{s-s}.$$  

ii) For any integral $n$ we have:

$$[H^{n}_{0,p}(J), H^{n+1}_{0,p}(J)]_{1/4} \subseteq H^{n+1/4}_{0,p}(J).$$

iii) For any real $s > 0$, $s \notin \mathbb{N} + \frac{1}{2}$:

$$H^{s}_{p}(J) \subseteq C^{m}\overline{J},$$

the space of continuous mapping defined over $\overline{J}$ whose derivative of order $\leq m$ are continuous over $\overline{J}$, with $m = s - \frac{1}{2}$.

The trace application defined from $C^{\infty}(\overline{J})$ into $\mathbb{R}^{2n}$:
can be extended to a continuous mapping from $H_{\rho}^{n+1/4+\varepsilon}(I)$ onto $\mathbb{R}^{2n}$ for any $\varepsilon > 0$.

iv) For any real $1/4 < s < 5/4$, $H_0^{s,\rho}(J)$ coincide with the subspace of $H_\rho^s(J)$ of functions vanishing at the real boundaries of $J$.

v) For any real $s > 1/2$, $H_\rho^s(J)$ is an algebra.

**THEOREM 2.2 (Grisvard [2]):**

For any $0 < q < s < p$, $H_\rho^s(J)$ satisfies the following double topological imbedding:

\[
[H_\rho^q(J), H_\rho^p(J)]_{\theta,1} \subset H_\rho^s(J) \subset [H_\rho^q(J), H_\rho^p(J)]_{\theta,\infty},
\]

with $\theta = \frac{s - p}{q - p}$, and the notation holds for the real interpolation (see Lions and Peetre [1]).

The two following results can be found in Canuto-Quarteroni [2] and Maday-Quarteroni [1].

**THEOREM 2.3:**

i) For any real $s \geq 1/4$, $H_\omega^s(I) \subset H_\omega^{s-1/4}(I)$.

ii) For any $0 \leq r < s$, the imbedding $H_\omega^s(I) \subset H_\omega^r(I)$ is compact.

In the next section we shall generalize the results (2.4) and (2.4').
Some New Results About Interpolation Between $H_{0,0}^p(I)$

This section is devoted to the proof of the following:

**THEOREM 2.4:** For any $0 < q < s < p$ not in $\mathbb{N} + 1/4$ we have:

$$[H_{0,0}^q(I), H_{0,0}^s(I)]_{[s-p]/[q-p]} = H_{0,0}^s(I).$$

This theorem is a consequence of the two following lemma:

**LEMMA 2.1:** For any integer $p \leq n$, we have:

$$u \in H_{0,0}^n(I) \Rightarrow \frac{d^p u}{dx^p} \in L^2_{\omega}4(n-p)+1(I).$$

**PROOF:** It is an easy matter to check that this result is a consequence of

$$(2.6) \quad u \in H^n_{0,\frac{1}{\sqrt{x}}}(0,1) \Rightarrow \frac{d^p u}{dx^p} \in L^2_{\omega}4(n-p)+1(0,1),$$

(we shift the difficulties at $\pm 1$ onto 0). So let $u$ be in $H^n_{0,\frac{1}{\sqrt{x}}}(0,1)$; from Theorem 2.1 (point iv) we have, for any $0 \leq p < n$:

$$\frac{d^p u}{dx^p}(0) = 0,$$

hence

$$\int_0^x \frac{d^{p+1} u(t)}{dx^{p+1}} dt = \frac{d^p u}{dx^p}(x).$$
Besides, from Lemma 6.2.1 of Nečas [1] we have, for any \( \alpha < 1 \) and any \( v \) such that \( \int_0^1 v^2(x)x^\alpha \, dx < \infty \):

\[
\int_0^1 \left( \int_0^x |v(x)| \right)^2 x^{\alpha-2} \, dx \leq \left( \frac{1}{1-\alpha} \right) \int_0^1 |v(x)|^2 x^\alpha \, dx,
\]

(2.7) taking then \( \alpha = -\frac{1}{2} - 2(n-(p+1)) \) and \( v = \frac{d^{p+1}u}{dx^{p+1}} \) we obtain:

\[
\int_0^1 \left( \frac{d^p u}{dx^p} \right)^2 x^{-1/2} - 2(n-p) \, dx \leq C \int_0^1 \left( \frac{d^{p+1}u}{dx^{p+1}} \right)^2 x^{-1/2} - 2(n-(p+1)) \, dx,
\]

and (2.6) holds by induction over \( p \).

**Lemma 2.2:** For any integer \( n > 0 \), the mapping \( u \mapsto u \omega^{1/2} \) is an homeomorphism from \( H^n_{0,\omega}(I) \) onto \( H^n_{0}(I) \).

**Proof:** Here again we prove the result for the weight \( \frac{1}{\sqrt{x}} \), say:

(2.8) \( u \mapsto ux^{-1/4} \) is an homeomorphism from \( H^n_{0,\frac{1}{\sqrt{x}}}(0,1) \) onto \( H^n_{0}(0,1) \).

Let \( \phi \in \mathcal{D}(0,1) \), then, for \( 0 \leq m \leq n \):

\[
\frac{d^m}{dx^m}(\phi x^{-1/4}) = \sum_{p=0}^m C^p_m \frac{d^p \phi}{dx^p} \frac{d^{m-p}(x^{-1/4})}{dx^{m-p}}
\]

\[
= \sum_{p=0}^m C^p_m p^p \frac{d^p \phi}{dx^p} x^{-1/4} -(m-p),
\]

with \( D^p_m = [-1/4 - (m-p+1)] D^{p+1}_m \) and \( D^m_m = 1 \). From Lemma 2.1 we then get:
\[ \| \frac{d^m}{dx^m} (\phi x^{-1/4}) \|_{0,1} \leq C \| \phi \|_{n,1} \leq \| \phi \|_{n,1}^{1/\sqrt{x}} ; \]

summing up these estimates for \( 0 \leq m \leq n \) we derive:

(2.9) \[ \| \phi x^{-1/4} \|_{n,1} \leq C \| \phi \|_{n,1}^{1/\sqrt{x}} . \]

Inversely, let us prove that, for any \( \phi \in \mathcal{D}(I) \):

(2.10) \[ \| \phi x^{1/4} \|_{n,1} \leq \| \phi \|_{n,1}^{1/\sqrt{x}} . \]

From Hardy's inequality (Lemma 2.5.1 of Nečas [1]) we derive by induction that, for any \( 0 \leq p \leq m \leq n \):

(2.11) \[ \| \frac{d^p}{dx^p} \phi \|_{0,x} \leq C \| \phi \|_{m,1}^{1/\sqrt{x}} \]

besides:

\[ \frac{d^m}{dx^m} (\phi x^{1/4}) = \sum_{p=0}^{m} \frac{C^p}{m} \frac{d^p}{dx^p} \phi \frac{d^{m-p}}{dx^{m-p}} (x^{1/4}) \]

\[ = \sum_{p=0}^{m} \frac{C^p}{m} D_m^{p} \frac{d^p}{dx^p} \phi \frac{x^{1/4}}{-m-p}, \]

with:

\[ D_m^{p} = [1/4 - (m - p + 1)] D_m^{p+1} \quad \text{and} \quad D_m^{m} = 1. \]

Then using (2.11) we get:
and (2.10) is derived by summing up these results for $0 \leq m \leq n$. We can now achieve (2.8) as a consequence of (2.9) and (2.10).

We can now prove the main result of this section.

**Proof of Theorem 2.4:** From (2.4) and Lemma 2.2 we deduce that the mapping $u \mapsto u^{1/2}$ is an homeomorphism from $H_0^s(I)$ onto $H_0^s(I)$ for any $s \geq 0$ not in $\mathbb{N} + 1/4 \cap \mathbb{N} + 1/2$ (see Lions-Magenes [1] for more details about the properties of spaces of interpolation).

Let us recall that, for any $q \leq s \leq p$ not in $\mathbb{N} + 1/2$ we have (see Lions-Magenes [1]):

$$(2.12) \quad H_0^s(I) = [H_0^q(I), H_0^p(I)]_{s-q}^{p-q}.$$ 

From the previous homeomorphism we deduce that, for any $q \leq s \leq p$ not in $\{ \mathbb{N} + 1/2 \} \cup \{ \mathbb{N} + 1/4 \}$:

$$(2.13) \quad H_0^{s,\omega}(I) = [H_0^{q,\omega}(I), H_0^{p,\omega}(I)]_{s-q}^{p-q}.$$ 

Let us remark now that the values of $p, q, s$ in $\mathbb{N} + 1/2$ have only been excluded due to (2.12), these values can now be recovered thanks to the reiteration theorem (Theorem 1.6.1 of Lions-Magenes [1]).
III. APPROXIMATION RESULTS OF PROJECTION OPERATOR IN WEIGHTED SOBOLEV SPACES

The previous theorem leads us to define over $H_{0,\omega}^r(I)$ a new scalar product. Indeed, for $p$ not in $\mathbb{N} + 1/4 \cup \mathbb{N} + 1/8$, $H_{0,\omega}^p(I)$ can be seen as the interpolate $1/2$ between $L_\omega^2(I)$ and $H_{0,\omega}^2(I)$ and for $p$ in $\mathbb{N} + 1/8$, $H_{0,\omega}^p(I)$ can be seen as the interpolate $1/3$ between $L_\omega^2(I)$ and $H_{0,\omega}^3(I)$.

If we consider the domain operator interpolation, this find expression in the existence of a selfadjoint operator $\Lambda_r$ such that:

* if $r \in \mathbb{N} + 1/8$, the domain $D(\Lambda_r^3)$ of the operator $\Lambda_r^3$ in $L_\omega^2(I)$ is $H_{0,\omega}^{3r}(I)$ if $r \notin \mathbb{N} + 1/8$, the domain of $D(\Lambda_r^2)$ of the operator $\Lambda_r^2$ in $L_\omega^2(I)$ is $H_{0,\omega}^{2r}(I)$.

* The domain $D(\Lambda_r)$ of the operator $\Lambda_r$ in $L_\omega^2(I)$ is $H_{0,\omega}^r(I)$ if $r \notin \mathbb{N} + 1/4$ and is included in $H_{0,\omega}^r(I)$ if $r \in \mathbb{N} + 1/4$.

Moreover:

\[
\text{(3.1)} \quad (u,v) \mapsto (((u,v)))_{r,\omega} = (\Lambda_r u, \Lambda_r v),
\]

is a scalar product whose associated norm is equivalent to the one defined in (2.2) if $r \notin \mathbb{N} + 1/4$.

Let us define now $P_{r,N}$ as the projection operator from $H_{0,\omega}^r(I)$ over $S_{N}^r$ with respect to the previous scalar product with:

\[
S_{N}^r = S_{N} \cap H_{0,\omega}^r(I),
\]

\[
S_{N} = \{ \phi \text{ defined over } I \mid \phi \text{ is a polynomial of degree } \leq N \}.
\]
Lemma 3.1: Let \( 0 \leq \nu \leq r \leq \sigma \) with \( \sigma \notin \mathbb{N} + 1/4 \) we have, for any \( \phi \in H^\sigma_\omega(I) \cap H^r_{0,\omega}(I) \):

\[
\|\phi - P_{r,N} \phi\|_{\nu,\omega} \leq C^{\nu-\sigma} \|\phi\|_{\sigma,\omega}.
\]

Remark 3.1: The case \( \nu = r = 0 \) has been studied in Canuto-Quarteroni [1], the case \( 0 \leq \nu \leq r = 1 \) has been looked at in Maday-Quarteroni [1] (note that the dependence of the constant is then \( C(\sigma) = C^*(\sigma!) \)). Moreover it is proved that no optimal bound was possible for \( H^\nu_{0,\omega}(I) \) norms with \( \nu > r \). Indeed, for example:

\[
\|\phi - P_{0,N} \phi\|_{\nu,\omega} \leq C^{2\nu-\sigma} \|\phi\|_{\sigma,\omega}.
\]

It is often necessary (see Canuto-Quarteroni [2], Maday-Metivet [2], Maday [1], and (4.14)) to obtain optimal results in higher norms.

Proof: We shall only consider the case \( r \notin \mathbb{N} + 1/8 \) for simplicity.

The proof is divided in two stages

1) We first prove (3.2) by induction over \( r \) in \( \mathbb{N} \). So, let us assume that (3.2) is true for \( s < r \) in \( \mathbb{N} \); let \( \phi \in H^r_{0,\omega}(I) \); then \( \phi_x \in H^{r-1}_{0,\omega}(I) \) and \( P_{r-1,N-1}(\phi_x) \in S^{r-1}_{N-1} \). Moreover if \( \phi(-1) = \phi(1) = 0 \) we have:

\[
\alpha = \int_{-1}^{1} P_{r-1,N-1}(\phi_x)(t)dt = \int_{-1}^{1} [P_{r-1,N-1}(\phi_x) - \phi_x](t)dt.
\]

From the Cauchy-Schwarz inequality we derive:
\[ |\alpha| \leq \left( \int_{-1}^{1} (P_{r-1,N-1}(\phi_x) - \phi_x)^2(t)\omega(t)dt \right)^{1/2} \left( \int_{-1}^{1} (\omega(t))^{-1} dt \right)^{1/2} \]

\[ \leq C \|P_{r-1,N-1}(\phi_x) - \phi_x\|_0, \omega; \]

hence, from the induction hypothesis:

(3.5) \[ |\alpha| \leq C N^{1-\sigma} \|\phi_x\|_{\sigma-1, \omega}. \]

Finally we have:

\[ R_N(x) = \int_{-1}^{x} \left[ P_{r-1,N-1}(\phi_x)(t) - \frac{\alpha(1 - t^2)^{r-1}}{1 \int_{-1}^{1} (1 - x^2)^{r-1} dx} \right] dt \in S_N^r. \]

Due to the Poincaré-like inequality, the polynomial satisfies the following:

\[ \|\phi - R_N\|_{r, \omega} \leq \|\phi - R_N\|_{r-1, \omega}, \]

the induction hypothesis, and (3.5) gives us:

\[ \|\phi - R_N\|_{r, \omega} \leq C \left( N^{(r-1)-(\sigma-1)} + N^{(1-\sigma)} \right) \|\phi_x\|_{\sigma-1, \omega} \]

\[ \leq C N^{r-\sigma} \|\phi\|_{\sigma, \omega}. \]

From the equivalence of the norms \( \| \|_{r, \omega} \) and \( \| \|_{r, \omega} \), and the identity:

\[ \|\phi - P_{r,N} \phi\|_{r, \omega} = \inf_{\phi_N \in S_N^r} \|\phi - \phi_N\|_{r, \omega}, \]

\[ \|\phi - R_N \phi\|_{r, \omega} = \inf_{\phi_N \in S_N^r} \|\phi - \phi_N\|_{r, \omega}, \]
we obtain for any $\phi$ in $L_\omega^p(I) \cap H_0^\sigma(I)$:

\[ |||\phi - P_{r,N} \phi|||_{r,\omega} \leq C N^{r-\sigma} |||\phi|||_{r,\omega}. \]

Besides, since the operator $\Lambda_r$ is selfadjoint, we have:

\[ |||\phi - P_{r,N} \phi|||_{0,\omega} = \inf_{\psi \in L_\omega^2(I)} \frac{(\phi - P_{r,N} \phi, \psi)_{\omega}}{|||\psi|||_{0,\omega}}. \]

\[ = \inf_{\psi \in L_\omega^2(I)} \frac{(\Lambda_r(\phi - P_{r,N} \phi), \Lambda_r^{-1} \psi)_{\omega}}{|||\psi|||_{0,\omega}}. \]

From (3.1) we then get:

\[ |||\phi - P_{r,N} \phi|||_{0,\omega} = \inf_{\psi \in L_\omega^2(I)} \frac{(|||\psi|||_{0,\omega})_{\omega}}{|||\psi|||_{0,\omega}} \]

By definition of $P_{r,N}$ we have, for any $\psi$ in $L_\omega^2(I)$:

\[ (((\phi - P_{r,N} \phi, P_{r,N}(\Lambda_r^{-2} \psi))_{r,\omega} = 0; \]

hence

\[ |||\phi - P_{r,N} \phi|||_{0,\omega} = \inf_{\psi \in L_\omega^2(I)} \frac{(|||\psi|||_{0,\omega})_{\omega}}{|||\psi|||_{0,\omega}} \]

\[ \leq |||\phi - P_{r,N} \phi|||_{r,\omega} \inf_{\psi \in L_\omega^2(I)} \frac{|||(\Lambda_r^{-2} \psi) - P_{r,N}(\Lambda_r^{-2} \psi)|||_{r,\omega}}{|||\psi|||_{0,\omega}}. \]
Due to (3.6) we then derive:

\[ \| \phi - P_r, N \phi \|^2_{0, \omega} \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega} N^{-r} \inf_{\psi \in L^2_\omega(I)} \frac{\| \Lambda^{-2}_r \psi \|^2_{0, \omega}}{\| \psi \|^2_{0, \omega}} 2r, \omega \]

\[ \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega} \inf_{\psi \in L^2_\omega(I)} \frac{\| \Lambda^2_r(\Lambda^{-2}_r \psi) \|^2_{0, \omega}}{\| \psi \|^2_{0, \omega}} \]

\[ \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega}. \]

Now, from the two estimates, valuable for any \( \phi \in H^\sigma(I) \cap H^r_0, \omega(I) \):

\[ \| \phi - P_r, N \phi \|^2_{r, \omega} \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega} \]

\[ \| \phi - P_r, N \phi \|^2_{0, \omega} \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega} \]

we derive that for any \( \theta \in ]0,1[ \):

\[ \| \phi - P_r, N \phi \|^2_{L^2_\omega(I), H^r_0, \omega(I)} \leq CN^{\theta r - \sigma} \| \phi \|^2_{\sigma, \omega}. \]

Due to (2.4) and (2.4') we deduce that, for any \( 0 \leq \nu \leq r \):

(3.8) \[ \| \phi - P_r, N \phi \|^2_{\nu, \omega} \leq CN^{-\sigma} \| \phi \|^2_{\sigma, \omega}. \]

ii) Let us now prove (3.3) for nonintegral values of \( r \). Let \( \phi \in \mathcal{D}(I) \), from step (i) we know that, for any \( \sigma \geq \frac{r + 1}{r} \), \( \sigma \in \mathbb{N} + \frac{1}{4} \)
Due to the interpolation of quotient spaces (see Lions-Magenes [1] Lemma I.13.2) we have, for any $\theta \in ]0,1[$:

$$
\|\phi\|_{H^{r+1}_0,\omega(I)/S_N^{r+1}} \leq C_N^{r+1-\sigma} \|\phi\|_{\sigma,\omega},
$$

$$
\|\phi\|_{H^r_0,\omega(I)/S_N^{r+1}} \leq C_N^{r-\sigma} \|\phi\|_{\sigma,\omega}.
$$

From (2.4), (2.4') we deduce (we take $\theta = r - \bar{r}$):

$$
\|\phi\|_{H^r_0,\omega(I)/S_N^{r+1}} \leq C_N^{r-\sigma} \|\phi\|_{\sigma,\omega},
$$
so that, for any $\sigma \geq \frac{r+1}{2} \sigma \in \mathbb{N} + \frac{1}{4}$:

\[(3.9) \quad \|\phi - P_{r,N}\phi\|_{r,\omega} < C\|\phi\|_{r,\omega} .\]

By definition we know that, for any $r \in \mathbb{N} + \frac{1}{4}$:

\[\|\phi - P_{r,N}\phi\|_{r,\omega} < C\|\phi\|_{r,\omega};\]

hence, (3.9) holds for any $\sigma \geq r, \sigma \in \mathbb{N} + \frac{1}{4}$ and any $r \in \mathbb{R}^\times$. The estimates in lower order norms are obtained following the same lines as in i).

We are now interested in the approximation of the spaces $H^r_\omega(I) \bigcap \tilde{H}^s_0,\omega(I)$ for $s \in \mathbb{N}, 0 \leq s \leq r$ by polynomials therein contained.

For simplicity of exposition we shall consider the case $s = 0$. From point iii) of Theorem 2.4 we can easily exhibit for any $\phi \in H^r_\omega(I)$ a polynomial $\phi_0$ of degree $\leq 2r - 1$ such that: $\phi - \phi_0 \in H^r_0,\omega(I)$ and for any real $p$:

\[(3.10) \quad \|\phi_0\|_{p,\omega} < C\|\phi\|_{r,\omega} .\]

Due to the previous lemma, we have, for any $0 \leq \nu \leq r \leq \sigma \in \mathbb{N} + \frac{1}{4}$:

\[\|\phi - \phi_0 - (P_{r,N}(\phi - \phi_0))\|_{\nu,\omega} \leq C\nu^{-\sigma} \|\phi - \phi_0\|_{\sigma,\omega} .\]

and (3.10) then implies:

\[\|\phi - (\phi_0 + P_{r,N}(\phi - \phi_0))\|_{\nu,\omega} \leq C\nu^{-\sigma} \|\phi\|_{\sigma,\omega} .\]
For $N$ large enough (more precisely $N \geq 2r - 1$) we then get the existence of an operator $\tilde{P}$ from $H^r_\omega(I)$ onto $S_N$ such that:

$$\|\phi - \tilde{P}(\phi)\|_{r,\omega} \leq C N^{\nu - \sigma} \|\phi\|_{\sigma,\omega}. $$

This estimate provides an answer to our question in the case $s = 0$. The same proof can be done to build an operator from $H^r_\omega(I) \cap H^s_{0,\omega}(I)$ onto $S^s_N$ satisfying analogous bounds. This leads us to state the main theorem of this paper:

**THEOREM 3.1:** Let $(\nu, r, \sigma) \in \mathbb{R}, \sigma \notin \mathbb{N} + \frac{1}{4}, s \in \mathbb{N}, 0 \leq s \leq r,$

$$0 \leq \nu \leq r \leq \sigma.$$ There exists an operator $\Pi^{s,0}_{r,N}$ from $H^r_\omega(I) \cap H^s_{0,\omega}(I)$ onto $S^s_N$ such that, for any $\phi \in H^\nu(I) \cap H^s_{0,\omega}(I)$ we have:

$$\|\phi - \Pi^{s,0}_{r,N} \phi\|_{r,N,\omega} \leq C N^{\nu - \sigma} \|\phi\|_{\sigma,\omega}. $$

**IV. AN APPLICATION**

**Definition of the Problem**

In order to explain how the previous results can be applied, we shall study an approximation of the very simple problem:

Find $\psi$ defined over $I$ such that:
(4.1)\[
\begin{cases}
\frac{d^4 \psi}{dx^4} = f \quad \text{over } I, \\
\psi = \frac{d\psi}{dx} = 0 \quad \text{at } \pm 1.
\end{cases}
\]

(This problem provides a first step for the analysis of Stokes and Navier-Stokes problems in the $\psi$-formulation; see Maday-Metivet [1], [2].) Let us define $H^{-2}(I)$ as follows:

$$H^{-2}(I) = \{ f \in \mathcal{D}'(I) \mid \exists g \in L^2_\omega(I) : f = \frac{d^2 g}{dx^2}\}.$$ 

We now want to prove the following:

**THEOREM 4.1:** Let $f \in H^{-2}(I)$; then there exists one and only one solution $\psi$ to the problem (4.1) in the space $H^{2}_{0, \omega}(I)$.

This theorem is a very simple consequence of Lax Milgram lemma and the two following lemmas.

**LEMA 4.1:** There exist two positive constants $\delta_1$ and $\delta_2$ such that, for any $\phi$ in $H^{2}_{0, \omega}(I)$:

$$\int_I \phi^2 \omega^9 \leq \delta_1 \int_I \frac{d\phi}{dx}^2 \omega^5 \leq \delta_2 \int_I \left(\frac{d^2\phi}{dx^2}\right)^2 \omega.$$ 

This lemma is a corollary of Lemma 2.1.
**Lemma 4.2:** There exist 3 positive constants $\alpha, \beta, \gamma$ such that for any $(\phi, \psi) \in H^2_\omega(I)^2 \times H^1_0,\omega(I)$:

\begin{align*}
(4.2) \quad & (\phi_{xx}, (\psi_{x})_{xx}) \geq \alpha \lVert \phi \rVert^2_{2, \omega}, \\
(4.3) \quad & \lVert \phi \rVert_{2, \omega} \leq \beta \lVert \phi_{xx} \rVert_{0, \omega}, \\
(4.4) \quad & (\phi_{xx}, (\psi_{xx})_{xx}) \leq \gamma \lVert \phi_{xx} \rVert_{0, \omega} \lVert \psi_{xx} \rVert_{0, \omega}.
\end{align*}

**Proof:**

i) We first note that (4.3) is an easy consequence of the previous lemma, and is an equivalent to the Poincaré inequality.

ii) Next, we get the following equalities, for any $\psi \in \mathcal{D}(I)$:

\begin{align*}
\int_I \phi_{xx} (\psi_{x})_{xx} \, dx &= \int_I \phi_{xx}^2 \omega \, dx + 2 \int_I \phi_{xx} \psi_x \omega_x \, dx + \int_I \psi_{xx} \psi_{xx} \, dx \\
&= \int_I \phi_{xx}^2 \omega \, dx + \int_I \left( \phi_{xx}^2 \right)_x \omega_x \, dx + \int_I \left( \psi_x \right)_x \psi_{xx} \, dx \\
&= \int_I \phi_{xx}^2 \omega \, dx - \int_I \phi_x^2 \omega_{xx} \, dx - \int_I \psi_x (\psi_{xx})_x \, dx \\
&= \int_I \phi_{xx}^2 \omega \, dx - 2 \int_I \phi_x^2 \omega_{xx} \, dx + \frac{1}{2} \int_I \psi_x^2 \omega_{xxxx} \, dx
\end{align*}
let us note that:

\[ \omega_{xx} = (1 + 2x^2)\omega^5, \]

\[ \omega_{xxxx} = (9 + 72x^2 + 24x^4)\omega^9; \]

hence:

(4.5)

\[ \int \psi_{xx}(\psi\omega)_{xx} \, dx = \int \psi_{xx}^2 \, dx - 2\int \psi_x^2(1 + 2x^2)\omega^5 \, dx + \frac{1}{2\int} \psi^2(9 + 72x^2 + 24x^4)\omega^9 \, dx. \]

Besides, let us set:

\[ P \equiv \int (\psi_{xx} \omega + 2x\psi_x \omega^3 + (2x^2 + 10^{-2})\omega^5)^2 \omega^{-1} \, dx; \]

\( P \) is \( \geq 0 \) and, an easy calculation gives:

\[ P \equiv \int \psi_{xx}(\psi\omega)_{xx} - 2 \cdot 10^{-2} \int \psi_x^2 \omega^5 - \int (5.78x^2 + 0.4839)\psi^2 \omega^9, \]

so that

\[ \int \psi_x^2 \omega^5 \leq 50\int \psi_{xx}(\psi\omega)_{xx}, \]

from which we derive:

\[ \int \psi_x^2(1 + 2x^2)\omega^5 \leq 150\int \psi_{xx}(\psi\omega)_{xx}. \]

Using that inequality in (4.5) we obtain \( \alpha > 0 \) such that:
\[ \int_I \psi_{xx}(\psi \omega)_{xx} \geq \alpha \beta \int_I \psi_{xx}^2 \omega, \]

and (4.2) is an consequence of (4.3).

iii) Finally, let us note that for any \((\phi, \psi)\) in \(H^2_\omega(I) \times H^2_0,\omega(I)\) we have:

\[ (4.6) \quad (\phi_{xx},(\psi)_{xx})_I = (\phi_{xx},\psi_{xx})_\omega + 2 \int_I \phi_{xx} \psi_x \omega_x + \int_I \phi_{xx} \psi_{xx}. \]

The following inequality is simple:

\[ (4.7) \quad |(\phi_{xx},\psi_{xx})_\omega| \leq \lVert \phi_{xx} \rVert_{0,\omega} \lVert \psi_{xx} \rVert_{0,\omega}. \]

Let us examine the second term:

\[ |\int_I \phi_{xx} \psi_x \omega_x| = |\int_I \phi_{xx}(\psi_x \omega_x^{-1}) \omega| \leq |\int_I \phi_{xx}^2 \omega|^{1/2} |\int_I \psi_x^2 \omega_x^2 \omega^{-1}|^{1/2}; \]

since \(\omega_x^2 \omega^{-1} = \omega^5\), we derive from Lemma 4.1 that:

\[ (4.8) \quad |\int_I \phi_{xx} \psi_x \omega_x| \leq C |\phi_{xx}|_{0,\omega} |\psi_{xx}|_{0,\omega}. \]

We obtain, in a similar way:

\[ (4.9) \quad |\int_I \phi_{xx} \psi_{xx}| \leq C |\phi_{xx}|_{0,\omega} |\psi_{xx}|_{0,\omega}, \]

so that (4.4) is a consequence of (4.6)-(4.9).
PROOF OF THEOREM 4.1: Let $f = g_{xx}$ be in $H^{-2}_0(I)$. Problem (4.1) is equivalent to the following:

\[
\begin{align*}
\text{Find } \psi \text{ in } V = H^2_{0,\omega}(I) \text{ such that, for any } \phi \text{ in } V:
\end{align*}
\]

\[
\begin{align*}
(4.10) & \quad \int_I \psi_{xx}(\phi_x)_{xx} = \int_I g(\phi_x)_{xx}.
\end{align*}
\]

The bilinear form $a$ defined by: For any $(\chi, \phi)$ in $V^2$:

\[
(4.11) \quad a(\phi, \chi) = \int_I \phi_{xx}(\chi_0)_{xx},
\]

is continuous and elliptic over $V$ (see Lemma 4.2), and Lax-Milgram lemma gives the existence and uniqueness of a solution of (4.10) hence of (4.1).

Approximation of Problem 4.1

We are interested in approximating the solution of (4.1) by a polynomial of degree $\leq N$. We use a Galerkin method approach known as Spectral Method (see Gottlieb-Orszag [1] for more details); hence from (4.10) we derive an approximate problem:

\[
\begin{align*}
\text{Find } \psi_N \text{ in } V_N = S^2_N \text{ such that, for any } \phi \text{ in } V_N:
\end{align*}
\]

\[
(4.12) \quad \begin{align*}
\int_I \psi_{Nxx}(\phi_x)_{xx} = \int_I g(\phi_x)_{xx}.
\end{align*}
\]
From Lemma 4.2 we know that problem (4.12) is wellposed in the sense that there exists one and only one solution. Moreover, we derive from (4.10) and (4.12):

\[ a(\psi - \psi_N, \phi) = 0 \quad \text{for any } \phi \in V_N, \]

so that (remind \( \Pi_{2,0}^N \psi \in V_N \)):

\[
(4.13) \quad a(\psi - \psi_N, \psi - \psi_N) = a(\psi - \psi_N, \phi - \Pi_{2,0}^N \psi).
\]

Due to Theorem 3.1 and Lemma 4.2, we then obtain the following

**THEOREM 4.2:** There exists one and only one solution \( \psi_N \) to problem (4.12); moreover it verifies, as soon as \( \psi \in H_0^2(I) \cap H_0^1(I) \):

\[
(4.14) \quad \psi - \psi_N \leq C N^{2-\sigma} \psi_{\sigma,\omega}.
\]

**REMARK 4.1:** The previous estimate is an optimal one in the sense that no polynomial of \( S_N^2 \) is asymptotically nearer from the solution \( \psi \) than the solution of the approximate problem.

**REMARK 4.2:** The previous theorem will be extended in a future paper where we shall consider a pseudospectral method (much more efficient from a computational point of view) for approximating a one-dimensional fourth order equation.
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We prove results concerning certain projection operators on the space of all polynomials of degree less than or equal to \( N \) with respect to a class of one-dimensional weighted Sobolev spaces. These results are useful in the theory of the approximation of partial differential equations with spectral methods.