Semiempirical Method of Determining Flow Coefficients for Pitot Rake Mass Flow Rate Measurements

Charles J. Trefny
Lewis Research Center
Cleveland, Ohio

November 1985
SEMIEMPIRICAL METHOD OF DETERMINING FLOW COEFFICIENTS
FOR PITOT RAKE MASS FLOW RATE MEASUREMENTS

Charles J. Trefny
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

SUMMARY

Pitot rakes are often used to measure mass flow rate in circular or annular ducts. Often the rakes are area weighted and a simple summation is used to determine the average velocity. Errors in flow rate measurement are inherent in this technique because of the discretization of the velocity profile. The error decreases as the number of tubes on the rakes increases, and resolution of the velocity profile improves. A study was conducted to determine the error in measuring mass flow rate with pitot rakes in an annulus. The ideal flow rate was determined by using a unique semiempirical analysis for fully developed, turbulent flow. The velocity profile obtained from this analysis was imposed on the pitot rake, and an area-weighted summation was used to determine the flow rate that the rake would indicate. Results in terms of flow coefficient, or the ratio of ideal to indicated flow rate, ranged from 0.903 for one probe placed at a radius dividing two equal areas to 0.984 for a 10-probe area-weighted rake. Flow coefficients were not a strong function of annulus hub-to-tip radius ratio for rakes with three or more probes.

INTRODUCTION

The measurement of airflow rate is of primary interest in the testing and development of aircraft propulsion systems. Several methods of measuring flow rate are available including choked-exit devices, venturis, and bellmouths. These methods generally require the installation of additional hardware for the specific purpose of measuring flow rate. Where installation of such hardware is impractical, other means of measuring flow rate must be devised. For inlet testing there is usually an array of total pressure rakes at the diffuser exit station to measure total pressure recovery and distortion. Such rakes are generally area weighted and may also be used to measure flow rate. Each ring (probe in each rake at a common radius) in the array is assigned an area of the circular or annular duct. If the rakes are area weighted, these are equal areas. This area combined with the measured ring total pressure, a representative static pressure, and the total temperature can be used to calculate a mass flow rate for that ring. Subsequent summation of all ring flow rates results in an estimate of the total mass flow rate. Because of the discrete nature of the measurements significant error exists in the indicated value of flow rate, especially in high velocity gradient regions such as boundary layers. Obviously the larger the number of probes on the rakes, the finer the resolution of the velocity profile and the smaller the error. To account for this discrepancy between the measured and ideal flow rates, experimentally determined flow coefficients applicable to a particular rake geometry and annulus hub-to-tip radius ratio have normally been used (ref. 1). However,
since the flow coefficient varies with the number of probes on the rakes and the hub-to-tip radius ratio of the annulus, a semiempirical procedure for determining an applicable flow coefficient would be desirable in many cases.

This investigation was conducted to determine the error in the rake mass flow rate measurement for turbulent flow in an annulus due to the discretization of the velocity profile and the variation of this error with the number of probes on an area-weighted rake. Also investigated was the effect of annulus hub-to-tip radius ratio on the error.

A semiempirical method for determining the velocity profile in fully developed, turbulent flow in an annulus is presented. The integral of this velocity profile is compared with summation of velocities as measured by an area-weighted rake subjected to the same velocity profile. Results are presented as flow coefficients, or the ratio of the integrated profile to the summed, for area-weighted rakes of up to 10 probes and a range of annulus hub-to-tip radius ratios.

SYMBOLS

\[ A_i \] area assigned to probe \( i \), m²

\[ A_n \] integration constant for region \( n \)

\[ B_n \] constant for region \( n \)

\[ C_F \] flow coefficient

\[ C_n \] constant for region \( n \)

\[ D_n \] constant for region \( n \)

\[ E_n \] constant for region \( n \)

\[ F(n) \] function of region \( n \)

\[ F(x) \] function of variable \( x \)

\( I \) number of probes on rake

\[ K_n \] dimensionless distance from origin to inner radius of region \( n \), \( r/R_t \)

\[ K'_n \] dimensionless distance from hub surface to inner radius of region \( n \), \((r - R_h)/(R_t - R_h)\)

\[ m_n \] ratio of turbulent to laminar viscosity in region \( n \)

\( N \) number of regions in semiempirical method

\( p \) static pressure gradient in flow direction, Pa/m

\( r \) distance from origin in radial direction, m
\[ R_h \] annulus hub radius, m
\[ R_t \] annulus tip radius, m
\[ u \] dimensionless distance from origin in radial direction, \( r/R_t \)
\[ v \] velocity in z-direction, m/sec
\[ V_{\text{max}} \] maximum velocity in z-direction, m/sec
\[ V_n \] velocity in z-direction at interface of regions \( n \) and \( n-1 \), m/sec
\[ \mu(\ell) \] coefficient of laminar viscosity, Pa sec/m\(^2\)
\[ \mu(t) \] coefficient of turbulent viscosity, Pa sec/m\(^2\)
\[ \tau_{rz}(\ell) \] laminar transport of z-momentum in r-direction, Pa/m\(^2\)
\[ \tau_{rz}(t) \] turbulent transport of z-momentum in r-direction, Pa/m\(^2\)

Subscripts:
\( h \) hub surface
\( i \) individual probe on rake
\( N \) outermost region in semiempirical method
\( n \) individual region in semiempirical method
\( r \) radial direction
\( t \) tip surface
\( z \) axial direction

Superscripts:
\( (\ell) \) laminar
\( (t) \) turbulent

**APPROACH**

**Derivation of Semiempirical Velocity Profiles**

To develop the turbulent velocity profiles, a semiempirical approach is used under the following assumptions:

1. Two-dimensional (axisymmetric) flow
2. Steady, fully developed flow
(3) Incompressible flow
(4) Linear pressure gradient in flow direction

Figure 1 depicts the flow situation and the coordinate system. The time-smoothed z-component of the momentum equation under the preceding assumptions becomes

\[ p = \frac{1}{r} \frac{d}{dr} \left[ r \left( \tau_{rz} + \tau_{rz}' \right) \right] \]  

Before equation (1) can be integrated to yield velocity profiles, an additional equation is needed to relate the laminar and turbulent flux of z-momentum in the radial direction to the radial velocity gradient. To develop this relation, the annular cross section is divided into an arbitrary number of annular regions \( N \). Within each region it is assumed that the negative of the radial velocity gradient is proportional to the sum of the laminar and turbulent momentum fluxes. Further the constant of proportionality in a particular region \( n \) is the sum of the laminar viscosity and a turbulent viscosity that varies from region to region. In equation form

\[ \tau_{rz} + \tau_{rz}' = - \left( \mu(t) + \nu(t) \right) \frac{dv}{dr} \]  

Expressed in terms of the ratio of turbulent to laminar viscosity, which is constant in a particular region \( n \), equation (2) becomes

\[ \tau_{rz} + \tau_{rz}' = - \mu(t) \left( 1 + m_n \right) \frac{dv}{dr} \]  

where

\[ m_n = \frac{\mu_n}{\mu(t)} \]  

Combining equations (1) and (3), there results for a region \( n \)

\[ p = - \frac{1}{r} \frac{d}{dr} \left[ r \mu(t) \left( 1 + m_n \right) \frac{dv}{dr} \right] \]  

Integrating over the radial direction yields after rearrangement

\[ \frac{dv}{dr} = - \mu(t) \frac{1}{(1 + m_n)} \frac{pr + A_n}{\frac{1}{2} + \frac{r}{2}} \]  

where \( A_n \) is the integration constant for region \( n \). Defining \( K_n \) as the ratio of the inner boundary radius of region \( n \) to the annulus tip radius \( R_t \) and defining \( V_n \) as the velocity at the inner boundary radius of region \( n \),
equation (6) can be integrated from the inner boundary of region \( n \) to an arbitrary radius within the region:

\[
\int_{V_n}^r dv = -\frac{1}{\mu(\lambda)(1 + m_n)} \int_{K_n R_t}^r \left( \frac{p r}{2} + \frac{A_n}{r} \right) dr
\]

Finally for any region \( n \)

\[
v(r) = v_n - \frac{1}{\mu(\lambda)(1 + m_n)} \left\{ \frac{pR_t^2}{4} \left[ \left( \frac{r}{R_t} \right)^2 - K_n^2 \right] + A_n \ln \left( \frac{r}{K_n R_t} \right) \right\}
\]

Note that \( K_1 \) is equal to the hub-to-tip radius ratio and \( K_{n+1} \) is equal to 1. At the interface of two adjacent regions the solutions are "spliced" together by equating the velocity gradients. Writing equation (6) for regions \( n \) and \( n + 1 \) at a common radius \( K_{n+1} \) and equating the two resulting expressions yields

\[
\frac{1}{1 + m_n} \left( \frac{pK_{n+1} R_t}{2} + \frac{A_n}{K_{n+1} R_t} \right) = \frac{1}{1 + m_{n+1}} \left( \frac{pK_{n+1} R_t}{2} + \frac{A_{n+1}}{K_{n+1} R_t} \right)
\]

Solving for \( A_{n+1} \) results in a recursion relation

\[
A_{n+1} = \left[ \frac{1 + m_{n+1}}{1 + m_n} \left( \frac{pK_{n+1} R_t}{2} + \frac{A_n}{K_{n+1} R_t} \right) - \frac{pK_{n+1} R_t}{2} \right] K_{n+1} R_t
\]

Velocities at the interfaces between regions can be computed by evaluating equation (8) at \( r = K_{n+1} R_t \):

\[
v_{n+1} = v(K_{n+1} R_t) = v_n - \frac{1}{\mu(\lambda)(1 + m_n)} \left[ \frac{pR_t^2}{4} \left( K_{n+1}^2 - K_n^2 \right) + A_n \ln \left( \frac{K_{n+1}}{K_n} \right) \right]
\]

Applying the no-slip boundary condition at the tip radius \( R \), equation (11) becomes

\[
0 = v_N - \frac{1}{\mu(\lambda)(1 + m_N)} \left[ \frac{pR_t^2}{4} \left( K_{N+1}^2 - K_N^2 \right) + A_N \ln \left( \frac{K_{N+1}}{K_N} \right) \right]
\]
Now $V_n$ can be successively replaced by using equation (11) until an equation of the form

$$V_1 = \sum_{n=1}^{N} F(n)$$

(13)

$$F(n) = \frac{1}{\mu(\xi)(1 + m_n)} \left[ \frac{pR_f^2}{4} (k_{n+1}^2 - k_n^2) + A_n \ln \left( \frac{k_{n+1}}{k_n} \right) \right]$$

results. Since $V_1$ is the velocity at the inner boundary of region 1, it is the velocity at the hub surface and is zero. Hence

$$0 = \sum_{n=1}^{N} F(n)$$

(14)

Equation (14) along with the $N - 1$ equations obtained from the recursion relation for $A_n$ (eq. (10)) can be arranged into the following form:

$$C_1 A_1 + C_2 A_2 + \ldots + C_{N-1} A_{N-1} + C_N A_N = B$$

(14)

$$E_1 A_1 - A_2 = -D_1$$

(10)

$$E_2 A_2 - A_3 = -D_2$$

(10)

$$\vdots$$

$$E_{N-1} A_{N-1} - A_N = -D_{N-1}$$

(10)

where

$$B = - \frac{pR_f^2}{4} \sum_{n=1}^{N} \frac{k_{n+1}^2 - k_n^2}{1 + m_n}$$

(15)

$$C_n = \frac{1}{1 + m_n} \ln \left( \frac{k_{n+1}}{k_n} \right)$$

(16)

$$D_n = \frac{p(K_{n+1}R_f)^2}{2} \left( \frac{1 + m_{n+1}}{1 + m_n} - 1 \right)$$

(17)

$$E_n = \frac{1 + m_{n+1}}{1 + m_n}$$

(18)
Cramer's rule can be used to solve for the integration constant in region 1 ($A_1$), after which repeated application of equation (10) yields all other $A_n$'s. The velocity profile for the entire annulus is now computed by using equation (8), beginning at the hub surface (where $V_1 = 0$) and progressing to the outer radius of region 1. At the outer radius of region 1 a solution for $V_2$ results, and equation (8) is applied again for region 2. In a similar manner the solution progresses across the annulus to the tip surface, where the no-slip boundary condition has previously been satisfied in equation (14).

Values of turbulent-to-laminar viscosity ratio $m_R$ and region boundaries, as well as the number of regions used $N$ in this investigation are depicted in figure 2. These constants were found to best fit available experimental velocity profile data over a wide range of hub-to-tip radius ratios with only the region 5 viscosity ratio $m_5$ varying with hub-to-tip radius ratio. In figure 3 nondimensional velocity profiles obtained by this procedure show good agreement with the experimental data of reference 2.

Calculation of Flow Coefficients

For the purpose of this analysis flow coefficient is defined as the ideal flow rate divided by the measured or indicated flow rate. The ideal flow rate is determined by using the semiempirical velocity profile. The indicated flow rate is determined by using an area-weighted summation of velocity values taken from the semiempirical profile at the rake probe locations. Under the assumption of incompressible flow the flow coefficient reduces to the following:

\[
C_F = \frac{1}{K_1} \frac{2 \int v(u)u \, du}{(1 - K_1^2) \sum_{i=1}^{I} A_i v_i}
\]

where $v_1$ is the velocity at probe location 1, $A_i$ is the annular area fraction associated with that probe, and $I$ is the number of probes on the rake. If the rake is area weighted, all $A_i$ are equal. Solutions to equation (19) were obtained by using the Fortran computer program listed in the appendix. The program evaluates the numerator by applying a sixth-order numerical integration technique known as Weddle's method to equation (8). Inputs to the program are annulus hub-to-tip ratio and the number of probes on the rake. A subroutine automatically locates the probes at area-weighted radii although provisions exist to input other rake geometries through an input data set. Turbulent-to-laminar viscosity ratios, region boundaries, and the number of regions used in conjunction with equation (8) are input through a separate input data set. All results presented were obtained by using the values in figure 2.
RESULTS

Figure 4 depicts the differences between the ideal velocity profile and the discretized profile that is summed in obtaining the indicated flow rate. For all probes compensating errors occur. However, probes near the hub and tip surfaces will clearly overpredict the flow rate because of the no-slip condition at these surfaces. The discretized profiles more closely approximate the ideal profile as the number of probes increases. Figure 5 presents the flow coefficients obtained with the Fortran program for area-weighted rakes with one to 10 probes over a range of annulus hub-to-tip radius ratios.

CONCLUSIONS

An investigation was conducted to determine the error in the rake mass flow rate measurement for turbulent flow in an annulus due to the discretization of the velocity profile and the variation of this error with the number of probes on an area-weighted rake. The following conclusions were drawn:

1. The semiempirical method presented for determining fully developed, turbulent velocity profiles in an annulus agreed adequately with experimental data.

2. Flow coefficients ranged from 0.903 for one probe placed at a radius dividing two equal areas to 0.984 for a 10-probe area-weighted rake.

3. Flow coefficients were not a strong function of annulus hub-to-tip radius ratio for rakes having three or more probes.
THIS PROGRAM COMPUTES SEMI-EMPIRICAL VELOCITY PROFILES FOR FULLY-DEVELOPED TURBULENT FLOW IN AN ANNULUS. THESE PROFILES ARE COMPARED TO DISCRETIZED PROFILES AS INDICATED BY A PILOT RAKE SUBJECT TO THE SEMI-EMPIRICAL FLOW FIELD. FLOW COEFFICIENTS ARE CALCULATED BY DIVIDING THE 'ACTUAL' FLOWRATE BY THE INDICATED FLOWRATE. THE 'ACTUAL' FLOWRATE IS OBTAINED BY A NUMERICAL INTEGRATION OF THE SEMI-EMPIRICAL PROFILE. THE INDICATED FLOWRATE IS OBTAINED BY AN AREA-WEIGHTED SUM OF VELOCITIES OBTAINED FROM THE SEMI-EMPIRICAL VELOCITY PROFILE AT THE RAKE PROBE LOCATIONS. RESULTS ARE OUTPUT GRAPHICALLY.

DOUBLE PRECISION CAPPACI),VISCOCI) DOUBLE PRECISION VISCOl,PRESS,RAOIUS
DOUBLE PRECISION BCI0),CCI0),DI,02,D3,DCI0),ECI0),BSUM,ATERM,ANUM,ADEN,ACl0)
DOUBLE PRECISION V(2000),VINTERCl0),RATIO,VREF,HCI0),HSUBCl0),VSUM,SUBSUMCl0),Pl,P2,P3,RV,COEFF7)

DIMENSION NINTCl0),VELC2000),RADC2000)
DIMENSION AREACl00),TLOCCl00),DISCl000),PLOCCl00),TVELCl00),PVELCl00)
DATA COEFF/l.,5.,1.,6.,1.,5.,1./
DIMENSION XVARSc100),YVARS100)
DIMENSION IVARS100)

CMMMMMINPUT REGION GEOMETRY AND FLOW VARIABLES
C
C
C
C
C

READCl5,1000) NREG
READCl5,1500) ALPHA
DO 100 I=I,NREG
READCl5,I00) CAPPACI),NINTCl),VISCOCI) CAPPACI)=ALPHA+Cl-ALPHA)MCAPPACI) CONTINUE
READCl5,1200) VISCOL READCl5,1200) PRESS READCl5,1200) RADIUS
CAPPACNREG+l) =1.

CMMMMMCOMPUTE INTEGRATION CONSTANT FOR REGION ONE
C
C
C
C

BSUM=O DO 220 I=I,NREG BSUM=BSUM-BCl)
220 CONTINUE
240 ATERM=1

CMMMMMNUMERICALLY INTEGRATE VELOCITY PROFILE FROM CAPPACl) TO 1
C
C
C
C

VSUM=O.

APPENDIX - COMPUTER PROGRAM LISTING
N\textsc{step}=N\textsc{init}(1)+N\textsc{init}(2)/2
\textsc{ratio}=\textsc{osqrt}(2\textsc{m}/\textsc{radiusmm}2)/2
\textsc{pl}=\textsc{visco}(1)+\textsc{visco}(2)
\textsc{p2}=\textsc{press} \times \textsc{radiusmm}2 / (\textsc{ratio}+2-\textsc{cappa}(1) \times 2)/4
\textsc{p3}=\textsc{ach}(1)+\textsc{dlod}(\textsc{ratio} \times \textsc{cappa}(1))
\textsc{vref}=\textsc{vinter}(n)-\textsc{pl}(\textsc{p2}+\textsc{p3})
\textsc{vavg}=\textsc{vsun}/(\textsc{vref}(1-\textsc{cappa}(1) \times 2))
\textsc{ckc20}
\textsc{pr2}(1)=\textsc{v}(1)/\textsc{vref}
\textsc{pr2}(2)=\textsc{v}(2)/\textsc{vref}
\textsc{pr2}(3)=\textsc{v}(3)/\textsc{vref}
\textsc{pr2}(4)=\textsc{v}(4)/\textsc{vref}
\textsc{pr2}(5)=\textsc{v}(5)/\textsc{vref}
\textsc{pr2}(6)=\textsc{v}(6)/\textsc{vref}
\textsc{pr2}(7)=\textsc{v}(7)/\textsc{vref}
\textsc{pr2}(8)=\textsc{v}(8)/\textsc{vref}
\textsc{pr2}(9)=\textsc{v}(9)/\textsc{vref}
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
\textsc{call} \textsc{yaxis}(1,1,\textsc{yvars})
\textsc{call} \textsc{corner}(1)
DO 700 I=1,NTUBE
READ(4,1400) TLOCC(I),AREA(I)
700 CONTINUE

CMMMMCOMPUTE POINTS OF DISCONTINUITY
C
ATOT=1-CAPPA(1)*M2
DISC(I)=CAPPA(I)

700 CONTINUE

CMMMMCOMPUTE VELOCITY AT EACH TUBE LOCATION AND DO WEIGHTED SUM
DO 800 J=1,NTUBE
DO 850 H=1,HREG
IF(CAPPA(H+1).GE.TLOCC(J))GO TO 875
850 CONTINUE
875 CONTINUE

RATIO=TLOCC(J)
PI=1./(VISCOLM1+VISCOLM2)*RATIO**2-CAPPA(H)**2)/4
P3=4.*LOG(RATIO/CAPPA(H))
VJ=(VINTER(H)-PI*(P2+P3))/VREF
TSUM=TSUM+AREA(J)*VJ

DO 900 I=1,HTUBE
II=III2 PVELCII)=VCI) PVELCII-1)=VCI)
900 CONTINUE

PLOCC(I)=CAPPA(I)
DO 950 I=1,NTUBE
II=IM2 PLOCCII)=DISC(I)+1) PLOCCII+1)=DISC(I+1)
950 CONTINUE

IVARS(2)=HPOINT
CALL GPLOT(PVEL,PLOC,IVARS)

IVARS(1)=6 IVARS(2)=HTUBE
IVARS(3)=3 IVARS(4)=62
IVARS(5)=1 IVARS(6)=20
CALL GPLOT(TVEL,TLOC,IVARS)
CALL NUMBER(4,CFLOW,6.4,PFLOW)
CALL CHARS(6,PFLOW,0.4,5,15)
CALL CHARS(12,'FLOW COEFF =',0.3,5,15)
PRINT1500.CFLOW
CALL DISPLA(1)
STOP

CMMMMFORMATS
1000 FORMAT(12)
1100 FORMAT(F6.4,2X,I2,2X,F10.5)
1200 FORMAT(D11.4)
1300 FORMAT(13)
1400 FORMAT(F10.5,2X,F10.5)
1500 FORMAT(F10.5)
C
END
REFERENCES


Figure 1. - Annulus nomenclature and coordinate system.

Figure 2. - Annular region geometry and turbulent-to-laminar viscosity ratios used in semilempirical scheme.
Figure 3. - Comparison of semiempirical velocity profiles with experimental data of reference 2.

Figure 4. - Comparison of discretized velocity profiles with "actual" profiles for five- and nine-probe rakes.
Figure 5. Flow coefficient as function of number of probes on area-weighted rake.
Flow coefficients applicable to area-weighted pitot rake mass flow rate measurements are presented for fully developed, turbulent flow in an annulus. A turbulent velocity profile is generated semiempirically for a given annulus hub-to-tip radius ratio and integrated numerically to determine the ideal mass flow rate. The calculated velocities at each probe location are then summed, and the flow rate as indicated by the rake is obtained. The flow coefficient to be used with the particular rake geometry is subsequently obtained by dividing the ideal flow rate by the rake-indicated flow rate. Flow coefficients ranged from 0.903 for one probe placed at a radius dividing two equal areas to 0.984 for a 10-probe area-weighted rake. Flow coefficients were not a strong function of annulus hub-to-tip radius ratio for rakes with three or more probes. The semiempirical method used to generate the turbulent velocity profiles is described in detail.
End of Document