SNR Estimation for the Baseband Assembly

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The expected value and the variance of the Baseband Assembly symbol signal-to-noise ratio (SNR) estimation algorithm are derived. The SNR algorithm treated here is designated as the Split Symbol Moments Estimator (SSME). It consists of averaging the first two moments of the integrated half symbols. The SSME is a biased, consistent estimator. The SNR degradation factor due to the jitter in the subcarrier demodulation and symbol synchronization loops is taken into account. Curves of the expected value of the SNR estimator versus the actual SNR are shown.

I. Introduction

The Baseband Assembly\(^1\) uses a Split Symbol Moments Estimator (SSME) algorithm to estimate the symbol signal-to-noise ratio (SNR) of the input signal. Here we describe the SSME algorithm and give the expected value and the variance of the SNR estimator. Two numerical examples corresponding to the Voyager and the Pioneer missions are included to illustrate its performance. As in previous Baseband Assembly analyses (Refs. 1 and 2), Nyquist sampling rate is assumed.

II. Statistics of the SNR Estimator

Figure 1 is a flow chart representation of the SSME algorithm. Referring to this figure, the input to the SNR estimator is a string of signal samples modeled as

\[ y_{ij} = s_{ij} + n_{ij} \]  

where

- \( s_{ij} \) = unitless random variable whose amplitude is proportional to the information signal voltage
- \( n_{ij} \) = unitless random variable whose amplitude is proportional to the rms noise voltage

The first two moments of \( y_{ij} \) are

\[ E\{y_{ij}\} = \sqrt{S} \]  
\[ E\{(y_{ij})^2\} = S + \sigma_n^2 \]

It is assumed that \( E\{n_{ij}\} = 0 \)

\[ i = 1, 2, \ldots, N_s \text{ Nyquist samples per symbol} \]
\[ j = 1, 2, \ldots, n \text{ symbols} \]

The variance of the noise process is assumed to be

\[ \sigma_n^2 = N_0 B_n \]  

where $N_0$ is the one-sided noise spectral density, and $B_n$ is the one-sided baseband noise-equivalent bandwidth.

As shown in Fig. 1, in the upper "arm" the samples from the first half of a symbol are summed to produce $Y_{a_j}$. In the lower "arm" the samples of the second half of a symbol are summed to produce $Y_{b_j}$. In this analysis, it will be assumed that the number of samples in both summers are equal at the instants when $Y_{a_j}$ and $Y_{b_j}$ are sampled. For this reason, $Y_{a_j}$ and $Y_{b_j}$ have identical statistics. Making $Y_{a_j} = Y_{b_j} = Y_j$, the mean value and the variance of $Y_j$ will be, assuming that the samples are independent,

$$\bar{Y}_j = E\{Y_{a_j}\} = E\{Y_{b_j}\} = \frac{N_j}{2} \sqrt{S} \sqrt{d_j} \quad (5)$$

$$\sigma_j^2 = E\{(Y_{a_j} - \bar{Y}_{a_j})^2\} = E\{(Y_{b_j} - \bar{Y}_{b_j})^2\} = \frac{N_j}{2} \sigma_n^2 \quad (6)$$

The factor $d_j$, designated as the SNR degradation factor, is due to the phase jitter and timing jitter in the subcarrier demodulation and symbol synchronization loops, respectively. In general,

$$0 < d_j < 1 \quad (7)$$

It can be shown that

$$d_j = \left(1 - \frac{\phi_j}{\pi/2}\right)^2 \left(1 - 2p_T \frac{\tau_j}{T_s}\right)^2 \quad (8)$$

where

- $\phi_j$ = phase error in the subcarrier demodulation loop during the $j$th symbol
- $\tau_j$ = timing error in the symbol synchronization loop during the $j$th symbol
- $p_T$ = probability of symbol transition
- $T_s$ = symbol time

In this preliminary analysis, it will be assumed that there is no doppler stress in the tracking loops and that $\phi_j$ and $\tau_j$ are functions of the phase and timing jitter only. With this assumption, $\phi_j$ and $\tau_j$ will be constant during one update interval, and the subscript $j$ can be dropped, i.e., we will assume that during the estimation interval

$$d_j = d_{j+1} = d \quad (9)$$

and, consequently, the statistics of $Y_j$ will be equal to those of $Y_{j+1}$.

In the SSME algorithm, the random variables $Y_a$ and $Y_b$ are combined to create two new random variables $X_p$ and $X_{ss}$ in the following way:

$$X_p = Y_a Y_b \quad (10)$$

and

$$X_{ss} = (Y_a + Y_b)^2 \quad (11)$$

Then, as shown in Fig. 1, $n$ samples of $X_p$ and $X_{ss}$ are averaged in the second pair of summers to produce $m'_p$ and $m'_{ss}$. Finally, $m'_p$ and $m'_{ss}$ are scaled and combined to produce the random variable $R^*$, which is the SNR estimator of the SSME algorithm, namely,

$$R^* = \frac{m'_p}{2 \left( \frac{1}{4} m'_{ss} - m'_p \right)} \quad (12)$$

The statistics of $R^*$ can be determined from the statistics of the random variables along the two paths in Fig. 1. These statistics are obtained in what follows.

Using Eqs. (6) and (7) and the fact that $Y_a$ and $Y_b$ are independent, the first two moments of their product defined in Eq. (10) will be

$$\bar{X}_p = N_2 S d/4 \quad (13)$$

$$\bar{X}_p^2 = (N_2 S d/4 + N_2 \sigma_n^2/2)^2 \quad (14)$$

The first two moments of $X_{ss}$ defined by Eq. (11) are obtained using Eq. (A-2) of Appendix A with $\mu = N_2 \sqrt{S} d$ and $\sigma^2 = N_2 \sigma_n^2$, namely,

$$\bar{X}_{ss} = E(X_{ss}^2) = N_2^2 S d + N_2 \sigma_n^2 \quad (15)$$

$$\bar{X}_{ss}^2 = E(X_{ss}^4) = 3N_2^2 \sigma_n^4 + 6N_2^3 S d \sigma_n^2 + N_2^4 S^2 d^2 \quad (16)$$

Referring to Fig. 1, and using Eq. (A-5), the first and second moments at the outputs of the second pair of summers will be

$$\bar{m}_p = \bar{X}_p \quad (17)$$
Having obtained the moments of \( m'_p \) and \( m'_s \), we are ready to determine the statistics of the estimator \( R^* \). Using Eq. (A-9), the expected value of \( R^* \) defined by Eq. (12) is

\[
\bar{R}^* = R^* + \frac{1}{2} \left[ \frac{\partial^2 R^*}{\partial m'_p^2} \text{var}(m'_p) + \frac{\partial^2 R^*}{\partial m'_s^2} \text{var}(m'_s) \right] m'_p \frac{\partial^2 R^*}{\partial m'_p \partial m'_s} \text{cov}(m'_p, m'_s) + \ldots
\]

Inserting Eqs. (A-16), (A-18), (A-19), (21), (22), and (26) in Eq. (27) and ignoring higher order terms, we obtain

\[
\bar{R}^* = \hat{R} + \frac{1}{n} (2\hat{R} + 1)
\]

where

\[
\hat{R} = \frac{\bar{m}_p}{\bar{m}'_s} = \frac{N_s S_d}{2\sigma_n^2} = R d
\]

is the degraded symbol SNR at the input to the SNR estimator. From Eq. (28) we observe that \( R^* \) is a biased but consistent estimator (i.e., the bias goes to zero when \( n \) goes to infinity).

The variance of \( R^* \) is obtained using Eq. (A-10), namely,

\[
\text{var}(R^*) = \left( \frac{\partial R^*}{\partial m'_p} \right)^2 \text{var}(m'_p) + \left( \frac{\partial R^*}{\partial m'_s} \right)^2 \text{var}(m'_s) + 2 \frac{\partial R^*}{\partial m'_p \partial m'_s} \text{cov}(m'_p, m'_s)
\]

\[
\text{var}(R^*) = \frac{N_s^2}{\bar{m}'_s^4} \left[ \frac{N_s^2}{\bar{m}'_s^4} \right] \text{var}(m'_p) + \left( \frac{\partial R^*}{\partial m'_s} \right)^2 \text{var}(m'_s) + 2 \frac{\partial R^*}{\partial m'_p \partial m'_s} \text{cov}(m'_p, m'_s)
\]
Inserting Eqs. (A-15), (A-17), (A-19), (15), (16), and (26) in Eq. (30), we obtain
\[
\text{var} (R^*) = \frac{1}{n} (1 + 4\hat{R} + 2\tilde{R}^2) \quad (31)
\]

By defining the SNR of our estimator as the ratio
\[
\text{SNR} (R^*) = \frac{(\hat{R}^*)^2}{\text{var} (R^*)} \quad (32)
\]

we see that
\[
\lim_{R \to 0} \text{SNR} (R^*) = \frac{1}{n} \quad (33)
\]
\[
\lim_{R \to \infty} \text{SNR} (R^*) \approx \frac{n}{2} + 2 \quad (34)
\]

### III. Evaluation of \( d \)

Assuming that there are no doppler or quantization errors, the SNR degradation factor defined in Eq. (8) is a function of the phase jitter in the subcarrier demodulation loop and the timing jitter in the symbol synchronization loop. Both jitter processes, \( \phi \) and \( \tau \), are modeled as Gaussian random variables having zero mean and variance \( \sigma_\phi^2 \) and \( \sigma_\tau^2 \), respectively.

According to Ref. 2, the variance of the phase error in the subcarrier demodulation loop at update instants is
\[
\sigma_\phi^2 = \left( \frac{T_L B_{L1}}{4K} \right) \frac{\pi^2}{\left( \frac{E_s}{N_0} \right)^2} \left( 1 + \frac{E_s}{N_0} \right) \quad (35)
\]

Repeating the steps of Ref. 2, it can be shown that the variance of the timing error in the symbol synchronization loop at update instants is
\[
\sigma_\tau^2 = \left( \frac{T_L B_{L2} a_2}{8 K a_1} \right) \frac{T_s^2}{\left( \frac{E_s}{N_0} \right)^2} \left[ 1 + 2 \left( \frac{E_s}{N_0} \right) (a_1 + a_2) \right] \quad (36)
\]

where
\[
B_{Lj} = \text{one sided noise-equivalent bandwidth, } j = 1 \text{ for subcarrier loop, } j = 2 \text{ symbol synch loop}
\]
\[
T_L = \text{loop update time, assumed to be identical for both loops}
\]
\[
K = \text{number of symbols between updates}
\]
\[
T_s = \text{symbol time} = 1/r
\]
\[
\frac{E_s}{N_0} = \text{ratio of energy per symbol to noise spectral density}
\]
\[
\Delta = R = \frac{NS}{2\sigma_n^2} \quad (37)
\]
\[
a_1 = M/N_s = \text{ratio of the width of the middle portion of a symbol to the total symbol length (typically 1/2)}
\]
\[
a_2 = L/N_s = \text{ratio of the width of the transition portion of a symbol to the total symbol length (typically 1/4)}
\]

The expected value of \( d \) in Eq. (8) will be
\[
d = \frac{1}{\sqrt{2\pi} \sigma_\phi} \int_{-\infty}^{\infty} \left( 1 - \left| \frac{\phi}{\pi} \right| \right)^2 \exp \left( -\frac{\phi^2}{2 \sigma_\phi^2} \right) d\phi
\]
\[
\times \frac{1}{\sqrt{2\pi} \sigma_\tau} \int_{-\infty}^{\infty} \left( 1 - 2p_T \frac{\tau}{T_s} \right)^2 \exp \left( -\frac{\tau^2}{2 \sigma_\tau^2} \right) d\tau
\]
improve our knowledge of $R$ if we compensate for the effects of the bias and the degradation factor in Eq. (28), i.e., we may assume that the actual input SNR is

$$\tilde{R} = \frac{\langle R^* \rangle}{\bar{d} \left(1 + \frac{2}{n}\right) + \frac{1}{n}} \quad (40)$$

where $\langle R^* \rangle$ is the average value of many $R_i^*$ and $\bar{d}$ is our estimate of $d$.

IV. Conclusions

In this article the expected value and the variance of the SSME SNR estimator was derived. This estimator was shown to be biased and consistent.

Figures 2 and 3 illustrate the numerical results for the Voyager and Pioneer missions. At high signal SNR, the positive bias of the estimator dominates over the degradation effect due to phase jitter in the tracking loops. At low SNR, it is the other way around. Figure 4 is for the ideal case when there is no jitter in the tracking loops ($d = 1$).

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References


\[
N_s = \frac{N_s}{2} + l \\
Y_{ij} \\
X_{sj} \\
\text{Fig. 1. Split symbol SNR estimator algorithm}
\]

\[
\text{Fig. 2. Mean value and SNR of SNR estimator vs actual SNR: Voyager mission}
\]

\[
\text{Fig. 3. Mean value and SNR of SNR estimator vs actual SNR: Pioneer mission}
\]

\[
\text{Fig. 4. Mean value and SNR of SNR estimator vs actual SNR: general}
\]
1. Relation Between Statistical and Probabilistic Moments

Given a random variable $x$ with Gaussian pdf $G(\mu, \sigma^2)$ and defining the $r$th moment as

$$\mu'_r = E \{x^r\} \quad \text{(A-1)}$$

the first four probabilistic moments of $x$ will be

$$\begin{align*}
\mu'_1 &= \mu \\
\mu'_2 &= \mu^2 + \sigma^2 \\
\mu'_3 &= 3\sigma^2 \mu + \mu^3 \\
\mu'_4 &= 3\sigma^4 + 6\sigma^2 \mu^2 + \mu^4
\end{align*} \quad \text{(A-2)}$$

Defining $m'_r$ as the $r$th statistical moment of a random variable

$$m'_r = \frac{1}{n} \sum_{j=1}^{n} (x_j)^r \quad \text{(A-3)}$$

and the variance of $m'_r$ as

$$\text{var} (m'_r) = E \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (x_j)^r - \mu'_r \right)^2 \right\} \quad \text{(A-4)}$$

Chapter 10 of Ref. 3 shows that

$$E \{m'_r\} = \mu'_r \quad \text{(A-5)}$$

and

$$\text{var} (m'_r) = \frac{1}{n} \left[ \mu^2_r - (\mu'_r)^2 \right] \quad \text{(A-6)}$$

This is an exact result.

Given a function $g$ of $K$ random variables $x_k$,

$$g(x) = g(x_1, x_2, \ldots, x_K) \quad \text{(A-7)}$$

with means

$$E \{x'_K\} = \theta_K$$

$$\theta \triangleq \theta_1, \theta_2, \ldots, \theta_K \quad \text{(A-8)}$$

it can be shown (Prob 10.17, Ref. 3) that

$$E \{g(x)\} = g(\theta) + \frac{1}{2} \sum_{k=1}^{K} \frac{\partial^2}{\partial x_k^2} g(x) \bigg|_{x = \theta} \text{var} (x_k) \quad \text{(A-9)}$$

$$+ \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\partial g(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} \bigg|_{x = \theta} \text{cov} (x_i, x_j) + \ldots$$

The variance of $g(x)$ will be (Eq. (10.12) of Ref. 3)

$$\text{var} \{g(x)\} = \sum_{k=1}^{K} \left[ \frac{\partial}{\partial x_k} g(x) \right]^2 \bigg|_{x = \theta} \text{var} (x_k) \quad \text{(A-10)}$$

$$+ \sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\partial g(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} \bigg|_{x = \theta} \text{cov} (x_i, x_j) + \ldots$$

2. Evaluation of the Derivatives of the Estimator

In the SSME algorithm $R^*$ is computed from $m'_p$ and $m'_ss$, namely,

$$R^* = \frac{m'_p}{2 \left( \frac{1}{4} m'_s - m'_p \right)} \quad \text{(A-11)}$$

It can be shown that

$$E \{m'_p\} = \frac{m'_p}{N_s} = \frac{1}{4} N_s^2 \text{sd} \quad \text{(A-12)}$$

$$E \{m'_ss\} = \frac{m'_ss}{N_s} = N_s^2 \text{sd} + N_s \sigma_n^2 \quad \text{(A-13)}$$
Let
\[
\hat{R} \triangleq \frac{E\{m_p'\}}{2 E\left\{\frac{1}{4} m_{ss}' - m_p'\right\}} = \frac{N_s Sd}{2a_n^2} \tag{A-14}
\]

The following derivatives of the estimator $R^*$ are evaluated:
\[
\frac{\partial R^*}{\partial m_p'} \bigg|_{m_p', m_{ss}'} = \frac{2}{a_n^2 N_s} (1 + 2\hat{R}) \tag{A-15}
\]
\[
\frac{\partial^2 R^*}{\partial m_p'^2} \bigg|_{m_p', m_{ss}'} = \left(\frac{4}{a_n^2 N_s}\right)^2 (1 + 2\hat{R}) \tag{A-16}
\]
\[
\frac{\partial^2 R^*}{\partial m_p'^2 \partial m_{ss}'} \bigg|_{m_p', m_{ss}'} = -2\left(\frac{1}{a_n^2 N_s}\right)^2 (1 + 4\hat{R}) \tag{A-19}
\]
\[
\frac{\partial^2 R^*}{\partial m_p' \partial m_{ss}'} \bigg|_{m_p', m_{ss}'} = -\frac{1}{a_n^2 N_s} \hat{R} \tag{A-17}
\]
Appendix B
Constants Used to Derive Figs. 2–4

In order to illustrate the performance of the SSME estimator, two cases are considered.

(1) Voyager

Data rate $r = 20,000$ symbols/second
Update time $T_L = 2.5$ seconds
$K = n = 2.5 \times 20,000$ symbols/loop update
$T_L B_L = 0.4153$ (from Table 1, Ref. 1 for both tracking loops
Noise bandwidth $B_n = 3.75$ MHz

(2) Pioneer

$r = 8$ symbols/second
$T_L = 2.5$ seconds
$K = n = 2.5 \times 8$ symbols/loop update
$T_L B_L = 0.4153$ for both tracking loops
$B_n = 135$ kHz

The performance of $R^*$ for the Magellan mission will be better than for Voyager.

Using Eqs. (35), (36), (39), (28), (29), (31), and (32), Figs. 2 and 3 are obtained.