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March 1986

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Bethlehem, Pennsylvania

THE NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GRANT NGR 39 007 011
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ABSTRACT

In this paper a partially reinforced cylinder containing an axial through crack is considered. The reinforcement is assumed to be fully bonded to the main cylinder. The composite cylinder is thus modelled by a nonhomogeneous shell having a step change in the elastic properties at the z=0 plane, z being the axial coordinate. Using a Reissner type transverse shear theory the problem is reduced to a pair of singular integral equations. In the special case of a crack tip touching the bimaterial interface it is shown that the dominant parts of the kernels of the integral equations associated with both membrane loading and bending of the shell reduce to the generalized Cauchy kernel obtained for the corresponding plane stress case. The integral equations are solved and the stress intensity factors are given for various crack and shell dimensions. A bonded fiberglass reinforcement which may serve as a crack arrester is used as an example.

1. Introduction

In studying the failure of structures for the purpose of calculating the fracture mechanics parameters, a very large variety of structural components may locally be modelled as relatively thin-walled plates or shells. Nearly all "pressure boundaries" and piping as well as some important parts of aerospace and hydrospace structures may be cited as examples of such components. From a viewpoint of structural integrity two of the important questions one may be concerned with in this respect are the life estimate based on the subcritical growth of an existing flaw and the residual strength or the load carrying capacity of the structure based on the criticality of a dominant flaw. In most cases the two questions may be adequately dealt with by idealizing the component with a plate or a shell and the flaw with a

(*) This study was supported by NSF under the Grant MEA-8414477 and by NASA-Langley under the Grant NGR-39-007-011.

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part-through or a through crack, by calculating the appropriate fracture mechanics parameter (e.g., the stress intensity factor), and by applying a suitable criterion along with the baseline characterization of the material.

In relatively thin-walled structures the inherently three-dimensional crack problem is approximated by a "plate" or a "shell" problem, that is, by suppressing the thickness coordinate through the use of a plate or a shell theory. In the earlier studies of the subject the classical plate and shell theories were used to solve the problem (see, for example, [1] for review). However, particularly in problems requiring the calculation of the stress intensity factors, the necessity of using a higher order theory has now been well-established. For example, it has been shown that by using a Reissner type transverse shear theory [2] (and hence by satisfying the boundary conditions on the crack surfaces for all stress and moment resultants separately) one could obtain an asymptotic stress state around the crack tips which is identical to that given by the in-plane and anti-plane elasticity solutions (see, for example, [3], [4] and [5]). Furthermore, it has also been shown that in the limiting case of small crack lengths these plate and shell results approach that of plane elasticity not only for the internal but also for the edge cracks [6], [7], [8]. Other results obtained by using a transverse shear theory for various crack-shell geometries and loading conditions may be found in [9]-[12].

With the exception of [7], [8] and [13], in all crack studies in shells that appeared in literature the shell is assumed to be "infinite" in the sense that the interaction of the perturbation field of the crack with the boundaries of or with other geometric discontinuities in the shell are assumed to be negligible. The interaction of the stress field around the crack with a stress-free boundary and with a fully clamped boundary in a cylindrical shell was considered in [7] and [8], respectively, where the special case of the crack intersecting the boundary has also been studied. In [13] the interaction of a crack field with a circumferential line stiffener in a cylindrical shell was studied by using the classical shell theory. In this paper we consider the somewhat more general problem of a nonhomogeneous cylindrical shell containing an axial crack. The problem studied is that of two relatively long cylinders having the same radii and different mechanical properties that are joined along their boundaries at a plane perpendicular to their
common axis (Fig. 1b). The model may be used to simulate composite cylinders, cylinders with reinforcing layers, and homogeneous cylinders having a step change in thickness (Fig. 1a and b). Such solutions are needed or may be very useful in, for example, crack arrest studies in pipes and containers with reinforcements.

2. General Formulation of the Problem

The analytical problem under consideration is described in Fig. 1b. The actual problem may arise, for example, from a reinforced shell shown in Fig. 1a. Since "shell theory" is used in formulating the problem, whatever the actual configuration and composition of the medium for \( \kappa_2 > 0 \), it has to be reduced to a homogeneous shell having the same radii as the semi-infinite cylinder occupying \( \kappa_2 < 0 \). The first step in the solution of the problem is, therefore, the determination of the elastic properties \( E_2, \nu_2 \) of an ("infinitely" long) equivalent shell of thickness \( h \) in terms of \( E_1, \nu_1, E', \nu', h, h_2 \) and \( R_1 \) where \( E' \) and \( \nu' \) may or may not be the same as \( E_1 \) and \( \nu_1 \). This step is briefly described in Appendix B where the composite cylinder is assumed to consist of fully-bonded thick-walled cylinders under axisymmetric plane strain condition and the equivalency of the radial displacements and axial strains is used to determine \( E_2 \) and \( \nu_2 \).

Details of the formulation of a homogeneous shallow shell containing a through crack by using a Reissner type transverse shear theory may be found, for example, in [9]-[11] and will not be repeated here. Thus, referring to Appendix A for the definition of normalized and dimensionless quantities, in terms of the displacement \( w \), stress function \( \phi \) and the auxiliary functions \( \psi \) and \( \Omega \), the basic equations for the nonhomogeneous cylindrical shell shown in Fig. 1b may be expressed as follows:

\[
\begin{align*}
\nabla^2 \phi - (\lambda_i/\mu_i) \frac{2}{\alpha y} \frac{\partial^2 w}{\partial y^2} &= 0, \quad (i=1, y<0; \ i=2, y>0) \\
\n(1) \\
\n\nabla^2 w + \lambda_i \mu_i \frac{2}{(1-\kappa_1 \nu_2)} \frac{\partial^2 \phi}{\partial y^2} &= 0, \quad (i=1, y<0; \ i=2, y>0) \\
\n(2) \\
\kappa_i \frac{\partial^2 \psi - \psi - w}{\partial y^2} &= 0, \quad (i=1, y<0; \ i=2, y>0) \\
\n(3) \\
\kappa_i (1-\nu_i) \frac{2}{\alpha y^2} \frac{\partial^2 \Omega - \Omega}{\partial y^2} &= 0, \quad (i=1, y<0; \ i=2, y>0) \\
\n(4)
\end{align*}
\]
where the constants $\lambda_i$, $\nu_i$ and $\kappa_i$ ($i=1,2$) are defined in Appendix A, $\nu_i$ is the Poisson's ratio, it is assumed that the cylindrical surfaces of the composite shell are free of tractions and the crack surface stress and moment resultants are the only nonzero external loads. The functions $\psi$ and $\Omega$ are related to the rotations as follows:

$$
\beta_x = \frac{\partial \psi}{\partial x} + \frac{\kappa_i (1-\nu_i)}{2} \frac{\partial \Omega}{\partial y}, \quad \beta_y = \frac{\partial \psi}{\partial y} - \frac{\kappa_i (1-\nu_i)}{2} \frac{\partial \Omega}{\partial x},
$$

\((i=1, y<0; i=2, y>0)\) \quad (5)

The normalized membrane, moment and transverse shear resultants are given by

$$
N_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad N_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y},
$$

\((6)\)

$$
M_{xx} = \frac{a}{h_{\mu_i}} \left( \frac{\partial \beta_y}{\partial x} + \nu_i \frac{\partial \beta_y}{\partial y} \right), \quad M_{yy} = \frac{a}{h_{\mu_i}} \left( \nu_i \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y} \right), \quad M_{xy} = \frac{a}{h_{\mu_i}} \left( \frac{1-\nu_i}{2} \left( \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \right) \right), (i=1, y<0; i=2, y>0),
$$

\((7)\)

$$
V_x = \frac{\partial w}{\partial x} + \beta_x, \quad V_y = \frac{\partial w}{\partial y} + \beta_y.
$$

\((8)\)

Eliminating $\phi$, from (1) and (2) it may be shown that

$$
\nu_i \nu_i w + \lambda_i^4 (1-\kappa_i \nu^2) \frac{\partial^2 w}{\partial y^2} = 0, (i=1, y<0; i=2, y>0).
$$

\((9)\)

If we express the solution of (9) by

$$
w(x,y) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(x,\alpha)e^{-i\alpha y}d\alpha + \frac{2}{\pi} \int_{0}^{\infty} f_2(y,\beta)\cos \beta x d\beta, (y<0), \\
\frac{2}{\pi} \int_{0}^{\infty} f_3(y,\beta)\cos \beta x d\beta, (y>0),
\end{array} \right.
$$

\((10)\)

and assume the solution of the ordinary differential equations resulting from (9) and (10) of the form
\[ f_1(x, \alpha) = R_1(\alpha)e^{m_1x} , \quad f_2(y, \beta) = R_2(\beta)e^{m_2y} , \]
\[ f_3(y, \beta) = R_3(\beta)e^{m_3y} , \quad (11) \]

The characteristic equations giving \( m_1, m_2 \) and \( m_3 \) may be obtained as follows:
\[ m_1 - 4\alpha^2m_1 + 6\alpha^4m_1^4 - (4\alpha^2+\kappa_1\lambda_1^4)\alpha^4m_1^2 + \alpha^4[\alpha^4 + \lambda_1^4(1+\kappa_1\alpha^2)] = 0 , \quad (12) \]
\[ m_2 - 4\beta^2 + \kappa_1\lambda_1^4)m_2^6 + (6\beta^4+\kappa_1\lambda_1^2\beta^2 + \lambda_1^4)m_2^4 - 4\beta^6m_2^2 + \beta^8 = 0 , \quad (13) \]
\[ m_3 - 4\beta^2 + \kappa_2\lambda_2^4)m_3^6 + (6\beta^4+\kappa_2\lambda_2^2\beta^2 + \lambda_2^4)m_3^4 - 4\beta^6m_3^2 + \beta^8 = 0 . \quad (14) \]

We designate the roots of (12)-(14) by \( m_{ij} \), \( i=1,2,3, \ j=1,...,8 \) and note that, properly ordered, they have the following properties:
\[ \text{Re}(m_{ij})<0, \quad m_{ij+4} = -m_{ij} , (i=1,2,3, j=1,2,3,4) . \quad (15) \]

Assuming that the composite shell shown in Fig. 1b is loaded symmetrically, it is sufficient to consider the problem for \( x>0 \) only. Also, since the external loads acting on the crack surfaces are statically self-equilibrating, the functions \( f_1, f_2 \) and \( f_3 \) must vanish at infinity and may, therefore be expressed as
\[ f_1(x, \alpha) = \sum_{j=1}^{4} R_{1j}(\alpha)e^{m_{ij}x} , \quad x>0 , \quad (16) \]
\[ f_2(y, \beta) = \sum_{j=5}^{8} R_{2j}(\beta)e^{m_{2j}y} , \quad y<0 , \quad (17) \]
\[ f_3(y, \beta) = \sum_{j=1}^{4} R_{3j}(\beta)e^{m_{3j}y} , \quad y>0 , \quad (18) \]

Similarly, if we let
\[
\phi(x, y) = \begin{cases} 
\frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(x, \alpha) e^{-i\alpha y} d\alpha + \frac{2}{\pi} \int_{0}^{\infty} g_2(y, \beta) \cos \beta x d\beta, & (y < 0) \\
\frac{2}{\pi} \int_{0}^{\infty} g_3(y, \beta) \cos \beta x d\beta, & (y > 0)
\end{cases}
\]

(19)

from (1), (2), (10) and (16)-(18) it may be shown that [10]

\[
g_1(x, \alpha) = -\frac{\lambda_1^2}{\mu_1} \alpha^2 \sum_{j=1}^{4} \frac{R_1j(\alpha)}{p_1j} e^{m_1jx}, \quad (x > 0),
\]

(20)

\[
g_2(y, \beta) = \frac{\lambda_2^2}{\mu_2} \delta \sum_{j=5}^{8} \frac{m_2j^2R_2j(\beta)}{p_2j^2} e^{m_2jy}, \quad (y > 0),
\]

(21)

\[
g_3(y, \beta) = \frac{\lambda_3^2}{\mu_3} \delta \sum_{j=1}^{4} \frac{m_3j^2R_3j(\beta)}{p_3j^2} e^{m_3jy}, \quad (y > 0),
\]

(22)

where

\[
p_{1j}^2 = m_{1j}^2 - \alpha^2, \quad p_{2j}^2 = m_{2j}^2 - \beta^2, \quad p_{3j}^2 = m_{3j}^2 - \beta^2.
\]

(23)

Assuming now the solution of (3) and (4) of the form

\[
\Omega(x, y) = \begin{cases} 
\frac{1}{2\pi} \int_{-\infty}^{\infty} h_1(x, \alpha) e^{-i\alpha y} d\alpha + \frac{2}{\pi} \int_{0}^{\infty} h_2(y, \beta) \sin \beta x d\beta, & (y < 0) \\
\frac{2}{\pi} \int_{0}^{\infty} h_3(y, \beta) \sin \beta x d\beta, & (y > 0)
\end{cases}
\]

(24)

\[
\psi(x, y) = \begin{cases} 
\frac{1}{2\pi} \int_{-\infty}^{\infty} k_1(x, \alpha) e^{-i\alpha y} d\alpha + \frac{2}{\pi} \int_{0}^{\infty} k_2(y, \beta) \cos \beta x d\beta, & (y < 0) \\
\frac{2}{\pi} \int_{0}^{\infty} k_3(y, \beta) \cos \beta x d\beta, & (y > 0)
\end{cases}
\]

(25)

It can be shown that
\[ h_1(x,\alpha) = A_1(\alpha) e^{r_1 x}, \quad r_1 = -[\alpha^2 + \frac{2}{\kappa_1(1-v_1)}]^{1/2}, \quad (x>0), \]

\[ h_2(y,\beta) = A_2(\beta) e^{r_2 y}, \quad r_2 = [\beta^2 + \frac{2}{\kappa_1(1-v_1)}]^{1/2}, \quad (y<0), \]

\[ h_3(y,\beta) = A_3(\beta) e^{r_3 y}, \quad r_3 = -[\beta^2 + \frac{2}{\kappa_2(1-v_2)}]^{1/2}, \quad (y>0), \]

\[ k_1(x,\alpha) = \sum_{j=1}^{4} \frac{R_{1j}(\alpha)}{\kappa_1 p_{1j}^{-1}} e^{m_{1j} x}, \quad (x>0), \]

\[ k_2(y,\beta) = \sum_{j=5}^{8} \frac{R_{2j}(\beta)}{\kappa_1 p_{2j}^{-1}} e^{m_{2j} y}, \quad (y<0), \]

\[ k_3(y,\beta) = \sum_{j=1}^{4} \frac{R_{3j}(\beta)}{\kappa_2 p_{3j}^{-1}} e^{m_{3j} y}, \quad (y>0). \]
\[ M_{xx}(0,y) = F_2(y), \quad (-d_1 < y < -b_1) , \]
\[ \beta_x(0,y) = 0, \quad (-\infty < y < -d_1, \quad -b_1 < y < 0) , \]  

where \( F_1 \) and \( F_2 \) are the crack surface tractions obtained from the solution of the uncracked shell under the given external loads. Note that for \( y > 0 \) the assumed solution has the proper symmetry and gives

\[ N_{xy}(0,y) = M_{xy}(0,y) = V_x(0,y) = \beta_x(0,y) = u(0,y) = 0, \quad (y > 0) . \]  

It is seen that once the functions \( w, \phi, \psi \) and \( \omega \) are determined, \( \beta_i, N_{ij}, M_{ij} \) and \( V_i, \) \( i,j=x,y \) may be expressed in terms of \( R_{ij} \) and \( A_i \) by using (5)-(8). To complete the formulation of the problem the displacements \( u \) and \( v \) need to be determined. This may be done by using the Hooke's law and the following kinematic relations

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \nu_{ij} \frac{\partial u_i}{\partial x_j} + \nu_{ji} \frac{\partial u_j}{\partial x_i} \right) , \quad (i,j=1,2) , \]

where the function \( Z(x_1,x_2) \) describes the middle surface of the shell. For the cylindrical shell under consideration \( Z, Z_1=0, Z_1=1/R \) and referring to Appendix A we find

\[ \frac{\partial^2 u}{\partial y^2} = 2(1+\nu_i) \frac{\partial N_{xy}}{\partial y} - \nu_{ij} \frac{\partial^2 w}{\partial y^2} + \nu_{ij} \frac{\partial N_{xx}}{\partial x} + \frac{\lambda_i^2}{\mu_{ij}} \frac{\partial^2 w}{\partial y^2} x , \]  

\[ \frac{\partial v}{\partial y} = N_{yy} - \nu_i N_{xx} , \]

where \( i=1 \) for \( y<0 \) and \( i=2 \) for \( y>0 \). From (42), (43) and the formulation given in this section it can be shown that

\[ u(x,y) = \frac{1}{2} \frac{\lambda_1^2}{\mu_1^2} \int_{-\infty}^{\infty} \frac{(2+\nu_i)\alpha^2 - m^2_{1j}}{p_{1j}^2} R_{1j}(\alpha) m_{1j} x - i\alpha y d\alpha \]
\[ + \frac{2}{\pi} \frac{\lambda_1^2}{\mu_1^2} \int_{0}^{\infty} \frac{8}{3} \frac{(2+\nu_i)m_{2j}^2 - \beta^2}{p_{2j}^2} R_{2j}(\beta) m_{2j} y \sin \beta x d\beta \]
\[ + \frac{\lambda_1^2}{\mu_1^2} x w(x,y), \quad (x>0, \quad y<0) , \]  

\[ -8- \]
\[ u(x,y) = \frac{2}{\pi} \frac{\lambda_2^2}{\mu_2^2} \int_0^\infty \sum_{j=1}^4 \frac{(2+n_2)m_{3j}^2 - \beta^2}{p_{3j}^2} R_{3j}(\beta) e^{m_{3j}y} \sin \beta x d\beta \]
\[ + \frac{\lambda_2^2}{\mu_2^2} x w(x,y), \quad (x>0, \ y<0), \quad (45) \]

\[ v(x,y) = -\frac{i}{2\pi} \frac{\lambda_1^2}{\mu_1^2} \int_{-\infty}^{\infty} \sum_{j=1}^4 \frac{\alpha}{p_{1j}} R_{1j}(\alpha)(m_{1j}^2 + n_1 \alpha^2)e^{m_{1j}x - i\alpha y} d\alpha \]
\[ -\frac{2}{\pi} \frac{\lambda_1^2}{\mu_1^2} \int_0^\infty \sum_{j=5}^8 \frac{m_{2j}}{p_{2j}^2} R_{2j}(\beta)(\beta^2 + n_2 m_{2j}^2)e^{m_{2j}y} \cos \beta x d\beta, \]
\[ (x>0, \ y<0), \quad (46) \]

\[ v(x,y) = -\frac{2}{\pi} \frac{\lambda_2^2}{\mu_2^2} \int_0^\infty \sum_{j=1}^4 \frac{m_{3j}}{p_{3j}^2} R_{3j}(\beta)(\beta^2 + n_2 m_{3j}^2)e^{m_{3j}y} \cos \beta x d\beta, \]
\[ (x>0, \ y>0). \quad (47) \]

3. The Integral Equations

Referring to the general formulation of the problem given in the previous section it is seen that the first thirteen conditions (33)-(37) are homogeneous and may be used to eliminate thirteen of the fifteen unknown functions \( R_{ij} \) and \( A_k \) (\( i=1,2,3; j=1,...,4; k=1,2,3 \)). The mixed boundary conditions (38) and (39) would then give the integral equations to determine the remaining two. To derive the integral equations we first introduce the following new unknown functions which are the complements of the known crack surface tractions \( F_1 \) and \( F_2 \):

\[ G_1(y) = \frac{3}{8y} u(+0,y), \quad G_2(y) = \frac{3}{8y} \beta_x(+0,y), \quad (y<0). \quad (48) \]

From the solution given in the previous section and from (48) it may be shown that

\[ G_1(y) = -\frac{i}{2\pi} \frac{\lambda_1^2}{\mu_1^2} \int_{-\infty}^{\infty} \sum_{j=1}^4 \frac{(1+n_1)\alpha^2 - p_{1j}}{p_{1j}^2} R_{1j}(\alpha)e^{-i\alpha y} d\alpha, \quad (49) \]
\[
G_2(y) = -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\alpha \sum_{j=1}^{4} \frac{\alpha m_{1j} R_{1j}(\alpha)}{\kappa_1 p_{1j}^{-1}} e^{-\alpha y} d\alpha \\
- \frac{\kappa_1 (1-\nu_1)}{4\pi} \int_{-\infty}^{\infty} \alpha^2 A_1(\alpha) e^{-\alpha y} d\alpha .
\] (50)

By inverting (49) and (50) and by using (33)-(37) one may express all fifteen unknown functions \( R_{ij} \) and \( A_k \) in terms of \( G_1 \) and \( G_2 \). From (38b), (39b) and (48) it may be observed that

\[
G_i(y) = 0, \quad (i=1,2; \quad -\infty < y < -d_1, \quad -b_1 < y < 0) ,
\] (51)

and (38b) and (39b) would be identically satisfied if

\[
\int_{-d_1}^{-b_1} G_i(y) dy = 0, \quad (i=1,2) .
\] (52)

From the formulation of the problem the conditions (38a) and (39a) may be expressed as

\[
N_{xx}(+0,y) = \lim_{x \to +0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\lambda_1}{\mu_1} (\frac{1}{2\pi})^2 \sum_{j=1}^{4} \frac{\alpha}{p_{1j}} R_{1j}(\alpha) e^{m_{1j} x-i\alpha y} d\alpha \\
+ 2 \frac{\lambda_1}{\mu_1} (\frac{1}{2\pi})^2 \sum_{j=5}^{8} \frac{m_{2j}}{p_{2j}} R_{2j}(\beta) e^{m_{2j} y} \cos\beta x d\beta = F_1(y) , \quad (-d_1 < y < -b_1) .
\] (53)

\[
M_{xx}(+0,y) = \lim_{x \to +0} \frac{a}{\mu_1} \int_{-\infty}^{\infty} \frac{\lambda_1}{\mu_1} (\frac{1}{2\pi})^2 \sum_{j=1}^{4} \frac{m_{1j}^{-\nu_1} \alpha^2}{\kappa_1 p_{1j}^{-1}} R_{1j}(\alpha) e^{m_{1j} x-i\alpha y} d\alpha \\
- \frac{i \kappa_1 (1-\nu_1)^2}{4\pi} \int_{0}^{\infty} \alpha r_1 A_1(\alpha) e^{r_1 x-i\alpha y} d\alpha - \frac{2}{\pi} \sum_{j=5}^{8} \frac{\beta^{2-\nu_1} m_{2j}^2}{\kappa_1 p_{2j}^{-1}} R_{2j}(\beta) e^{m_{2j} y} \\
- \frac{\kappa_1 (1-\nu_1)^2}{2} \beta r_2 A_2(\beta) e^{r_2 y} \cos \beta x d\beta = F_2(y) , \quad (-d_1 < y < -b_1) .
\] (54)
If we now write $R_{ij}$ and $A_j$ in terms of $G_1$ and $G_2$ and take into account (51), (53) and (54) become

$$\sum_{j=1}^{2} H_{kj}(y,t)G_j(t)dt = F_k(y), \quad (k=1,2, \ -d_1 < y < -b_1),$$

where $B_{kj}$ and $C_{kj}, (k,j=1,2)$ are known functions. The derivation of these functions are rather lengthy but quite straightforward and will not be reproduced in this paper. From the viewpoint of obtaining the correct singular behavior of $G_1$ and $G_2$ near and at the end points $y=-d_1$ and $y=-b_1$ and a sufficiently accurate solution of the system of integral equations (55), it is essential that the dominant parts of the kernels given by (56) be separated. This can be done by examining the asymptotic behavior of $B_{kj}(x,\alpha)$ and $C_{kj}(y,\beta), (k,j=1,2)$, for $|\alpha| \to \infty$ and $\beta \to \infty$, respectively, by separating the asymptotic terms $B_{kj_{\infty}}$ and $C_{kj_{\infty}}$ and by evaluating the corresponding integrals in closed form. A key step in this process is the determination of the roots of the characteristic equations (12)-(14) (see, for example, [10]). It may then be shown that for large values of $|\alpha|$ and $\beta$ the roots have the form

$$m_{1j} = -|\alpha|(1 + \frac{p_{1j}}{2\alpha^2} - \frac{p_{1j}^2}{8\alpha^4} + \ldots), \quad (j=1,\ldots,4),$$

$$m_{2j} = \beta(1 + \frac{p_{2j}}{2\beta^2} - \frac{p_{2j}^2}{8\beta^4} + \ldots), \quad (j=5,\ldots,8),$$

$$m_{3j} = -\beta(1 + \frac{p_{3j}}{2\beta^2} - \frac{p_{3j}^2}{8\beta^4} + \ldots), \quad (j=1,\ldots,4).$$

Also, from (26)-(28) it may be seen that for $|\alpha| \gg 1$ and $\beta \gg 1$ we have

$$r_1 = -|\alpha|(1 + \frac{1}{\kappa_1(1-\nu_1)\alpha^2} - \ldots), \quad r_2 = \beta(1 + \frac{1}{\kappa_1(1-\nu_1)\beta^2} - \ldots),$$

$$r_3 = -\beta(1 + \frac{1}{\kappa_2(1-\nu_2)\beta^2} - \ldots).$$

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After evaluating the dominant kernels the integral equations (55) may be expressed as

\[
\begin{align*}
\int_{-d_1}^{-b_1} & \left\{ k_{1s}(y,t) + k_{11}(y,t) \right\} G_1(t) + k_{12}(y,t) G_2(t) \, dt = 2\pi F_1(y), \quad (-d_1 < y < -b_1), \\
\int_{-d_1}^{-b_1} & \left( 1 - v_1^2 \right) \left\{ k_{21}(y,t) G_1(t) + [k_{2s}(y,t) + k_{22}(y,t)] G_2(t) \right\} \, dt \\
& = 2\pi \frac{h_{14}^u}{a} F_2(y), \quad (-d_1 < y < -b_1),
\end{align*}
\]  

(61)

where

\[
\begin{align*}
k_{1s}(y,t) & = k_{2s}(y,t) = \frac{1}{t-y} - \frac{c_1(3c_3 + 2c_4)}{c_1c_5} \frac{1}{t+y} + \frac{6c_3}{c_5} \frac{y}{(t+y)^2} \\
& - \frac{4c_3}{c_6} \frac{y^2}{(t+y)^3}
\end{align*}
\]  

(63)

and the kernels \( k_{ij}(y,t), (i,j=1,2) \) are bounded in the closed interval \(-d_1 \leq (y,t) \leq -b_1 \) (including the case \( b_1 = 0 \)). The dominant or singular kernels \( k_{is} \) and the bounded kernels \( k_{ij}, (i,j=1,2) \) are obtained from the integrals of the form:

\[
k_{1s}(y,t) = \lim_{x \to +0} \left( \int_{-\infty}^{\infty} B_{1s}(x,\alpha)e^{i(t-y)\alpha} \, d\alpha + \int_{0}^{\infty} C_{1s}(y,\beta) \cos \beta x \, d\beta \right),
\]  

(65)

\[
k_{1j}(y,t) = \lim_{x \to +0} \left( \int_{-\infty}^{\infty} [B_{1j}(0,\alpha) - B_{1j}(0,\alpha)]e^{i(t-y)\alpha} \, d\alpha \\
+ \int_{0}^{\infty} [C_{1j}(y,\beta) - C_{1j}(y,\beta)] \, d\beta \right), \quad (j=1,2).
\]  

(66)

-12-
The important point to observe about the integral equations is that, as in the homogeneous shells, only the diagonal kernels contain singular terms and the dominant kernels \( k_{1s} \) and \( k_{2s} \) corresponding to membrane and bending loads are identical. This physically expected result is possible again because of the use of a transverse shear theory in formulating the problem. Also, it can easily be shown that the dominant kernel \( k_{1s} \) found for the nonhomogeneous shell in this study is identical to that obtained for two dissimilar bonded half planes with a crack perpendicular to the interface under plane stress conditions which is given in [14]. The plane stress problem is, of course, a limiting case of the shell problem and this, too, is the expected result.

4. The Stress Intensity Factor

After solving the integral equations, clearly any desired field quantity may be obtained from integrals with appropriate kernels having \( G_1 \) and \( G_2 \) as the density functions. For example, it may be observed that before going to the limit, (53) and (54) give \( N_{xx}(x,y) \) and \( M_{xx}(x,y) \) everywhere in the shell. In particular we note that (55) or (61) and (62) are valid for \( x=0 \) outside as well as within the cut \(-d_1<y<-b_1\). Thus, through a simple asymptotic analysis of (61) and (62) one can obtain the stress intensity factors at the crack tips which, for \( b>0 \), are defined by (Fig. 1b)

\[
k_1(-b,x_3) = \lim_{x_2 \to b+0} \sqrt{2(x_2+b)} \sigma_{11}(0,x_2,x_3), \quad (67)
\]

\[
k_1(-d,x_3) = \lim_{x_2 \to d-0} \sqrt{-2(x_2+d)} \sigma_{11}(0,x_2,x_3). \quad (68)
\]

In the shells the in-plane stress components are obtained by combining membrane and bending stresses as follows:

\[
\sigma_{ij}(x_1,x_2,x_3) = \sigma_{ij}^m + \sigma_{ij}^b, \quad (i,j=1,2)
\]

\[
\sigma_{ij}^m = \frac{1}{h} N_{ij}(x_1,x_2), \quad \sigma_{ij}^b = \frac{12x_3}{h^3} M_{ij}(x_1,x_2), \quad (i,j=1,2).
\]
Now for $b>0$ from (61)-(63) it is seen that the dominant kernels $k_{1s}$ and $k_{2s}$ consist of $(t-y)^{-1}$ only and hence the solution of the integral equations may be expressed as [15]

$$G_j(y) = \frac{P_j(y)}{[-(y+d_1)(y+b_1)]^{\frac{1}{2}}} , \quad (j=1,2; -d_1<y<-b_1) \tag{71}$$

where $P_1$ and $P_2$ are unknown bounded functions. By substituting from Appendix A and (71) into (61) and (62) and using (67)-(70), it can be shown that

$$k_1(-b,x_3) = -\frac{E_1}{2} \lim_{x_2 \rightarrow -b-0} \sqrt{2(x_2+b)} \frac{3}{3x_2} [u_1(+0,x_2)+x_2\delta_{11}(+0,x_2)]$$

$$= -\frac{E_1}{2} \sqrt{a} [P_1(-b_1) + \frac{x_3}{a} P_2(-b_1)] , \tag{72}$$

$$k_1(-d,x_3) = \frac{E_1}{2} \lim_{x_2 \rightarrow -d+0} \sqrt{2(x_2+d)} \frac{3}{3x_2} [u_1(+0,x_2)+x_2\delta_{11}(+0,x_2)]$$

$$= \frac{E_1}{2} \sqrt{a} [P_1(-d_1) + \frac{x_3}{a} P_2(-d_1)] . \tag{73}$$

In the case of $b=0$ from (63) it is seen that the dominant part of the kernel is a generalized Cauchy kernel, that is it contains, in addition to $(t-y)^{-1}$, terms which become unbounded as the variables $y$ and $t$ approach the end point $-b_1=0$ (Fig. 1b). The contribution of these terms to the singular behavior of the solution at $y=0$ can be studied by assuming the solution of the integral equations (61) and (62) as

$$G_j(y) = S_j(y)(-y)^\gamma(y+d_1)^\omega , \quad (-1<\text{Re}(\gamma,\omega)<0) , \tag{74}$$

and by following the function theoretic method (see, for example, [14] and [15]). Thus, by substituting from (74) into (61) and (62), the characteristic equations giving $\gamma$ and $\omega$ may be obtained as follows:

$$\cos \pi \gamma - \frac{2c_3}{c_6} \gamma(\gamma+2) - \left( \frac{3c_3+2c_4}{c_6} + \frac{2c_2c_5}{c_1c_6} \right) = 0 , \tag{75}$$
At the crack tip $y=-d_1$ which is embedded in a homogeneous medium (76) gives $\omega=-1/2$. The characteristic equation (75) found for $\gamma$ is identical to that given in [14] for the plane stress case and its examination would show that for all material combinations the equation has only one root satisfying $-1<\text{Re}(\gamma)<0$ and this root is always real. In [14] it was found that for the plane problem in the small neighborhood of the singular point $y=0$ located at the interface the stress state has the form

$$ \sigma_{ij}(r,\theta) = \frac{k_1}{\sqrt{2}} r^\gamma f_{ij}(\theta), \quad i,j=r,\theta, \quad (77) $$

where $r$ and $\theta$ are the polar coordinates and the functions $f_{ij}$ are dependent on the bimaterial constants $E_k, \nu_k, (k=1,2)$ and are given in [14]. The constant $k_1$ is again defined as the stress intensity factor and is obtained from the calculated values of $\sigma_{\theta\theta}(r,0)$ by normalizing $f_{\theta\theta}(0)=1$. Thus, in the present shell problem the stress intensity factor at the crack tip $y=0$ may be defined as

$$ k_1(0,x_3) = \lim_{x_2 \to 0} \sqrt{2} x_2^{-\gamma} \sigma_{11}(0,x_2,x_3). \quad (78) $$

To evaluate the stress intensity factor $k_1(0,x_3)$ the asymptotic expression of $\sigma_{11}$ for $x_2>0$ is needed. This can again be obtained in terms of $G_1$ and $G_2$ from the basic formulation of the shell given in this paper. After somewhat lengthy but straightforward analysis it may be shown that

$$ k_1(0,x_3) = - \frac{E_2}{\sqrt{2}} \left( \frac{3c_6+c_1+2\gamma(c_1+c_6)}{c_1 c_6 \sin \pi \gamma} \right) \lim_{x_2 \to 0} (-x_2)^{-\gamma} \frac{3}{3x_2} [u_1(+0,x_2) $$

$$ + x_3 \beta_{11}(+0,x_2)] $$

$$ = - E_2 a^{-\gamma} \left( \frac{3c_6+c_1+2(c_1+c_6)\gamma}{c_1 c_6 \sin \pi \gamma} \right)[S_1(0) + \frac{x_3}{a} S_2(0)], \quad (79) $$
Also, at the crack tip $x_2 = -d$ the expression

$$ k_1(-d, x_3) = \frac{E_1}{2} \lim_{x_2 \to -d+0} \frac{\sqrt{2(x_2+d)}}{\partial x_2} \left[ u_1(+0, x_2) + x_3 \delta_{11}(+0, x_2) \right] $$  \hspace{1cm} (80)

is still valid and by using (48) and (71) can be written as

$$ k_1(-d, x_3) = \frac{E_1}{2^{1-\gamma}} \sqrt{a} \left[ S_1(-d_1) + \frac{x_3}{a} S_2(-d_1) \right]. $$  \hspace{1cm} (81)

5. Results

For $b > 0$ the integral equations (61) and (62) subject to (52) may easily be solved by using the Gauss-Chebyshev quadrature formulas described in, for example, [14] by assuming the solution in the form (71). The stress intensity factors may then be obtained from (72) and (73). From (72) and (73) it is seen that the values of the bounded functions $P_1$ and $P_2$ at the end points $-d_1$ and $-b_1$ are associated with respectively the membrane and the bending components of the stresses near the crack tips. For the pressurized shell shown in Fig. 1b, one can, therefore, define the following normalized stress intensity factors:

$$ k_m(r_j) = \frac{k_1(r_j, 0)}{(pR_i/h)^{1-\gamma}} \gamma k_b(r_j) = \frac{k_1(r_j, h/2) - k_1(r_j, 0)}{(pR_i/h)^{1-\gamma}}, $$  \hspace{1cm} (82)

where $r_1 = -b$ and $r_2 = -d$ and the stress intensity factors $k_1(r_j, x_3)$, ($j = 1, 2$) are given by (72) and (73). For $b = 0$ the definition of the normalized stress intensity factors at $r_2 = -d$ would remain the same as in (82) and may be obtained from (81). At $r_1 = -b = 0$ we have

$$ k_m(0) = \frac{k_1(0, 0)}{(pR_i/h)^{1-\gamma}} \gamma k_b(0) = \frac{k_1(0, h/2) - k_1(0, 0)}{(pR_i/h)^{1-\gamma}} $$  \hspace{1cm} (83)

where $k_1$ is given by (79). In (82) and (83) $R_i$ is the inner radius of the cylinder.

The reinforced shell shown in Fig. 1a is considered as an example. The main shell having properties $E_1, \nu_1$ is steel and the reinforcing shell
is a fiber reinforced composite. The dimensions of four different cylinders used in the analysis are shown in Table 1. The table also shows the elastic constants $E_2, v_2$ of the equivalent homogeneous shell shown in Fig. 1b. The derivation of $E_2, v_2$ and the solution of the nonhomogeneous shell in the absence of any cracks giving the crack surface loads $F_1$ and $F_2$ are given in Appendix B (see, (61) and (62)). It should be noted that $F_1$ and $F_2$ are functions of $x=X^2/a$. One may also note that the effective elastic constants $E_2, v_2$ as well as $F_1, F_2$ are dependent on the axial constraint in the cylinder. In the examples it is assumed that either there is no axial constraint (labeled as a cylinder with "open ends") or the ends of the cylinder are "closed", corresponding to the total axial force in the internally pressurized cylinder $P = 0$ and $P = \pi R^2 p$, respectively (Eq. B6).

Table 1. Dimensions and the effective material constants $(E_2, v_2)$ of the composite shell used in numerical examples.

<table>
<thead>
<tr>
<th>Shell #</th>
<th>1</th>
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<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$ {psi}</td>
<td>3x10^7</td>
<td>3x10^7</td>
<td>3x10^7</td>
<td>3x10^7</td>
</tr>
<tr>
<td>E (GPa)</td>
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<td>207</td>
<td>207</td>
<td>207</td>
</tr>
<tr>
<td>$v_1$</td>
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<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>$E'_1$ (reinforcing shell) {ksi}</td>
<td>2x10^7</td>
<td>2x10^7</td>
<td>2x10^7</td>
<td>2x10^7</td>
</tr>
<tr>
<td>E (ksi)</td>
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<td>138</td>
<td>138</td>
<td>138</td>
</tr>
<tr>
<td>$v'_1$ (reinforcing shell)</td>
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<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>h (thickness) {in}</td>
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<td>0.615</td>
<td>0.5</td>
<td>0.404</td>
</tr>
<tr>
<td>h (in)</td>
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<td>0.0156</td>
<td>0.0127</td>
<td>0.0103</td>
</tr>
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<td>R (mean radius) {in}</td>
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<td>$h_2$ (reinforcing shell) {in}</td>
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<td>0.5</td>
<td>0.404</td>
</tr>
<tr>
<td>$h_2$ (in)</td>
<td>0.0254</td>
<td>0.0156</td>
<td>0.0127</td>
<td>0.0103</td>
</tr>
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<td>R/h</td>
<td>100</td>
<td>38.524</td>
<td>35.5</td>
<td>29.203</td>
</tr>
</tbody>
</table>

Open Ends:

| $E_2$ {psi} | 50.2x10^6 | 49.72x10^6 | 49.66x10^6 | 49.48x10^6 |
| E (ksi)     | 346       | 343       | 342       | 341       |
| $v_2$       | 0.2214    | 0.2174    | 0.2168    | 0.2153    |
| $\gamma_2$ | 0.449518  | 0.450677  | 0.450827  | 0.451265  |

Closed Ends:

| $E_2$ {psi} | 50.06x10^6 | 49.37x10^6 | 49.28x10^6 | 49.03x10^6 |
| E (ksi)     | 345       | 340       | 340       | 338       |
| $v_2$       | 0.2272    | 0.2321    | 0.2327    | 0.2345    |
| $\gamma_2$ | 0.440538  | 0.450728  | 0.450886  | 0.451323  |
For these two cases the values of $E_2$ and $v_2$ calculated from Appendix B are given in Table 1.

The stress intensity factors obtained by using dimensions and properties of four different shells shown in Table 1 are given in Tables 2-5. The normalized stress intensity factors given in the tables are defined by (82) for $b>0$ (or $c>a$) and by (81) and (83) for $b=0$ (or $c=a$). One may observe that as the relative distance of the crack to the interface $c/a$ increases the stress intensity factors approach those given for the homogeneous shell [9]. The tables also show that the membrane component $k^m$ of the stress intensity factor (which is by far the dominant part) decreases with the decreasing crack distance to the boundary. Even though the results found for "open" and "closed" ended cylinders are different, for the axial through crack geometry under consideration the differences seem to be relatively insignificant. It should be emphasized that the tables show the stress intensity factors normalized with respect to $(pR_1/h)\sqrt{a}$ which is the corresponding flat plate (or plane stress) result under the same membrane loading as the shell. Thus the variation in the stress intensity factors as a function of $a/h$ and $R/h$ is entirely due to curvature and thickness effects.

In Tables 2-5 the membrane and bending components of the stress intensity factors are given separately. In all cases, the calculated bending components were such that the stress intensity factors on the outside surface $x_3=h/2$ were greater than that on the inside surface. Figures 2-5 show the stress intensity factor $k_1(c_1,h/2)$, $(c_1=-d,-b,0)$ obtained from (72), (73), (79) and (81) for some selected shell-crack geometries. The effect of reinforcement may be clearly observed from Figures 2 and 3. The asymptotic behavior of $k_1(-b,h/2)$ as $c\to a$ is due to different definitions of $k_1$ for $b>0$ and $b=0$ as given by (67) and (78). For these two cases the cleavage stress in the close neighborhood of the crack tip may be expressed as (Fig. 1b)

$$\sigma_{11}(0,x_2) = \frac{k_1(-b,h/2)}{\sqrt{2r}}, \ (r=x_2+b, \ b>0, \ x_2>-b), \ (84)$$
Thus, since $1/2+\gamma>0$ (see Table 1), the stress intensity factor defined on the basis of the conventional square root singularity becomes

$$\lim_{b\to 0} k_1(-b, h/2) = \lim_{r\to 0} \sqrt{2r} \sigma_{11}(0, x_2) = \lim_{r\to 0} k_1(0, h/2)r^{1/2+\gamma} = 0. \quad (86)$$

Figures 4 and 5 show the normalized stress intensity factors on the outside surface $x_3=h/2$ for $b=0$ obtained from (79) and (81). Here, too, it may be seen that even for a relatively very shallow shell (shell no. 1), there is considerable curvature and thickness effect.

References


References (cont.)


Table 2. Membrane component $k_m$ of the normalized stress intensity factor in a pressurized composite shell with open ends which contains an axial crack.

<table>
<thead>
<tr>
<th>Shell No.</th>
<th>$c/a$</th>
<th>$k_m(-b)$</th>
<th>$k_m(-d)$</th>
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</thead>
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</tr>
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<td></td>
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</table>
Table 3. Bending component $k_b$ of the normalized stress intensity factor in a pressurized composite shell with open ends which contains an axial crack.

| Shell No. | $c/a$ | $a/h$ | $k_m(-b)$ | | $k_m(-d)$ | |
|-----------|------|------|-----------| |-----------| |
|           | 1.0  | 1.1  | 1.5       | 2    | 10       | 1.0  | 1.1  | 1.5  | 2    | 10       |
| 1         | 0.005| 0.002| 0.008     | 0.014| 0.039    | 0.015| 0.016| 0.020| 0.024| 0.037    |
| 2         | 0.047| 0.021| 0.035     | 0.044| 0.046    | 0.042| 0.044| 0.050| 0.055| 0.043    |
| 3         | 0.087| 0.042| 0.062     | 0.075| 0.064    | 0.069| 0.072| 0.078| 0.082| 0.064    |
| 10        | 0.225| 0.134| 0.202     | 0.212| 0.197    | 0.191| 0.196| 0.204| 0.203| 0.198    |
| 2         | 0.033| 0.015| 0.026     | 0.034| 0.043    | 0.043| 0.035| 0.041| 0.046| 0.039    |
| 2         | 0.101| 0.046| 0.069     | 0.081| 0.071    | 0.076| 0.079| 0.085| 0.089| 0.070    |
| 3         | 0.154| 0.074| 0.108     | 0.123| 0.110    | 0.111| 0.114| 0.121| 0.123| 0.110    |
| 10        | 0.164| 0.124| 0.211     | 0.207| 0.198    | 0.200| 0.202| 0.207| 0.203| 0.198    |
| 3         | 0.037| 0.017| 0.027     | 0.036| 0.044    | 0.036| 0.038| 0.043| 0.047| 0.040    |
| 2         | 0.107| 0.050| 0.080     | 0.086| 0.074    | 0.079| 0.082| 0.089| 0.092| 0.074    |
| 3         | 0.161| 0.078| 0.117     | 0.128| 0.114    | 0.115| 0.119| 0.125| 0.127| 0.114    |
| 10        | 0.148| 0.108| 0.197     | 0.200| 0.192    | 0.194| 0.197| 0.201| 0.197| 0.193    |
| 4         | 0.046| 0.021| 0.032     | 0.043| 0.047    | 0.041| 0.043| 0.049| 0.053| 0.042    |
| 2         | 0.120| 0.055| 0.082     | 0.096| 0.083    | 0.088| 0.091| 0.097| 0.099| 0.082    |
| 3         | 0.176| 0.086| 0.124     | 0.140| 0.126    | 0.126| 0.129| 0.135| 0.136| 0.125    |
| 10        | 0.102| 0.099| 0.186     | 0.172| 0.172    | 0.178| 0.179| 0.181| 0.177| 0.172    |
Table 4. Membrane component $k_m$ of the normalized stress intensity factor in a pressurized composite shell with closed ends which contains an axial crack.

<table>
<thead>
<tr>
<th>Shell No.</th>
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Table 5. Bending component $k_b$ of the normalized stress intensity factor in a pressurized composite shell with closed ends which contains an axial crack.

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<th>$k_b$ (-d)</th>
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APPENDIX A
Normalized quantities used in the formulation of the composite shell problem

\[ x = x_1/a, \ y = x_2/a, \ z = x_3/a, \ b_1 = b/a, \ c_1 = c/a, \ d_1 = d/a, \]  
\[ u = u_1/a, \ v = u_2/a, \ w = u_3/a, \ \beta_x = \beta_1, \ \beta_y = \beta_2, \]

\[ \phi(x,y) = \begin{cases} 
F(x_1,x_2)/(E_1a^2), & y < 0, \\
F(x_1,x_2)/(E_2a^2), & y > 0, 
\end{cases} \]  
\[ \sigma_{\alpha\beta}(x,y) = \begin{cases} 
\sigma_{ij}/E_1, & y < 0, \\
\sigma_{ij}/E_2, & y > 0, 
\end{cases} \]  
\[ N_{\alpha\beta}(x,y) = \begin{cases} 
N_{ij}/(E_1h), & y < 0, \\
N_{ij}/(E_2h), & y > 0, 
\end{cases} \]  
\[ M_{\alpha\beta}(x,y) = \begin{cases} 
M_{ij}/(E_1h^2), & y < 0, \\
M_{ij}/(E_2h^2), & y > 0, 
\end{cases} \]  
\[ V_\gamma(x,y) = \begin{cases} 
V_k/(B_1h), & y < 0, \\
V_k/(B_2h), & y > 0, 
\end{cases} \]  
\[ B_i = 5E_i/(12(1+\nu_i)), \ \kappa_i = E_i/(B_i\mu_i^4), \ (i=1,2), \]  
\[ \mu_i^4 = 12(1-\nu_i^2)a^2/h^2, \ \lambda_i^4 = 12(1-\nu_i^2)a^4/(R^2h^2), \ (i=1,2). \]
APPENDIX B
Solution of the uncracked composite shell problem

The first problem here is the determination of elastic constants $E_2$ and $v_2$ of a cylindrical shell which has the same stiffness as a long layered cylinder (see Fig. 1a and 1b for $x_2 > 0$). The second problem is the solution of the bonded shells shown in Fig. 1b in the absence of any cracks in order to determine the crack surface tractions $F_1(y) = N_{11}(0,x_2)/(hE_1)$ and $F_2(y) = M_{11}(0,x_2)/(h^2E_1)$ which are used in the perturbation solution of the cracked shell as the input functions. In the first problem we use the following basic (axisymmetric) solution of a thick-walled cylinder under plane strain conditions:

$$\sigma_{rr}(r) = A + \frac{B}{r^2}, \quad \sigma_{ \theta \theta}(r) = A - \frac{B}{r^2},$$

$$w(r) = \frac{1-v-2v^2}{E} Ar - \frac{1+v}{E} B r - \nu \varepsilon_{zz} r, \quad (a<r<b), \quad (B1a-c)$$

where $A$ and $B$ are unknown constants and $w$ is the radial displacement. For the three cylinders we have (Fig. 1)

$$a_1 = R_i, \quad b_1 = R_i + h, \quad A = A_1, \quad B = B_1, \quad \nu = \nu_1, \quad E = E_1, \quad (B2)$$

$$a_2 = R_i + h, \quad b_2 = R_i + h + h_2, \quad A = A_2, \quad B = B_2, \quad \nu = \nu', \quad E = E', \quad (B3)$$

$$a_3 = R_i, \quad b_3 = R_i + h, \quad A = A_3, \quad B = B_3, \quad \nu = \nu_2, \quad E = E_2, \quad (B4)$$

where the dimensions $R_i$, $h$, $h_2$ and the elastic constants $E_1$, $\nu_1$, $E'$ and $\nu'$ are known. The boundary and continuity conditions

$$\sigma_{1 rr}(a_1) = -p, \quad \sigma_{2 rr}(b_2) = 0, \quad \sigma_{1 rr}(b_1) = \sigma_{2 rr}(a_2),$$

$$w_1(b_1) = w_2(a_2), \quad \sigma_{3 rr}(a_3) = -p, \quad \sigma_{3 rr}(b_3) = 0 \quad (B5)$$

are used to eliminate $A_i$ and $B_i$, $(i=1,2,3)$. Additional information needed to account for $\varepsilon_{zz}$ is
\[ \varepsilon_{zz} = \left( \sigma_{zz} - \nu_{rr} \sigma_{rr} - \nu_{\theta\theta} \right) / E , \quad \int_{A} \sigma_{zz} \, dA = P \quad (B6a,b) \]

where \( P \) is the total axial force. If the cylinder is fully constrained \( \varepsilon_{zz} = 0 \), if there is no axial constraint, \( P = 0 \) and, if the ends of the cylinder are closed and if it is subjected to an internal pressure \( p \), then \( P = \pi R_{1}^{2} p \). In this problem it is assumed that the cylinders 1 and 2 are fully bonded along \( r = b_{1} = a_{2} \) giving \( \varepsilon_{1zz} = \varepsilon_{2zz} \).

The two shells are said to be equivalent if

\[ \varepsilon_{3zz} = \varepsilon_{1zz} , \quad w_{3}(R) = w_{1}(R) , \quad (B7a,b) \]

where \( R = R_{1} + h/2 \). After eliminating \( A_{i} \) and \( B_{i} \) \((i=1,2,3)\), \((B7a)\) and \((B7b)\) would give the equivalent (or effective) elastic constants \( E_{2} \) and \( \nu_{2} \) in terms of \( E_{1} , \nu_{1} , E' , \nu' , R_{1} , h \) and \( h_{2} \).

The next problem is the evaluation of the stresses in the bonded shell shown in Fig. 1b in the absence of any cracks. Referring to \([16]\) and Fig. 1b, the solution of the "open-ended" or axially unconstrained cylindrical shell problem may be expressed as

\[ u_{3}(x_{2}) = \begin{cases} e^{\beta_{1}x_{2}}(C_{1}\cos\beta_{1}x_{2}+C_{2}\sin\beta_{1}x_{2}) - \frac{pR^{2}}{E_{1}h} , & x_{2} < 0 , \\ -\beta_{2}x_{2}(C_{3}\cos\beta_{2}x_{2}+C_{4}\sin\beta_{2}x_{2}) - \frac{pR^{2}}{E_{2}h} , & x_{2} > 0 , \end{cases} \quad (B8) \]

where

\[ \beta_{1} = 3(1-\nu_{1}^{2})/R_{1}^{2}h^{2} , \quad \beta_{2} = 3(1-\nu_{2}^{2})/R_{2}^{2}h^{2} , \quad (B9) \]

and the constants \( C_{1}, ..., C_{4} \) are determined from

\[ u_{3}(-0) = u_{3}(+0) , \quad \frac{d}{dx_{2}} u_{3}(-0) = \frac{d}{dx_{2}} u_{3}(+0) , \quad (B10) \]

\[ D_{1} \frac{d^{2}}{dx_{2}^{2}} u_{3}(-0) = D_{2} \frac{d^{2}}{dx_{2}^{2}} u_{3}(+0) , \quad D_{1} \frac{d^{3}}{dx_{2}^{3}} u_{3}(-0) = D_{2} \frac{d^{3}}{dx_{2}^{3}} u_{3}(+0) , \quad (B11) \]
\[ D_1 = \frac{E_1 h^3}{12(1-\nu_1^2)}, \quad D_2 = \frac{E_2 h^3}{12(1-\nu_2^2)}. \]  
(B12)

Note that (B11) represents the continuity of \( M_{22} \) and \( V_2 \). Equations (B8), (B10) and (B11) may be shown to reduce

\[ C_1 - C_3 = pR^2 (E_2 - E_1)/(hE_1E_2), \]
\[ C_1 + C_2 = (C_4 - C_3) \beta_2/\beta_1, \]
\[ C_2 = -D_2 \beta_2^2 C_4/(D_1 \beta_1^2), \]
\[ C_2 - C_1 = (C_4 + C_3) D_2 \beta_2^3/(D_1 \beta_3^3). \]  
(B13a-d)

After determining \( C_1, \ldots, C_4 \) from (B13), by observing that [16]

\[ N_{11}(x_2) = -\frac{E_1 h}{R} u_3(x_2), \quad M_{11}(x_2) = -\nu_1 D_1 \frac{d^2}{dx_2^2} u_3(x_2), \]
\[ (x_2<0), \]  
(B14)

the input functions \( F_1 \) and \( F_2 \) may be expressed as (see (38a), (39a) and Appendix A)

\[ F_1(y) = \frac{1}{R} e^{\beta_1 ay} [C_1 \cos \beta_1 ay + C_2 \sin \beta_1 ay] - \frac{pR}{hE_1}, \quad (y<0), \]  
(B15)

\[ F_2(y) = \frac{2\nu_1 D_1 \beta_1^2}{h^2 E_1} e^{\beta_1 ay} [-C_1 \sin \beta_1 ay + C_2 \cos \beta_1 ay], \quad (y<0). \]  
(B16)

Similarly, if the ends of the pressurized composite cylinder shown in Fig. 1b are closed, following [17] we find

(*) Note that \( F_1 \) and \( F_2 \) used in the perturbation problem, correspond to the membrane and bending resultants obtained from the uncracked cylinder with the opposite sign.
\[ u_3(x_2) = \begin{cases} 
 e^{\beta_1 x_2} \left[ B_1 \cos \beta_1 x_2 + B_2 \sin \beta_1 x_2 \right] + \left( \frac{\nu_1 - 2}{2} \right) \frac{p R^2}{E_1 h}, & (x_2 < 0) \\
 e^{-\beta_2 x_2} \left[ B_3 \cos \beta_2 x_2 + B_4 \sin \beta_2 x_2 \right] + \left( \frac{\nu_2 - 2}{2} \right) \frac{p R^2}{E_2 h}, & (x_2 > 0) 
\end{cases} \]

(B17)

\[ F_1(y) = \frac{1}{R} e^{\beta_1 a y} \left( B_1 \cos \beta_1 a y + B_2 \sin \beta_1 a y \right) - \frac{p R}{E_1 h}, & (y < 0) \]

(B18)

\[ F_2(y) = \frac{2 \nu_1 D_1 \beta_1^2}{h^2 E_1} e^{\beta_1 a y} \left( -B_1 \sin \beta_1 a y + B_2 \cos \beta_1 a y \right), & (y > 0) \]

(B19)

where \( B_1, \ldots, B_4 \) are obtained from the solution of the following system:

\[ B_1 - B_3 = \frac{p R^2}{h} \left( \frac{2 - \nu_1}{2 E_1} - \frac{2 - \nu_2}{2 E_2} \right), \]

(B20a)

\[ B_1 + B_2 = (B_4 - B_3) \beta_2 / \beta_1, \]

(B20b)

\[ B_2 = -B_4 D_2 \beta_2 / (D_1 \beta_1), \]

(B20c)

\[ B_2 - B_1 = (B_4 + B_3) D_2 \beta_2^3 / (D_1 \beta_1^3). \]

(B20d)
Fig. 1  Geometry and notation for a reinforced cylindrical shell containing an axial crack.
Fig. 2 Normalized stress intensity factors on the outside surface of an "open-ended" reinforced cylindrical shell containing an axial through crack (Shell No. 1, R/h = 100).
Fig. 3  Normalized stress intensity factors on the outside surface of an "open-ended" reinforced cylindrical shell containing an axial through crack (Shell No. 3, R/h = 35.5).
Fig. 4  Normalized stress intensity factor at the crack tip $x_2=0$ on the outside surface $x_3=h/2$ in "open-ended" reinforced cylindrical shells (see Table 1 for the dimensions of the shells and for the values of $\alpha=-\gamma$ for each shell).

$$k_1(0, h/2) k_0$$

$$a_{11}(0, x_2, h/2) = \frac{k_1(0, h/2)}{\sqrt{2} x_2^{\alpha}}$$

$$k_0 = \frac{p R_i}{h} a^\alpha, \ (0 < \alpha < 0.5)$$
Fig. 5  Normalized stress intensity factor at the far end $x_2=-d$ of a through crack intersecting the interface in "open-ended" reinforced cylindrical shells (see insert in Fig. 4 for geometry and Table 1 for dimensions of the shells).