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Absorbing Boundary Conditions for Exterior Problems

S. I Hariharan

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ABSORBING BOUNDARY CONDITIONS FOR EXTERIOR PROBLEMS

by

S. I. Hariharan*

1. Introduction

Many formulations arising from physical nature yield problems in unbounded regions Mathematical formulations of such problems yield governing partial differential equations in or near a given domain in such a fashion that: i) the equations may be linear but with non-constant coefficients, or ii) the equations may be nonlinear, but at large distances essentially behave linearly and with constant coefficients. This note presents a survey of the treatment of such problems, when the desired solutions are governed by elliptic or hyperbolic partial differential equations These problems are called exterior problems and commonly arise in the fields of aerodynamics, meteorology, electromagnetic scattering, and atmospheric acoustical wave propagation. The main difficulties with these problems are the boundary conditions that need to be prescribed at large distances from the region of interest. Usually only an asymptotic behavior is known Such conditions may be sufficient to check the well-posedness of the problem however, if one wants to compute the solutions of these problems numerically, infinite distances need to be truncated to finite distances. The boundary conditions imposed on these finite distance boundaries should dictate the behavior at infinity and be accurate with the interior numerical scheme. Furthermore, the shorter the distances, the more efficient the solutions in terms of computation time required. This consideration is the essential need in several problems, depending on the problem and the kind of computer that is used. Typical of such cases are most three dimensional problems. The intention here is to present some available techniques to overcome this difficulty, including application to some model problems.

To begin, let us illustrate some concepts using simple one dimensional

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model problems. First let us consider an elliptic problem where the field equations are reduced wave equations.

Consider a slab of thickness L with a varying index of refraction $n(x)$ Left of this slab, let the media be homogeneous with index of refraction 1 and right of the slab $(x > L)$ let the index of refraction be $n_0\,(> 1)$, which is a constant Such problems are common in optics as well as geophysical waves. This situation is illustrated in Figure 6.1 The governing equations are as follows.

$$\begin{array}{lll} u'' + k^2 u & = 0 & x < 0 \\ u'' + k^2 n^2(x) u & = 0 & 0 < x < L \\ u'' + k^2 n_0^2 u & = 0 & x > L \end{array} \tag{1.1}$$

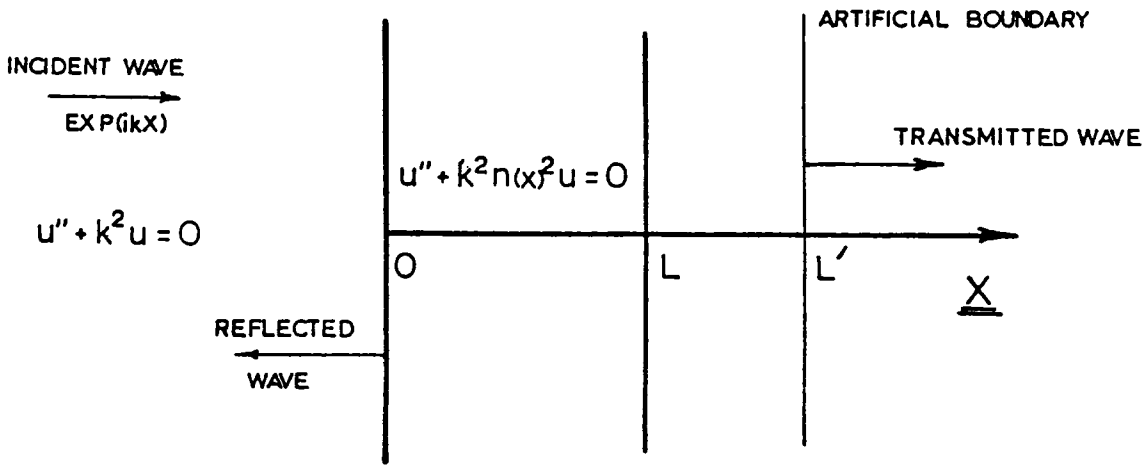

Figure 1 One dimensional model

Then (1.1) needs to be solved in $[0, L']$, $(L' > L)$ say with boundary conditions

$$B_1 u(0) = g \tag{1.2}$$

$$B_2 u(L') = 0. \tag{1.3}$$

u, u' continuous on interfaces, (1.4)

where B_1 and B_2 are boundary operators and g are given are to be chosen according to the physics of the problem. For example, if there is a unit

amplitude wave traveling from $-\infty$ incident on the slab, then for $x < 0$ the solution may be written as

$$u(x) = e^{ikx} + R(k)e^{-ikx} \tag{1.5}$$

where $R(k)$ is the reflection coefficient and measures the part of the wave which is reflected from the boundary $x = 0$ Eliminating $R(k)$ from (1.5) yields

$$u'(0) + iku(0) = 2ik \tag{1.6}$$

which is called an inflow condition. Thus in view of (1.2) $B_1 \equiv d/dx + ik$ and $g = 2ik$. Now we can do a similar calculation to obtain a boundary condition at L' For $x > L$ all the waves transmit and do not reflect back as there are no other boundaries for $x > L$. In this region the solution may be written as

$$u(x) = T(k)e^{ikn_0 x} \tag{1.7}$$

where $T(k)$ is the transmission coefficient which measures the transmitted part of the wave. Eliminating $T(k)$ in (1.7) we obtain

$$u'(L') - ikn_0 u(L') = 0. \tag{1.8}$$

Again comparing with (1.3) we see that the operator B_2 is

$$B_2 \equiv \frac{d}{dx} - ikn_0 . \tag{1.9}$$

The boundary condition (1.8) is the desired absorbing boundary condition for this problem, which is exact.

We note that this condition could have been applied exactly at $x = L$, and this, as we will see, does not always hold in higher dimensions. There is another technical difficulty in this type of problem. One can find that even if $n^2(x) \equiv$ constant there is a countable set of wave numbers $\{k_n\}$ called the *interior resonant values* for which the continuous problem does not have solutions. When one deals with higher dimensional problems, it is not easy to calculate such frequencies when one has a general shaped region where the solution is sought.

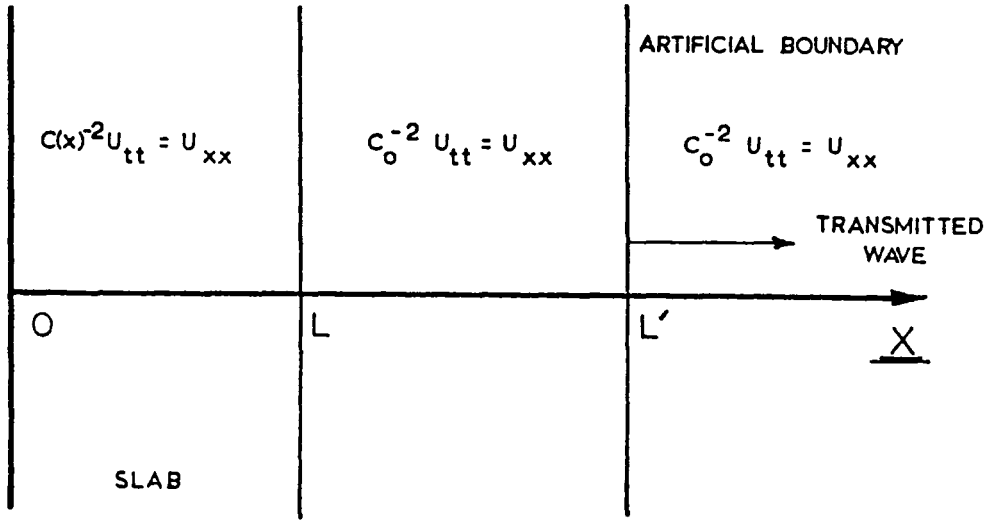

Figure 2 One dimensional model

As an example of hyperbolic problems and associated boundary conditions, let us consider again a one dimensional model In particular let us consider the following problem (see figure 2):

$$\begin{aligned} \frac{1}{c^2(x)} u_{tt} &= u_{xx} \qquad 0 < x < L \\ \frac{1}{c_0^2} u_{tt} &= u_{xx} \qquad x > L \end{aligned} \tag{1.10}$$

$$u, u_x, u_t \text{ continuous on } x = L \tag{1.11}$$

$$u(0,t) = f(t) \tag{1.12}$$

$$Bu(L',t) = 0 \quad (L' > L). \tag{1.13}$$

This problem tells us that the wave is propagating from a source at $x = 0$ and the wave speed changes in $0 < x < L$ and for $x > L$ the waves do not reflect in this region. Equation (1.13) dictates a no reflection condition in this region and B is an operator which is to be determined to fit this physical nature. For $x > L$ the outgoing wave solution can be written as

$$u(x,t) = \phi(x - c_0 t) \tag{1.14}$$

which yields

$$c_0 u_x + u_t = 0 . \tag{1.15}$$

Thus the absorbing condition to be imposed at $x = L'$ is (1.15) and

$$B \equiv c_0 \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \tag{1.16}$$

We see from both one dimensional examples that the absorbing boundary conditions are exact and easy to obtain. The difficulties arise in higher dimensions as we will see. Nevertheless the first problem we described here serves as a model for inverse problems that can be investigated numerically to calculate $n^2(x)$, if the no reflection condition $B_2 u = 0$ is given. This is described in Dunn and Hariharan[6].

The approach taken is as follows: In section two we describe a local boundary condition procedure for exterior elliptic problems. In section three we describe a nonlocal boundary condition for similar problems. In section four we describe a new method that combines the previous two procedures but is more accurate than either. Finally, in section five we describe the methods known for time dependent problems and describe their use for a practical problem.

2. Local Boundary Conditions Procedure

We pointed out in the last section that the absorbing boundary conditions are exact in one dimension and do not hold in higher dimensions. We want to investigate this statement a little further and provide a treatment for this, that of Bayliss, Grunzburger, and Turkel.[2] For illustrative purposes let us consider the following problem that commonly arises in two dimensional acoustic scattering.

Let Ω be a simply connected, bounded domain with boundary Γ and its exterior Ω^+. Then the problem is (see figure 3):

$$\Delta u + k^2 u = 0 \text{ in } \Omega \tag{2.1}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma \tag{2.2}$$

u satisfies a radiation condition (2.3).

Briefly this problem has a physical meaning that there is a wave incident on Ω whose boundary Γ is a perfect reflector and the reflected waves all decay at infinity. u measures the scattered part of the wave and g is the contribution from the incident wave. Regardless of the boundary condition on Γ, the issue here is the radiation condition. Note that the radiation condition is another

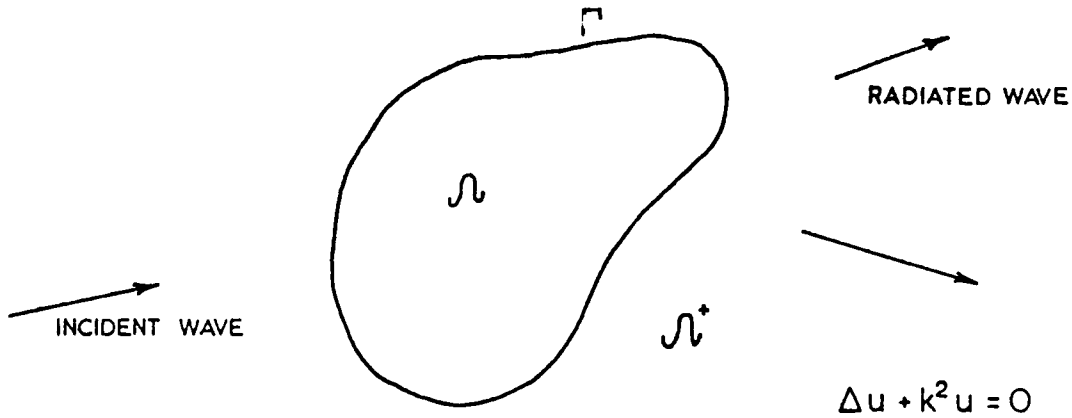

Figure 3 Two dimensional scattering problem

name for the absorbing condition It is known that at large distances the solution of (2.1) behaves like

$$u \sim \frac{e^{ikr}}{\sqrt{r}}\left(a_0(\theta) + \frac{a_1(\theta)}{r} + \frac{a_2(\theta)}{r^2} + \cdots\right) + \frac{e^{-ikr}}{\sqrt{r}}\left(b_0(\theta) + \frac{b_1(\theta)}{r} + \frac{b_2(\theta)}{r^2} + \cdots\right) \tag{2.4}$$

The first part of this solution dictates outgoing waves and the second part dictates incoming waves, which can easily be seen from the signs $e^{\pm ikr}$ Thus in order for the radiation condition to be satisfied we must pick only the first part, i.e.,

$$u \sim \frac{e^{ikr}}{\sqrt{r}}\left(a_0(\theta) + \frac{a_1(\theta)}{r} + \frac{a_2(\theta)}{r^2} + \cdots\right) \tag{2.5}$$

Comparing this to the one dimensional case in (1.7), shows that instead of a single transmission coefficient we have many coefficients $a_i, i = 0, 1, 2$ Thus the one dimensional way of eliminating the transmission coefficient does not exactly work in this situation This shows the difficulty in higher dimensions. However, if we settle for loss of accuracy, in particular eliminating $a_0(\theta)$,we have

$$\begin{aligned} u_r - iku &= -\frac{e^{ikr}}{2r^{\frac{1}{2}}}\left(a_0(\theta) + \frac{a_1(\theta)}{r} + \frac{a_2(\theta)}{r^2} + \cdots\right) \\ &\quad - \frac{e^{ikr}}{\sqrt{r}}\left(\frac{a_1(\theta)}{r^2} + \frac{2a_2(\theta)}{r^3} + \cdots\right). \end{aligned} \tag{2.6}$$

Thus we see that

$$r^{\frac{1}{2}}(u_r - iku) = 0(1/r) \tag{2.7}$$

or simply $\lim_{r\to\infty} r^{\frac{1}{2}}(u_r - \imath ku) = 0$ which is Sommerfeld's radiation condition in two dimensions.

Taking a closer look at the right hand side of (2.6), in particular, the first expression, we see that it is nothing more than $-u/2r$. Thus we have

$$u_r - \imath ku + \frac{1}{2r}u = 0\left(1/r^{5/2}\right). \tag{2.8}$$

Comparing (2 7) and (2 8) shows the error introduced in the boundary condition at large distances is reduced by a factor of $1/r$ This was first studied in connection with numerical implementation for time dependent problems by Bayliss and Turkel[3] and also by Engquist and Majda[7] For time harmonic cases with time dependence of the form $e^{-\imath kt}$, both of their boundary conditions reduce to the form (2.8). Bayliss, Gunzburger,and Turkel[2] generalized these conditions. They defined the boundary operator on the left side of (2.8) by

$$B_1 \equiv \frac{\partial}{\partial r} - ik + 1/2r. \tag{2.9}$$

When (2.8) is explicitly written it has the form

$$B_1 u = -\frac{e^{\imath kr}}{\sqrt{r}}\left(\frac{a_1(\theta)}{r^2} + \frac{2a_2(\theta)}{r^3} + \cdots\right) \tag{2.10}$$

They observed that now the second coefficient $a_1(\theta)$ can also be eliminated If one does that by setting $v = B_1 u$ it is readily verified that

$$\frac{\partial v}{\partial r} - ikv + \frac{5}{r}v = 0\left(1/r^{9/2}\right)$$

or

$$B_2 u = \left(\frac{\partial}{\partial r} - \imath k + \frac{5}{2r}\right)\left(\frac{\partial}{\partial r} - ik + \frac{1}{2r}\right)u = 0\left(1/r^{9/2}\right). \tag{2 11}$$

This process can be repeated to generalize

$$B_m u = 0\left(1/r^{2m+1/2}\right) \tag{2 12}$$

where

$$B_m = \prod_{j=1}^{m} \left(\frac{\partial}{\partial r} + \frac{(2j - \frac{3}{2})}{r} - ik \right) . \tag{2.13}$$

Thus in order to implement this approach the problem can be considered in a truncated region (figure (4)) as follows:

Seek w such that

$$\Delta w + k^2 w = 0 \text{ in } \Omega_T \tag{2.14}$$

$$\frac{\partial w}{\partial r} = g \text{ on } \Gamma \tag{2.15}$$

$$B_m w = 0 \text{ on } \Gamma_\infty \tag{2.16}$$

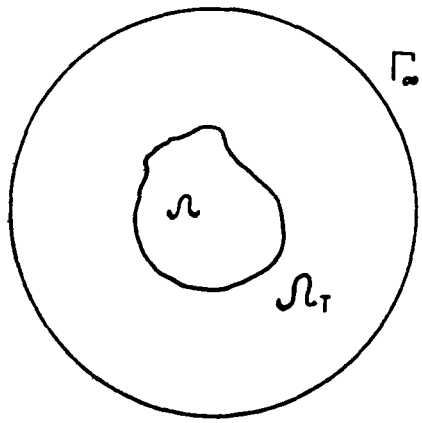

Figure 4 Computational domain

Where Γ_∞ is a circle where the approximate boundary conditions are to be imposed.

In the region Ω_T one can use a finite element formulation of the problem similar to the one used in the next section to seek the solution But there are some difficulties.

First, the governing equation is a second order differential equation Even the second order boundary condition involves second order derivatives in the radial direction. And all the higher order conditions (2.13) have higher order radial operators. This can be overcome by appealing to the differential equation itself. Rewriting the equation in cylindrical coordinates

$$(\Delta + k^2)w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + k^2 w = 0 . \tag{2.17}$$

From (2.17) we see that second order radial operators can be translated into single first order radial derivatives plus derivatives in the tangential direction.

The second difficulty is the following. In comparing (2.1) - (2.3) and (2.14) - (2.16) we see that there is an *a priori* error introduced (irrespective of the numerical scheme used) of the order $0(1/R^{2m+1/2})$, where R is the distance of Γ_∞ from the origin. This of course depends on the order of the operator too. The question is what can we say about $\| u - w \|$ in a suitable norm The answer is not available in two dimensions. However, Bayliss, Gunzburger,and Turkel were able to prove for corresponding operators B_m in three dimensions for $m = 1$ and $m = 2$ the following theorem. Let $\overline{w} = u - w$, then

$$\| \overline{w} \|_{(\Gamma)} = C/r^{m+1}, \tag{2.18}$$

where C depends on k and Γ_∞ and the norm is defined on the surface as

$$\| w \|^2_{(\Gamma)} = \int_\Gamma | w |^2 \, dA \, .$$

This theorem tells that the error on the artificial boundary in that given norm is inversely proportional to $1/r^{m+1}$ and this error remains the same in the interior boundary too. The key point here is that this bound is dominated by C which depends on the wave number k. For three dimensions as $k \to 0$ this estimate is still valid. However, in two dimensions. even though it is possible to get such an estimate as the wave number becomes smaller, this constant can grow larger. In fact, in two dimensions there is a logarithmic branch point as $k \to 0$ and the constant C can grow very large. For this reason, for low frequency cases the problem must be examined very carefully. Such a treatment is technical and it is available in Hariharan and MacCamy [12]

These difficulties are pertinent to the reduced wave equations. However. other strongly elliptic cases can be handled without difficulties. For example if we consider Laplace's equation $\Delta u = 0$, then the solution at infinity either has to be bounded or behaves logarithmically with a given behavior. To apply the above process let us consider the following problem:

$$\Delta u = 0 \quad \text{in } \Omega^+ \tag{2.19}$$

$$u = g \quad \text{on } \Gamma \tag{2.20}$$

$$u \sim \log r \text{ as } r = \sqrt{x^2 + y^2} \to \infty. \tag{2.21}$$

Solutions for this problem can be written as

$$u(r,\theta) = \log r + \sum_{n=0}^{\infty} \frac{a_n}{r^n} \cos n\theta \,. \tag{2.22}$$

This can be used to obtain boundary conditions on the artificial boundary Γ_∞. Set $v = u - \log r - a_0$. Eliminating a_1 we see that,

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0(\frac{1}{r^3}) \,. \tag{2.23}$$

Similarly eliminating a_2 we obtain

$$\left(\frac{\partial}{\partial r} + \frac{3}{r}\right)\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = 0\left(\frac{1}{r^5}\right) \,. \tag{2.24}$$

This process can be repeated to obtain higher order conditions. The constant a_0 can be obtained by averaging the solution on Γ_∞. In section 4 we provide an alternative treatment to resolve the difficulties. In fact we implemented the conditions (2.23) and (2.24) along with ours to compare the efficiency of our procedure.

3. Nonlocal Boundary Conditions Procedure

This procedure is a little more difficult to handle than the previous one. But, it does not introduce an *a priori* error due to placing the artificial boundary at finite distances. There are two different versions of this procedure. The first one originated in the work of Fix and Marin[9] which was done for situations of wave guides and was made general in two dimensions by MacCamy and Marin[15] The second version is due to Johnson and Nedelec[13], which was done independently but has similarities in the approach. Extension to three dimensions was done by Aziz and Kellogg[1] All these procedures are done in view of implementing finite element methods As a result, a reasonable amount of analysis is available for this method. It is almost impossible to summarize all of it here. We choose the method of MacCamy and Marin and describe how the nonlocal conditions are treated. The

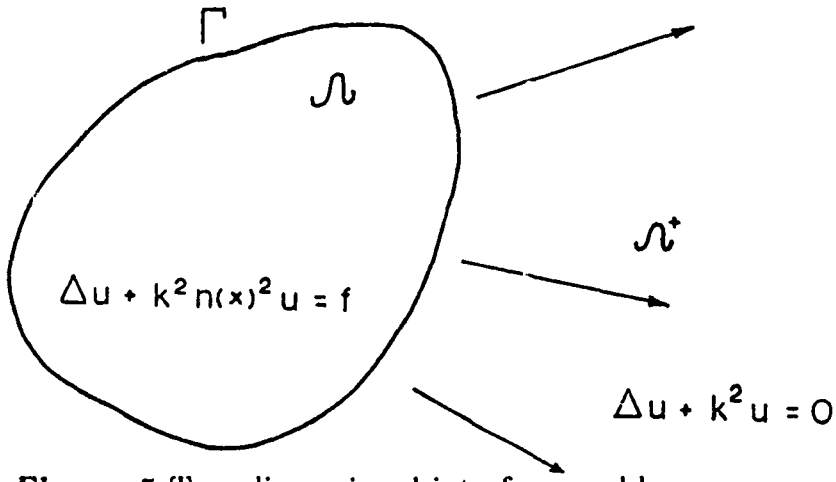

Figure 5 Two dimensional interface problem

attractive feature of this method is that it is appropriate to interface problems analogous to the one dimensional model that we described in section one. Let us present the model problem in two dimensions. Again this is a scattering type of a problem, but this time the scatterer is an inhomogeneous penetrable body. The problem is as follows.

$$\Delta u + k^2 n^2(\underline{x})u = f \quad \text{in } \Omega \tag{3.1}$$

$$\Delta u + k^2 u = 0 \quad \text{in } \Omega^+ \tag{3.2}$$

$$u^- = u^+ \text{ on } \Gamma \tag{3.3}$$

$$\frac{\partial u^-}{\partial n} = \frac{\partial u^+}{\partial n} \text{ on } \Gamma \tag{3.4}$$

$$u - u_i \text{ satisfies Sommerfeld's radiation at infinity.} \tag{3.5}$$

Equations (3.1) and (3.2) say that the index of refraction of the media in Ω is $n^2(\mathbf{x}), \mathbf{x} = (x, y)$ and outside is just a constant. In (3.5) $u_i(\mathbf{x})$ is a prescribed incident field which satisfies (3.2). (3.3) and (3.4) are continuity conditions of the solution on Γ In many applications they may not be continuous, but rather may have jumps. These jumps can be treated without much difficulty To describe the procedure let us extend $n^2(\mathbf{x})$ in $\mathcal{R}^2$ so that

$$\mu(x, y) = \begin{cases} n^2(x, y) & \text{in } \Omega \\ 1 & \text{in } \Omega^+ \end{cases}$$

and extend f in a similar manner, so that

$$g(x, y) = \begin{cases} f(x, y) & \text{in } \Omega \\ 0 & \text{in } \Omega^+ \end{cases} .$$

The governing equations will thus be

$$\Delta u + k^2 \mu u = g \qquad \text{in } \mathcal{R}^2. \tag{3.6}$$

To do numerical calculations let us truncate the infinite region by a circle Γ_∞ into a finite one as depicted in Figure 6. Denote the truncated region by Ω_T.

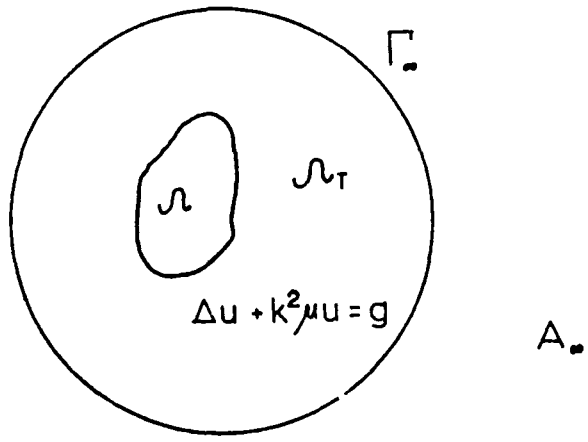

Figure 6 Computational domain

A standard way of handling this problem is to use variational methods Suppose u is a solution and v is an arbitrary differentiable function in Ω_T If we multiply (3.6) by v, integrate over Ω_T and use integration by parts we find

$$-\int_{\Omega_T} \nabla u \cdot \nabla v + k^2 \int_{\Omega_T} \mu u v + \int_{\Gamma_\infty} v \frac{\partial u}{\partial n} = \int_{\Omega_T} g v. \tag{3.7}$$

This variational form is the basis of a numerical method. But, in order to implement this numerically, one should have enough information about $\partial u/\partial n$ on Γ_∞. This in fact requires the knowledge of the solution exterior to Γ_∞ in the region A_∞, recalling that in this region u satisfies

$$\Delta u + k^2 u = 0 \ .$$

This suggests that one may seek an integral representation of the solutions. In particular let us consider

$$u(\mathbf{x}) = \int_{\Gamma_\infty} \sigma(\mathbf{y})\, G_k(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} + u_i(\mathbf{x}) \qquad \mathbf{x} \in A_\infty \tag{3.8}$$

where G_k is the free space Green's function and has the form

$$G_k(\mathbf{x},\mathbf{y}) = -\frac{i}{4} H_0^{(1)}(k \mid \mathbf{x}-\mathbf{y} \mid). \tag{3.9}$$

The representation (3.8) is known as the simple layer representation, which satisfies the Helmholtz equation and satisfies the radiation condition at infinity. These two facts are easily verified by applying the Helmholtz operator to (3.8) and by realizing the asymptotic behavior of (3.9) for large values of $\mid \mathbf{x} \mid$ which yields the form (2.5). In order to calculate the normal derivative of u on Γ_∞, (3.8) together with the standard jump relation in potential theory yields

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \frac{1}{2}\sigma(\mathbf{x}) + \int_{\Gamma_\infty} \sigma(\mathbf{y}) \frac{\partial}{\partial n} G_k(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}} + \frac{\partial u_i(\mathbf{x})}{\partial n} \quad \mathbf{x} \in \Gamma_\infty. \tag{3.10}$$

Thus $\partial u/\partial n$ can be calculated once σ is known. One can use (3.8) to obtain a singular integral equation of the first kind to determine σ when $\mathbf{x}$ is on Γ_∞.

$$u_s \equiv u - u_i = \int_{\Gamma_\infty} \sigma(\mathbf{y}) G_k(\mathbf{x},\mathbf{y}) ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma_\infty \tag{3.11}$$

Let us denote this equation in an operator form

$$u_s(x) = G_k \circ \sigma(\mathbf{x}), \qquad \mathbf{x} \in \Gamma_\infty \ . \tag{3.12}$$

Then σ can be formally inverted to obtain its value of the form

$$\sigma = G_k^{-1} \circ u_s \ . \tag{3.13}$$

But there is a technical difficulty here. G_k^{-1} exists provided $-k^2$ is not an eigenvalue of the operator Δ with zero Dirichlet condition on Γ_∞. This result is familiar in the diffraction theory and it is analogous to the statement that we made for the corresponding one dimensional problem. That is to say, such values of k will be the interior resonant values of Ω_T. But there is a way to treat this problem. Ursell[18] proposed to seek solutions in the exterior not only by a simple layer operator as in (3.8), but combined with a double layer representation. This slightly complicates the explanation of the present method, but nevertheless can be done. For the moment we shall

assume k is not a resonant value so that (3.13) holds. Then (3.10) takes the form (for $\mathbf{x} \in \Gamma_\infty$):

$$\frac{\partial u}{\partial n}(\mathbf{x}) = \frac{1}{2} G_k^{-1} \circ u_s(\mathbf{x}) + \int_{\Gamma_\infty} G_k^{-1} \circ u_s(\mathbf{y}) \frac{\partial G_k}{\partial n}(\mathbf{x}, \mathbf{y})\, ds_{\mathbf{y}} + \frac{\partial u_i(\mathbf{x})}{\partial n}. \quad (3.14)$$

The right hand side of (3.14) is another integral operator acting on u, so that (3.14) can be written (for $\mathbf{x} \in \Gamma_\infty$) as.

$$\partial u / \partial n = T_k(u) - T_k(u_i) - \frac{\partial u_i}{\partial n}. \quad (3.15)$$

Thus, the normal derivative of u is a functional of u on Γ_∞ A closer look at (3.14) shows that to calculate $\partial u/\partial n$ at each point on Γ_∞ we need the knowledge of u on the entire boundary Γ_∞. Thus the boundary condition is nonlocal. Properties of the operator T_k are discussed in detail in reference[15] In particular it is a bounded linear operator.

Returning to the variational formulation, (3.7) together with (3.15) one has

$$a(u, v) \equiv -\int_{\Omega_T} \nabla u \cdot \nabla v + k^2 \int_{\Omega_T} \mu u v + \int_{\Gamma_\infty} v\, T_k(u) =$$

$$\int_{\Omega_T} g v + \int_{\Gamma_\infty} (T_k(u_i) + \frac{\partial u_i}{\partial n}) v \equiv F(v). \quad (3.16)$$

To provide a brief numerical implementation of (3.16) we seek approximate solutions u^h such that

$$a(u^h, v_h) = F(v_h) \quad (3.17)$$

$$\text{for all } v_h \in S^h.$$

where u^h and v_h are in a finite dimensional subspace S^h of an infinite dimensional space S where u is sought. Then (3.17) is made equivalent to a matrix problem by selecting a basis $\{\varphi_1, \varphi_2, \quad \varphi_N\}$ for S^h. This says u can be approximated by

$$u^h = \sum_{j=1}^{N} q_j \varphi_j \quad (3.18)$$

which satisfies

$$a(u^h, \varphi_i) = F(\varphi_i), \; i = 1, \cdots N. \quad (3.19)$$

Then (3.19) is the matrix problem

$$\mathbf{k}\,\mathbf{q} = \mathbf{f} \tag{3.20}$$

where $\mathbf{q} = (q_1, q_2, \cdots, q_N)^T$ is the vector of weights in (3.20) and $\mathbf{f}$ is the source term.

$$\mathbf{k}\big|_{ij} = a(\varphi_j, \varphi_i) = -\int_{\Omega_T} \nabla\varphi_j \cdot \nabla\varphi_i + k^2 \int_{\Omega_T} \mu\varphi_j\varphi_i + \int_{\Gamma_\infty} \varphi_i T_k(\varphi_j). \tag{3.21}$$

We can now describe how the nonlocal boundary condition (3.15) is used. We see from (3.21) that calculating (3.15) is equivalent to computing the integrals

$$\int_{\Gamma_\infty} \varphi_i T_k(\varphi_j) ds \tag{3.22}$$

for the basis functions $\varphi_1, \varphi_2, \cdots \varphi_N$ of the approximation space S^h. This computation is carried out in a straightforward manner using (3.12) and (3.11). First solve for $\sigma_i, i = 1, \cdots, N$.

$$\int_{\Gamma_\infty} \sigma_i(\mathbf{y}) G_k(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} = \varphi_i(\mathbf{x}) \ , \ \mathbf{x} \in \Gamma_\infty \tag{3.23}$$

and compute $T_k(\varphi_i)(\mathbf{x})$ (for $\mathbf{x} \in \Gamma_\infty$) from

$$T_k(\varphi_i)(\mathbf{x}) = \frac{1}{2}\sigma_i(\mathbf{x}) + \int_{\Gamma_\infty} \sigma_i(\mathbf{y}) \frac{\partial}{\partial n} G_k(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} \,. \tag{3.24}$$

There are effective procedures to implement (3.23) and (3.24) in two dimensions which can be found in MacCamy and Marin[15], and also in conjunction with an integral equation treatment to this problem when $n^2(\mathbf{x})$ is a constant found in Hariharan and MacCamy[12]

Some final remarks are needed. We assumed u and $\partial u/\partial n$ are continuous on the boundary Γ. If they have jump discontinuities of the form

$$\alpha_1 \left(\frac{\partial u}{\partial n}\right)^- = \alpha_2 \left(\frac{\partial u}{\partial n}\right)^+ . \tag{3.25}$$

The variational form should be modified. These modifications can be found in reference[15] and in Bielak, MacCamy, and McGhee[4] together with numerical implementation.

4. Infinite order radiation condition procedure

In this section we provide a description of a new method due to Canuto, Hariharan, and Lustman[5]. From the discussions of local boundary conditions procedure, we saw that errors introduced in computations are twofold The first one is the error which varies according to the farfield distance and the order of the boundary condition used. The second one is due to implementation of finite element method. In the nonlocal boundary condition procedure we see that error is essentially only due to the finite element method and the solution of the integral equation on the boundary Γ_∞. No other error is introduced. The local conditions can be improved by either increasing the distance of the artificial boundary or increasing the order of the operator B_m so that the dominating error will be essentially due to the finite element implementation. Error estimates, in such procedures are usually at the best of $O(h^2)$ for this type of problem. A question which then can be asked is whether we can improve the accuracy of the solutions by a method which does not have an a priori error due to placing the artificial boundary and at the same time achieve error in computation $O(h^M)$, where M is the number of elements used. It is possible, if one uses spectral methods, and a brief theoretical discussion of obtaining such accuracy is found in Lustman [14] for simple one-dimensional problems. The method we will describe here does not have any theoretical error estimates yet. At this point we only have numerical evidences. Theoretical error estimates are available only in one dimenison, and extension to higher dimensions is still much in need of work. The procedure we are going to describe here works for a general second-order elliptic problem, exterior to a given domain in two dimensions. The structure is as follows:

$$Lu = f \quad \text{in } \Omega^+ \tag{4.1}$$

$$u = g \quad \text{on } \Gamma \tag{4.2}$$

$$B_\infty u = 0 \quad \text{as } |\mathbf{x}| = \sqrt{x^2 + y^2} \to \infty, \tag{4.3}$$

where Ω^+ and Γ are same as that described in section 2. B_∞ is a boundary

operator at ∞. For illustrative purposes and to appraise numerical results let us consider the problem given through (2.19) - (2.21), which we repeat here.

$$\Delta u = 0 \quad \text{in } \Omega^+ \tag{4.4}$$

$$u = g \quad \text{on } \Gamma \tag{4.5}$$

$$u \sim \log r \quad \text{as } r = \sqrt{x^2 + y^2} \to \infty \tag{4.6}$$

Again the goal here is treat the condition (4.6) numerically Solution in the farfield may be sought through separation of variables of the form

$$u(r, \varphi) = \log r + \sum_k \frac{a_k}{r^{|k|}} e^{ik\varphi} \tag{4.7}$$

where (r, φ) are the polar coordinates in the plane. Note that the right hand side of (4.8) satisfies the radiation condition (4.6) at infinity The coefficients a_k are unknown. The approach here consists of expressing each coefficient a_k as a functional of u, rather than eliminating a finite number of them using differential operations described in section two. From (4.7) we see that for any $r > 0, a_k/r^{|k|}$ is the k-th Fourier coefficient of the periodic function of $\varphi \mapsto u(r, \varphi)$. Thus we can invert the coefficient to obtain

$$\frac{a_k}{r^{|k|}} = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, e^{-ik\theta}\, d\theta = \hat{u}_k(r). \tag{4.8}$$

If we differentiate (4.7) with respect to r

$$u_r(r, \varphi) = \frac{1}{r} \left[1 - \sum_k |\, k \,| \frac{a_k}{r^{|k|}} e^{ik\varphi} \right] \tag{4.9}$$

and use (4.8), we obtain an integro-differential relation on circle of radius r, as follows:

$$u_r(r, \varphi) = \frac{1}{r} \left[1 - \frac{1}{2\pi} \int_0^{2\pi} \sum_k |\, k \,| \, e^{ik(\varphi - \theta)}\, u(r, \theta)\, d\theta \right] \tag{4.10}$$

or

$$u_r = \frac{1}{r} - \frac{1}{r} K \circ u \tag{4.11}$$

where $K \circ u$ is the convolution of u with singular kernel

$$K(\eta) = \frac{1}{\pi} \sum_{k=1}^{\infty} k \cos k\eta \ . \tag{4.12}$$

Again we consider a truncated region Ω_T with an artificial boundary Γ_∞ where the condition corresponding to that of (4.11) will be imposed. Note at this point that (4.11) is similar to the condition (3.15) except it is imposed on a circle of radius $r = R_\infty$. It is not a necessary restriction that Γ_∞ should be a circle. But the restriction guarantees accuracy, if the approximate solution is a trigonometric polynomial on Γ_∞, which is the case if a spectral Fourier method is used in the angular direction. Suppose the approximate solution on Γ_∞ to be u^N, which is a trigonometric polynomial of degree N

$$u^N(R_\infty, \varphi) = \sum_{|k| \le N}^{*} |\, k \,|\ \hat{u}_k^N\,(R_\infty)\ e^{ik\varphi} \ . \tag{4.14}$$

The asterisk in the summation indicates periodicity in the φ direction (i.e., $\hat{u}_N^N(R_\infty) = \hat{u}_{-N}^N(R_\infty)$). Then (4.16) says that the integral operator K produces a new polynomial of degree N, whose Fourier coefficients are obtained from those of $u^N(R_\infty, \cdot)$ by multiplication by the modulus of the wave number $|\, k \,|$. If u^N is known on Γ_∞ through its values $u^N(R_\infty, \varphi_j)$ at equally spaced points $j\pi/N$, $j = 0, 1, \cdots, 2N-1$, then $K \circ u^N$ can be computed at the same nodes exactly and efficiently, first, by Fourier transforming the values of u^N to get its coefficients, then multiplying by the modulus of the wave numbers and finally by Fourier transforming back to get the point values of $K \circ u^N$. This takes order of $N \log_2 N$ operations if one uses fast Fourier transforms, which are always used in this type of calculations. Thus the spectral solution is required to satisfy the radiation condition

$$u_r^N = \frac{1}{R_\infty}\,[1 - K \circ u^N] \quad \text{on } \Gamma_\infty \ . \tag{4.15}$$

Unlike the family of conditions described in section 2, this is precisely the same boundary condition satisfied by the exact solution except for the truncation error which comes from using a finite number of modes. No other error

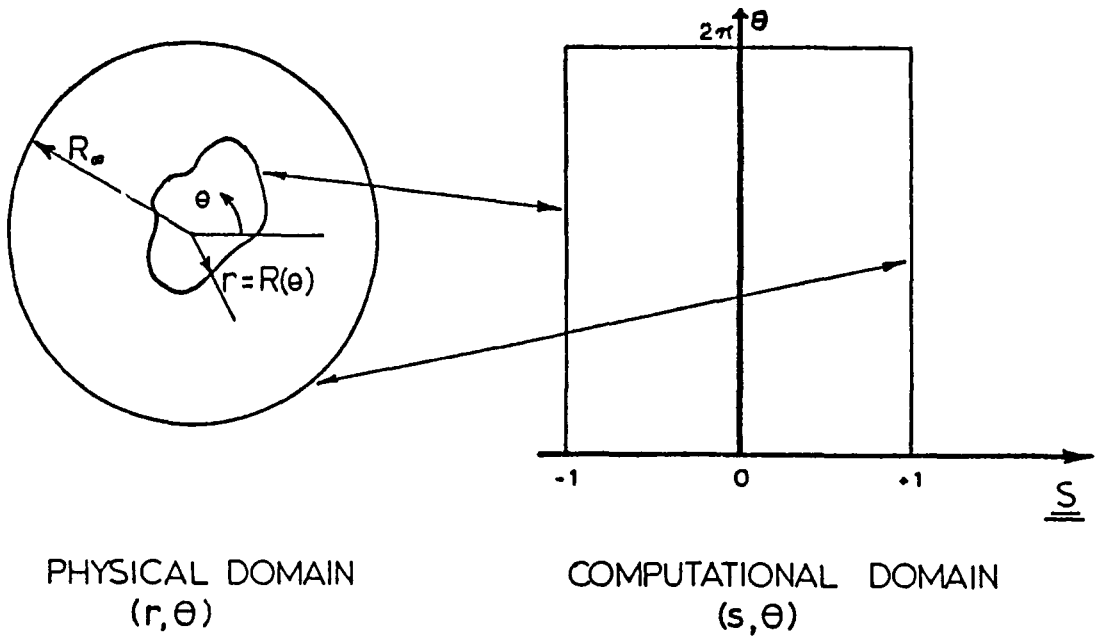

Figure 7 Mapping of physical domain to computational domain

is introduced, and this is why we call this form the infinite order radiation condition.

The actual implementation in a simplified form is as follows:

Suppose the boundary Γ is polar representable as $r = R(\theta)$. We use Chebyshev polynomial approximation in the radial direction and Fourier decomposition in the angular direction. For this we need to make a coordinate transform as depicted in figure 7. In particular the transformation is

$$\begin{aligned} \varphi &= \theta \\ r &= r(s,\theta). \end{aligned} \tag{1.16}$$

So that

$$\begin{aligned} r(-1,\theta) &= R(\theta) \\ r(1,\theta) &= R_\infty \end{aligned} \quad 0 \le \theta \le 2\pi\,. \tag{1.17}$$

he side (1) in the transformed plane will correspond to the boundary Γ and side (2) will be the boundary Γ_∞. Such a transformation can be done in several ways. In particular, the following stretched one is very effective

$$r(s,\theta) = R(\theta)\, e^{(s+1)\alpha(\theta)}\,. \tag{1.18}$$

So that at $s = -1$, $r(-1, \theta) = R(\theta)$. Demanding $r(1, \theta) = R_\infty$, we determine the stretching parameter $\alpha(\theta)$ as

$$\alpha(\theta) = \frac{1}{2} \log(R_\infty / R(\theta)) \ . \tag{4.19}$$

This transformation in turn changes the Laplace equation into the form

$$Lu \equiv \Delta u = a u_{ss} + b u_{s\theta} + c u_{\theta\theta} + d u_s = 0 \tag{4.20}$$

where a, b, c and d are functions of s and θ obtained from chain rule. Seek approximate solutions of the form

$$u^N(s, \theta) = \sum_{m=0}^{M} \sum_{|k| \le N}^{*} \hat{u}_{mk} \, T_m(s) \, e^{ik\theta} \tag{4.21}$$

where T_m is the mth Chebyshev polynomial of the first kind, defined by

$$T_m(\cos \theta) = \cos(m\theta) \tag{4.22}$$

and u^N is determined by u^N_{ij} at the $(M+1) \times 2N$ Chebyshev-Fourier nodes

$$(s_i, \theta_j) = (\cos i\pi/M, j\pi/N) \tag{4.23}$$

$$i = 0, \cdots M$$

$$j = 0, \cdots 2N - 1 \, .$$

At $i = M$ the given Dirichlet condition should be imposed. At $i = 0, \partial u / \partial s$ obtained from $\partial u / \partial r$ should be updated. Placing the inhomogeneous terms arising from the boundary conditions in a forcing term f, let us rewrite the final spectral operator as

$$\left(L_{sp} u^N - f\right)(s_i, \theta_j) = 0. \tag{4.24}$$

The matrix formed by $L_{sp} u^N$ is large and does not have sparse structures. Thus in order to solve (4.24), an iterative procedure is desirable. We outline only the brief idea of implementing this procedure. Dropping the superscript N in (4.24) the method is as follows:

$$u_{n+1} = u_n + \alpha_n (L_{sp} u_n - f) \tag{4.25}$$

where α_n is chosen so that the ℓ_2 norm of the residual

$$r_{n+1} = f - L_{sp}u_n \tag{4.26}$$

is minimum. This gives

$$\alpha_n = -\frac{(r_n, L_{sp}r_n)}{(L_{sp}r_n, L_{sp}r_n)} \tag{4.27}$$

where (,) denotes an ℓ_2 inner product

As a sample comparison we can generate a situation where the exact solution is

$$u(x,y) = \log\,[(x - \cdot 7)^2 + y^2]. \tag{4.28}$$

For the geometry of a circle ($r = R(\theta) = 1$) and for a 33×32 grid, ℓ_2 errors are listed for different values of R_∞. Solutions are compared against Bayliss, Gunzburger, and Turkel's procedure (BGT) with their first order (FO) and second order (SO) boundary conditions given in section two (Equations 2.23). IO CHL denotes the infinite order radiation condition of the present method. (Canuto, Hariharan and Lustman).

R_∞	FO BGT	SO BGT	IO CHL
1.2	$.16 \times 10^{-2}$	$.031 \times 10^{-2}$	$.00037 \times 10^{-2}$
3	$.15 \times 10^{-3}$	$.025 \times 10^{-3}$	01×10^{-3}
5	$.14 \times 10^{-4}$	$.16 \times 10^{-4}$	11×10^{-4}

TABLE 1

We see from this table that in all cases the infinite order radiation condition is superior, especially when the artificial boundary is near the body, which is desired anyway. As R_∞ increases the first order and second order conditions become better and comparable to the infinite order condition. This is because the grid size is fixed If the grid size is increased, the infinite order condition will improve. Further illustrations can again be found in reference[5].

Note further that this procedure can be easily extended to other types of problems. If we consider the Helmholtz equation $\Delta u + k^2 u = 0$ with radiation condition at infinity, we can write the outgoing solutions as follows:

$$u(r,\theta) = \sum_{n=0}^{\infty} a_n H_n^{(1)}(kr)\, e^{in\theta} \tag{4.30}$$

where $H_n^{(1)}$ are the Hankel functions of the first kind and order n.

If we take the radial derivative of (4.30) we have

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} a_n k\; H_n^{(1)\prime}\,(kr)\, e^{in\theta}. \tag{4.31}$$

Again we note that $a_n H_n^{(1)}(kr)$ is the nth Fourier coefficient of $u(r,\cdot)$, which can be inverted to give

$$a_n H_n^{(1)}(kr) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)\, e^{-in\varphi} d\varphi\ . \tag{4.32}$$

Substitution of this in (4.31) gives

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} \frac{k H_n^{(1)\prime}(kr)}{H_r^{(1)}(kr)} \int_0^{2\pi} u(r,\varphi)\, e^{in(\theta-\varphi)}\, d\varphi \tag{4.33}$$

or

$$\frac{\partial u}{\partial r} = K \circ u\,. \tag{4.34}$$

This is similar to what we had in equation (4.11) except that a finite number of values of $H_n^{(1)}$ and its derivatives must be calculated. To do this there are well known recurrence relations and numerical evaluations available in the literature.

5. Boundary Conditions for Time Dependent Problems

This has been the most difficult problem to handle numerically. It is still an open question if a nonlocal condition that is suitable for numerical calculations can be obtained. We saw in section 1 that it is relatively easy

to obtain an absorbing condition for a simple wave equation in one dimension. We proceed in the same fashion to obtain boundary conditions in two dimensions. This process results in the procedure introduced by Engquist and Majda[7],[8]. Before we proceed to discuss the two dimensional case, note that in three dimensions D'Alambert's solution holds. If we consider

$$\frac{1}{c^2} u_{tt} = u_{xx} + u_{yy} + y_{zz} \tag{5.1}$$

then the general solution for spherically radiating and incoming waves can be written as

$$u(r,t) = \frac{1}{r} f(r - ct) + \frac{1}{r} g(r + ct). \tag{5.2}$$

This representation is well known and may be found in Morse and Feshback[16] The first part indicates outgoing waves. If one wants to impose a radiation condition, this part may be used to represent the solution as follows:

$$u(r,t) = \frac{1}{r} f(r - ct). \tag{5.3}$$

If we assume the sound speed is normalized to one, then it is easy to check from (5.3) that

$$u_r + u_t + \frac{u}{r} = 0 \tag{5.4}$$

which is local, both in space and in time, but provides an exact description of spherically radiating waves.

Thus in three dimensions a radiation condition similar to what we discussed in sections 2 through 5, in the time domain, will become exact if one imposes (5.4) at sufficiently far distances. In two as well as in three dimensions analogous to (5.2) one wants a cylindrically or spherically radiating waves at finite distances and that causes problems, as we will see a little later.

Let us begin the discussion of two dimensional wave equations. Consider a wave traveling from left incident on the boundary $x = L (L > 0)$ without reflecting (see figure 8) and governed by

$$u_{tt} = u_{xx} + u_{yy} \ . \tag{5.5}$$

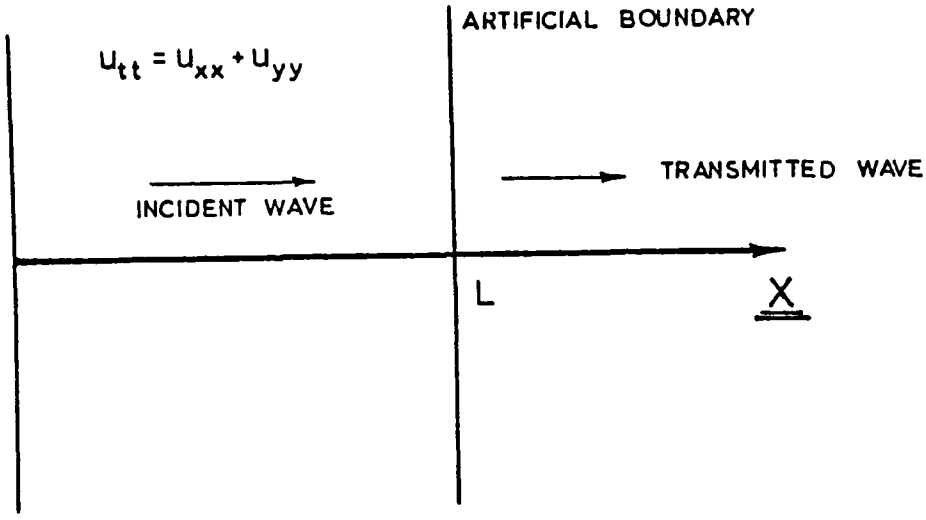

Figure 8 Two dimensional model

Let us pause for a moment and consider the one dimensional case discussed in section 1. The equation for $C_0 \equiv 1$ is

$$u_{tt} = u_{xx} \,. \tag{5.6}$$

This can be written as

$$\underbrace{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)}_{\longrightarrow} \underbrace{\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)}_{\longleftarrow} u = 0. \tag{5.7}$$

The operator $B = \partial/\partial x + \partial/\partial t$ dictated the outgoing part of the wave given by (1.16). The operator $\partial/\partial x - \partial/\partial t$ dictates the incoming part of the wave. To obtain a non reflective condition we set $Bu = 0$. Let us examine if it is possible to do an analogous argument for two dimensions. Equation (5.5) can be written as

$$\underbrace{\left(\frac{\partial}{\partial x} + \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\right)}_{\longrightarrow} \underbrace{\left(\frac{\partial}{\partial x} - \sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}\right)}_{\longleftarrow} u = 0 \,. \tag{5.8}$$

We have indicated by arrows as in (5.7) that in (5.8) the resulting operators may have a similar meaning. Unfortunately we have a square root of an operator. But this can be easily explained using the theory of pseudo differential operators. For this purpose let us take Fourier transforms of equations

(5.5) with respect to t and y and call the corresponding dual variables τ and ς. This gives

$$\hat{u}_{xx} + (\tau^2 - \varsigma^2)\hat{u} = 0 \ . \tag{5.9}$$

Equation (5.9) is a reduced wave equation in the x direction, whose solutions may be written as

$$\hat{u}(x;\tau,\varsigma) = \underline{A(\tau,\varsigma)\, e^{i\sqrt{\tau^2-\varsigma^2}x}} + \underline{B(\tau,\varsigma)\, e^{-i\sqrt{\tau^2-\varsigma^2}x}} \ . \tag{5.10}$$

Arrows in (5.10) indicate the waves moving in the right and left directions In order to impose an outgoing solution we see that we must choose

$$\hat{u}(x;\tau,\varsigma) = A(\tau,\varsigma)\, e^{i\sqrt{\tau^2-\varsigma^2}x} \ . \tag{5.11}$$

Now inverting the Fourier transform in (5.11), we have

$$u(x,y,t) = \int\int A(\tau,\varsigma)\, e^{-i\tau t + i\varsigma y + i\sqrt{\tau^2-\varsigma^2}x}\, d\tau d\varsigma \ . \tag{5.12}$$

Even though (5.12) gives a perfectly nonreflecting condition at $x = L$ it is rather impractical to impose computationally. However, in the pseudo-differential operator terminology $i\sqrt{\tau^2-\varsigma^2}$ is nothing but the symbol of the pseudo differential operator $\sqrt{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2}}$. Engquist and Majda proceeded with this guideline that the approximations of this symbol $i\sqrt{\tau^2-\varsigma^2}$ yield a family of approximate boundary conditions. In particular, approximations around $\varsigma = 0$ (this has the physical meaning of near normal incidence) gives the boundary conditions that we are seeking. For example, the zeroth order approximation of $i\sqrt{\tau^2-\varsigma^2}$ is

$$i\sqrt{\tau^2-\varsigma^2} \approx i\tau \ . \tag{5.13}$$

Now differentiate (5.12) with respect to x.

$$u_x(x,y,t) = \int\int i\sqrt{\tau^2-\varsigma^2}\, A(\tau,\varsigma)\, e^{-i\tau t + i\varsigma y + i\sqrt{\tau^2-\varsigma^2}x}\, d\tau d\varsigma . \tag{5.14}$$

Substitute (5.13) into (5.14) and evaluate at $x = L$ to obtain

$$u_x(L,y,t) = \int\int i\tau\, A(\tau,\varsigma)\, e^{-i\tau t + i\varsigma y + i\sqrt{\tau^2-\varsigma^2}L}\, d\tau d\varsigma \ . \tag{5.15}$$

The right hand side of (5.15) is nothing but the Fourier transform of $-u_t$. Thus (5.15) can be rewritten as

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) u\Big|_{x=L} = 0 \ , \tag{5.16}$$

which is the one dimensional condition (1.15). Obviously this will not be accurate enough for high resolution to handle two dimensional effects. Engquist & Majda proceeded to obtain higher order approximations of the symbol $i\sqrt{\tau^2 - \varsigma^2}$. The next order Taylor approximation is

$$i\sqrt{\tau^2 - \varsigma^2} \approx i\left(\tau - \frac{1}{2}\frac{\varsigma^2}{\tau}\right). \tag{5.17}$$

Substitution of (5.17) in (5.14) yields

$$u_x(L, u, t) = \int\int i\left(\tau - \frac{1}{2}\frac{\varsigma^2}{\tau}\right) A(\varsigma, \tau)\, e^{-i\tau t + i\varsigma y + i\sqrt{\tau^2 - \varsigma^2}\, L}\, d\tau d\varsigma \tag{5.18}$$

Differentiating (5.18) with respect to t we see that

$$u_{xt} = u_{tt} - \frac{1}{2} u_{yy} \ , \ \text{ at } x = L \ . \tag{5.19}$$

In this way higher order boundary conditions can be generated. One warning should be given that the higher order Taylor approximants of the symbol $i\sqrt{\tau^2 - \varsigma^2}$ do not always yield stable boundary conditions. The proof of this statement is difficult and will be found in reference[7]. Instead these authors proposed Pade' approximants and found out that they are stable for all approximate boundary conditions. The second Taylor approximation coincides with the first Pade' approximation and from physical reasoning both the boundary conditions (5.16) and (5.14) are stable

There is another independent work due to Reynolds [20], parallel to the work of Engquist and Majda which derives the boundary condition (5.19) as a special case. Moreover, this work describes in detail on reducing edge reflections. Recently, Keys[21] proposed a new method again by decomposing the wave equation into incoming and outgoing components to obtain a family

of boundary conditions. In this work, he derived Engquist and Majda's as well as Reynolds's conditions as special cases.

Now suppose cylindrically radiating boundary conditions are imposed. Then it is necessary to change the equation (5.5) into cylindrical polar coordinates:

$$u_{tt} = u_{rr} + \frac{1}{r^2}\, u_{\theta\theta} + \frac{1}{r}\, u_r \,. \tag{5.20}$$

This has nonconstant coefficients and needs sufficient modification to apply the above theory. Again variable coefficient theory pseudo differential operators can be used. It is difficult to summarize this procedure; however, there is a similar approach to that which we have given above. First Pade approximation of the resulting symbol yields the boundary condition

$$\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{1}{2r}\right) u = 0 \tag{5.21}$$

to be imposed at sufficiently large distance r.

It is interesting to note that this condition can be obtained from separation of variables analogous to that of the spherically outgoing solution (5.3) In this case we do not have a simple form of D'Alembert's solution, rather it has the form

$$u(r,\theta,t) = \frac{f(t-r)}{\sqrt{r}} \left(a_0(\theta) + \frac{a_1(\theta)}{r} + \cdots \right). \tag{5.22}$$

Eliminating $a_0(\theta)$ in (5.22) we see that

$$\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{1}{2r}\right) u = 0\left(1/r^{5/2}\right). \tag{5.23}$$

We see that boundary condition (5.21) agrees with the physical one (5.23) Eliminating $a_1(\theta), a_2(\theta)$ etc., Bayliss & Turkel[2] obtained a family of radiation conditions, which agree only with (5.23) of Engquist & Majda Other higher order conditions differ from each other. Higher order conditions are very appealing, but difficult to implement. It will be of interest to both applied mathematicians as well as engineers to see how effective are these

boundary conditions. Hariharan and Bayliss[10] implemented three dimensional version of (5.23), i.e., (5.1) to a practical problem described below. In three dimensions this boundary condition is asymptotically accurate to $0(\frac{1}{r^3})$.

The problem is to solve for sound radiation into atmosphere from a cylindrical pipe. There is an incident wave on the left end of the pipe, and sound radiates into atmosphere from open end of the pipe (figure 9).

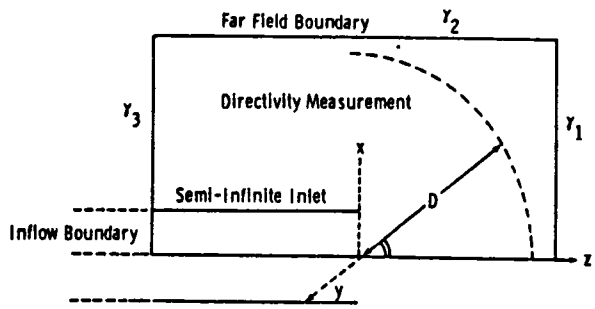

Figure 9 Computational plane of sound radiation problem

The incident pressure wave, in particular, has the form

$$p(r,\theta,\varphi) \simeq f(r,\theta)\, e^{im\varphi} \tag{5.24}$$

where m is called the mode number. The governing wave equations have the form

$$\frac{\partial p}{\partial t} + u_z + v_r + \frac{v + imw}{r} = 0 \tag{5.25}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0 \tag{5.26}$$

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial r} = 0 \tag{5.27}$$

$$\frac{\partial w}{\partial t} + \frac{im}{r}\, p = 0\ . \tag{5.28}$$

These equations are derived from Euler equations of the associated flow problem and are available in reference[10] Due to cylindrical symmetry

(which one can recognize from these equations) it is sufficient to solve the problem in one plane. In this plane as in figure 5.2 the truncated boundaries are γ_1, γ_2,and γ_3 together with the axis of the pipe and the inflow boundary where a source term is to be imposed.

There are boundary conditions on the inflow boundary, axis, and on the walls of the pipe. But we shall be concerned only with the radiation condition on γ_1, γ_2,and γ_3 For rest of the details, the reader is referred to reference[10] again. When the condition (5.23) is imposed for p, the acoustic pressure, we have

$$\frac{\partial p}{\partial r} + \frac{\partial p}{\partial t} + \frac{p}{2r} = 0. \tag{5.29}$$

Suppose a point on $\gamma_i (i = 1, 2, 3)$ is at a distance $R = \sqrt{z^2 + r^2}$ from the origin and the line from the origin to the point makes an angle α with the axis. Then

$$\frac{\partial p}{\partial R} = \frac{\partial p}{\partial z} \cos \alpha + \frac{\partial p}{\partial r} \sin \alpha. \tag{5.30}$$

But from (5.26) and (5.27) we see that

$$\frac{\partial p}{\partial z} = -\frac{\partial u}{\partial t}$$
$$\frac{\partial p}{\partial r} = -\frac{\partial v}{\partial t}.$$

Thus (5.29) becomes

$$\frac{\partial p}{\partial t} - (v \cos \alpha + v \sin \alpha) + \frac{p}{R} = 0 \ . \tag{5.30}$$

This condition, together with other boundary conditions, was used to solve the system (5.25) through (5.28) by a fourth order finite difference scheme to obtain solutions reported in figure 5.32. In this figure the vertical axis measures the sound pressure levels (dB) and the horizontal axis gives angles θ where the calculations were made at a distance of 10 diameters of the pipe. For this situation when the inflow has a time dependence of the form $e^{-i\omega t}$, Weiner-Hopf solutions can be computed. We compared our

procedure with the work of Savkar and Edelfelt[17] which uses the Weiner-Hopf technique. Also, we compared the solutions with some experimental data from Ville and Silcox[19]. The comparison is shown in figure 5.3 for a wave number $k = 3.37$ and for azimuthal angular dependence of the solution $e^{i2\theta}$. Weiner-Hopf theory and the numerical solutions agree well. However, there is a discrepancy with the experimental results, especially near the axis for small values of θ. The reason is due to certain uncontrollable factors, such as plane wave and lower order mode of $e^{i\theta}$ dependence, which arise in the experimental situations. The main point of emphasis here is that the radiation condition (5.23) is a suitable condition for a practical problem.

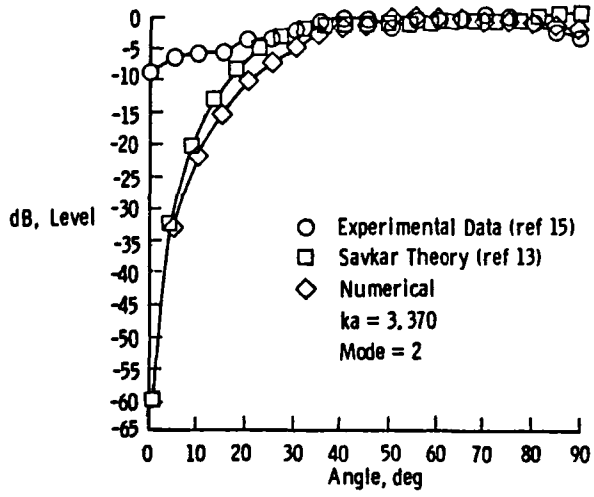

Figure 10 Comparison of results

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