NOTES ON DRIFT THEORY

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1. Alternative Derivation of Drift Velocity. A particle at position \( \mathbf{x} \), with momentum \( \mathbf{p} \), velocity \( \mathbf{v} \), in a magnetic field \( \mathbf{B} \) has a gyroradius \( R = \mathbf{b} \times \mathbf{p} / q \), where \( \mathbf{b} = B / B^2 \). Its guiding centre is at \( x_g = \mathbf{x} - R \). Differentiation of \( x_g \) w.r.t. time in the steady state (\( \partial / \partial t = 0 \)), with employment of the Lorentz force \( \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \) then gives

\[
\langle v_g \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{b} + \langle \mathbf{p} \times (\mathbf{v} \cdot \mathbf{v}) \mathbf{b} \rangle / q, \quad (1)
\]

for the instantaneous, average drift velocity of particles inside a volume element \( \mathrm{d}^3 \mathbf{x} \) around \( \mathbf{x} \). For a particle distribution function \( F(\mathbf{x}, \mathbf{p}, t) \) with density \( N \), this average is defined as

\[
\langle \ldots \rangle = (1/N) \int \ldots \mathbf{F} \mathrm{d}^3 \mathbf{p} = (1/N) \int \mathbf{d}^3 \mathbf{p} \int \ldots \mathbf{F} \mathrm{d} \Omega,
\]

where \( \mathrm{d} \Omega = \sin \theta \mathrm{d} \theta \mathrm{d} \phi \). We shall also use differential (omni-directional) averages in momentum space in the interval \( (\mathbf{p}, \mathbf{p} + \mathrm{d} \mathbf{p}) \) denoted by

\[
\langle \ldots \rangle = \int \ldots \mathbf{F} \mathrm{d}^3 \mathbf{p} / \int \mathbf{F} \mathrm{d} \Omega
\]

In terms of directions in momentum space, \( \mathbf{p} \) may be written as

\[
\mathbf{p} = p(\cos \theta_1 + \sin \theta \cos \phi \mathbf{e}_2 + \sin \theta \sin \phi \mathbf{e}_3),
\]

with \( \theta \) the pitch and \( \phi \) the phase angle relative to \( \mathbf{B} \). In index notation the last term of \( (1) \) is \( \varepsilon_{ijk} p_j \mathbf{v}_k / q \) and determination of \( \langle v_g \rangle \) or \( \langle v_g \rangle \Omega \) only requires finding the components \( p_j \mathbf{v}_k \) of the \( \langle \mathbf{p} \mathbf{v} \rangle \) tensor, given a particular \( F \). In BMW it is shown that the most complicated \( F \) which gives a tractable expression for \( v_g \) is

\[
F(\mathbf{x}, \mathbf{p}, t) = F_{10} + F_{11} \sin \theta \cos \phi + F_{12} \sin \phi, \quad (4)
\]

where \( p_\perp \) are the Legendre polynomials. The \( \langle \mathbf{p} \mathbf{v} \rangle \) tensor is of the form:

\[
\langle \mathbf{p} \mathbf{v} \rangle = \begin{bmatrix}
\langle p_\parallel \mathbf{v}_\parallel \rangle & 0 & 0 \\
0 & \langle p_\perp \mathbf{v}_\perp \rangle & 0 \\
0 & 0 & \langle p_\perp \mathbf{v}_\perp \rangle
\end{bmatrix}, \quad \langle p_\parallel \mathbf{v}_\parallel \rangle = \frac{1}{3} \langle \mathbf{p} \mathbf{v} \rangle, \quad \langle p_\perp \mathbf{v}_\perp \rangle = \frac{1}{3} \langle \mathbf{p} \mathbf{v} \rangle
\]

\[
\frac{1}{3} \langle \mathbf{p} \mathbf{v} \rangle = \frac{1}{3} \langle \mathbf{p} \mathbf{v} \rangle (1 - \frac{2}{5} F_{10}) \quad (5)
\]
For this form of $F$, the average drift velocity \( \langle \mathbf{v} \rangle \) becomes
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + \frac{1}{3q} \left( \langle \mathbf{p} \rangle \mathbf{v} - \frac{1}{2} \langle \mathbf{p} \mathbf{v} \rangle B \mathbf{B} \right)
\] (6)

There are six noteworthy points about this seemingly familiar result: (i) It is the instantaneous average guiding centre velocity of particles in the volume element $d^3x$ in terms of the fields $E$ and $B$ inside $d^3x$. The usual "single particle" derivations (e.g. Rossi and Olbert, 1970) give a similar result, but for the time (gyroperiod) averaged drift velocity in terms of the fields $E_0$ and $B_0$ at the guiding centre. (ii) Our analysis does not need the usual "single particle" condition that $R < L$ (where $L$ is the smallest scale length of variation of $E$ and $B$). But, this does not mean that (6) is valid for any ratio $R/L$, as is discussed in Section 3. (iii) The result (6) is usually interpreted to hold only for gyrotrropic distributions, in which $F_1 = F_1, F_2 = 0$ in (4), (e.g. Rossi and Olbert, 1970, Lee and Fisk, 1981). This too strict condition on $F$ stems from imprudent mixing of the $\langle \mathbf{p} \mathbf{v} \rangle$ tensor with the pressure tensor $P = (N/3) \rho_v$ is the scalar pressure. Because of the error mentioned in (iii), this is usually quoted as an approximate result, strictly valid for isotropic distributions only. (iv) If $F$ is of the simple form $F = (1 + \cos \theta)^2$, the underlined term in (6) exceeds the first one in $[7]$ only if $\gamma > 3.56$. This represents a much higher anisotropy than usually found in Astrophyysical applications, because then fully 95% of the particles have pitch angles in the forward range $0 > \theta > \pi/2$. (v) For first order anisotropies, $F_{\perp}(z) = 0$, the exact guiding centre velocity is form (5):
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + \frac{1}{3q} \left( \langle \mathbf{p} \rangle \mathbf{v} - \frac{1}{2} \langle \mathbf{p} \mathbf{v} \rangle B \mathbf{B} \right)
\] (7a)

or
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + \frac{1}{3q} \left( \langle \mathbf{p} \rangle \mathbf{v} - \frac{1}{2} \langle \mathbf{p} \mathbf{v} \rangle B \mathbf{B} \right)
\] (7b)

where $P = (N/3) \rho_v$ is the scalar pressure. Because of the error mentioned in (iii), this is usually quoted as an approximate result, strictly valid for isotropic distributions only. (vi) Since
\[
\mathbf{v} \times \mathbf{B} = \frac{1}{B} \left[ \frac{\mathbf{B} \times \mathbf{V}}{B^2} + \nabla \times \mathbf{B} \right]
\] (8)

the last term in (7) is the sum of the usual gradient and curvature drift.

### 2. Average Particle Velocity $\langle \mathbf{v} \rangle$ and Nomenclature.

From the first moment of the Vlasov Equation $\partial F / \partial t + \mathbf{v} \cdot \nabla F + \mathbf{E} \cdot \mathbf{B} F = 0$, it is readily shown that in the steady state the average particle velocity for a distribution of the form (4) is
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + \frac{1}{qN} \left( \langle \mathbf{p} \rangle \mathbf{v} - \frac{1}{2} \langle \mathbf{p} \mathbf{v} \rangle B \mathbf{B} \right)
\] (9)

The underlined term is identical to the one in (6). For first order anisotropies this reduces to the well-known form
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + 1/(qN) \mathbf{b} \times \mathbf{v},
\] (10a)

or in the differential case
\[
\langle \mathbf{v} \rangle = \langle \mathbf{v} \rangle + \mathbf{E} \times \mathbf{B} + \frac{\rho v}{3qU} \mathbf{b} \times \mathbf{v} \mathbf{v}
\] (10b)

where $U = p^2 (1/4) / F d\Omega$ is the differential number density in $(p, p+dp)$, and $C = 1-1/(3U) \partial^2 \rho / \partial p \partial u$ the Compton-Getting factor. The results (10) are standard in cosmic ray literature.

From the definition $x_d = x - R$ it follows that $\langle x_d \rangle = \langle x \rangle - \langle dR / dt \rangle$. The average circulation velocity $\langle dR / dt \rangle$ is usually called the diamagnetic drift, and together with $\langle x \rangle$, these two velocities are frequently referred to as collective plasma drifts. This is in opposition to the guiding centre drift which is called a "single particle drift". The average guiding centre drift seems equally collective to us. We also argue that the guiding centre drift is the only drift, because it is designed to tell you where the distribution goes after the trouble's some, typically dominant circular motion of each particle has been subtracted.

In BNM these arguments are extended to the non-steady state $(a / at = 0)$. After multiplication of all velocities with $qN$ it is easily shown that for first order anisotropies...
\[ j = j_g + j_b + j_p = j_g + v \times M + \alpha B / \alpha t, \quad (11) \]

where \( M = -S \hat{p} \), \( v \times B \) is the diamagnetic moment per unit volume and \( \Pi = B \times N \hat{p} \) the polarization vector of the plasma. The total plasma current density is therefore the sum of the guiding centre (or free) current density, the diamagnetic (or bound) current density and the polarization current density. When standard "single particle" drift analyses are used, the equivalent result for (11) can only be written down readily for the perpendicular components (e.g., Parker, 1957). Northrop (1961, 1963) also included the parallel components but he needed the introduction of a distribution function \( F_0 \) for guiding centres in addition to \( F \), and elaborate algebra to do it. By contrast our result is essentially true per definition.

3. Neutral Sheet Drift. Standard drift theory breaks down at neutral sheets (\( B = 0 \)) because the condition \( R/L < 1 \) is violated. Our derivations are independent of this ratio but they still break down. The reason follows from Figure 1 which shows a field \( \mathbf{B} = \mathbf{B}(x) \mathbf{e}_z \), with \( \mathbf{B}(x) \) homogeneous in \( x \), and \( dB/dx \) any arbitrary large positive value. The drift of particles such as (a) and (b) is perfectly well described by the last term of (7), independent of the magnitude of \( dB/dx \). However (8) shows that at \( x = 0 \), \( \mathbf{v} \times \mathbf{B} = 0 \), while Figure 1 suggests that individuals drift with finite velocity. Consequently \( \langle v_\perp \rangle \) cannot be infinite (as it was used by Jokipii and Thomas (1981, and references therein)). The obvious reason for this breakdown is that the drift velocity at \( x = 0 \) depends on the point angle at the point of crossing.

The drift in and around a neutral sheet is readily calculated if \( \mathbf{B} \) is given by \( \mathbf{B}(x) = 2H(x-1) \mathbf{B}_0 \mathbf{e}_z \), where \( \mathbf{B}_0 \) is a constant and \( H \) the unit step function at \( x = 0 \). Figure 2 shows the drift of a particle with velocity \( \mathbf{v} = (V, \theta, \phi) \), projected onto the plane \( \theta = \pi/2 \). It crosses the sheet at a point under phase angle \( \phi_c \), and will again do so at point \( b \) after having completed a projected arc length \( \ell = 2R_0 (\pi - \phi_c) \sin \theta \), where \( R_0 = \rho / (qB_0) \). At that time the guiding centre abruptly jumps a distance \( \ell = 2R_0 \sin \theta \sin \phi_c \) in the direction \( -\mathbf{e}_y \). Therefore the component of the (time averaged) guiding centre velocity in the direction \( -\mathbf{e}_y \) may be taken as

\[ v_{gy} = \ell / \Delta t = (\ell / s) v \sin \theta = \left[ v / (\Pi - \phi_c) \right] \sin \theta \sin \phi_c \quad (12) \]

From the relation \( \cos \phi_c = \cos \phi + x / (R_0 \sin \theta) \) it follows that of all the particles momentarily inside \( d^3x \), at a distance \( x \) from the sheet, only those with

\[ G_1 < \phi < \Pi - G_1 \] and \( \phi_1 < \phi < 2\Pi - \phi_1 \]

where \( \sin G_1 = x / (2R_0) \) and \( \cos \phi_1 = 1 - x / (R_0 \sin \theta) \) (13)

can cross the sheet before completing a full orbit. Therefore the directionally averaged drift velocity over an isotropic particle distribution inside \( d^3x \) is

\[ \langle v_\perp \rangle = -e_y (v / 4\Pi) \int d\sin^2 \theta \int d\sin \phi_c (\theta, \phi) / \left[ \Pi - \phi_c (\theta, \phi) \right] \]

(14)

The x-component of the drift velocity oscillates and does not contribute to the average, while the z-component of the average is zero for an isotropic distribution. The integral (14) can be evaluated as a series at \( x = 0 \) to give \( \langle v_\perp \rangle = -0.463 e_y \) while its numerically calculated value in the range \(-2R_0 < x < 2R_0 \) is shown in Figure 2.

This interpretation of guiding centre drift at a neutral sheet differs considerably from previous ones. Firstly, there is no sheet current density of the form \( j_0 = j_0 (x) \mathbf{e}_y \) as is sometimes suggested. In fact the assumed isotropy implies \( j = 0 \) everywhere, even in a volume element \( d^3x \) at \( x = 0 \). Secondly, the distribution in the entire range \(-2R_0 < x < 2R_0 \) progresses (drifts) in the direction \(-\mathbf{e}_y \) instead of the infinite sheet drift current \( j_0 = j_0 (x) \mathbf{e}_y \) derived by Jokipii et al. (1977) from the \( \mathbf{v} \times \mathbf{B} / B^2 \) term in (F).

As an example we consider the motion of cosmic rays in the interplanetary magnetic field.
where \( \Omega \) is the angular velocity of the Sun, \( V \) is the radial solar wind velocity and the function \((1-2H)\) models a flat neutral plane at \( \Theta=\pi/2 \). The neutral sheet drift (14) will be experienced by cosmic ray particles within an angle \( \Delta \Theta=2R_0/r \) from the plane. If \( V=400 \mbox{ km/s} \), \( \Omega=(2\pi/27) \mbox{ day}^{-1} \), \( r=1 \mbox{ A.U.} \), and \( B_0=5/2 \mbox{ nT} \) (\( |B|=5 \mbox{ nT at Earth} \)), it is readily shown that

\[
\Delta \Theta \approx 0.72 \left( \frac{P}{P_0} \right) r (r^2 + r_0^2)^{-1/2} \mbox{ degrees},
\]

where \( P \) is particle rigidity with \( P_0=1 \mbox{ GV} \). Typical cosmic ray protons with kinetic energy \( T=250 \mbox{ MeV} \), \( P=0.75 \mbox{ GV} \) are therefore convected along the neutral plane in a region with thickness \( 2\Theta=1 \mbox{ degree} \). This neutral plane drift effect is not included in existing drift models of cosmic ray modulation. Jokipii and Thomas (1981, and references therein) used the \( \delta \)-function type drift velocity at the neutral plane, derived from the \( \mathbf{v} \times \mathbf{B}/\mathbf{B}\) term. Potgieter and Moraal (1985) used a finite neutral plane drift velocity pattern, similar to that of Figure 2, but not based on the fundamental arguments used here. In their model the angular region \( \Delta \Theta \) in (16) could also be varied in an empirical and unjustified way, and it is significant that cosmic ray observations were better explained with finite neutral plane drift in a region \( \Delta \Theta \geq 1^\circ \), than with a \( \delta \)-function type drift at \( \Theta=\pi/2 \) only.

References
Northrop, T.G.: 1963, Adiabatic Motion of Charged Particles, Interscience, N.Y.