SECOND-ORDER COMPTON-GETTING EFFECT ON ARBITRARY INTENSIY DISTRIBUTION

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1 INTRODUCTION Theoretical studies of energetic particles in space are often referred to a special frame of reference. To compare theory with experiment, one has to transform the particle distribution from the special frame to the observer's frame, or vice versa. Various methods have been derived to obtain the directional distribution in the comoving frame from the directional fluxes measured on a spacecraft (e.g., Erdos, 1981; Sanderson *et al.*, 1985, and references therein). These methods have become progressively complicated as increasingly detailed directional particle data become available.

We present here a set of 2nd order correct formulae for the transformation of an arbitrary differential intensity distribution, expressed as a series of spherical harmonics, between any two frames in constant relative motion. These formulae greatly simplify the complicated procedures currently in use for the determination of the differential intensity distribution in a comoving frame.

2. THE DESIRED FORM OF THE TRANSFORMATION In the observer's frame, let the particle differential intensity w.r.t. momentum (or rigidity) be expressed in terms of the spherical harmonics C_n^m, S_n^m as :

$$j_p^S(p_S,\theta_S,\phi_S) = \sum_{n=0} \sum_{m=0}^{n} \{A_{nm}^S(p_S)C_n^m(\theta_S,\phi_S)+B_{nm}^S(p_S)S_n^m(\theta_S,\phi_S)\}. \tag{1}$$

We wish to transform this to the frame moving with velocity $\underline{W}$ relative to the observer, *also in terms of spherical harmonics* :

$$j_p(p,\theta,\phi) = \sum_n \sum_m \{A_{nm}(p)C_n^m(\theta,\phi) + B_{nm}(p)S_n^m(\theta ,\phi)\}. \tag{2}$$

Here $$C_n^m(\theta,\phi) = P_n^m(\cos \theta)\cos(m\phi),\ S_n^m(\theta,\phi) = P_n^m(\cos\theta)\sin(m\phi), \tag{3}$$

$P_n^m(\ )$ = associated Legendre function, p = particle momentum, and θ, ϕ are the polar and azimuthal angles of the particle *arrival* direction. The superscript or subscript S indicates that the quantity is referred to the frame of the observer/spacecraft.

In short, our purpose is to find the relation between the coefficients A_{nm}, B_{nm} in the comoving frame and the coefficients A_{jk}^S, B_{jk}^S and their p-derivatives in the observer's frame, and *vice versa.*

We denote below the components of $\underline{W}$ along and normal to the magnetic field by $\underline{W}_\parallel$ and $\underline{W}_\perp$ respectively, and define $\hat{W}_\parallel = W_\parallel /W$, $\hat{W}_\perp = W_\perp/W$, and $\epsilon = W/v$, with v the particle speed. We choose the polar coordinate axis to point along the magnetic field, and measure ϕ from the direction of $\underline{W}_\perp$. Hence, θ, ϕ denote the particle pitch-angle and gyrophase respectively. We will refer to this coordinate system as the standard one.

3. SOME CURRENT SOLUTIONS Gleeson and Axford (1968) and Forman (1970) obtained 1st order transformations from (2) to (1), for $j_p=A_{oo}(p)$. Forman's elegant approach uses the Lorentz invariance of the phase-space distribution function. Balogh *et al.* (1973) obtained a 2nd order result for the isotropic case and a 1st order result for the anisotropic case that includes the n=1 terms. Ng (1984) obtained a 2nd order transformation for gyrotropic distributions ($j_p = \sum_n A_{no}(p)P_n(\cos\theta)$) valid for relativistic particles.

Another approach, taken by Sanderson *et al.* (1985), is based on a method by Ipavich (1974). This is outlined below. One may use eqn (12) of Ipavich (1974) to transform each term in the series (1), *provided one pays the price of using a non-standard coordinate system* in which the polar axis points along the velocity $\underline{W}$. Denote the non-standard polar and azimuthal angles by θ^*, $\emptyset^*$. This allows any term $A^s_{nm}(p_s)C^m_n(\theta^*_s,\emptyset^*_s)$ *for the phase-space density* to be transformed to:

$$A^s_{nm}[p(1-2\varepsilon\cos\theta^*+\varepsilon^2)^{\frac{1}{2}}]C^m_n\{\tan^{-1}[\sin\theta^*/(\cos\theta^*-\varepsilon),\phi^*]\}, \quad (4)$$

and similarly for $B^s_{nm}(p_s)C^m_n(\theta^*_s,\emptyset^*_s)$. Following Ipavich and Sanderson *et al.*, one assumes a power-law spectrum $(p_s d/dp_s)\log A^s_{nm}(p_s) = -2r_n$, and obtains from (4),

$$(1-2\varepsilon\cos\theta^*+\varepsilon^2)^{-r_n-1}A^s_{nm}(p)C^m_n\{\tan^{-1}[\sin\theta^*/(\cos\theta^*-\varepsilon),\phi^*]\}. \quad (5)$$

Eqn (5) may be compared to Sanderson *et al.*'s eqn (A11). There remains the task of converting (5) to the form (2), including a change from $\theta^*,\emptyset^*$ back to the standard angular coordinates $\theta,\emptyset$.

Sanderson *et al.* accomplish the above by *numerically integrating* the transformed intensity obtained from (5) (their eqns (A11) and (A12)) over each sector, each energy interval and each telescope to produce an *equivalent set of counts in the comoving frame*, and then *fitting* (2) to the equivalent set of counts.

Although the assumption of a power-law spectrum may be avoided by using (4), the numerical integration and the second fitting are time-consuming and unnecessary for say, >10 KeV protons, because we are able to obtain the transformation posed in Section 2, correct to $O(W/v)^2$ and for *arbitrary* direction of $\underline{W}$.

4. DERIVATION OF THE TRANSFORMATION FORMULAE We begin with

$$j_p(p,\theta,\phi) = \sum_{nm}\sum\{\frac{vp^2}{v_s p_s^2}A^s_{nm}(p_s)C^m_n(\theta_s,\phi_s)+\frac{vp^2}{v_s p_s^2}B^s_{nm}(p_s)S^m_n(\theta_s,\phi_s)\}, \quad (6)$$

and consider each term in the series. First we transform the spherical harmonics $C^m_n(\theta_s,\emptyset_s)$. Galilean transformation of momentum gives

$$\cos\theta_s = \cos\theta - \varepsilon(\hat{W}_{||}-\zeta\cos\theta) + \tfrac{1}{2}\varepsilon^2[(3\zeta^2-1)\cos\theta - 2\hat{W}_{||}\zeta] + O(\varepsilon^3), \quad (7)$$

$$\begin{aligned}\sin^m\theta_s\cos m\phi_s &= \sin^m\theta\cos m\phi + m\varepsilon[\zeta\sin^m\theta\cos m\phi - \hat{W}_\perp\sin^{m-1}\theta\cos(m-1)\phi]+\\ &+ \tfrac{1}{2}\varepsilon^2\{[m(m+2)\zeta^2-m]\sin^m\theta\cos m\phi - 2m^2\hat{W}_\perp\zeta\sin^{m-1}\theta\cos(m-1)\phi +\\ &+ \hat{W}^2_\perp(m-1)m\sin^{m-2}\theta\cos(m-2)\phi\} + O(\varepsilon^3),\end{aligned} \quad (8)$$

where $\zeta = -\underline{\hat{v}}.\underline{\hat{W}} = \hat{W}_{||}\cos\theta + \hat{W}_\perp\sin\theta\cos\phi$, (9)

$$\zeta\cos m\phi = \hat{W}_{||}\cos\theta\cos m\phi + \tfrac{1}{2}\hat{W}_\perp\sin\theta[\cos(m-1)\phi + \cos(m+1)\phi], \quad (10)$$

$$\zeta^2\cos m\phi = (\hat{W}^2_{||}\cos^2\theta + \tfrac{1}{2}\hat{W}^2_\perp\sin^2\theta)\cos m\phi + \hat{W}_{||}\hat{W}_\perp\sin\theta\cos\theta\ x$$

$$\times\ [\cos(m-1)\phi + \cos(m+1)\phi] + \tfrac{1}{4}\hat{W}_1^2 \sin^2\theta[\cos(m-2)\phi + \cos(m+2)\phi]. \qquad (11)$$

Using (7) – (10), and various properties of P_n^m, we obtain

$$C_n^m(\theta_s,\phi_s) = C_n^m(\theta,\phi)+\varepsilon(n+1)U_{n-1}^{mc}(\theta,\phi) +\varepsilon n V_{n+1}^{mc}(\theta,\phi) +\varepsilon^2(n-1)(n+1)X_{n-2}^{mc}(\theta,\phi)+ + \varepsilon^2 n(n+1)Y_n^{mc}(\theta,\phi) + \varepsilon^2 n(n+2)Z_{n+2}^{mc}(\theta,\phi) + O(\varepsilon^3), \qquad (12)$$

where $$(2n+1)U_{n-1}^{mc} = \tfrac{1}{2}\hat{W}_1(n+m-1)(n+m)C_{n-1}^{m-1} - \hat{W}_{\parallel}(n+m)C_{n-1}^{m} - \tfrac{1}{2}\hat{W}_1 C_{n-1}^{m+1}, \qquad (13)$$

$$(2n+1)V_{n+1}^{mc} = \tfrac{1}{2}\hat{W}_1(n-m+1)(n-m+2)C_{n+1}^{m-1} + \hat{W}_{\parallel}(n-m+1)C_{n+1}^{m} - \tfrac{1}{2}\hat{W}_1 C_{n+1}^{m+1}, \qquad (14)$$

$$(2n-1)(2n+1)X_{n-2}^{mc} = \tfrac{1}{8}\hat{W}_1^2\frac{(n+m)!}{(n+m-4)!}C_{n-2}^{m-2} - \tfrac{1}{2}\hat{W}_{\parallel}\hat{W}_1\frac{(n+m)!}{(n+m-3)!}C_{n-2}^{m-1} + + \tfrac{1}{4}(2\hat{W}_{\parallel}^2-\hat{W}_1^2)\frac{(n+m)!}{(n+m-2)!}C_{n-2}^{m} + \tfrac{1}{2}\hat{W}_{\parallel}\hat{W}_1(n+m)C_{n-2}^{m+1} + \tfrac{1}{8}\hat{W}_1^2 C_{n-2}^{m+2}, \qquad (15)$$

$$(2n-1)(2n+3)Y_n^{mc} = \tfrac{1}{4}\hat{W}_1^2\frac{(n+m)!(n-m+2)!}{(n+m-2)!(n-m)!}C_n^{m-2} + \tfrac{1}{2}(2m-1)\hat{W}_{\parallel}\hat{W}_1(n-m+1)(n+m)C_n^{m-1} + - \tfrac{1}{2}[(2n^2+2n-2m^2-1)\hat{W}_{\parallel}^2 + (n^2+n+m^2-1)\hat{W}_1^2]C_n^m + \tfrac{1}{2}(2m+1)\hat{W}_{\parallel}\hat{W}_1 C_n^{m+1} + \tfrac{1}{4}\hat{W}_1^2 C_n^{m+2}, \qquad (16)$$

$$(2n+1)(2n+3)Z_{n+2}^{mc} = \tfrac{1}{8}\hat{W}_1^2\frac{(n-m+4)!}{(n-m)!}C_{n+2}^{m-2} + \tfrac{1}{2}\hat{W}_{\parallel}\hat{W}_1\frac{(n-m+3)!}{(n-m)!}C_{n+2}^{m-1} + + \tfrac{1}{4}(2\hat{W}_{\parallel}^2-\hat{W}_1^2)\frac{(n-m+2)!}{(n-m)!}C_{n+2}^{m} - \tfrac{1}{2}\hat{W}_{\parallel}\hat{W}_1(n-m+1)C_{n+2}^{m+1} + \tfrac{1}{8}\hat{W}_1^2 C_{n+2}^{m+2}. \qquad (17)$$

Note that the functions U, V, X, Y, and Z are all expressed in terms of the spherical harmonics $C_n^m(\theta,\phi)$, and that

$$P_n^{-m}(\theta,\phi) = (-1)^m P_n^m(\theta,\phi)(n-m)!/(n+m)!, \qquad (18)$$

$$\zeta C_n^m = V_{n+1}^{mc} - U_{n-1}^{mc}, \qquad (19)$$

$$\zeta[(n-2)U_{n-1}^{mc} + (n+3)V_{n+1}^{mc}] - \frac{1}{2}C_n^m = (4-2n)X_{n-2}^{mc} - 3Y_n^{mc} + (2n+6)Z_{n+2}^{mc}. \qquad (20)$$

Secondly, we expand $A_{nm}^s(p_s)$ about $p = p_s$:

$$A_{nm}^s(p_s) = A_{nm}^s(p)+\varepsilon\zeta pA_{nm}'^s(p) + \tfrac{1}{2}\varepsilon^2[(1-\zeta^2)pA_{nm}'^s(p) + \zeta^2p^2A_{nm}''^s(p)]+O(\varepsilon^3). \qquad (21)$$

Then we multiply (21) and (12), and use (10), (11), (19) and (20) to express $A_{nm}^s(p_s)C_n^m(\theta_s,\phi_s)$ in terms of a series of $C_n^m(\theta,\phi)$ correct to $O(\varepsilon^3)$. This expression actually applies to the transformation of phase-space distribution function, but will not be exhibited here.

Finally, multiplying

$$vp^2/(v_sp_s^2) = 1 + 3\varepsilon\zeta - (3\varepsilon^2/2)(1-5\zeta^2) + O(\varepsilon^3) \qquad (22)$$

into the above expression, and simplifying as before, we obtain

$$[vp^2/(v_sp_s^2)]A_{nm}^s(p_s)C_n^m(\theta_s,\phi_s) = A_{nm}^sC_n^m + \varepsilon[(n-2)A_{nm}^s + pA_{nm}'^s]U_{n-1}^{mc} + \varepsilon[(n+3)A_{nm}^s + -pA_{nm}'^s]V_{n+1}^{mc} + \varepsilon^2[(n-4)(n-2)A_{nm}^s + (2n-5)pA_{nm}'^s + p^2A_{nm}''^s]X_{n-2}^{mc} + \varepsilon^2[(n-2)(n+3)A_{nm}^s + +4pA_{nm}'^s + p^2A_{nm}''^s]Y_n^{mc} + \varepsilon^2[(n+3)(n+5)A_{nm}^s - (2n+7)pA_{nm}'^s + p^2A_{nm}''^s]Z_{n+2}^{mc} + O(\varepsilon^3), \qquad (23)$$

the right hand side being evaluated at $(p,\theta,\emptyset)$. Eqns (8), (10) - (17), (19) - (20) and (23) hold when we replace all the cosines dependent on $\emptyset$ by sines, C^m_n by S^m_n, U^{mc}_{n1} by U^{ms}_{n1}, etc. Substituting (23) into (6), we obtain the desired result in the form (2), as well as the relation between A_{nm} and A^s_{jk} and their derivatives. These, however, are not exhibited here for lack of space. For the inverse transformation, we merely need to interchange quantities labelled and unlabelled with S, and reverse the signs of $\hat{W}_{\parallel}$ and $\hat{W}_{\perp}$ in the above.

5. SOME IMPLICATIONS For >10 KeV protons, once A^s_{jk} and their 1st and 2nd derivatives are experimentally determined, one need only substitute these into the formulae to obtain the relevant A_{nm} in the comoving frame, bypassing the complicated procedures mentioned in Section 3.

The formulae are useful in other regards. For example, we have

$$A^S_{n0} = A_{n0} + \varepsilon[(n+1)/(2n+3)][n-1+pd/dp][\tfrac{1}{2}\hat{W}_1(n+2)A_{n+1,1} - \hat{W}_{\parallel}(n+1)A_{n+1,0}] + \varepsilon[n/(2n-1)][n+2-pd/dp][\tfrac{1}{2}\hat{W}_1(n-1)A_{n-1,1} + \hat{W}_{\parallel}\, nA_{n-1,0}] + O(\varepsilon^2). \quad (24)$$

For highly anisotropic solar particles, the gyrotropic terms A_{k0} >> the non-gyrotropic term A_{k1}. The former may be eliminated in (24) by choosing the $\underline{E}\times\underline{B}$ drift frame ($W_{\parallel} = 0$). If in addition, $A_{n-1,1}$ and $A_{n+1,1}$ are neglible, then $A_{n0} - A^s_{n0} = O(\varepsilon^2)$.

Next consider the 1st harmonic terms.

$$A^S_{10} = A_{10} + \varepsilon\hat{W}_{\parallel}(3A_{00} - pA'_{00} - 2pA'_{20}/5) - \varepsilon\hat{W}_1 pA'_{21} + O(\varepsilon^2),$$
$$-A^S_{11} = -A_{11} + \varepsilon\hat{W}_1(3A_{00} - pA'_{00} + pA'_{20}/5 - 6pA'_{22}/5) + 3\varepsilon\hat{W}_{\parallel} pA'_{21}/5 + O(\varepsilon^2),$$
$$-B^S_{11} = -B_{11} + 3\varepsilon\hat{W}_{\parallel} pB'_{21}/5 - 6\varepsilon\hat{W}_1 pB'_{22}/5 + O(\varepsilon^2). \quad (25)$$

Even if all the non-gyrotropic terms vanish, if $pA'_{20} \neq 0$, the *1st order* 1st harmonic anisotropy vector is *not* aligned with $\underline{W}$'

6. REMARKS The new formulae are useful in the study of transverse anisotropies. The derivation in Section 4 is being extended to cover relativistic particles (see also Ng, 1984). These and other considerations and a more complete account of the present work will be presented elsewhere.

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