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NOTES ON DRIFT THEORY

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ABSTRACT

It is shown that there is a simpler way to derive the average guiding centre drift of a distribution of particles than via the so-called "single particle" analysis. Based on this derivation it is shown that the entire drift formalism can be considerably simplified, and that results for low order anisotropies are more generally valid than is usually appreciated. This drift analysis leads to a natural alternative derivation of the drift velocity along a neutral sheet. Full derivations are given in Burger, Moraal and Webb (1985; referred to as BMW), of which copies will be available at the Conference.

1. Alternative Derivation of Drift Velocity. A particle at position $\underline{x}$, with momentum $\underline{p}$, velocity $\underline{v}$, in a magnetic field $\underline{B}$ has a gyroradius $\underline{R} = \underline{b} \times \underline{p}/q$, where $\underline{b} = \underline{B}/B^2$. Its guiding centre is at $\underline{x}_g \equiv \underline{x} - \underline{R}$. Differentiation of $\underline{x}_g$ w.r.t. time in the steady state $(\partial/\partial t = 0)$, with employment of the Lorentz force $\dot{\underline{p}} = q(\underline{E} + \underline{v} \times \underline{B})$ then gives

$$< \underline{v}_g > = < \underline{v}_{\parallel} > + \underline{E} \times \underline{b} + < \underline{p} \times (\underline{v}\cdot\underline{\nabla})\underline{b} >/q, \tag{1}$$

for the instantaneous, average drift velocity of particles inside a volume element d^3x around $\underline{x}$. For a particle distribution function $F(\underline{x}, \underline{p}, t)$ with density N, this average is defined as

$$<\cdots> = (1/N) \int \cdots F d^3p = (1/N) \int dp p^2 \int \cdots F d\Omega, \tag{2a}$$

where $d\Omega = \sin\Theta d\Theta d\phi$. We shall also use differential (omnidirectional) averages in momentum space in the interval $(p, p + dp)$ denoted by

$$<\cdots>_{\Omega} = \int \cdots F d\Omega / \int F d\Omega \tag{2b}$$

In terms of directions in momentum space, $\underline{p}$ may be written as

$$\underline{p} = p(\cos\Theta \underline{e}_1 + \sin\Theta \cos\phi \, \underline{e}_2 + \sin\Theta \sin\phi \, \underline{e}_3), \tag{3}$$

with Θ the pitch and ϕ the phase angle relative to $\underline{B}$. In index notation the last term of (1) is $\varepsilon_{ijk} < p_j v_\ell > b_{k,\ell}/q$ and determination of $< \underline{v}_g >$ or $< \underline{v}_g >_\Omega$ only requires finding the components $< p_j v_\ell >$ of the $< \underline{p}\underline{v} >$ tensor, given a particular F. In BMW it is shown that the most complicated F which gives a tractable expression for $\underline{v}_g$ is

$$F(\underline{x},\underline{p},t) = F_{00} + \{F_{10}\cos\Theta + F_{11}\sin\Theta\cos\phi + F_{1,-1}\sin\Theta\sin\phi\} + \sum_{\ell=2}^{\infty} F_{\ell 0} P_\ell(\cos\Theta), \tag{4}$$

where P_ℓ are the Legendre polinomials. The $< \underline{p}\,\underline{v} >$ tensor is of the form:

$$< \underline{p}\,\underline{v} > = \begin{bmatrix} <p_\parallel v_\parallel> & 0 & 0 \\ 0 & \tfrac{1}{2}<p_\perp v_\perp> & 0 \\ 0 & 0 & \tfrac{1}{2}<p_\perp v_\perp> \end{bmatrix}, \quad \begin{aligned} < p_\parallel v_\parallel > &= \frac{1}{3} < pv\left(1 + \frac{2}{5}\frac{F20}{F00}\right) > \\ \tfrac{1}{2}<p_\perp v_\perp> &= <p_2 v_2> = <p_3 v_3> \\ &= \frac{1}{3} < pv \left(1 - \frac{1}{5}\frac{F20}{F00}\right) > \end{aligned} \tag{5}$$

For this form of F, the average drift velocity (1) becomes

$$\langle \underline{v}_g \rangle = \langle \underline{v}_{\parallel} \rangle + \underline{E}\times\underline{b} + \frac{1}{q}\left[\tfrac{1}{2}\langle p_\perp v_\perp \rangle \underline{\nabla}\times\underline{b} + \underline{\left(\langle p_{\parallel} v_{\parallel} \rangle - \tfrac{1}{2}\langle p_\perp v_\perp \rangle\right)\frac{1}{B}\left(\underline{\nabla}\times\frac{\underline{B}}{B}\right)_\perp} \right] \quad (6)$$

There are six noteworthy points about this seemingly familiar result: (i) It is the *instantaneous* average guiding centre velocity of particles in the volume element d^3x in terms of the fields $\underline{E}$ and $\underline{B}$ *inside* d^3x. The usual "single par= ticle" derivations (e.g. Rossi and Olbert, 1970) give a similar result, but for the *time* (gyroperiod) averaged drift velocity in terms of the fields $\underline{E}_g$ and $\underline{B}_g$ *at the guiding centres*. (ii) Our analysis does not need the usual "single par= ticle condition that R<<L (where L is the smallest scale length of variation of $\underline{E}$ and $\underline{B}$). *But*, this does not mean that (6) is valid for any ratio R/L, as is discussed in Section 3. (iii) The result (6) is usually interpreted to hold only for gyrotropic distributions, in which $F_{11}=F_{1,-1}=0$ in (4), (e.g. Rossi and Olbert, 1970, Lee and Fisk, 1981). This too strict condition on F stems from imprudent mixing of the $\langle \underline{p}\ \underline{v} \rangle$ tensor with the pressure tensor $\underline{P}=N(\langle \underline{p}\ \underline{v}\rangle - \langle \underline{p}\rangle\langle \underline{v}\rangle)$. (iv) If F is of the simple form $F\propto(1+\cos\Theta)^\gamma$, the underlined term in (6) exceeds the first one in [ ] only if $\gamma\geq 3.56$. This represents a much higher anisotropy than usually found in Astrophysical applications, because then fully 95% of the particles have pitch angles in the forward range $0>\Theta>\Pi/2$. (v) For first order anisotropies, $F_{\ell j}(\ell\geq 2)=0$, the exact guiding centre velocity is form (5):

$$\langle \underline{v}_g \rangle = \langle \underline{v}_{\parallel} \rangle + \underline{E}\times\underline{b} + (P/qN)\ \underline{\nabla}\times\underline{b} \quad (7a)$$

or

$$\langle \underline{v}_g \rangle_\Omega = \langle \underline{v}_{\parallel} \rangle_\Omega + \underline{E}\times\underline{b} + (pv/3q)\ \underline{\nabla}\times\underline{b} \quad (7b)$$

where $P = (N/3)\langle pv \rangle$ is the scalar pressure. Because of the error mentioned in (iii), this is usually quoted as an approximate result, strictly valid for iso= tropic distributions only. (vi) Since

$$\underline{\nabla}\times\underline{b} = \frac{1}{B}\left[\frac{\underline{B}\times\underline{\nabla}B}{B^2} + \underline{\nabla}\times\frac{\underline{B}}{B}\right], \quad (8)$$

the last term in (7) is the sum of the usual gradient and curvature drift.

2. Average Particle Velocity $\langle \underline{v} \rangle$ and Nomenclature. From the first moment of the Vlasov Equation $\partial F/\partial t + \underline{v}\cdot\underline{\nabla}F + \dot{\underline{p}}\cdot\underline{\nabla}_p F = 0$, it is readily shown that in the steady state the average particle velocity for a distribution of the form (4) is

$$\langle \underline{v} \rangle = \langle \underline{v}_{\parallel} \rangle + \underline{E}\times\underline{b} + \frac{1}{q}\left[\frac{\underline{b}}{N}\times\underline{\nabla}(N\langle \tfrac{1}{2}p_\perp v_\perp \rangle) + \underline{\left(\langle p_{\parallel} v_{\parallel}\rangle - \tfrac{1}{2}\langle p_\perp v_\perp \rangle\right)\frac{1}{B}\left(\underline{\nabla}\times\frac{\underline{B}}{B}\right)_\perp}\right] \quad (9)$$

The underlined term is identical to the one in (6). For first order anisotro= pies this reduces to the well-known form

$$\langle \underline{v} \rangle = \langle \underline{v}_{\parallel} \rangle + \underline{E}\times\underline{b} + 1/(qN)\ \underline{b}\times\underline{\nabla}P, \quad (10a)$$

or in the differential case

$$\langle \underline{v} \rangle_\Omega = \langle \underline{v}_{\parallel} \rangle_\Omega + C\underline{E}\times\underline{b} + pv/(3qU)\ \underline{b}\times\underline{\nabla}U \quad (10b)$$

where $U = p^2(1/4\pi)\int F d\Omega$ is the differential number density in (p,p+dp), and $C = 1-(1/3U)\partial/\partial p(pU)$ the Compton-Getting factor. The results (10) are standard in cosmic ray literature.

From the definition $\underline{x}_g=\underline{x}-\underline{R}$ it follows that $\langle \underline{v}_g \rangle = \langle \underline{v} \rangle - \langle d\underline{R}/dt \rangle$. The average circulation velocity $\langle d\underline{R}/dt \rangle$ is usually called the diamagnetic drift, and to= gether with $\langle \underline{v} \rangle$, these two velocities are frequently referred to as *collective plasma drifts*. This is in opposition to the guiding centre drift which is cal= led a "single particle drift". The average guiding centre drift seems equally collective to us. We also argue that the guiding centre drift is the *only* drift, because it is designed to tell you where the distribution goes after the trouble= some, typically dominant circular motion of each particle has been subtracted.

In BMW these arguments are extended to the non-steady state ($\partial/\partial t=0$). After multiplication of all velocities with qN it is easily shown that for first or= der anisotropies

$$\underline{J} = \underline{J}_g + \underline{J}_b + \underline{J}_p = \underline{J}_g + \underline{\nabla} \times \underline{M} + \partial\underline{\Pi}/\partial t, \qquad (11)$$

where $\underline{M}=-N\langle\tfrac{1}{2}p_\perp v_\perp\rangle\underline{b}$ is the diamagnetic moment per unit volume and $\underline{\Pi}=\underline{b}\times N\langle\underline{p}\rangle$ the polarisation vector of the plasma. The total plasma current density is therefore the sum of the guiding centre (or free) current density, the diamagnetic (or bound) current density and the polarisation current density. When standard "single particle" drift analyses are used, the equivalent result for (11) can only be written down readily for the perpendicular components (e.g. Parker, 1957). Northrop (1961, 1963) also included the parallel components but he needed the introduction of a distribution function F_g for guiding centres in addition to F, and elaborate algebra to do it. By contrast our result is essentially true per definition.

3. Neutral Sheet Drift. Standard drift theory breaks down at neutral sheets (B=0) because the condition R/L<<1 is violated. Our derivations are independent of this ratio but they still break down. The reason follows from Figure 1 which shows a field $\underline{B}=B(x)\underline{e}_z$, with B(x) homogeneous in x, and dB/dx any arbitrary large positive value. The drift of particles such as (a) and (b) is perfectly well described by the last term of (7), independent of the magnitude of dB/dx. However (8) shows that at x=0, $\underline{\nabla}\times\underline{b}\to\infty$, while Figure 1 suggests that individuals drift with finite velocity. Consequently $\langle\underline{v}_g\rangle$ cannot be infinite (as it was used by Jokipii and Thomas (1981, and references therein)). The obvious reason for this breakdown is that the drift velocity at x=0 *depends on the phase angle at the point of crossing.*

The drift in and around a neutral sheet is readily calculated if $\underline{B}$ is given by $\underline{B}(x)=[2H(x)-1]B_0\underline{e}_z$, where B_0 is a constant and H the unit step function at x=0. Figure 2 shows the orbit of a particle with velocity $\underline{v}=(v,\Theta,\phi)$, projected onto the plane $\Theta=\pi/2$. It last crossed the sheet at point a under phase angle ϕ_c, and will again do so at point b after having completed a projected arc length $s=2R_0(\pi-\phi_c)\sin\Theta$, where $R_0=p/(qB_0)$. At that time the guiding centre abruptly jumps a distance $\ell=2R_0\sin\Theta\sin\phi_c$ in the direction of $-\underline{e}_y$. Therefore the component of the (time averaged) guiding centre velocity in the direction $-\underline{e}_y$ may be taken as

$$v_{gy} = \ell/\Delta t = (\ell/s)\,v\sin\Theta = [v/(\Pi-\phi_c)]\sin\Theta\sin\phi_c \qquad (12)$$

From the relation $\cos\phi_c=\cos\phi+x/(R_0\sin\Theta)$ it follows that of all the particles momentarily inside d^3x, at a distance x from the sheet, only those with

$$\Theta_1<\Theta<\Pi-\Theta_1 \text{ and } \phi_1<\phi<2\Pi-\phi_1, \text{ where } \sin\Theta_1=x/(2R_0) \text{ and } \cos\phi_1=1-x/(R_0\sin\theta) \qquad (13)$$

can cross the sheet before completing a full orbit. Therefore the directionally averaged drift velocity over an isotropic particle distribution inside d^3x is

$$\langle\underline{v}_g\rangle_\Omega = -\underline{e}_y(v/4\Pi)\int_{\Theta_1}^{R-\Theta_1} d\Theta\sin^2\Theta \int_{\phi_1}^{2\Pi-\phi_1} d\phi\sin\phi_c(\Theta,\phi)/[\Pi-\phi_c(\Theta,\phi)] \qquad (14)$$

The x-component of the drift velocity oscillates and does not contribute to the average, while the z-component of the average is zero for an isotropic distribution. The integral (14) can be evaluated as a series at x=0 to give $\langle\underline{v}_g\rangle=-0{,}463\underline{e}_y$ while its numerically calculated value in the range $-2R_0\le x\le 2R_0$ is shown in Figure 2.

This interpretation of guiding centre drift at a neutral sheet differs considerably from previous ones. Firstly, there is no sheet current density of the form $\underline{J}(x)=J_0\delta(x)\underline{e}_y$ as is sometimes suggested. In fact the assumed isotropy implies $\underline{J}=0$ everywhere, even in a volume element d^3x at x=0. Secondly, the distribution in the entire range $-2R_0<x<2R_0$ progresses (drifts) in the direction $-\underline{e}_y$, instead of the infinite sheet drift current $\underline{J}_g=J_{g0}\delta(x)\underline{e}_y$ derived by Jokipii *et al.* (1977) from the $\underline{\nabla}\times B/B^2$ term in (F).

As an example we consider the motion of cosmic rays in the interplanetary magnetic field.

$$\underline{B} = B_0(r_0/r)^2\left[e_r - (\Omega r \sin\Theta/V)\underline{e}_\phi\right]\left[1 - 2H(\Theta-\pi/2)\right], \quad (15)$$

where $\underline{\Omega}$ is the angular velocity of the Sun, $\underline{V}$ is the radial solar wind velocity and the function (1-2H) models a flat neutral plane at $\Theta=\pi/2$. The neutral sheet drift (14) will be experienced by cosmic ray particles within an angle $\Delta\Theta=2R_0/r$ from the plane. If $V=400\ \mathrm{kms}^{-1}$, $\Omega=(2\pi/27)\ \mathrm{day}^{-1}$, $r=1$ A.U., and $B_0=5/\sqrt{2}$ nT ($|B|= 5$ nT at Earth), it is readily shown that

$$\Delta\Theta \approx 0.72\ (P/P_0)r(r^2 + r_0^2)^{-\frac{1}{2}} \text{ degrees}, \quad (16)$$

where P is particle rigidity with $P_0=1$ GV. Typical cosmic ray protons with kine= tic energy T=250 MeV, P=0.75 GV are therefore convected along the neutral plane in a region with thickness $2\Delta\Theta\approx1$ degree. This neutral plane drift effect is not included in existing drift models of cosmic ray modulation. Jokipii and Thomas (1981, and references therein) used the δ-function type drift velocity at the neutral plane, derived from the $\underline{\nabla}\times\underline{B}/B^2$ term. Potgieter and Moraal (1985) used a finite neutral plane drift velocity pattern, similar to that of Figure 2, but not based on the fundamental arguments used here. In their model the angular region $\Delta\Theta$ in (16) could i.a. be varied in an empirical and unjustified way, and it is significant that cosmic ray observations were better explained with finite neu= tral plane drift in a region $\Delta\Theta \gtrsim 1°$, than with a δ-function type drift at $\Theta=\pi/2$ only.

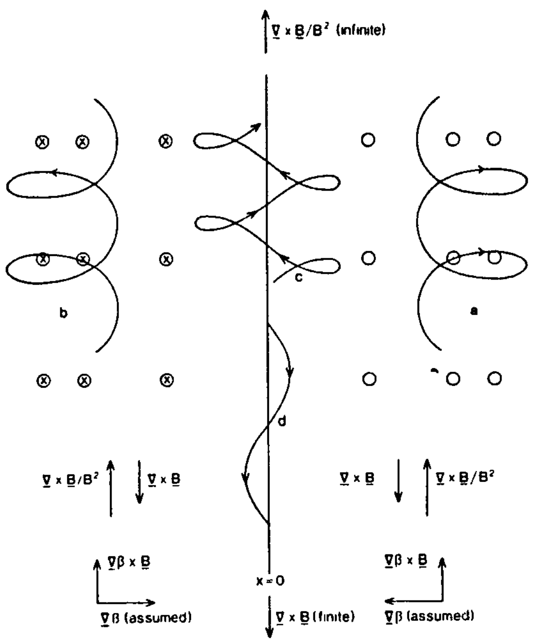

Figure 1

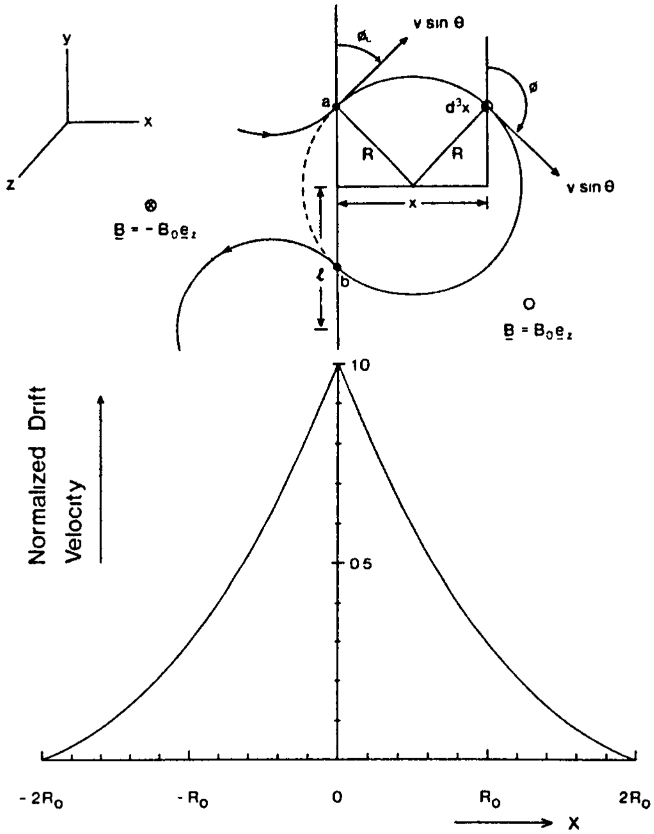

Figure 2

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