LINEAR AND NONLINEAR DYNAMIC ANALYSIS OF
REDUNDANT LOAD PATH BEARINGLESS ROTOR SYSTEMS

by

V.R. Murthy, Principal Investigator

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ABSTRACT

The bearingless rotorcraft offers reduced weight, less complexity and superior flying qualities. The bearingless rotors are presently being developed and it is most likely that the next generation rotorcraft would be equipped with these rotors. Almost all practical designs of bearingless rotors include multiple load paths and the one that was flight tested by the Boeing Vertol has three load paths. The determination of natural vibration characteristics is basic to any dynamic design and they form the basis for all practical aeroelastic stability analyses of rotor blades. Almost all the current industrial structural dynamic programs of conventional rotors which consist of single load path rotor blades employ the transfer matrix method because this method is ideally suited for one-dimensional chain-like structures. In this report, this method is extended to multiple load path rotor blades without resorting to an equivalent single load path approximation. Unlike the conventional blades, it is necessary to introduce the axial-degree-of-freedom into the solution process to account for the differential axial displacements in the different load paths. With the present extension, the current rotor dynamic programs can be modified with relative ease to account for the multiple load paths without resorting to the equivalent single load path modeling. The results obtained by the transfer matrix method are validated by comparing with the finite-element solutions. A differential stiffness matrix due to blade rotation is derived to facilitate the finite-element solutions.
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NOMENCLATURE

A

Area of cross-section

B\_1, B\_2

Cross-section integrals, reference 8

C\_1, C\_2

Cross-section integrals, reference 8

e

Mass centroid offset from elastic axis, positive when in front of elastic axis

e\_A

Distance between area centroid and elastic axis, positive for centroid forward

E

Young's modulus

G

Shear modulus

I\_y

Cross-section moment of inertia about y-axis

I\_y'\_o

Reference moment of inertia for non-dimensionalization

I\_z

Cross-section moment of inertia about z-axis

J

Torsional rigidity

k\_A

Polar radius of gyration of cross-sectional area effective in carrying tensile stresses about elastic axis

k\_m

Mass radius of gyration of blade cross section

\[ k^2 = k_{m_1}^2 + k_{m_2}^2 \]

k\_m_1, k\_m_2

Cross-section integrals, reference 8

l

Length of the load paths

m

Mass per unit length

m\_o

Reference mass for non-dimensionalization

M\_x

Twisting moment about x-axis

M\_y, M\_z

Bending moments about y and z axes, respectively

n

Number of load paths

N

Axial force

R

Radius of the rotor

T

Tension
\( [T] \) Transfer matrix of the blade

\( [T^i] \) Transfer matrix of the ith load path

\( u, v, w \) Elastic displacements in the \( x, y, z \) directions, respectively

\( V_y, V_z \) Shear forces along \( y \) and \( z \) directions, respectively

\( x, y, z \) Mutually perpendicular axis system with \( x \) along the undeformed blade and \( y \) towards the leading edge

\( \{z\} \) State vector

\( \psi \) Slope of deflection curve normal to plane of rotation

\( \nu \) Slope of deflection curve in the plane of rotation

\( \theta \) Pretwist angle

\( \phi \) Elastic twist about the elastic axis

\( \omega \) Frequency of vibration

\( \Omega \) Blade rotational speed

\( \beta_{pc} \) Precone angle

**Superscripts**

' Differentiation with respect to \( x \)

. Differentiation with respect to time

- Non-dimensional quantity

\( i \) Quantities corresponding to the ith load path

**Subscripts**

1 Quantities at the root

2 Quantities at the clevis

3 Quantities at the blade tip

\( i \) Quantities corresponding to the ith load path
I. INTRODUCTION

The bearingless rotorcraft offers simplicity of the design and superior flying qualities. The original purpose of introducing hinges is to relieve the blades from high stresses and this is an important design problem for bearingless rotors and a solution to this problem lies in the optimization of blade root stiffness distribution and the use of advanced composite materials. The bearingless rotor technology was successfully applied by Sikorsky in Blackhawk and S-76 helicopters for their tail rotors (Ref. 1). Boeing Vertol built the first successful bearingless main rotor (Refs. 2 and 3) and this flew first in 1978. During a recent study (Ref. 4) for concept definition of the Integrated Technology Rotor/Flight Research Rotor (ITR/FRR), thirty-three hub concepts were proposed. Twenty-one out of these thirty-three concepts were bearingless designs and hence it is very likely that the next generation rotorcraft would be equipped with a bearingless rotor.

Basically, the structural design of bearingless rotors include multiple load paths and also some kinematic couplings are introduced intentionally through design for specific reasons. It is evident from Refs. 4 to 7 that several flexbeam and pitch control configurations are possible for bearingless main rotors and the variations in the configuration have significant effects on the aeroelastic stability of the rotors. Therefore, it is necessary to have accurate analytical methods to predict the natural vibration and aeroelastic stability characteristics of rotor blades including the multiple load paths.
The determination of natural vibration characteristics is basic to any dynamic design of the rotor system. Also, they form the basis for almost all practical aeroelastic stability analyses of rotor blades. Almost all the current structural dynamic programs of single load path rotor blades employ the transfer matrix method because the method is ideally suited for one-dimensional chain-like structures. In this report, this method is extended to multiple load path rotor blades without resorting to the equivalent blade modeling. Any equivalent single load path approximation may not simulate completely the dynamic behavior of multiple load path systems. With the present extension, the current rotor dynamic programs can be modified with relative ease to account for the multiple load paths without resorting to the equivalent single load path approximation. The results obtained by the transfer matrix formulation are validated by comparing with the finite-element solutions. The finite-element solutions are generated by adding a differential stiffness matrix associated with the rotation to the non-rotating stiffness matrix of the blade. To facilitate these calculations, a general purpose differential stiffness matrix due to rotation is derived analytically. This matrix will be very useful both for the generation of finite-element solutions and validation of the transfer matrix solutions.
II. BASIC EQUATIONS

The nonlinear equations of motion for the elastic bending and torsion of rotor blades are given below from Ref. 8. For algebraic conciseness, the terms $e_1, B_1, B_2, C_1, C_2$ and $k_A$ are treated as zero. However, they do not affect the general nature of the formulation presented here. Also, the aerodynamic terms $L_u, L_v,$ and $L_w$ are omitted from the equations of motion. To account for the differential axial displacements in the load paths the axial degree-of-freedom is included in the equations of motion.

\[
\begin{align*}
- (E_A(u' + \frac{y''^2}{2} + \frac{w''^2}{2}))'' & - 2\Omega m \ddot{v} \\
+ \mu \ddot{\phi} - \Omega^2 \mu u & = \Omega^2 m x \\
- (T_v')' + [(E_{I_{z}} \cos^2(\theta+\phi) + E_{I_{y}} \sin^2(\theta+\phi))] v'' \\
+ (E_{I_{z}} - E_{I_{y}}) \cos(\theta+\phi) \sin(\theta+\phi) w'')'' \\
+ 2\Omega \mu \ddot{u} + \mu \ddot{v} - m e \phi \sin \theta - 2m e (\dot{v'} \cos \theta + \dot{w'} \sin \theta) \\
- m \ddot{\Omega}^2 [v + e \cos(\theta+\phi)] & - 2m \Omega \dot{\phi} \ddot{w} \\
- \{m e [\ddot{\Omega}^2 x \cos(\theta+\phi) + 2\Omega \dot{v} \cos \theta]\}' & = 0
\end{align*}
\]

\[
\begin{align*}
- (T_w') + \{(E_{I_{z}} - E_{I_{y}}) \cos (\theta+\phi) \sin (\theta+\phi) v'' \\
+ [E_{I_{z}} \sin^2 (\theta+\phi) + E_{I_{y}} \cos^2 (\theta+\phi)] w'')'' \\
+ \mu \ddot{w} + m e \ddot{\phi} \cos \theta + 2m \Omega \dot{\phi} \ddot{v} \\
- \{m e [\ddot{\Omega}^2 x \sin (\theta+\phi) + 2\Omega \dot{v} \sin \theta]\}' & = - m \ddot{\Omega}^2 \ddot{x}
\end{align*}
\]
\[-(GJ\phi')' + (E_l z' - E_l y')[(w''^2 - v''^2) \cos \theta \sin \theta
\]
\[+ v'' w'' \cos 2\theta] + \frac{m k^2}{m_2} + m \Omega^2 \phi (k^2_{m_2} - k^2_{m_1}) \cos 2\theta
\]
\[+ m e [\Omega^2 x (w' \cos \theta - v' \sin \theta) - (v - \Omega^2 v) \sin \theta
\]
\[+ w' \cos \theta]) = - \frac{\omega^2}{m} (k^2_{m_2} - k^2_{m_1}) \cos \theta \sin \theta - m e \Omega^2 \beta_p \sin \theta \cos \theta \quad (4)
\]

where \( T = EA \left( u' + \frac{v'}{2} + \frac{w'}{2} \right) \) \quad (5)

Equations (1) to (4) are coupled nonlinear partial differential equations in variables \( u, v, w \) and \( \phi \). Substitute Eq. (5) into Eq. (1) and integrate as shown below

\[
T(x) = \int_0^X T'(x) dx = - \int_0^X \Omega^2 m x dx - \int_0^X m(2\Omega \dot{\nu} - \ddot{u} + \Omega^2 u) dx + k_1 + k_2 \quad (6)
\]

where \( k_1 \) and \( k_2 \) are arbitrary constants.

The boundary condition for tension \( T(x) \) is given by

\[
T(R) = 0 \quad (7)
\]

Substitute Eq. (6) into Eq. (7)

\[
- \int_0^R \Omega^2 m x dx - \int_0^R m(2\Omega \dot{\nu} - \ddot{u} + \Omega^2 u) dx + k_1 + k_2 = 0 \quad (8)
\]

To satisfy Eq. (8), take \( k_1 \) and \( k_2 \) as

\[
k_1 = \int_0^R \Omega^2 m x dx \quad (9)
\]

\[
k_2 = \int_0^R m(2\Omega \dot{\nu} - \ddot{u} + \Omega^2 u) dx \quad (10)
\]

Substitute Eqs. (9) and (10) into Eq. (6)
\[ T(x) = \int_{x}^{R} \Omega^2 m \, dx + \int_{x}^{R} m(2\Omega\dddot{u} - \dddot{u} + \Omega^2 u) \, dx \] (11)

From Eq. (5), \( u' \) is given by

\[ u' = \frac{T}{EA} - \frac{v'^2}{2} - \frac{w'^2}{2} \] (12)

Substitute Eq. (11) into Eq. (12)

\[ u'(x) = \frac{\Omega^2}{EA} \int_{x}^{R} m \, dx + \frac{1}{EA} \int_{x}^{R} m(2\Omega\dddot{u} - \dddot{u} + \Omega^2 u) \, dx - \frac{v'^2}{2} - \frac{w'^2}{2} \] (13)

Integrate Eq. (13) with respect to \( x \)

\[ u(x) = \frac{\Omega^2}{EA} \int_{0}^{x} \int_{x}^{R} m \, dx + \frac{1}{EA} \int_{0}^{x} \int_{x}^{R} m(2\Omega\dddot{u} - \dddot{u} + \Omega^2 u) \, dx \]

\[ - \frac{1}{2} \int_{0}^{x} (v'^2 + w'^2) \, dx \] (14)

The constant integration is zero because \( u(0) = 0 \). Differentiate Eq. (14) with respect to time

\[ \ddot{u} = \frac{1}{EA} \int_{0}^{x} \int_{x}^{R} m(2\Omega\dddot{u} - \dddot{u} + \Omega^2 \dddot{u}) \, dx - \int_{0}^{x} (v'\dddot{v}' + w'\dddot{w}') \, dx \] (15)

Now, by substituting Eqs. (11) and (15) into Eqs. (2) and (3) for the underlined terms, two equations can be obtained in terms of \( v, w \) and \( \phi \). These two equations plus Eq. (4) are the necessary equations to solve for the unknowns \( v, w \) and \( \phi \). Once \( v \) and \( w \) are known, \( u \) can be obtained from Eq. (15) and \( T(x) \) can be obtained from Eq. (11). In essence, the bending and torsional equations are decoupled from the axial equation by the above formulation.

For determination of natural vibration characteristics, one is interested in the linear, homogeneous, undamped equations of motion. The process of obtaining these equations involves the following steps:
1. Substitution of Eqs. (11) and (15) into Eqs. (2) and (3).

2. Substitution of the following relations for small values of $\phi$.

\[
\begin{align*}
\cos(\theta + \phi) &= \cos \theta - \phi \sin \theta \\
\cos^2(\theta + \phi) &= \cos^2 \theta - \phi \sin 2\theta \\
\sin(\theta + \phi) &= \sin \theta + \phi \cos \theta \\
\sin^2(\theta + \phi) &= \sin 2\theta + \phi \cos 2\theta
\end{align*}
\]

3. Dropping of nonlinear terms, that is, terms containing the products of $u, v, w, \phi$ and/or their derivatives.

4. Dropping of damping type terms, that is, terms containing $\dot{u}, \dot{v}, \dot{w}, \dot{\phi}$ and their derivatives.

5. Dropping of nonhomogeneous terms, that is, terms not containing $u, v, w, \phi$ and/or their derivatives.

The above steps reduces Eqs. (1) to (4) to the following equations for simple harmonic time dependency of $u, v, w$ and $\phi$ with frequency $\omega$.

\[
\begin{align*}
(EAu')' + (\omega^2 + \Omega^2)m_u &= 0 \quad (16) \\
-(Tv')' + ((E_l \cos^2 \theta + E_y \sin^2 \theta)v'' \\
+ (E_l - E_y) \cos \theta \sin \theta \omega''' \omega m v \\
+ \omega^2 me \phi \sin \theta - \mu \Omega^2 (v-e \sin \theta \phi) \\
+ (me \Omega^2 x \sin \theta \phi)' &= 0 \quad (17) \\
-(Tw')' + ((E_l - E_y) \cos \theta \sin \theta v'' \\
+ (E_l \sin^2 \theta + E_y \cos^2 \theta)w'' \omega m w \\
- \omega^2 me \cos \theta \phi - (me \Omega^2 x \cos \theta \phi)' &= 0 \quad (18)
\end{align*}
\]
\[
\begin{align*}
- (GJ\phi)' &- \omega^2 mk^2_\phi + \Omega^2 m (k^2_{\text{m}_2} - k^2_{\text{m}_1}) \cos 2\theta \phi \\
&+ me\Omega^2 x (\cos \theta w' - \sin \theta v') + me (\omega^2 + \Omega^2) \sin \theta v \\
&- \omega^2 me \cos \theta w = 0 \\
\end{align*}
\]

where

\[
T = \Omega^2 \int_0^R m x dx
\]

The above equations can be reduced to the following twelve non-dimensional first-order differential equations:

\[
\begin{align*}
\frac{d\bar{u}}{d\bar{x}} &= (EI_y \omega_o / EAR^2) \bar{N} \\
\frac{d\bar{w}}{d\bar{x}} &= \bar{\psi} \\
\frac{d\bar{v}}{d\bar{x}} &= \bar{\nu} \\
\frac{d\bar{\psi}}{d\bar{x}} &= \bar{c}_{12} \bar{M}_z + \bar{c}_{11} \bar{M}_y \\
\frac{d\bar{v}}{d\bar{x}} &= \bar{c}_{22} \bar{M}_z + \bar{c}_{21} \bar{M}_y \\
\frac{d\bar{\phi}}{d\bar{x}} &= (EI_y / GJ) \bar{\mu}_x \\
\frac{d\bar{M}_x}{d\bar{x}} &= -\bar{\omega}^2 \bar{m} \bar{e} \cos \theta \bar{w} + (\bar{\omega}^2 + \bar{\Omega}^2) \bar{m} \bar{e} \sin \theta \bar{v} \\
&+ \bar{\Omega}^2 \bar{m} \bar{e} \bar{x} \cos \theta \bar{\psi} - \bar{\Omega}^2 \bar{m} \bar{e} \bar{x} \sin \theta \bar{\nu} \\
&+ (\bar{\Omega}^2 \bar{m} (k^2_{\text{m}_2} - k^2_{\text{m}_1}) \cos 2\theta - \omega^2 mk^2_{\text{m}_2}) \bar{\phi} \\
\frac{d\bar{M}_z}{d\bar{x}} &= \bar{T} \bar{v} - \bar{\Omega}^2 \bar{m} \bar{e} \bar{x} \sin \theta \bar{\phi} - \bar{\nu} \bar{y} \\
\frac{d\bar{M}_y}{d\bar{x}} &= -\bar{T} \bar{\psi} - \bar{\Omega}^2 \bar{m} \bar{e} \cos \theta \bar{\phi} + \bar{\nu} \bar{z} \\
\frac{d\bar{v}}{d\bar{x}} &= - (\bar{\omega}^2 + \bar{\Omega}^2) \bar{m} \bar{v} + (\bar{\omega}^2 + \bar{\Omega}^2) \bar{m} \bar{e} \sin \theta \bar{\phi} \\
\frac{d\bar{v}}{d\bar{x}} &= - \bar{\omega}^2 \bar{m} \bar{w} - \bar{\omega}^2 \bar{m} \bar{e} \cos \theta \bar{\phi} \\
\frac{d\bar{N}}{d\bar{x}} &= - (\bar{\omega}^2 + \bar{\Omega}^2) \bar{\mu} \bar{u}
\end{align*}
\]
where
\[
\begin{align*}
\ddot{x} &= x/R; \quad \ddot{u} = u/R; \quad \ddot{v} = v/R; \quad \ddot{w} = w/R; \quad \ddot{\phi} = \psi; \quad \ddot{\psi} = \nu; \\
\ddot{\theta} &= \phi; \quad \ddot{M}_x = M_x R/\text{EI}_y; \quad \ddot{M}_y = M_y R/\text{EI}_y; \quad \ddot{M}_z = M_z R/\text{EI}_y; \\
\ddot{N} &= N R^2/\text{EI}_y; \quad \ddot{V}_y = V_y R^2/\text{EI}_y; \quad \ddot{V}_z = V_z R/\text{EI}_y; \quad \ddot{m} = m/\text{m}_o; \\
\ddot{e} &= e/R; \quad \ddot{k}_{m_1}^2 = k_{m_1}^2 R^2; \quad \ddot{k}_{m_2}^2 = k_{m_2}^2 R^2; \quad \ddot{k}_m = k_m^2 R^2; \\
\ddot{\omega}^2 &= \omega_m^2 R^4/\text{EI}_y; \quad \ddot{\Omega}^2 = \Omega_m^2 R^4/\text{EI}_y; \quad \ddot{C}_{11} = C_{11} \text{EI}_y; \\
\ddot{C}_{12} &= C_{12} \text{EI}_y; \quad \ddot{C}_{22} = C_{22} \text{EI}_y; \quad \ddot{C}_{21} = C_{21} \text{EI}_y.
\end{align*}
\]

\(\text{EI}_y\) = Reference Stiffness; \(\text{m}_o\) = Reference Mass

\[
\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}^{-1}
\]

\[
a_{11} = -(\text{EI}_y \cos^2 \theta + \text{EI}_z \sin^2 \theta)
\]

\[
a_{12} = -a_{21} = (\text{EI}_y - \text{EI}_z) \cos \theta \sin \theta
\]

\[
a_{22} = (\text{EI}_y \sin^2 \theta + \text{EI}_z \cos^2 \theta)
\]

\[
\ddot{\Theta} = \int_0^1 \frac{\ddot{\Omega}^2 - \ddot{m}_x^2}{\ddot{x}} dx
\]

Equations (21) to (32) can be arranged into a matrix differential equation of the following form:

\[
\{\ddot{z}(\ddot{x})\} = [A(\ddot{x})] \{\ddot{z}(\ddot{x})\}
\]

(33)

The transfer matrix relating the state vector at any location to the initial state vector is defined by

\[
\{\ddot{z}(\ddot{x})\} = [\ddot{T}(\ddot{x})] \{\ddot{z}(0)\}
\]

(34)
It can be shown that the transfer matrix defined in the above equation is governed by the following equation (Ref. 9):

\[ [\tilde{T}'(\tilde{x})] = [A(\tilde{x})][\tilde{T}(\tilde{x})] \] (35)

\[ [\tilde{T}(0)] = [-1, 0], \text{ identity matrix} \] (36)

where \([A(\tilde{x})]\) is defined in Eq. (33).
III. NATURAL VIBRATION CHARACTERISTICS

Almost all the practical bearingless rotor designs include multiple load paths. For instance, the bearingless main rotor that was flight tested by the Boeing Vertol has three load paths, viz., two fiberglass flexbeams and a filament-wound graphite torque tube (Ref. 7). The flexbeams and torque tube are connected to the blade through a rigid clevis. The idealization of a three load path rotor blade is shown in Fig. 1.

3.1. Equilibrium Across the Clevis:

Consider the free-body diagram for the clevis as shown in Fig. 2. Let \((h_{y_1}, h_{z_1})\) be the location of the \(i\)th load path with reference to a coordinate system located at the blade (point '0').

Force equilibrium requires the following relations to be satisfied:

\[
N_2 = \sum_{i=1}^{n} N_{i2}
\]  

\[
V_{y2} = \sum_{i=1}^{n} V_{y_{i2}}
\]  

\[
V_{z2} = \sum_{i=1}^{n} V_{z_{i2}}
\]

Moment equilibrium requires the following equation to be satisfied.

\[
(\sum_{i=1}^{n} (i \ M_{x_{i2}} + j \ M_{y_{i2}} + k \ M_{z_{i2}}) - \sum_{i=1}^{n} (i \ M_{x_{i2}} + j \ M_{y_{i2}} + k \ M_{z_{i2}}) \quad (37)
\]

\[
- \sum_{i=1}^{n} (j \ h_{y_{i1}} + k \ h_{z_{i1}}) \times (i \ N_{i2} + j \ V_{y_{i2}} + k \ V_{z_{i2}})
\]

The above equation can be written in the component form as shown below:
FIG. 1 MODEL FOR A TRIPLE LOAD PATH BLADE
FIG. 2 FREE-BODY DIAGRAM OF THE CLEVIS

A. Forces

B. Moments

\[
\begin{align*}
M_{x_{12}} & \quad M_{y_{12}} \\
M_{x_{22}} & \quad M_{y_{22}} \\
M_{x_{32}} & \quad M_{y_{32}} \\
N_{12} & \quad V_{y_{12}} \\
N_{22} & \quad V_{y_{22}} \\
N_{32} & \quad V_{y_{32}}
\end{align*}
\]

12.
\[ M_x = \sum_{i=1}^{n} (M_{x_{12}} - h_{yi} y_{12} + y_{yi} z_{12}) \]  
(40)

\[ M_y = \sum_{i=1}^{n} M_{y_{12}} + h_{zi} N_{12} \]  
(41)

\[ M_z = \sum_{i=1}^{n} M_{z_{12}} - h_{yi} N_{12} \]  
(42)

Equations (37) to (42) can be arranged into a matrix equation as shown below.

\[ \{f_2\} = \sum_{i=1}^{n} [A_i] \{f_{12}\} \]  
(43)

where

\[ \{f_2\}^T = \begin{bmatrix} M_{x_2} & M_{z_2} & M_{y_1} & V_{y_1} & V_{z_1} & N_{2} \end{bmatrix} \]

\[ \{f_{12}\}^T = \begin{bmatrix} M_{x_{12}} & M_{z_{12}} & M_{y_{12}} & V_{y_{12}} & V_{z_{12}} & N_{12} \end{bmatrix} \]

and

\[ [A_i] = \begin{bmatrix} 1 & 0 & 0 & -h_{yi} & h_{yi} & 0 \\ 0 & 1 & 0 & 0 & 0 & -h_{yi} \\ 0 & 0 & 1 & 0 & 0 & h_{zi} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

3.2. Compatibility Across the Clevis

Consider the plane of the clevis as shown in Fig. 3. Let \((x,0,0)\) be the location of the blade and \((x, h_{yi}, h_{zi})\) be the location of the \(i\)th load path before deformation. Assume that the plane of the clevis is rigid,
FIG. 3  PLANE OF THE CLEVIS

location of the blade

location of the load path
that is, rotates about x, y and z axes without any elastic deformation. After deformation, the x-coordinates of the blade and the ith load path are given by the second line of the following table.

<table>
<thead>
<tr>
<th>State</th>
<th>Axial Coordinate</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Before Deformation</td>
<td>Blade</td>
<td>ith Load Path</td>
</tr>
<tr>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>After Deformation</td>
<td>x + u₂</td>
<td>x + u₂ - h₁ψ₂ - h₁v₂</td>
</tr>
</tbody>
</table>

The axial displacement of ith load path is given by subtracting the x-coordinate before deformation from the x-coordinate after deformation as shown below.

\[ u_{12} = (x + u₂ - h₁ψ₂ - h₁v₂) - x \]

Rearrangement of the above equation yields

\[ u₂ = u_{12} + h₁ψ_{12} + h₁v_{12} \] (44)

The other compatibility conditions consistent with the rigid clevis are

\[ w₂ = w_{12} ; \quad v₂ = v_{12} ; \quad ψ₂ = ψ_{12} ; \quad v₂ = v_{12} \]

The above equations together with Eq. (44) can be arranged into a matrix equation of the following form.
3.3 Frequency Determinant

Define the following relations between the state vectors from Fig. 1 and the definition of transfer matrix

\[ \{ z_3 \} = [T] \{ z_2 \} \]  \hspace{1cm} (46)

Equation (46) relates the state vectors of the blade at Stations 2 (clevis) and 3 (tip). Rewrite this equation into the following partitioned form.

\[ \begin{pmatrix} \{ d_3 \} \\ \{ f_3 \} \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} \{ d_2 \} \\ \{ f_2 \} \end{pmatrix} \]  \hspace{1cm} (47)

where \( d \) stands for deflections and \( f \) stands for forces.

Extract the following equation for forces from Eq. (47)

\[ \{ f_3 \} = [T_{21}] \{ d_2 \} + [T_{22}] \{ f_2 \} \]  \hspace{1cm} (48)
Similarly, the transfer matrix relation for the \( i \)th load path can be written as (see Fig. 1)

\[ \{z_{12}\} = [T^i] \{z_{11}\} \quad (49) \]

Rewrite the above equation into partitioned form as

\[
\begin{bmatrix}
d_{12} \\
f_{12}
\end{bmatrix} =
\begin{bmatrix}
T^i_{11} & T^i_{12} \\
T^i_{21} & T^i_{22}
\end{bmatrix}
\begin{bmatrix}
d_{11} \\
f_{11}
\end{bmatrix}
\quad (50)
\]

The displacement and force vectors can be written in terms of \( \{f_{11}\} \) by virtue of boundary condition \( \{d_{11}\} = \{0\} \) as shown below.

\[ \{f_{12}\} = [T^i_{22}] \{f_{11}\} \quad (51) \]

\[ \{d_{12}\} = [T^i_{12}] \{f_{11}\} \quad (52) \]

From Eq. (52)

\[ \{f_{11}\} = [T^i_{12}]^{-1} \{d_{12}\} \quad (53) \]

Substitute Eq. (45) into Eq. (53)

\[ \{f_{11}\} = [T^i_{12}]^{-1} \left[ B_1 \right]^{-1} \{d_{2}\} \quad (54) \]

Substitute Eq. (54) into Eq. (51)

\[ \{f_{12}\} = [T^i_{22}] [T^i_{12}]^{-1} \left[ B_1 \right]^{-1} \{d_{2}\} \quad (55) \]

Substitute Eq. (55) into Eq. (43)

\[ \{f_2\} = \sum_{i=1}^{n} \left[ A_i \right] [T^i_{22}] [T^i_{12}]^{-1} \left[ B_1 \right]^{-1} \{d_{2}\} \quad (56) \]

Substitute Eq. (56) into Eq. (48)
\{f_3\} = ([T_{21}] + [T_{22}] \sum_{i=1}^{n} [A_i] [T_{22}^i] [T_{12}^i]^{-1} [B_i])^{-1} \{d_2\} \quad (57)

The vector \{f_3\} = \{0\} by virtue of the boundary conditions and for nontrivial solutions of \{d_2\}, the determinant of the coefficient matrix should be zero and this condition yields the following frequency equation to determine the natural frequencies.

\[ \text{det} ([T_{21}] + [T_{22}] \sum_{i=1}^{n} [A_i] [T_{22}^i] [T_{12}^i]^{-1} [B_i]) = 0 \quad (58) \]

### 3.4 Mode Shapes

The state vector at the clevis from Eq. (47) can be written as

\[
\begin{bmatrix}
\{d_2\} \\
\{f_2\}
\end{bmatrix}
= \begin{bmatrix}
\bar{T}_{11} & \bar{T}_{12} \\
\bar{T}_{21} & \bar{T}_{22}
\end{bmatrix}
\begin{bmatrix}
\{d_3\} \\
\{f_3\}
\end{bmatrix}
\quad (59)
\]

where

\[
\begin{bmatrix}
\bar{T}_{12} & \bar{T}_{12} \\
\bar{T}_{21} & \bar{T}_{22}
\end{bmatrix}
= \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}^{-1}
\]

From the above equation, the displacement vectors at the clevis and the tip of the blade can be related as shown below by virtue of the boundary condition \{f_3\} = \{0\}.

\[
\{d_2\} = [\bar{T}_{11}] \{d_3\} \quad (60)
\]

Substitute Eq. (60) into Eq. (57)

\[
\{f_3\} = [C] \{d_3\} \quad (61)
\]

where

\[
[C] = ([T_{21}] + [T_{22}] \sum_{i=1}^{n} [A_i] [T_{22}^i] [T_{12}^i]^{-1} [B_i]) [\bar{T}_{11}]
\]
Assume \( w_3 = 1 \) arbitrarily and rewrite rows 2 to 6 of Eq. (61) as

\[
\begin{bmatrix}
C_{21} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}
\begin{bmatrix}
u_3 \\
v_3 \\
\psi_3 \\
v_3 \\
\phi_3
\end{bmatrix} =
\begin{bmatrix}
u_3 \\
v_3 \\
\psi_3 \\
v_3 \\
\phi_3
\end{bmatrix} =
\begin{bmatrix}
-C_{22} \\
-C_{32} \\
-C_{42} \\
-C_{52} \\
-C_{62}
\end{bmatrix}
\] (62)

By solving the above equation, \( u_3, v_3, \psi_3, v_3 \) and \( \phi_3 \) are known and together with \( w_3 = 1 \) the entire \( \{d_3\} \) is known. Once, \( \{d_3\} \) is known, the state vector at the clevis can be determined from Eq. (59) as shown below

\[
\begin{bmatrix}
d_2 \\
f_2
\end{bmatrix} =
\begin{bmatrix}
\tilde{T}_{11} \\
\tilde{T}_{21}
\end{bmatrix}
\begin{bmatrix}
d_3
\end{bmatrix}
\] (63)

Once the state vector at the clevis is known, the deflection vectors in the blade and the load paths can be obtained as follows:

**Blade:** By definition of the transfer matrix the state vector at any location \( x \), is given by

\[
\begin{bmatrix}
d_x \\
f_x
\end{bmatrix} =
\begin{bmatrix}
T_{11} & T_{12} \\
\tilde{T}_{21} & \tilde{T}_{22}
\end{bmatrix}
\begin{bmatrix}
d_2 \\
f_2
\end{bmatrix}
\] (64)

From the above equation

\[
\{d_x\} = [T_{11}]_x \{d_2\} + [T_{12}]_x \{f_2\}
\] (65)

**Load path:** By definition of the transfer matrix the state vector at any location, \( x \), in the load path is given by
From the above equation

\[ \{d_{lx}\} = \{T_{12}\} \{f_{l1}\} \]  \hspace{1cm} (67)

by virtue of the boundary condition (\(\{d_{l1}\} = \{0\}\)).

Substitute Eq. (54) into Eq. (67)

\[ \{d_{lx}\} = \{T_{12}\}^{-1} \{T_{12}\}^{-1} \{B_{l}\} \{d_{2}\} \]  \hspace{1cm} (68)

The mode shapes can be computed from Eqs. (65) and (68).

3.5 Tension Coefficients

The preceding formulation assumes that the transfer matrices for the blade and the load paths are either known or can be calculated. The transfer matrices can be calculated if all the coefficients of the differential equations of motion are known and tension appears as coefficient in these equations. The calculation of tensions in the blade is straightforward and it is calculated from Eq. (20).

The following procedure is employed for calculation of the tension coefficients in the load paths. Draw a free-body diagram for the clevis as shown in Fig. 4. The tension 'T' applied by the blade to the clevis is known from Eq. (20). From the free-body diagram the following equilibrium equation can be obtained

\[ \{b\} = \sum_{i=1}^{n} [A_i] \{f_{i2}\} \]  \hspace{1cm} (69)
FIG. 4 TENSION CALCULATIONS IN THE LOAD PATHS
where
\[
\{b\}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & T \end{bmatrix}
\]

\([A_1]\) is defined in Eq. (43)

Let \([T^i]\) be the transfer matrix of the \(i\)th load path corresponding to the static case \((\omega = 0)\). By definition, one can write that

\[
\begin{pmatrix}
\{d_{12}\} \\
\{f_{12}\}
\end{pmatrix}
= \begin{bmatrix}
T_{11}^i & T_{12}^i \\
T_{21}^i & T_{22}^i
\end{bmatrix}
\begin{pmatrix}
\{d_{11}\} \\
\{f_{11}\}
\end{pmatrix}
\]

From the above equation

\[
\{f_{12}\} = [T_{22}^i] \{f_{11}\}
\]

(71)

Substitute Eq. (71) into Eq. (69)

\[
\{b\} = \sum_{i=1}^{n} [A_1] [T_{22}^i] \{f_{11}\}
\]

(72)

From Eq. (70)

\[
\{d_{12}\} = [T_{12}^i] \{f_{11}\}
\]

(73)

by virtue of the boundary condition \((\{d_{11}\} = \{0\})\).

Substitute Eq. (73) into Eq. (72)

\[
\{b\} = \sum_{i=1}^{n} [A_1] [T_{22}^i] [T_{12}^i]^{-1} \{d_{12}\}
\]

(74)

Substitute Eq. (45) into Eq. (74)

\[
\{b\} = \sum_{i=1}^{n} [A_1] [T_{22}^i] [T_{12}^i]^{-1} [B_i]^{-1} \{d_2\}
\]

(75)
From the above equation

\[ \{d_2\} = [D]^{-1} \{b\} \]  

where

\[ [D] = \sum_{i=1}^{n} [A_i] [T_{22}^i] [T_{12}^i]^{-1} [B_i]^{-1} \]  

From Eq. (45)

\[ \{d_{12}\} = [B_1]^{-1} \{d_2\} \]  

From Eq. (73)

\[ \{f_{11}\} = [T_{12}^i]^{-1} \{d_{12}\} \]  

Substituting Eqs. (76), (77) and (78) into Eq. (71)

\[ \{f_{12}\} = [T_{22}^i] [T_{12}^i]^{-1} [B_i]^{-1} [D]^{-1} \{b\} \]  

The last or sixth row in the above equation is \( N_{12} \). Once \( N_{12} \) is known, the tension in the \( i \)th load path can be obtained from the following equation

\[ T_i(x) = \frac{q^2}{2} \int x^2 \text{dx} + N_{12} \]  

where

\[ l = \text{span of the load path.} \]

The above formulation assumes that the transfer matrices (static) of the load paths are known which depend on the tensions to be calculated. The following iterative scheme is used to solve the problem. The initial tensions in the load paths are calculated by assuming that the load paths
are coincident with the blade, i.e., \( h_1 = h_2 = 0 \). The tensions corresponding to this case are calculated as follows. The transfer matrix for axial motion of a beam for the static case is given by

\[
[T(x)] = \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}
\]  

(81)

where

\[
a = \int_0^x \frac{dx}{EA}
\]

By definition of the transfer matrix

\[
\begin{bmatrix}
u_{12} \\
N_{12}
\end{bmatrix} = \begin{bmatrix}
1 & a_1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
u_{11} = 0 \\
N_{11}
\end{bmatrix}
\]

(82)

From the above equation

\[
\begin{aligned}
u_{12} &= a_1 N_{11} \\
N_{12} &= N_{11}
\end{aligned}
\]

(83)

Consider the following two cases:

Case 1: 2 Load Paths

Expanding Eq. (83)

\[
\begin{aligned}
u_{12} &= a_1 N_{12} \\
u_{22} &= a_2 N_{22}
\end{aligned}
\]

(84)

For coincident nodes \((u_{12} = u_{22})\), Eq. (84) becomes
\[ a_1 N_{12} - a_2 N_{22} = 0 \]  

For equilibrium

\[ N_{12} + N_{22} = T \]

By solving Eqs. (85) and (86)

\[ N_{12} = \frac{a_2}{a_1 + a_2} T \]

\[ N_{22} = \frac{a_1}{a_1 + a_2} T \]

\[ \text{Case 2: 3 Load Paths} \]

Expanding Eq. (83) for coincident nodes

\[ a_1 N_{12} = a_2 N_{22} = a_3 N_{32} \]

For equilibrium

\[ N_{12} + N_{22} + N_{32} = T \]

By solving Eqs. (88) and (89)

\[ N_{12} = \frac{a_2 a_3 T}{D} \]

\[ N_{22} = \frac{a_3 a_1 T}{D} \]

\[ N_{32} = \frac{a_1 a_2 T}{D} \]

where

\[ D = (a_1 a_2 + a_2 a_3 + a_3 a_1). \]
Now the above result can be generalized to the n load path case as shown below.

\[
N_{12} = \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{a_j}{a_i} \right) \quad (91)
\]

where

\[
\prod_{j=1}^{n} a_j = a_1 a_2 \ldots a_n / a_i
\]

Now, the tension in the ith load path corresponding to the coincident load path case is given by

\[
T_i^c(x) = \Omega^2 \int_0^L m x \, dx + N_{12} \quad (92)
\]

The tensions obtained from coincident nodes are used to obtain the transfer matrices of the noncoincident problem. At the most, one more iteration may be required to obtain the convergence for practical bearingless rotor designs.
IV. RESULTS

4.1 Computer Program

A general purpose computer program has been developed to determine the natural vibration characteristics of rotating multiple load path rotor blades based on the transfer matrix formulation. The listing of the computer program and the user's instructions are given in appendices A and B respectively. The significant features of the computer program are as follows.

1. The non-uniform properties including the pretwist for the blades as well as for the load paths can be handled by the program.

2. The coupled flapwise bending, chordwise bending, torsion and axial stretching degrees-of-freedom are included in the program. In the case of single load path blade the axial stretching is decoupled similar to the conventional analysis.

3. No equivalent single load path approximation is made in the program.

4. Maximum number of load paths allowed in the program is three but can be extended very easily to "n" number of load paths.

5. Continuous system model is used in the program and a fourth-order Runge-Kutta integration scheme is used to determine the transfer matrices. If a discrete model is used just like in the case of Bell Helicopter's C-81 program, the corresponding transfer matrices can be used in place of the continuous system transfer matrices and the formulation is independent of the model.
6. The natural frequencies are computed by a frequency scanning technique. Sign changes in the values of frequency determinant are detected by starting from an initial value and incrementing at steps of specified "h" until the required sign changes are detected or a final prescribed value is reached. If two frequencies are closer than the increment "h", then there is a chance of missing those frequencies. In such cases, the frequencies are detected by the fact that if any three consecutive frequency determinants have the same sign and the absolute value of the middle determinant is the smallest of the three, then there are two frequencies in that range. In this case smaller increments are taken to bracket the roots.

4.2 Numerical Results

The following data is used for numerical calculations to validate the formulation and the computer program.

1. Radius of the rotor = 260 in.
2. Distance of the clevis from the root = 52 in.
3. Length of the blade = 208 in.
4. Rotational speed (Ω) = 360 RPM
5. Flapwise bending stiffness (EI_y) = 0.2977 x 10^8 lb/in^2
6. Chordwise bending stiffness (EI_z) = 10 x 10^8 lb/in^2
7. Torsional stiffness (GJ) = 0.2 x 10^8 lb/in^2
8. Axial stiffness (EA) = 10^{11} lb/in^2
9. Mass per unit length = 0.0015 lb-sec^2/in^2
For two load path blade

\[ h_{y_1} = 0.0; \quad h_{z_1} = -1.0 \text{ in}; \quad h_{y_2} = 0.0; \quad h_{z_2} = 3.0 \text{ in}. \]

For three load path blade

\[ h_{y_1} = 1.0 \text{ in}; \quad h_{z_1} = 3.0 \text{ in}; \quad h_{y_2} = -1.0 \text{ in}; \quad h_{z_2} = -1.0 \text{ in}; \quad h_{y_3} = 2.0 \text{ in}; \quad h_{z_3} = -2.0 \text{ in}. \]

The natural frequencies obtained from the computer program are presented in Table 1 for load paths 1 and 2 respectively corresponding to the rotational speed \( \Omega = 360 \text{ RPM} \). The non-rotating and rotating (\( \Omega = 360 \text{ RPM} \)) natural frequencies for a three load path case are presented in Table 2 and in this case out-of-plane locations are assumed for the load paths for validation of a general case. A mode shape corresponding to the two load path case is presented in Table 3. All the results obtained from the transfer matrix formulation are compared with the finite-element solutions for validation of the present formulation. The finite-element solutions are obtained from the NASTRAN program using the 80 beam elements. An external stiffness matrix associated with the centrifugal forces was computed separately and added to the non-rotating stiffness matrix calculated by the NASTRAN program. The derivation of the differential stiffness matrix associated with the blade rotation is given in the next section. From the results presented in Tables 1 to 3, it is clear that the transfer matrix formulation
Table 1. Comparison of natural frequencies, single and two load path cases, rad/sec

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Single Load Path Case</th>
<th>Two Load Path Case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Transfer matrix method</td>
<td>Finite-element method</td>
</tr>
<tr>
<td>1</td>
<td>36.7738 F</td>
<td>36.8062 F</td>
</tr>
<tr>
<td>2</td>
<td>48.1092 C</td>
<td>48.1511 C</td>
</tr>
<tr>
<td>3</td>
<td>104.9309 F</td>
<td>104.9170 F</td>
</tr>
<tr>
<td>4</td>
<td>138.2931 T</td>
<td>138.2254 T</td>
</tr>
<tr>
<td>5</td>
<td>202.4001 F</td>
<td>202.3235 F</td>
</tr>
<tr>
<td>6</td>
<td>280.5927 C</td>
<td>280.4453 C</td>
</tr>
<tr>
<td>7</td>
<td>336.3352 F</td>
<td>336.1560 F</td>
</tr>
<tr>
<td>8</td>
<td>402.5505 T</td>
<td>402.5403 T</td>
</tr>
<tr>
<td>9</td>
<td>507.5868 F</td>
<td>507.2334 F</td>
</tr>
<tr>
<td>10</td>
<td>669.1642 T</td>
<td>667.9858 F,T</td>
</tr>
</tbody>
</table>

F = Predominantly flapwise bending  
C = Predominantly chordwise bending  
T = Predominantly torsion
Table 2. Comparison of natural frequencies, non-coplanar three load paths case

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Ω = 0</th>
<th>Ω = 360 RPM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Transfer matrix method</td>
<td>Finite-element method</td>
</tr>
<tr>
<td>1</td>
<td>11.4292 F</td>
<td>11.4292 F</td>
</tr>
<tr>
<td>2</td>
<td>66.1632 C</td>
<td>66.2114 C</td>
</tr>
<tr>
<td>3</td>
<td>70.1983 F</td>
<td>70.2009 F</td>
</tr>
<tr>
<td>4</td>
<td>153.6332 T</td>
<td>153.6552 T</td>
</tr>
<tr>
<td>5</td>
<td>181.4035 F</td>
<td>181.3978 F</td>
</tr>
<tr>
<td>6</td>
<td>264.3820 F</td>
<td>264.4106 F</td>
</tr>
<tr>
<td>7</td>
<td>405.9868 C</td>
<td>406.1411 C</td>
</tr>
<tr>
<td>8</td>
<td>418.2305 F</td>
<td>418.1628 F</td>
</tr>
<tr>
<td>9</td>
<td>445.6527 T</td>
<td>445.9648 T</td>
</tr>
<tr>
<td>10</td>
<td>--</td>
<td>665.0317 F</td>
</tr>
</tbody>
</table>

F = Predominantly flapwise bending
C = Predominantly chordwise bending
T = Predominantly torsion
Table 3. a. Comparison of mode shapes, two load path case, third mode, Omega = 360 rpm.
(Flapwise Deflection)

<table>
<thead>
<tr>
<th>x/R</th>
<th>Transfer Matrix Method</th>
<th>Finite Element Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Load Path I</td>
<td>Load Path II</td>
</tr>
<tr>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.10</td>
<td>-0.0743</td>
<td>-0.0744</td>
</tr>
<tr>
<td>0.160</td>
<td>-0.1315</td>
<td>-0.1318</td>
</tr>
<tr>
<td>0.200</td>
<td>-0.1468</td>
<td></td>
</tr>
<tr>
<td>0.312</td>
<td>-0.2839</td>
<td></td>
</tr>
<tr>
<td>0.408</td>
<td>-0.4600</td>
<td></td>
</tr>
<tr>
<td>0.504</td>
<td>-0.5732</td>
<td></td>
</tr>
<tr>
<td>0.600</td>
<td>-0.5639</td>
<td></td>
</tr>
<tr>
<td>0.712</td>
<td>-0.3533</td>
<td></td>
</tr>
<tr>
<td>0.808</td>
<td>+0.0008</td>
<td></td>
</tr>
<tr>
<td>0.904</td>
<td>+0.4747</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>+1.0000</td>
<td></td>
</tr>
</tbody>
</table>
Table 3. b. Comparison of mode shapes, two load path case, third mode, Omega = 360 rpm. (Chordwise Deflection)

<table>
<thead>
<tr>
<th>x/R</th>
<th>Transfer Matrix Method</th>
<th>Finite Element Method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Load Path I</td>
<td>Load Path II</td>
</tr>
<tr>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0210</td>
<td>0.0209</td>
</tr>
<tr>
<td>0.160</td>
<td>0.0380</td>
<td>0.0379</td>
</tr>
<tr>
<td>0.200</td>
<td></td>
<td>0.0436</td>
</tr>
<tr>
<td>0.312</td>
<td></td>
<td>0.0835</td>
</tr>
<tr>
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<td></td>
<td>0.1308</td>
</tr>
<tr>
<td>0.504</td>
<td></td>
<td>0.1590</td>
</tr>
<tr>
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<td></td>
<td>0.1524</td>
</tr>
<tr>
<td>0.712</td>
<td></td>
<td>0.0890</td>
</tr>
<tr>
<td>0.808</td>
<td></td>
<td>0.0132</td>
</tr>
<tr>
<td>0.904</td>
<td></td>
<td>-0.1482</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td>-0.2972</td>
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</tbody>
</table>
Table 3. c. Comparison of mode shapes, two load path case, third mode, Omega = 360 rpm. (Torsional Deflection)

<table>
<thead>
<tr>
<th>x/R</th>
<th>Transfer Matrix Method</th>
<th>Finite Element Method</th>
</tr>
</thead>
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<tr>
<td></td>
<td>Load Path I</td>
<td>Load Path II</td>
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<tr>
<td>0.00</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0015</td>
<td>0.0015</td>
</tr>
<tr>
<td>0.160</td>
<td>0.0023</td>
<td>0.0023</td>
</tr>
<tr>
<td>0.200</td>
<td>0.0029</td>
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<td>0.312</td>
<td>0.0061</td>
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</tr>
<tr>
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<td>0.0086</td>
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</tr>
<tr>
<td>0.504</td>
<td>0.0109</td>
<td></td>
</tr>
<tr>
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<td>0.0126</td>
<td></td>
</tr>
<tr>
<td>0.712</td>
<td>0.0140</td>
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</tr>
<tr>
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<td>0.0147</td>
<td></td>
</tr>
<tr>
<td>0.904</td>
<td>0.0150</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>0.0151</td>
<td></td>
</tr>
</tbody>
</table>
yields very accurate results for multiple load path rotor blades. In fact, the slight discrepancies noticed in these results is due to the approxima-
tions made in the finite-element solutions, for instance, a constant tension is assumed within each finite-element and actual variation is used in the transfer matrix solution.

4.3 Differential Stiffness Matrix Due to Rotation

The work done by centrifugal forces due to combined flapwise bending, chordwise bending, torsion and axial stretching is given by (Ref. 10)

\[
P = \frac{1}{2} \int_0^l T(v'^2 + w'^2) dx - \frac{1}{2} \int_0^l \Omega^2 (u^2 + v^2) dx
\]

\[
- \frac{1}{2} \int_0^l \Omega^2 m (\cos \beta - \phi \sin \beta) v dx
\]

\[
+ \int_0^l \Omega^2 m \{x e (v' \cos \beta + w' \sin \beta)
\]

\[
+ \phi x e (-v' \sin \beta + w' \cos \beta) \} dx
\]

\[
+ \frac{1}{2} \int_0^l \Omega^2 m (k^2_{m2} - k^2_{m1}) (\sin 2\beta + \phi \cos 2\beta) \phi dx
\]

Consider the following displacement functions for flapwise bending, chordwise bending, torsion and axial displacement in terms of the shape functions

\[
w(x) = w_1 (1 - 3x^2/\ell^2 + 2x^3/\ell^3) + \theta_{y_1} (-x + 2x^2/\ell - x^3/\ell^2)
\]

\[
+ w_2 (3x^2/\ell^2 - 2x^3/\ell^3) + \theta_{y_2} (x^2/\ell - x^3/\ell^2)
\]
\begin{align*}
v(x) &= v_1 \left(1 - 3x^2/\ell^2 + 2x^3/\ell^3\right) + \theta_{z_1} (x - 2x^2/\ell + x^3/\ell^2) \\
&\quad + v_2 \left(3x^2/\ell^2 - 2x^3/\ell^3\right) + \theta_{z_2} (-x^2/\ell + x^3/\ell^2) \quad (95) \\
\phi(x) &= \phi_1 \left(1 - x/\ell\right) + \phi_2 \left(x/\ell\right) \quad (96) \\
u(x) &= u_1 \left(1 - x/\ell\right) + u_2 \left(x/\ell\right) \quad (97)
\end{align*}

Substitute Eqs. (94) to (97) into Eq. (93) and arrange the resulting equation into the following form

\begin{equation}
P = \frac{1}{2} \{d\}^T [k] \{d\} \quad (98)
\end{equation}

where

\begin{equation}
\{d\}^T = \begin{bmatrix}
u_1 & w_1 & \phi_1 & \theta_{y_1} & \theta_{z_1} & u_2 & v_2 & w_2 & \phi_2 & \theta_{y_2} & \theta_{z_2}
\end{bmatrix}
\end{equation}

The matrix \([k]\) in Eq. (98) is the differential stiffness matrix due to rotation. The final form of this matrix is given in Appendix C.
References


THIS PROGRAM COMPUTES THE NATURAL VIBRATION CHARACTERISTICS OF MULTIPLE LOAD PATH ROTOR BLADES (UP TO THREE LOAD PATHS) UNDERGOING BENDING - BENDING - TORSION - AXIAL VIBRATIONS.

DECLARATION STATEMENTS

IMPLICIT REAL*8 (A - H, O - Z)
REAL*8 MASS, MASS1(3,21), MASS2(101), KM1S1(3,21)
REAL*8 KM2S1(3,21), KM1S2(101), KM2S2(101)
DIMENSION STA1(21), STA2(101), E1Y1(3,21),
1 E1Y2(101), EA1(3,21), EA2(101), E1Z1(3,21),
1 E1Z2(101), G1J1(3,21), GJ2(101), E1(3,21),
1 E2(101), BETA1(3,21), BETA2(101), W1(3,11),
1 W2(51), V1(3,11), V2(51), PHI1(3,11), PHI2(51),
1 SL1(51), SL2(51), STA(101), FD(3,2),
1 U(3), BB(12), AT(3), TEN1(3,21),
1 TEN2(101), D11(3,21), D12(3,21),
1 D13(3,21), D14(3,21), D15(3,21),
1 D16(3,21), D17(3,21), D18(3,21),
1 D19(3,21), D191(3,21), D192(3,21),
1 D21(101), D22(101), D23(101),
1 D24(101), D25(101), D26(101),
1 D27(101), D28(101), D29(101),
1 D291(101), D292(101), FREQUENCY(10),
1 TF1(3,12,12), TF2(12,12), XA(6,6), XB(6,6), XC(6,6),
1 XD(6,6), XE(6,6), XR(6,6)
COMMON/X1/FREQUENCY, H, H1, H2, IJK
COMMON/X2/FACT
COMMON/X3/STA, NS
COMMON/X4/NPATH, ICPL
COMMON/X5/E1Y1, TEN2, EA1, EA2, D11, D21, D12, D22, D13,
1 D23, D14, D24, D15, D25, D16, D26, D17, D27, D18, D28, D19,
1 D29, D191, D291, D192, D292, OMEGAN
COMMON/X6/SL1, SL2, CPM, FREQUENCY, HERTZ, BLANK, DOT, STAR, J1, IPLOT
COMMON/X7/ HH1, HH2
COMMON/X8/ BB, IND
COMMON/X9/ FD
COMMON/X10/ TF1, TF2
COMMON/X11/ XA, XB, XC, XD, XE, XR
COMMON/X12/ CON
THIS SECTION READS THE DATA OF THE SYSTEM

READ(20,100) NPATH, ISTAGE, I PLOT, NS1, NS2
READ(20,105) SPAN1, SPAN2, SCH, OMEGA
READ(20,105) (STA1(I), I=1,NS1)
READ(20,105) (STA2(I), I=1,NS2)
DO 400 I=1,NPATH
   READ(20,105) (MASS1(I,J), J=1,NS1)
   READ(20,105) (EIY1(I,J), J=1,NS1)
   READ(20,105) (EIZ1(I,J), J=1,NS1)
   READ(20,105) (GJ1(I,J), J=1,NS1)
   READ(20,105) (E1(I,J), J=1,NS1)
   READ(20,105) (KMS1(I,J), J=1,NS1)
   READ(20,105) (KM2S1(I,J), J=1,NS1)
   READ(20,105) (EA1(I,J), J=1,NS1)
400 CONTINUE
READ(20,105) (MASS2(I), I=1,NS2)
READ(20,105) (EIY2(I), I=1,NS2)
READ(20,105) (EIZ2(I), I=1,NS2)
READ(20,105) (GJ2(I), I=1,NS2)
READ(20,105) (E2(I), I=1,NS2)
READ(20,105) (BETA2(I), I=1,NS2)
READ(20,105) (KM1S2(I), I=1,NS2)
READ(20,105) (KM2S2(I), I=1,NS2)
READ(20,105) (EA2(I), I=1,NS2)
READ(20,105) (FD(I,J), J = 1,2), I = 1, NPATH
IF(NPATH.GT.1) READ(20,105) ((FD(I,J), J = 1,2), I = 1, NPATH)
IF(NPATH.GT.1) READ(20,100) ITR
IF(ISTAGE.NE.1) READ(20,100) NF
IF(ISTAGE.EQ.4) READ(20,105) (FREQEN(J), J = 1,NF)
IF(IPLOT.EQ.1) READ(20,110) BLANK, DOT, STAR
THIS SECTION PRINTS THE DATA OF THE SYSTEM

SPAN = SPAN1 + SPAN2
WRITE(22,200)
WRITE(22,205)
WRITE(22,347) ISTAGE
WRITE(22,348) I PLOT
IF (ISTAGE .NE. 1) WRITE(22,210) NF
WRITE(22,310) H1
WRITE(22,215) H
WRITE(22,220) H2
WRITE(22,225) SPAN
WRITE(22,230) SPAN1
WRITE(22,235) SPAN2
WRITE(22,236) SCH
WRITE(22,240) OMEGA
WRITE(22,245) NS1
WRITE(22,250) NS2
WRITE(22,315) NPATH
DO 405 I = 1, NPATH
  IF(I .EQ. 1) WRITE(22,255)
  IF(I .EQ. 2) WRITE(22,260)
  IF(I .EQ. 3) WRITE(22,265)
  WRITE(22,270)
  WRITE(22,275) (STA1(J), J=1,NS1)
  WRITE(22,280)
  WRITE(22,275) (MASS1(I,J), J=1,NS1)
  WRITE(22,285)
  WRITE(22,275) (EIY1(I,J), J=1,NS1)
  WRITE(22,350)
  WRITE(22,275) (EIZ1(I,J), J=1,NS1)
  WRITE(22,352)
  WRITE(22,275) (GJ1(I,J), J=1,NS1)
  WRITE(22,354)
  WRITE(22,275) (EI(I,J), J=1,NS1)
  WRITE(22,356)
  WRITE(22,275) (BETA1(I,J), J=1,NS1)
  WRITE(22,358)
  WRITE(22,275) (KM1S1(I,J), J=1,NS1)
  WRITE(22,360)
  WRITE(22,275) (KM2S1(I,J), J=1,NS1)
  WRITE(22,286)
  WRITE(22,275) (EA1(I,J), J=1,NS1)
  IF(NPATH .GT. 1) WRITE(22,321) I, FD(I,1), FD(I,2)
  IF(NPATH .GT. 1) WRITE(22,375) ITR
405 CONTINUE
WRITE(22,290)
WRITE(22,270)
WRITE(22,275) (STA2(I), I=1,NS2)
WRITE(22,280)
WRITE(22,275) (MASS2(I), I=1,NS2)
WRITE(22,285)
WRITE(22,275) (EIY2(I), I=1,NS2)
WRITE(22,350)
WRITE(22,275) (EIZ2(I), I=1,NS2)
WRITE(22,352)
WRITE(22,275) (GJ2(I), I=1,NS2)
WRITE(22,354)
WRITE(22,275) (E2(I), I=1,NS2)
WRITE(22,356)
WRITE(22,275) (BETA2(I), I=1,NS2)
WRITE(22,358)
WRITE(22,275) (KM1S2(I), I=1,NS2)
WRITE(22,360)
WRITE(22,275) (KM2S2(I), I=1,NS2)
WRITE(22,343)
WRITE(22,275) (EA2(I), I=1,NS2)
WRITE(22,205)

C
This section interpolates the data

WRITE(22, 200)
WRITE(22, 205)
WRITE(22, 295)

\[ \text{NS} = \text{NS1} \]
\[ \text{STA}(J) = \text{STA1}(J) \]

411 CONTINUE

\[ \text{HH} = \text{SPAN1}/20.0 \]

DO 420 I = 1, NPATH

DO 410 J = 1, NS1

\[ \text{D21}(J) = \text{MASS1}(I, J) \]
\[ \text{D22}(J) = \text{EIY1}(I, J) \]
\[ \text{D23}(J) = \text{EIZ1}(I, J) \]
\[ \text{D24}(J) = \text{GJ1}(I, J) \]
\[ \text{D25}(J) = \text{E1}(I, J) \]
\[ \text{D26}(J) = \text{BETA1}(I, J) \]
\[ \text{D27}(J) = \text{KM1S1}(I, J) \]
\[ \text{D28}(J) = \text{KM2S1}(I, J) \]
\[ \text{TEN2}(J) = \text{EA1}(I, J) \]

410 CONTINUE

CALL INTPOL(21, D21, HH)
CALL INTPOL(21, D22, HH)
CALL INTPOL(21, D23, HH)
CALL INTPOL(21, D24, HH)
CALL INTPOL(21, D25, HH)
CALL INTPOL(21, D26, HH)
CALL INTPOL(21, D27, HH)
CALL INTPOL(21, D28, HH)
CALL INTPOL(21, TEN2, HH)

DO 415 J = 1, 21

\[ \text{MASS1}(I, J) = \text{D21}(J) \]
\[ \text{EIY1}(I, J) = \text{D22}(J) \]
\[ \text{EIZ1}(I, J) = \text{D23}(J) \]
\[ \text{GJ1}(I, J) = \text{D24}(J) \]
\[ \text{E1}(I, J) = \text{D25}(J) \]
\[ \text{BETA1}(I, J) = \text{D26}(J) \]
\[ \text{KM1S1}(I, J) = \text{D27}(J) \]
\[ \text{KM2S1}(I, J) = \text{D28}(J) \]
\[ \text{EA1}(I, J) = \text{TEN2}(J) \]

415 CONTINUE

IF(I .EQ. 1) WRITE(22, 255)
IF(I .EQ. 2) WRITE(22, 260)
IF(I .EQ. 3) WRITE(22, 265)

WRITE(22, 280)
WRITE(22, 275) (D21(J), J=1,21)
WRITE(22, 285)
WRITE(22, 275) (D22(J), J=1,21)
WRITE(22, 350)
WRITE(22, 275) (D23(J), J=1,21)
WRITE(22, 352)
WRITE(22, 275) (D24(J), J=1,21)
WRITE(22, 354)
WRITE(22,275) (D25(J), J=1,21)
WRITE(22,356)
WRITE(22,275) (D26(J), J=1,21)
WRITE(22,358)
WRITE(22,275) (D27(J), J=1,21)
WRITE(22,360)
WRITE(22,275) (D28(J), J=1,21)
WRITE(22,286)
WRITE(22,275) (TEN2(J), J=1,21)  

420 CONTINUE
NS = NS2
DO 421 J = 1, NS2
   STA(J) = STA2(J)
421 CONTINUE

HH = SPAN2/100.0
CALL INTPOLU01, MASS2, HH
CALL INTPOLU01, EIY2, HH
CALL INTPOLU01, EIZ2, HH
CALL INTPOLU01, GJ2, HH
CALL INTPOLU01, E2, HH
CALL INTPOLU01, BETA2, HH
CALL INTPOLU01, KM1S2, HH
CALL INTPOLU01, KM2S2, HH
CALL INTPOLU01, EA2, HH
WRITE(22,300)
WRITE(22,280)
WRITE(22,275) (MASS2(J), J=1,101)
WRITE(22,285)
WRITE(22,275) (EIY2(J), J=1,101)
WRITE(22,350)
WRITE(22,275) (EIZ2(J), J=1,101)
WRITE(22,352)
WRITE(22,275) (GJ2(J), J=1,101)
WRITE(22,354)
WRITE(22,275) (E2(J), J=1,101)
WRITE(22,356)
WRITE(22,275) (BETA2(J), J=1,101)
WRITE(22,358)
WRITE(22,275) (KM1S2(J), J=1,101)
WRITE(22,360)
WRITE(22,275) (KM2S2(J), J=1,101)
WRITE(22,343)
WRITE(22,275) (EA2(J), J=1,101)
THIS SECTION NONDIMENSIONALIZES THE DATA AND DETERMINES THE COEFFICIENTS OF THE FIRST ORDER DIFFERENTIAL EQUATIONS

\[ PI = 4.0 \times \text{DATAN}(1.0D+00) \]
\[ OMEGA = OMEGA \times PI/30.0 \]
\[ OMEGAS = OMEGA \times OMEGA \]
\[ \text{IF}(\text{NPATH} \ .EQ. 1) \ FD(1,1) = 0.0 \]
\[ \text{IF}(\text{NPATH} \ .EQ. 1) \ FD(1,2) = 0.0 \]
\[ ICPL = 0.0 \]
\[ \text{DO 422} \ I = 1, \text{NPATH} \]
\[ \text{DO 422} \ J = 1,2 \]
\[ \text{IF}(\text{FD}(I,J) \ .GE. 1.0E-10) \ ICPL = 1 \]

\[ \text{CONTINUE} \]
\[ \text{STEP} = \text{SPAN2}/100.0 \]
\[ X = \text{SPAN1} \]
\[ H5 = \text{STEP}/24.0 \]
\[ \text{DO 425} \ I = 1, 101 \]
\[ D21(I) = \text{MASS2}(I) \times H5 \times X \]
\[ X = X + \text{STEP} \]

\[ \text{CONTINUE} \]
\[ \text{TEN2}(101) = 0.0 \]
\[ \text{TEN2}(100) = (9.0 \times D21(101) + 19.0 \times 1) \]
\[ D21(100) - 5.0 \times D21(99) + D21(98) \times OMEGAS \]
\[ \text{DO 426} \ I = 2, 99 \]
\[ J = 101 - I \]
\[ \text{TEN2}(J) = \text{TEN2}(J+1) + ( -D21(J+2) + 13.0 \times 1 \times D21(J+1) + D21(J) - D21(J-1) \times OMEGAS \]

\[ \text{CONTINUE} \]
\[ \text{TEN2}(1) = \text{TEN2}(2) + D21(4) - 5.0 \times D21(3) + 19.0 \times D21(2) + 9.0 \times D21(1) \times OMEGAS \]
\[ \text{EIY} = \text{EIY2}(1) \]
\[ \text{MASS} = \text{MASS2}(1) \]
\[ \text{SPANS} = \text{SPAN} \times \text{SPAN} \]
\[ \text{FA} = \text{SPANS} / \text{EIY} \]
\[ \text{FACT} = \text{DSQRT}(\text{FA} \times \text{SPANS} \times \text{MASS}) \]
\[ \text{CON} = \text{SPAN} / \text{SCH} \]
\[ \text{OMEGAN} = \text{OMEGAS} \times \text{FACT} \times \text{FACT} \]
\[ \text{HH} = \text{SPAN1} / (20.0 \times \text{SPAN}) \]
\[ \text{DO 4251} \ I = 1, \text{NPATH} \]
\[ X = 0.0 \]
\[ \text{DO 4251} \ J = 1,21 \]
\[ \text{KM1S1}(I,J) = \text{KM1S1}(I,J) / \text{MASS1}(I,J) \]
\[ \text{KM2S1}(I,J) = \text{KM2S1}(I,J) / \text{MASS1}(I,J) + \text{E1}(I,J) \times \text{E1}(I,J) \]
\[ \text{EA1}(I,J) = 1.0 / ( \text{EA1}(I,J) \times \text{FA}) \]
\[ \text{PITCH} = \text{BETA1}(I,J) \times PI / 180.0 \]
\[ \text{CO} = \text{DCOS}(\text{PITCH}) \]
\[ \text{SI} = \text{DSIN}(\text{PITCH}) \]
\[ \text{CS} = \text{CO} \times \text{CO} \]
\[ \text{SS} = \text{SI} \times \text{SI} \]
\[ \text{A11} = -\text{EIZ1}(I,J) \times \text{SS} - \text{EIY1}(I,J) \times \text{CS} \]
\[ \text{A12} = -\text{CO} \times \text{SI} \times (\text{EIZ1}(I,J) - \text{EIY1}(I,J)) \]
\[ \text{A22} = \text{EIZ1}(I,J) \times \text{CS} + \text{EIY1}(I,J) \times \text{SS} \]
\[ \text{DE} = \text{A11} \times \text{A22} + \text{A12} \times \text{A12} \]
\[ \text{D19}(I,J) = -\text{A12} \times \text{EIY} / \text{DE} \]
D191(I,J) = A22 * E1(I,J) / DE
D192(I,J) = A11 * E1(I,J) / DE
D11(I,J) = E1I/G1(I,J)
D12(I,J) = MASS1(I,J)/MASS
D13(I,J) = D12(I,J) * CO * E1(I,J) / SPAN
D14(I,J) = D12(I,J) * SI * E1(I,J) / SPAN
D15(I,J) = D13(I,J) * X * OMEGAN
D16(I,J) = D14(I,J) * X * OMEGAN
D17(I,J) = OMEGAN * D12(I,J) * (KM1S(I,J) - 1)
D18(I,J) = D12(I,J) * (KM1S(I,J) + KM2S(I,J)) / SPANS

X = X + HH

4251 CONTINUE
X = SPAN1 / SPAN
HH = SPAN2 / (100.0 * SPAN)
DO 428 J = 1, 101
KM1S2(J) = KM1S2(J) / MASS2(J)
KM2S2(J) = KM2S2(J) / MASS2(J) + E2(J) * E2(J)
EA2(J) = 1.0 / (EA2(J) * FA)
PITCH = BETAT2(J) * PI / 180.0
CO = DCOS(PITCH)
SI = DSIN(PITCH)
CS = CO * CO
SS = SI * SI
A12 = -CO * SI * (EI2(J) - EIY2(J))
A11 = -EI2(J) * SS - E1I2(J) * CS
A22 = EI2(J) * CS + EIY2(J) * SS
DE = A11 * A22 + A12 * A12
D29(J) = -A12 * E1I / DE
D291(J) = A22 * E1I / DE
D292(J) = A11 * E1I / DE
D21(J) = E1I / G2(J)
D22(J) = MASS2(J) / MASS
D23(J) = D22(J) * CO * E2(J) / SPAN
D24(J) = D22(J) * SI * E2(J) / SPAN
D25(J) = D23(J) * X * OMEGAN
D26(J) = D24(J) * X * OMEGAN
D27(J) = OMEGAN * D22(J) * (KM2S2(J) - 1)
           KM1S2(J) ) * (CS - SS) / SPANS
D28(J) = D22(J) * (KM1S2(J) + KM2S2(J)) / SPANS
TEN2(J) = FA * TEN2(J)
          X = X + HH

428 CONTINUE
HH1 = SPAN1 / (10.0 * SPAN)
HH2 = SPAN2 / (50.0 * SPAN)
STEP = SPAN1 / (20.0 * SPAN)
H5 = STEP / 24.0
GO TO (4281, 4282, 4282), NPATH
4281
GO TO 4286
4282 DO 4284 J = 1,NPATH
AT(J) = (9.0 * EA1(J,1) + 19.0 * EA1(J,2) - 5.0
           * EA1(J,3) + EA1(J,4) ) * H5
DO 4283 K = 1,18
AT(J) = AT(J) + H5 * ( -EA1(J,K) + 13.0 * EA1(J,K+1)
1 + 13.0 * EA1(J,K+2) - EA1(J,K+4) )
CONTINUE
AT(J) = AT(J) + H5 * ( EA1(J,18) - 5.0 * EA1(J,19)
+ 9.0 * EA1(J,20) + 9.0 * EA1(J,21) )

CONTINUE
IF(NPATH .EQ. 3) GO TO 4285
BB(1) = AT(1) * TEN2(1) / ( AT(1) + AT(2) )
BB(2) = AT(2) * TEN2(1) / ( AT(1) + AT(2) )
GO TO 4286

PR = AT(1) * AT(2) + AT(2) * AT(3) + AT(3) * AT(1)
BB(1) = TEN2(1) * AT(2) * AT(3) / PR
BB(2) = TEN2(1) * AT(3) * AT(1) / PR
BB(3) = TEN2(1) * AT(1) * AT(2) / PR

CONTINUE
WRITE(22,380)
WRITE(22,275) ((BB(I) / FA),I = 1,NPATH)
DO 4290 JJ = 1, NPATH
X = 0.0
DO 4287 I = 1, 21
MASS2(I) = MASS1(JJ,I) * H5 * X / MASS
X = X + STEP
4287 CONTINUE
TEN1(JJ,21) = 0.0
TEN1(JJ,20) = (9.0 * MASS2(21) + 19.0 * MASS2(20)
1 - 5.0 * MASS2(19) + MASS2(18) ) * OMEGAN
DO 429 I = 2, 19
J = 21 - I
TEN1(JJ,J) = TEN1(JJ,J+1) + ( -MASS2(J+2) + 13.0 *(MASS2(J+1)
1 + MASS2(J)) - MASS2(J-1) ) * OMEGAN
429 CONTINUE
TEN1(JJ,1) = TEN1(JJ,2) + ( MASS2(4) - 5.0 * MASS2(3)
1 + 19.0 * MASS2(2) + 9.0 * MASS2(1) ) * OMEGAN

CONTINUE
IF(ICPL .EQ. 0) ITR = 1
DO 4291 I = 1, NPATH
DO 4291 J = 1, 21
EIY1(I,J) = TEN1(I,J) + BB(I)
4291 CONTINUE
WRITE(22,376)
DO 4292 I = 1,NPATH
WRITE(22,377) I
WRITE(22,275) ((EIY1(I,J)/FA),J = 1,21)
4292 CONTINUE
DO 4318 IK = 1, ITR
DO 431 J = 1, 6
DO 431 K = 1, 6
XR(J,K) = 0.0
431 CONTINUE
CALL TRAMAT(0.0D+00)
DO 4314 I = 1, NPATH
DO 4310 J = 1, 6
DO 4310 K = 1, 6
XC(J,K) = TF1(I,J,K+6)
XD(J,K) = TF1(I,J+6,K+6)
4310 CONTINUE
CALL SOLUTN(XC,6,-1,6)
DO 4311 J = 1, 6
XC(J,4) = XC(J,4) - FD(I,2) * XC(J,1)
XC(J,5) = XC(J,5) - FD(I,1) * XC(J,1)

4311 CONTINUE
CALL MATMUL(6,6,6,XD,XC,XE)
DO 4312 J = 1,6
   DO 4312 K = 1,6
      XC(J,K) = XE(J,K)
      IF(I .EQ. 1) XA(J,K) = XE(J,K)
      IF(I .EQ. 2) XB(J,K) = XE(J,K)
   4312 CONTINUE
DO 4313 J = 1,6
   XE(1,J) = XE(1,J) - FD(I,2) * XE(4,J) + FD(I,1) * XE(5,J)
   XE(2,J) = XE(2,J) - FD(I,1) * XE(6,J)
   XE(3,J) = XE(3,J) + FD(I,2) * XE(6,J)
4313 CONTINUE
CALL SOLUTN(XR,6,-1,6)
DO 4315 J = 1,6
   XD(J,1) = XR(J,6) * TEN2(1)
4315 CONTINUE
DO 4316 I = 1, NPATH
   IF(I .EQ. 1) CALL MATMUL(6,6,1,XA,XD,XE)
   IF(I .EQ. 2) CALL MATMUL(6,6,1,XB,XD,XE)
   IF(I .EQ. 3) CALL MATMUL(6,6,1,XC,XD,XE)
   BB(I) = XE(6,1)
4316 CONTINUE
DO 4317 I = 1, NPATH
   DO 4317 J = 1, 21
      EIY1(I,J) = TEN1(I,J) + BB(I)
4317 CONTINUE
WRITE(22,378) IK
DO 4318 I = 1, NPATH
   WRITE(22,377) I
   WRITE(22,275) ((EIY1(I,J)/FA), J= 1,21)
4318 CONTINUE
WRITE(22,205)
H = H * FACT
H1 = H1 * FACT
H2 = H2 * FACT
IF(NPATH .LE. 1) GO TO 452
DO 451 J = 1, NPATH
   DO 451 K = 1, 2
      FD(J,K) = FD(J,K)/SPAN
451 CONTINUE
HH1 = SPAN1 / (10.0 * SPAN )
HH2 = SPAN2 / (50.0 * SPAN )
IF(ISTAGE .EQ. 4) GO TO 471
IF(ISTAGE .NE. 1) GO TO 460
This section computes the frequency determinants if ISTAGE = 1

```fortran
WRITE(22,200)
WRITE(22,205)
WRITE(22,305)

455  P = H1 * H1
     IF(H1 .GT. H2) GO TO 505
     FR = H1/FACT
     F = DET(P)
     WRITE(22,275) FR, H1, F
     H1 = H + H1
     GO TO 455
```

C
THIS SECTION COMPUTES THE NATURAL VIBRATION CHARACTERISTICS

CALL NATFRE(NF)
IF(IJK .EQ. 0) GO TO 500
WRITE(22,200)
WRITE(22,205)
WRITE(22,370)
WRITE(22,205)
DO 461 J = 1, IJK
FRE = FREQEN(J) / FACT
HERTZ = FRE / (2.0 * PI)
WRITE(22,371) J, FRE, HERTZ
461 CONTINUE
WRITE(22,205)
IF(ISTAGE .EQ. 3) GO TO 505
ST = SPAN1/(SPAN * 10.0)
DO 465 J = 1, 11
SL1(J) = FLOAT(J-1) * ST
465 CONTINUE
ST = SPAN2/(SPAN * 50.0)
DO 470 J = 1, 51
SL2(J) = SL1(11) + FLOAT(J-1) * ST
470 CONTINUE
GO TO 473
471 DO 472 J = 1, NF
FREQEN(J) = FREQEN(J) * FACT
472 CONTINUE
IJK = NF
473 DO 495 IJ = 1, IJK
   J1 = IJ
   PP = FREQEN(IJ)
   P = PP * PP
   CALL SHAPES(P,W1,W2,V1,V2,PHI1,PHI2,U,UC,PSIC,ANUC)
   AMAX = W1(1,1)
   DO 475 I = 1,NPATH
   DO 475 J = 1, 11
      IF(DABS(AMAX) .LT. DABS(W1(I,J))) AMAX = W1(I,J)
      IF(DABS(AMAX) .LT. DABS(V1(I,J))) AMAX = V1(I,J)
      IF(DABS(AMAX) .LT. DABS(PHI1(I,J))) AMAX = PHI1(I,J)
   475 CONTINUE
   DO 480 J = 1, 51
      IF(DABS(AMAX) .LT. DABS(W2(J))) AMAX = W2(J)
      IF(DABS(AMAX) .LT. DABS(V2(J))) AMAX = V2(J)
      IF(DABS(AMAX) .LT. DABS(PHI2(J))) AMAX = PHI2(J)
   480 CONTINUE
   DO 485 I = 1, NPATH
   DO 485 J = 1, 11
      U(I) = U(I)/AMAX
      W1(I,J) = W1(I,J)/AMAX
      V1(I,J) = V1(I,J)/AMAX
      PHI1(I,J) = PHI1(I,J)/AMAX
   485 CONTINUE
   DO 490 J = 1, 51
      W2(J) = W2(J)/AMAX
   490 CONTINUE
V2(J) = V2(J)/AMAX
PHI2(J) = PHI2(J)/AMAX

CONTINUE
FRE = PP / FACT
HERTZ = FRE / (2.0 * PI)
CPM = HERTZ * 60.0
CALL PLOT(1, W2, W1)
CALL PLOT(2, V2, V1)
CALL PLOT(3, PHI2, PHI1)

DO 496 J = 1, NPATH
   U(J) = U(J) / AMAX
   WRITE(22,372) J, U(J)
496 CONTINUE

PSIC = PSIC / AMAX
WRITE(22,373) PSIC
ANUC = ANUC / AMAX
WRITE(22,374) ANUC
495 CONTINUE

500 IF(IJK .LT. NF) WRITE(22,320) IJK
505 IF(ISTAGE .EQ. 1) WRITE(22,205)
FORMATS

100 FORMAT(5IS)
105 FORMAT(5E14.7)
110 FORMAT(3A1)
200 FORMAT(1H1)
205 FORMAT(//2X, '============================================',
1 '============================================')
210 FORMAT(//5X, 'NUMBER OF FREQUENCIES REQUIRED = ',I5)
215 FORMAT(//5X, 'FREQUENCY INCREMENTS (RAD/SEC) = ',E14.7)
220 FORMAT(//5X, 'ENDING FREQUENCY (RAD/SEC) = ',E14.7)
225 FORMAT(//5X, 'RADIUS OF THE ROTOR (INCHES) = ',E14.7)
230 FORMAT(//5X, 'LENGTH OF INBOARD SEGMENTS (IN) = ',E14.7)
235 FORMAT(//5X, 'LENGTH OF THE BLADE (INCHES) = ',E14.7)
236 FORMAT(//5X, 'SEMI-CHORD (INCHES) = ',E14.7)
240 FORMAT(//5X, 'ROTATIONAL SPEED (RPM) = ',E14.7)
245 FORMAT(//5X, 'NUMBER OF DATA POINTS FOR INBOARD SEGMENTS = ',I5)
250 FORMAT(//5X, 'NUMBER OF DATA POINTS FOR BLADE = ',I5)
255 FORMAT(//5X, 'PROPERTIES OF THE FIRST LOAD PATH')
260 FORMAT(//5X, 'PROPERTIES OF THE SECOND LOAD PATH')
265 FORMAT(//5X, 'PROPERTIES OF THE THIRD LOAD PATH')
270 FORMAT(//5X, 'DATA POINT LOCATIONS IN INCHES')
275 FORMAT(//5X, 'DATA POINT LOCATIONS IN INCHES')
277 FORMAT(4(6X,E14.7))
280 FORMAT(//5X, 'MASS PER UNIT LENGTH (LB-SEC**2/IN**2)')
285 FORMAT(//5X, 'FLAPWISE BENDING STIFFNESS (LB-IN**2)')
286 FORMAT(//5X, 'AXIAL STIFFNESS(LB)')
290 FORMAT(//5X, 'PROPERTIES OF THE BLADE')
295 FORMAT(//5X, 'INTERPOLATED VALUES FOR INBOARD SEGMENTS, 21',
1 'EQUIDISTANT VALUES')
300 FORMAT(//5X, 'INTERPOLATED VALUES FOR THE BLADE, 101',
1 'EQUIDISTANT VALUES')
305 FORMAT(//5X, 'FREQUENCY (RAD/SEC) NON-DIMENSIONAL FREQUENCY',
1 'DETERMINANT')
310 FORMAT(//5X, 'STARTING FREQUENCY (RAD/SEC) = ',E14.7)
315 FORMAT(//5X, 'NUMBER OF LOAD PATHS = ',I5)
320 FORMAT(//5X, 'NUMBER OF FREQUENCIES DETECTED WITHIN RANGE',
1 'AND THE BLADE(IN) ARE = ',F6.2,5X,F6.2)
321 FORMAT(//5X, 'Y, Z DISTANCES BETWEEN LOAD PATH NO: ',I2,
1 'AND THE BLADE(IN) ARE = ',F6.2,5X,F6.2)
343 FORMAT(//5X, 'DISTANCE BETWEEN MASS AND ELASTIC AXIS(IN) = ')
372 FORMAT(//5X,'AXIAL DISPLACEMENT OF LOAD PATH # ',I3,2X,
1/5X, 'AT THE CLEVIS = ',E11.4)
373 FORMAT(//5X,'FLAPWISE BENDING SLOPE AT THE CLEVIS = ',E11.4)
374 FORMAT(//5X,'CHORDWISE BENDING SLOPE AT THE CLEVIS = ',E11.4)
375 FORMAT(//5X,'NUMBER OF TENSION ITERATIONS = ',I5)
376 FORMAT(//5X,'THE FOLLOWING ARE THE STARTING TENSIONS')
377 FORMAT(//5X,'THE TENSION COEFFICIENTS IN LOAD'
1/5X,'PATH #',I5,' ARE')
378 FORMAT(//5X,'THE TENSIONS AFTER ITERATION #',I5,' ARE')
379 FORMAT(//5X,'THE TENSIONS IN THE BLADE ARE')
380 FORMAT(//5X,'STARTING TENSIONS IN THE LOAD PATHS AT THE'
1 /5X,'CLEVIS ARE')
STOP
END
C
FUNCTION DET(P)

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION TF1(3,12,12), TF2(12,12), A(6,6), B(6,6), C(6,6), D(6,6), E(6,6), R(6,6), FD(3,2)
COMMON/X4/NPATH,ICPL
COMMON/X9/FD
COMMON/X10/TF1, TF2
COMMON/XS2/DETER
CALL TRAMAT(P)
DO 10 I = 1, 6
  DO 10 J = 1, 6
    A(I,J) = TF2(H-6, J)
    B(I,J) = TF2(I+6, J+6)
    R(I,J) = 0.0
  10 CONTINUE
DO 20 I = 1, NPATH
  DO 15 J = 1, 6
    DO 15 K = 1, 6
      C(J,K) = TF1(I, J, K+6)
      D(J,K) = TF1(I, J+6, K+6)
    15 CONTINUE
  CALL SOLUTN(C, 6, -1, 6)
  DO 16 J = 1, 6
    C(J,4) = C(J,4) - FD(I,2) * C(J,1)
    C(J,5) = C(J,5) - FD(I,1) * C(J,1)
  16 CONTINUE
CALL MATMUL(6, 6, 6, D, C, E)
  DO 18 J = 1, 6
    E(1,J) = E(1,J) - FD(I,2) * E(4,J) + FD(I,1) * E(5,J)
    E(2,J) = E(2,J) - FD(I,1) * E(6,J)
    E(3,J) = E(3,J) + FD(I,2) * E(6,J)
  18 CONTINUE
  DO 20 J = 1, 6
    DO 20 K = 1, 6
      R(J,K) = R(J,K) - E(J,K)
    20 CONTINUE
CALL MATMUL(6, 6, 6, B, R, D)
  DO 25 J = 1, 6
    DO 25 K = 1, 6
      B(J,K) = D(J,K) + A(J,K)
  25 CONTINUE
CALL SOLUTN(B, 6, -1, 6)
DET = DETER
RETURN
END
SUBROUTINE INTPOL(N, A, H)

C This subroutine interpolates for the required values

IMPLICIT REAL*8 (A - H, 0 - Z)
DIMENSION A(101), STA(101), TABLE(101,1), B(101)
COMMON/X3/STA, NS
NN = N - 1
A(N) = A(NS)
NM1 = NS - 1
DO 20 I = 1, NM1
20 TABLE(I,1) = (A(I+1) - A(I))/(STA(I+1) - STA(I))
XARG = H
DO 35 I = 2, NN
   DO 25 J = 1, NS
      IF(J .EQ. NS .OR. XARG .LE. STA(J)) GO TO 30
   25 CONTINUE
30 MAX = J
   IF(MAX .LE. 2) MAX = 2
   ISUB = MAX - 1
   YEST = TABLE(ISUB,1)
   B(I) = YEST * (XARG - STA(ISUB)) + A(ISUB)
35 XARG = XARG + H
DO 40 J = 2, NN
40 A(J) = B(J)
RETURN
END
SUBROUTINE MATMUL(L, M, N, A, B, C)

MATRIX MULTIPLICATION

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION A(6,6), B(6,6), C(6,6)
DO 10 I = 1, L
  DO 10 J = 1, N
    C(I,J) = 0.0
  DO 10 K = 1, M
    C(I,J) = C(I,J) + A(I,K) * B(K,J)
  10 CONTINUE
RETURN
END
SUBROUTINE MROOT(SDT, H, FRE, IJ, ICOUNT)

C THIS SUBROUTINE CALCULATES THE MISSING FREQUENCIES, IF ANY

C

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION SDT(2,2000), FREMIS(10), FRE(10)
COMMON/X2/FACT
INDEX = 0
N = ICOUNT - 2
IJ = 0
DO 10 I = 1, N
   A = SDT(2,I)
   B = SDT(2,I+1)
   C = SDT(2,H-2)
   D = A * B
   IF(D .LT. 0.0) GO TO 10
   D = B * C
   IF(D .LT. 0.0) GO TO 10
   D = DABS(A) - DABS(B)
   IF(D .LT. 0.0) GO TO 10
   D = DABS(C) - DABS(B)
   IF(D .LT. 0.0) GO TO 10
   INDEX = INDEX + 1
   IF(INDEX .GT. 5) GO TO 15
   FREMIS(INDEX) = SDT(1,I)
10 CONTINUE
IF(INDEX .EQ. 0) GO TO 90
15 CONTINUE
DO 60 I = 1, INDEX
   NS = 1
   HH = H/(FLOAT(NS) * 10.0)
   PP = FREMIS(INDEX)
   P = PP * PP
   F = DET(P)
   F = DSIGN(1.0D+00,F)
   A = FREMIS(INDEX) + 2.0 * H
   PP = PP + HH
   IF(PP .LT. A) GO TO 50
   P = PP * PP
   G = DET(P)
   G = DSIGN(1.0D+00,G)
   IF(F*G .GT. 0.0) GO TO 40
   IJ = IJ + 1
   P = (PP-HH) * (PP-HH)
   C = DET(P)
   P = PP * PP
   D = DET(P)
   FRE(IJ) = PP - D * HH/(D - C)
   IF((IJ/2*IJ) .EQ. 0) GO TO 60
40   F = G
   GO TO 30
50   NS = NS + 10
IF(NS .GT. 100) GO TO 70
GO TO 20
60 CONTINUE
70 WRITE(22,100)
WRITE(22,110)
DO 80 I = 1, INDEX
   A = FREMIS(I)/FACT
   B = A + 2.0 * H/FACT
   WRITE(22,120) A, B
80 CONTINUE
90 CONTINUE
100 FORMAT(1H1)
110 FORMAT(10X,'CHECK FOR TWO FREQUENCIES BETWEEN EACH OF THE 1',/10X,'FOLLOWING SETS. OTHERWISE, THEY ARE MISSING')
120 FORMAT(2(10X,E14.7))
RETURN
END

C
SUBROUTINE NATFRE(N)

THIS SUBROUTINE SCANS THE FREQUENCY DETERMINANT WITH RESPECT TO THE FREQUENCY UNTIL THE SPECIFIED NUMBER OF SIGN CHANGES ARE DETECTED, STARTING FROM ZERO FREQUENCY. USES THE "DET" ROUTINE

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION FREQEN(10), JKL(10), SDT(2,2000), FRE(10), STO(20)
COMMON/X1/FREQEN, H, H1, H2, IJK
IJK = 0
IJ = 0
ICOUNT = 1
DO 5 J = 1, N
   JKL(J) = 0
   PP = H1
   P = PP * PP
   F = DET(P)
   SDT(1,ICOUNT) = PP
   SDT(2,ICOUNT) = F
   IF(DABS(F) .GT. 0.0001) GO TO 10
   IJK = IJK + 1
   JKL(IJK) = 1
   FREQEN(IJK) = PP
   PP = PP + H
   P = PP * PP
   F = DET(P)
   ICOUNT = ICOUNT + 1
   SDT(1,ICOUNT) = PP
   SDT(2,ICOUNT) = F
10 F = DSIGN(1.0D+00,F)
15 PP = PP + H
   IF(PP .GT. H2) GO TO 55
   P = PP * PP
   G = DET(P)
   ICOUNT = ICOUNT + 1
   SDT(1,ICOUNT) = PP
   SDT(2,ICOUNT) = G
   IF(DABS(G) .GT. 0.0001) GO TO 20
   IJK = IJK + 1
   JKL(IJK) = 1
   FREQEN(IJK) = PP
   IF(IJK .EQ. N) GO TO 55
   PP = PP + H
   P = PP * PP
   F = DET(P)
   ICOUNT = ICOUNT + 1
   SDT(1,ICOUNT) = PP
   SDT(2,ICOUNT) = F
   F = DSIGN(1.0D+00,F)
   GO TO 15
20 G = DSIGN(1.0D+00,G)
   IF(F.GT.0.0) GO TO 25
IJK = IJK + 1
FREQEN(IJK) = PP - H
IF(IJK .EQ. N) GO TO 55
25 F = G
GO TO 15
55 IF(IJK .EQ. 0) GO TO 65
HS = H/10.0
DO 50 J = 1, IJK
   IF(JKL(J) .EQ. 1) GO TO 50
   PP = FREQEN(J)
   P = PP * PP
   F = DET(P)
   F = DSIGN(1.0D+00,F)
35 PP = PP + HS
   P = PP * PP
   G = DET(P)
   IF(DABS(G) .GT. 0.0001) GO TO 40
   JKL(J) = 1
   FREQEN(J) = PP
   GO TO 50
40 G = DSIGN(1.0D+00,G)
   IF(F*G .GT. 0.0) GO TO 45
   FREQEN(J) = PP - HS
   GO TO 50
45 F = G
GO TO 35
50 CONTINUE
DO 60 J = 1, IJK
   IF(JKL(J) .EQ. 1) GO TO 60
   PP = FREQEN(J)
   P = PP * PP
   F = DET(P)
   PP = PP + HS
   P = PP * PP
   G = DET(P)
   DIFF = G - F
   FREQEN(J) = PP - G*HS/DIFF
60 CONTINUE
65 CONTINUE
CALL MRROOT(SDT, H, FRE, IJ, ICOUNT)
IF(IJ .EQ. 0) GO TO 100
N2 = IJK + IJ
IF(IJK .EQ. 0) GO TO 75
DO 70 I = 1, IJK
70 STO(I) = FREQEN(I)
75 N1 = IJK + 1
DO 80 I = N1, N2
80 STO(I) = FREQEN(I-IJK)
DO 90 I = 2, N2
   DO 90 J = I, N2
   IF(STO(I-1) - STO(J)) 90, 90, 85
85 STORE = STO(J)
   STO(J) = STO(I-1)
   STO(I-1) = STORE
90 CONTINUE
IJK = N2
IF(N2 .GT. N) IJK = N
DO 95 I = 1, IJK
 95 FREQEN(I) = STO(I)
100 CONTINUE
RETURN
END
SUBROUTINE PLOT(NN, C, B)

PRIMTS AND PLOTS NATURAL VIBRATION CHARACTERISTICS

IMPLICIT REAL*8 (A - H, O - Z)
REAL*8 LINE(61)
DIMENSION C(51), B(3,11), SL1(11), SL2(51), A(61), SL(61)
COMMON/X4/NPATH,ICPL
COMMON/X6/SL1,SL2,CPM,FRE,HERTZ,BLANK,DOT,STAR,J1,IPLCT
WRITE(22,10)
WRITE(22,20)
WRITE(22,30) J1, FRE, HERTZ, CPM
WRITE(22,20)
IF(NN .EQ. 1)WRITE(22,39)
IF(NN .EQ. 2)WRITE(22,40)
IF(NN .EQ. 3)WRITE(22,41)
DO 50 J = 1, 11
   SL(J) = SL1(J)
   A(J) = B(1,J)
50 CONTINUE
DO 60 J = 2, 51
   SL(J+10) = SL2(J)
   A(J+10) = C(J)
60 CONTINUE
WRITE(22,70)
DO 80 J = 1, 11
   WRITE(22,90) SL(J),A(J),SL(J+21),A(J+21),SL(J+42),A(J+42)
80 CONTINUE
DO 81 J = 12, 19
   WRITE(22,90) SL(J),A(J),SL(J+21),A(J+21),SL(J+42),A(J+42)
81 CONTINUE
WRITE(22,70)
DO 120 I = 2, NPATH
   IF(NN .EQ. 1) WRITE(22,100) I
   IF(NN .EQ. 2) WRITE(22,101) I
   IF(NN .EQ. 3) WRITE(22,102) I
      WRITE(22,70)
      DO 120 J = 1, 3
         WRITE(22,90) SL1(J),B(I,J),SL1(J+4),B(I,J+4),SL1(J+8), B(I,J+8)
120 CONTINUE
WRITE(22,90) SL1(4), B(I,4), SL1(8), B(I,8)
130 CONTINUE
WRITE(22,10)
DO 150 J = 1, 61
LINE(J) = DOT

150 CONTINUE
J = 30.0 * (A(1) + 1.0) + 1.5
LINE(J) = STAR
WRITE(22,110) (LINE(J), J = 1, 61)
DO 160 J = 1, 61
   LINE(J) = BLANK
160 CONTINUE
LINE(31) = DOT
DO 170 JJ = 2, 61
   J = 30.0 * (A(JJ) + 1.0) + 1.5
   IF(J .GT. 61) GO TO 170
   LINE(J) = STAR
   WRITE(22,190) (LINE(JV), JV = 1,61)
   LINE(J) = BLANK
LINE(31) = DOT

170 CONTINUE
180 CONTINUE
10 FORMAT(1H1)
20 FORMAT(/2X, '******************************************************************************************',
     1 '******************************************************************************************')
30 FORMAT(/5X,'MODE NUMBER = ',I2,8X,'FREQ. RAD/SEC = ',F10.4,8X,
     1/5X,'FREQ. HERTZ = ',F10.4,8X,'FREQ. CYCLES/MINUTE = ',F10.4)
39 FORMAT(/15X,' FLAPWISE DEFLECTION/SEMI-CHORD ')
40 FORMAT(/15X,' CHORDWISE DEFLECTION/SEMI-CHORD ')
41 FORMAT(/15X,' TORSIONAL DEFLECTION (RAD) ')
70 FORMAT(/4X,'STA X/L',3X,'DEFLN',9X,'STA X/L',3X,'DEFLN',9X,
     1'STA X/L',3X,'DEFLN')
90 FORMAT(/3(2X,F8.4,3X,E11.4))
100 FORMAT(/5X,'FLAPWISE DEFLECTION/SEMI-CHORD LOAD PATH # ',I5)
101 FORMAT(/5X,'CHORDWISE DEFLECTION/SEMI-CHORD LOAD PATH # ',I5)
102 FORMAT(/5X,'TORSIONAL DEFLECTION (RAD) LOAD PATH # ',I5)
110 FORMAT(/4X,61A1)
190 FORMAT(4X,61A1)
1101 RETURN
END
FUNCTION RUNGE(Y, F, J, M, HH)

FOURTH ORDER RUNGE-KUTTA METHOD

IMPLICIT REAL*8 (A - H, O - Z)
INTEGER RUNGE
DIMENSION Y(12), F(12), PHI(12), SAVEY(12)
NN = 12
M = M + 1
GO TO(5, 10, 20, 30, 40), M
5  RUNGE = 1
    RETURN
10 DO 15 JJ = 1, NN
    SAVEY(JJ) = Y(JJ)
    PHI(JJ) = F(JJ)
    Y(JJ) = SAVEY(JJ) + HH * F(JJ) / 2.0
15 CONTINUE
   J = J + 1
   RUNGE = 1
   RETURN
20 DO 25 JJ = 1, NN
    PHI(JJ) = PHI(JJ) + 2.0 * F(JJ)
    Y(JJ) = SAVEY(JJ) + HH * F(JJ) / 2.0
25 CONTINUE
   RUNGE = 1
   RETURN
30 DO 35 JJ = 1, NN
    PHI(JJ) = PHI(JJ) + 2.0 * F(JJ)
    Y(JJ) = SAVEY(JJ) + HH * F(JJ)
35 CONTINUE
   J = J + 1
   RUNGE = 1
   RETURN
40 DO 45 JJ = 1, NN
45 Y(JJ) = SAVEY(JJ) + (PHI(JJ) + F(JJ)) * HH / 6.0
   M = 0
   RUNGE = 0
   RETURN
   END
SUBROUTINE SHAPES(P,W1,W2,V1,V2,PHI1,PHI2,U,UC,PSIC,ANUC)

C THIS SUBROUTINE COMPUTES THE NATURAL MODE SHAPES

C

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION W1(3,11),W2(51),V1(3,11), V2(51),
1 PHI1(3,11), PHI2(51),U(3),TF1(3,12,12),
1 TF2(12,12),TT1(3,11,3,12),TT2(51,3,12),
1 A(6,6),B(6,6),C(6,6),D(6,6),E(6,6),
1 FD(3,2),R(6,6),BB(12),TEM(12,12) , G(5,5)
COMMON/X4/NPATH,ICPL
COMMON/X8/BB,IND
COMMON/X9/FD
COMMON/X10/TF1, TF2
COMMON/X12/CON
COMMON/XS3/TT1, TT2
CALL TRAMAT(P)
DO 5 I = 1,12
DO 5 J = 1,12
TEM(I,J) = TF2(I,J)
5 CONTINUE
DO 10 I = 1,6
DO 10 J = 1,6
A(I,J) = TF2(I+6,J)
B(I,J) = TF2(I+6,J+6)
R(I,J) = 0.0
10 CONTINUE
DO 20 I = 1,NPATH
DO 15 J = 1,6
DO 15 K = 1,6
C(J,K) = TF1(I,J,K+6)
D(J,K) = TF1(I,J+6,K+6)
15 CONTINUE
CALL SOLUTN(C,6,-1,6)
DO 16 J = 1,6
C(J,4) = C(J,4) - FD(I,2) * C(J,1)
C(J,5) = C(J,5) - FD(I,1) * C(J,1)
16 CONTINUE
CALL MATMUL(6,6,6,D,C,E)
DO 18 J = 1,6
E(1,J) = E(1,J) - FD(I,2) * E(4,J)+ FD(I,1) * E(5,J)
E(2,J) = E(2,J) - FD(I,1) * E(6,J)
E(3,J) = E(3,J) + FD(I,2) * E(6,J)
18 CONTINUE
DO 20 J = 1,6
DO 20 K = 1,6
R(J,K) = R(J,K) + E(J,K)
20 CONTINUE
CALL MATMUL(6,6,6,B,R,D)
DO 25 J = 1,6
DO 25 K = 1,6
E(J,K) = D(J,K) + A(J,K)
25 CONTINUE
CALL SOLUTN(TEM,12,-1,12)
  DO 30 J = 1,6
  DO 30 K = 1,6
    D(J,K) = TEM(J,K)
30 CONTINUE
CALL MATMUL(6,6,6,E,D,A)
WRITE(24,111) ((A(I,J),J = 1,6),I = 1,6)
111 FORMAT(/(6(E11.4,1X)))
IF(ICPL .NE. 0) GO TO 310
C(1,1) = 0.0
C(2,1) = 1.0
DO 32 K = 1,4
  DO 31 J = 1,4
    G(J,K) = A(J,K+2)
31 CONTINUE
G(K,5) = -A(K,2)
32 CONTINUE
CALL SOLUTN(G,4,1,5)
DO 33 J = 3,6
  C(J,1) = BB(J-2)
33 CONTINUE
GO TO 375
310 DO 36 K = 1,5
  KK = K
  IF(K .GE. 2) KK = K + 1
  DO 35 J = 1,5
    D(J,K) = A(J,KK)
35 CONTINUE
D(K,6) = -A(K,2)
36 CONTINUE
CALL SOLUTN(D,5,1,6)
C(1,1) = BB(1)
C(2,1) = 1.0
DO 37 J = 3,6
  C(J,1) = BB(J-1)
37 CONTINUE
375 CONTINUE
376 DO 376 I = 1,6
  A(I,1) = 0.0
  B(I,1) = 0.0
  DO 376 K = 1,6
    A(I,1) = A(I,1) + TEM(I,K) * C(K,1)
    B(I,1) = B(I,1) + TEM(I+6,K) * C(K,1)
376 CONTINUE
DO 38 I = 1,51
  DO 39 J = 1,3
    STO = STO + TT2(I,J,K) * A(K,1) + TT2(I,J,K+6) * B(K,1)
38 CONTINUE
PHI2(I) = STO
IF(J .EQ. 1) W2(I) = STO * CON
IF(J .EQ. 2) W2(I) = STO * CON
39 CONTINUE
DO 48 J = 2,6
C(J,1) = A(J,1)
48 CONTINUE
DO 60 I = 1,NPATH
C(1,1)=A(1,1)-FD(I,2)*A(4,1)-FD(I,1)*A(5,1)
DO 50 J = 1,6
DO 50 K = 1,6
B(J,K) = TFi(I,J,K+6)
50 CONTINUE
CALL SOLUTN(B,6,-1,6)
CALL MATMUL(6,6,1,B,C,D)
DO 55 J = 1,11
W1(I,J) = 0.0
V1(I,J) = 0.0
PHI1(I,J) = 0.0
DO 55 K = 1,6
W1(I,J) = W1(I,J) + ( TT1(I,J,1,K+6) * D(K,1) ) * CON
V1(I,J) = V1(I,J) + ( TT1(I,J,2,K+6) * D(K,1) ) * CON
PHI1(I,J) = PHI1(I,J) + TT1(I,J,3,K+6) * D(K,1)
55 CONTINUE
U(I) = C(1,1)
60 CONTINUE
UC = A(1,1)
PSIC = A(4,1)
ANUC = A(5,1)
RETURN
END
SUBROUTINE SOLUTION(A, N, INDIC, NRC)

IMPLICIT REAL*8 (A - H, O - Z)
DIMENSION A(NRC,NRC), X(12), IROW(12), JCOL(12), JORD(12), Y(12)
COMMON/XS2/DETER
COMMON/X8/X, IND
IND = 0
MAX = N
IF(INDIC .GE. 0) MAX = N + 1
DETER = 1.0
DO 80 K = 1, N
   KM1 = K - 1
   DO 60 I = 1, N
      DO 60 J = 1, N
      IF(K .EQ. 1) GO TO 55
      DO 50 ISCAN = 1, KM1
         DO 50 JSCAN = 1, KM1
         IF (I .EQ. IROW(ISCAN)) GO TO 60
         IF(J .EQ. JCOL(JSCAN)) GO TO 60
      50 CONTINUE
      55 IF(DABS(A(I,J)) .LE. DABS(PIVOT)) GO TO 60
         PIVOT = A(I,J)
         IROW(K) = I
         JCOL(K) = J
   60 CONTINUE
   IF (DABS (PIVOT) .GT. 0.1E-20) GO TO 65
   DETER = 0.0
   IND = 1
   RETURN
   65 IROWK = IROW(K)
   JCOLK = JCOL(K)
   DETER = DETER * PIVOT
   DO 70 J = 1, MAX
      A(IROWK, J) = A(IROWK, J)/PIVOT
      A(IROWK, JCOLK) = 1.0/PIVOT
      DO 80 I = 1, N
         AIJCK = A(I, JCOLK)
         IF (I .EQ. IROWK) GO TO 80
         A(I, JCOLK) = -AIJCK/PIVOT
         DO 75 J = 1, MAX
            IF(J .NE. JCOLK) A(I,J) = A(I,J) - AIJCK * A(IROWK,J)
   70 CONTINUE
   DO 85 I = 1, N
      IROWI = IROW(I)
      JCOLI = JCOL(I)
      JORD(IROWI) = JCOLI
      IF (INDIC .GE. 0) X(JCOLI) = A(IROWI, MAX)
   80 CONTINUE
   CONTINUE
   INTCH = 0
   NM1 = N-1
   DO 90 I = 1, NM1
      IP1 = I + 1
      DO 90 J = IP1, N

IF(JORD(J) .GE. JORD(I)) GO TO 90
JTEMP = JORD(J)
JORD(J) = JORD(I)
JORD(I) = JTEMP
INTCH = INTCH + 1

90 CONTINUE
IF(INTCH/2*2 .NE. INTCH) DETER = -DETER
IF(INDIC .LE. 0) GO TO 94
RETURN

94  DO 100 J = 1, N
    DO 95 I = 1, N
      IROWI = IROW(I)
      JCOLI = JCOL(I)
      Y(JCOLI) = A(IROWI,J)
    CONTINUE
  95 CONTINUE
  100 CONTINUE
  A(I,J) = Y(I)

  105 CONTINUE
  DO 110 J = 1, N
    DO 105 I = 1, N
      IROWJ = IROW(J)
      JCOLJ = JCOL(J)
      Y(IROWJ) = A(I,JCOLJ)
    CONTINUE
  105 CONTINUE
  A(I,J) = Y(J)
  110 CONTINUE
RETURN
END

C
SUBROUTINE TRAMAT(P)

THIS SUBROUTINE COMPUTES THE TRANSFER MATRIX

IMPLICIT REAL*8 (A - H, O - Z)

INTEGER RUNGE

DIMENSION TF1(3,12,12), TF2(12,12), TT1(3,11,3,12),
TT2(51,3,12), T(12), V(12), TEN1(3,21), TEN2(101), EA1(3,21),
EA2(101), D11(3,21), D21(101), D12(3,21), D22(101), D13(3,21),
D123(101), D14(3,21), D24(101), D15(3,21), D25(101), D16(3,21),
D126(101), D17(3,21), D27(101), D18(3,21), D28(101), D19(3,21),
D129(101), D191(3,21), D291(101), D192(3,21), D292(101)

COMMON/X4/NPATH, ICPL

COMMON/X5/TEN1, TEN2, EA1, EA2, D11, D21, D12, D22, D13,
D123, D14, D24, D15, D25, D16, D26, D17, D27, D18, D28, D19,
D129, D191, D291, D192, D292, OMEGAN

COMMON/X7/HH1, HH2

COMMON/X10/TF1, TF2

COMMON/XS3/TT1, TT2

HH = HH1

PP = P + OMEGAN

DO 35 I = 1, NPATH

DO 35 J = 1, 12

DO 10 K = 1, 12

V(K) = 0.0

CONTINUE

V(J) = 1.0

M = 0

NJ = 1

DO 25 L = 1, 10

K = RUNGE(V, T, NJ, M, HH)

IF(K .NE. 1) GO TO 20

T(1) = EA1(I,NJ)*V(12)

T(2) = V(4)

T(3) = V(5)


T(6) = D11(I,NJ) * V(7)

T(7) = -P * D13(I,NJ) * V(2) + PP * D14(I,NJ) * V(3)

+ D15(I,NJ) * V(4) - D16(I,NJ) * V(5)

+ (D17(I,NJ) - P * D18(I,NJ)) * V(6)

T(8) = TEN1(I,NJ) * V(5) - D16(I,NJ) * V(6) - V(10)


T(10) = -PP * D12(I,NJ) * V(3) + PP * D14(I,NJ) * V(6)

T(11) = -P * D12(I,NJ) * V(2) - P * D13(I,NJ) * V(6)

T(12) = -PP * D12(I,NJ) * V(1)

GO TO 15

TT1(I,L+1,1,J) = V(2)

TT1(I,L+1,2,J) = V(3)

TT1(I,L+1,3,J) = V(6)

CONTINUE

DO 30 IJ = 1, 12
TF1(I,IJ,J) = V(IJ)

30 CONTINUE
35 CONTINUE
HH = HH2
DO 65 J = 1, 12
   DO 40 K = 1, 12
      V(K) = 0.0
   CONTINUE
   V(J) = 1.0
   M = 0
   NJ = 1
   DO 55 L = 1, 50
      K = RUNGE(V, T, NJ, M, HH)
      IF(K .NE. 1) GO TO 50
      T(1) = EA2(NJ)*V(12)
      T(2) = V(4)
      T(3) = V(5)
      T(4) = D29(NJ) * V(8) + D291(NJ) * V(9)
      T(5) = D292(NJ) * V(8) - D29(NJ) * V(9)
      T(6) = D21(NJ) * V(7)
      T(7) = -P * D23(NJ) * V(2) + PP * D24(NJ) * V(3)
           + D25(NJ) * V(4) - D26(NJ) * V(5)
           + (D27(NJ) - P * D28(NJ) ) * V(6)
      T(8) = TEN2(NJ) * V(5) - D26(NJ) * V(6) - V(10)
      T(9) = -TEN2(NJ) * V(4) - D25(NJ) * V(6) + V(11)
      T(10) = -PP * D22(NJ) * V(3) + PP * D24(NJ) * V(6)
      T(11) = -P * D22(NJ) * V(2) - P * D23(NJ) * V(6)
      T(12) = -PP * D22(NJ) * V(1)
      GO TO 45
   CONTINUE
   TT2(L+1,1,J) = V(2)
   TT2(L+1,2,J) = V(3)
   TT2(L+1,3,J) = V(6)
50 CONTINUE
   DO 60 IJ = 1,12
      TF2(IJ,J) = V(IJ)
   CONTINUE
60 CONTINUE
   DO 75 I = 1, NPATH
      DO 70 J = 1,3
         DO 70 K = 1,12
            TT1(I,1,J,K) = 0.0
         CONTINUE
         TT1(I,1,1,2) = 1.0
         TT1(I,1,2,3) = 1.0
         TT1(I,1,3,6) = 1.0
      CONTINUE
      TT2(1,J,K) = 0.0
   CONTINUE
   DO 80 J = 1,3
      DO 80 K = 1,12
         TT2(1,J,K) = 0.0
      CONTINUE
     TT2(1,1,2) = 1.0
     TT2(1,2,3) = 1.0
     TT2(1,3,6) = 1.0
   RETURN
END
APPENDIX B

USER'S INSTRUCTIONS

This program computes the natural frequencies and the associated mode shapes of a nonuniform pretwisted multiple load path rotating rotor blade with coupled flapwise bending, chordwise bending and torsional degrees of freedom with differential axial displacements in the load paths.

I DATA CARD

* NPATH, ISTAGE, IPLOT, NS1, NS2

* FORMAT(5 I 5)

NPATH: Number of load paths, maximum = 3.

ISTAGE: Program performs four functions:

1 - computes the values for frequency determinants only.

2 - computes natural frequencies and mode shapes.

3 - computes natural frequencies only.

4 - computes mode shapes corresponding to given natural frequencies.
IPLOT: 0 - no mode shape plots.

1 - plots mode shapes.

It is preferable to provide data such that the variation of the properties of the blade are linear or uniform between any two stations. For a uniform blade, it is enough to provide the data at the axis of rotation and the clevis for the load paths and clevis and the tip for the blade.

Note: Load paths - First station should correspond to the axis of rotation and the last station should correspond to the clevis.

Blade - First station should correspond to the clevis and the last station should correspond to the tip of the blade.
II DATA CARD SET

* SPAN1, SPAN2, SCH, OMEGA

* FORMAT(5 E 14.7)

SPAN1: span of the load paths (inches).

SPAN2: span of the blade (inches).
Note: SPAN1 * SPAN2 = Radius of the rotor.

SCH: semi-chord of the blade (inches).

OMEGA: rotational speed, (rpm).

The following data should be provided in the given order. The format is (5 E 14.7) unless otherwise specified.

1. STA1: station locations for load paths, (inches).

2. STA2: station locations for blade, (inches).

The following data corresponds to a single load path system:

3. MASS1: mass/unit length (lb-sq.sec/sq.in).

4. EIY1: flapwise bending stiffness (lb-sq.in).
5. \( E_{IZ1} \): chordwise bending stiffness (lb-sq.in).

6. \( G_{J1} \): torsional stiffness (lb-sq.in).

7. \( E_1 \): distance between mass and elastic axis (inches).

8. \( \beta_{1} \): twist of the load path including the collective (degrees).

9. \( K_{M1S1} \): mass moment of inertia about the chordwise axis. (lb-sq.sec).

10. \( K_{M2S1} \): mass moment of inertia about an axis perpendicular to the chordwise axis through the c.g. (lb-sq.sec).

11. \( E_{A1} \): axial stiffness (lb).

12. REPEAT THE DATA 3 TO 11 FOR LOAD PATHS 2 AND 3.

The following data corresponds to the blade:

13. \( M_{ASS2} \): Mass/unit length (lb-sq.sec/sq.in).

14. \( E_{IY2} \): flapwise bending stiffness (lb-sq.in).

15. \( E_{IZ2} \): chordwise bending stiffness (lb-sq.in).

16. \( G_{J2} \): torsional stiffness (lb-sq.in).

17. \( E_2 \): distance between mass and elastic axis (inches).
18. BETA2: twist of the blade including the collective (degrees).

19. KM1S2: mass moment of inertia about the chordwise axis. (lb-sq.sec).

20. KM2S2: mass moment of inertia about an axis perpendicular to the chordwise axis through the c.g. (lb-sq.sec).


III DATA CARD

* H1, H, H2

* FORMAT(3 E 14.7)

H1: starting frequency (rad/sec).

H: frequency increment (rad/sec).

H2: ending frequency (rad/sec).

CASE 1: ISTAGE = 1

In this case, only frequency determinants are computed for various frequencies. It starts with frequency H1, increments by H until it reaches the value H2.
CASE 2: ISTAGE = 2

In this case, the natural frequencies and mode shapes are computed by the frequency scanning technique. Values of frequency determinant are scanned starting from the value $H_1$ at steps of $H$ until the required sign changes are detected or the value $H_2$ is reached. So the required number of frequencies or the number of frequencies that lie between $H_1$ and $H_2$ whichever is less, are computed. If two frequencies are closer than increment $H$, then there is a possibility of missing these frequencies. In such cases the frequencies are detected by the fact that if any three consecutive determinants have the same sign and the absolute value of the middle determinant is the smallest of the three then there are two frequencies in that range. In this case smaller increments are taken to bracket the missing frequencies. However there exists a remote possibility of missing these roots also. In such a case a closer scanning of the frequency determinants is required.
IV DATA CARD

* FD(1,J)

* FORMAT(5 E 14.7)

This card is required if NPATH is greater than 1 and gives the location of the j-th load path with respect to the clevis (inches).

\[ FD(1,J) = h_y^j \]

\[ FD(2,J) = h_z^j \]

V DATA CARD

* ITR - number of tension iterations.

* FORMAT(I 5)

This card is required if NPATH > 1. ITR = 2 will be enough.
VI DATA CARD

* NF

* FORMAT(1 5)

This card is required if ISTAGE is not equal to 1.

NF = number of frequencies, maximum = 10

VII DATA CARD

* FREQEN

* FORMAT(5 E 14.7)

This card is required if ISTAGE = 4.

FREQEN(J) = natural frequencies.
VIII DATA CARD

* BLANK, DOT, STAR

* FORMAT(3 A 1)

This card is required if I_PLOT = 1.

BLANK = blank space

DOT = .

STAR = *
THE NON-ZERO ELEMENTS OF THE STIFFNESS MATRIX ARE:

\[ K_{11} = -\Omega_{j\alpha} \cdot M \cdot (L + 2 \cdot C9 \cdot L2 + C9 \cdot C9 \cdot L3) \]

\[ K_{12} = 4 \cdot C5 \cdot C5 \cdot L3 + 12 \cdot C5 \cdot C6 \cdot L4 + 9 \cdot C6 \cdot C6 \cdot L5 \]
\[ \text{TEM} = L - 2 \cdot C5 \cdot L3 - 2 \cdot C6 \cdot L4 + C5 \cdot C5 \cdot L5 + 2 \cdot C5 \cdot C6 \cdot L6 + 6 \cdot C6 \cdot C6 \cdot L7 \]

\[ K_{22} = C11 \cdot K_{22} - \Omega_{j\alpha} \cdot M \cdot \text{TEM} \]

\[ K_{33} = C1 \cdot C1 \cdot L3 + 2 \cdot C1 \cdot C2 \cdot L4 + C2 \cdot C2 \cdot L5 \]

\[ K_{44} = L + 2 \cdot C9 \cdot L2 + C9 \cdot C9 \cdot L3 \]

\[ K_{45} = \Omega_{j\alpha} \cdot (K_{M25} - K_{M15}) \cdot C_{TBETA} \cdot K_{44} \]

\[ K_{55} = L - 4 \cdot C3 \cdot L2 + 2 \cdot (2 \cdot C3 \cdot C3 - C4) \cdot L3 + 4 \cdot C3 \cdot C4 \cdot L4 + C4 \cdot C4 \cdot L5 \]

\[ K_{66} = L + 8 \cdot C7 \cdot L2 + (16 \cdot C7 \cdot C7 + 6 \cdot C8) \cdot L3 + 24 \cdot C7 \cdot C8 \cdot L4 + 9 \cdot C8 \cdot C8 \cdot L5 \]

\[ \text{TEM} = L + 4 \cdot C7 \cdot L4 + (4 \cdot C7 \cdot C7 + 2 \cdot C8) \cdot L5 + 4 \cdot C7 \cdot C8 \cdot L6 + C8 \cdot C8 \cdot L7 \]

\[ K_{66} = C11 \cdot K_{66} - \Omega_{j\alpha} \cdot M \cdot \text{TEM} \]

\[ K_{71} = -\Omega_{j\alpha} \cdot M \cdot C10 \cdot (L2 + C9 \cdot L3) \]

\[ K_{77} = -\Omega_{j\alpha} \cdot M \cdot C10 \cdot C10 \cdot L3 \]

\[ K_{82} = -C11 \cdot (4 \cdot C5 \cdot C5 \cdot L3 + 12 \cdot C5 \cdot C6 \cdot L4 + 9 \cdot C6 \cdot C6 \cdot L5) \]
\[ \text{TEM} = C5 \cdot L3 + C6 \cdot L4 - C5 \cdot C5 \cdot L5 - 2 \cdot C5 \cdot C6 \cdot L6 - C6 \cdot C6 \cdot L7 \]

\[ K_{82} = K_{82} - \text{TEM} \cdot \Omega_{j\alpha} \cdot M \]

C-1
K84 = -OMEGAS * M * E * SBETA * (C5 * L3 + (2. * C6 + C5 * C9) * L4 + 2. * C6 * C9 * L5)
K84 = K84 - OMEGAS * M * E * SBETA * TEM

TEM = C5 * C5 * L4 + (2. * C5 * C7 + C6) * L5 + (C5 * C8 + 2. * C6 * C7) * L6 + C6 * C8 * L7
K86 = C11 * K86 - OMEGAS * M * TEM

TEM = C5 * C5 * L5 + 2. * C5 * C6 * L6 + C6 * C6 * L7
K88 = C11 * K88 - OMEGAS * M * TEM


K94 = C1 * L3 + (C1 * C9 + C2) * L4 + C2 * C9 * L5
TEM = XJ * (C1 * L2 + (C1 * C9 + C2) * L3 + C2 * C9 * L4)
K94 = OMEGAS * M * E * CBETA * (K94 + TEM)

K95 = (K95 + C2 * C4 * L5) * C11


K102 = L2 + C5 * L4 + 2. * C6 * L5
TEM = 2. * C5 * L3 + 3. * C6 * L4
K102 = OMEGAS * M * E * SBETA * C10 * (K102 + XJ * TEM)

K103 = -OMEGAS * M * E * CBETA * C10 * (C1 * L4 + C2 * L5 + XJ * (C1 * L3 + C2 * L4))

K104 = OMEGAS * (KM2S - KM1S) * C10 * CTBETA * (L2 + C9 * L3)

K105 = OMEGAS * M * E * CBETA * C10 * K105

K106 = 2. * (C7 * L4 + C8 * L5)
TEM = L2 + 4. * C7 * L3 + 3. * C8 * L4
K106 = -OMEGAS * M * E * SBETA * C10 * (K106 + XJ * TEM)

K108 = C5 * L4 + 2. * C6 * L5
TEM = 2. * C5 * L3 + 3. * C6 * L4
K108 = -OMEGAS * M * E * SBETA * C10 * (K108 + XJ * TEM)

K109 = OMEGAS * M * E * CBETA * C10 * (C1 * L4 + C2 * L5 + XJ * (C1 * L3 + C2 * L4))

K1010 = OMEGAS * (KM2S - KM1S) * C10 * CTBETA * L3

K113 = -C11 * (C1 * C3 * L3 + (C1 * C4 + C2 * C3) * L4 + C2 * C4 * L5)

K114 = C3 * L3 + (C4 + C3 * C9) * L4 + C4 * C9 * L5
TEM = C3 * L2 + (C4 + C3 * C9) * L3 + C4 * C9 * L4
K114 = OMEGAS * M * E * CBETA * (K114 + XJ * TEM)


K119 = C11 * (C1 * C3 * L3 + (C1 * C4 + C2 * C3) * L4 + C2 * C4 * L5)
\[ K_{1110} = C_3L_4 + C_4L_5 + XJ(C_3L_3 + C_4L_4) \]
\[ K_{1110} = \Omega \text{M}E \text{CBETA}C_{10}K_{1110} \]

\[ K_{1111} = C_{11}(C_3C_3L_3 + 2.0C_3C_4L_4 + C_4C_4L_5) \]

\[ K_{122} = -C_{11}(4.0C_5C_7L_3 + 6.0(C_6C_7 + C_5C_8)L_4 + 9.0C_6C_8L_5) \]
\[ \text{TEM} = C_7L_3 + C_8L_4 - C_5C_7L_5 - (C_6C_7 + C_5C_8)L_6 - C_6C_8L_7 \]
\[ K_{122} = K_{122} - \Omega \text{M} \text{TEM} \]

\[ K_{124} = C_7L_3 + (2.0C_8 + C_7C_9)L_4 + 2.0C_8C_9L_5 \]
\[ \text{TEM} = XJ(2.0C_7L_2 + (3.0C_8 + 2.0C_7C_9)L_3 + 3.0C_8C_9L_4) \]
\[ K_{124} = -\Omega \text{M} \text{E} \text{SBETA}(K_{124} + \text{TEM}) \]

\[ K_{126} = 2.0C_7L_2 + (8.0C_7C_7 + 3.0C_8)L_3 + 18.0C_7C_8L_4 + 9.0C_8C_8L_5 \]
\[ \text{TEM} = C_7L_4 + (2.0C_7C_7 + C_8)L_5 + 3.0C_7C_8L_6 + C_8C_8L_7 \]
\[ K_{126} = C_{11}K_{126} - \Omega \text{M} \text{TEM} \]

\[ K_{128} = 4.0C_5C_7L_3 + 6.0(C_6C_7 + C_5C_8)L_4 + 9.0C_6C_8L_5 \]
\[ \text{TEM} = C_5C_7L_5 + (C_6C_7 + C_5C_8)L_6 + C_6C_8L_7 \]
\[ K_{128} = C_{11}K_{128} - \Omega \text{M} \text{TEM} \]

\[ K_{1210} = C_7L_4 + 2.0C_8L_5 + XJ(2.0C_7L_3 + 3.0C_8L_4) \]
\[ K_{1210} = -\Omega \text{M} \text{E} \text{SBETA}C_{10}K_{1210} \]

\[ K_{1212} = 4.0C_7C_7L_3 + 12.0C_7C_8L_4 + 9.0C_8C_8L_5 \]
\[ \text{TEM} = C_7C_7L_5 + 2.0C_7C_8L_6 + C_8C_8L_7 \]
\[ K_{1212} = C_{11}K_{1212} - \Omega \text{M} \text{TEM} \]
where

\[
\text{OMEGAS} = \omega^2; \quad C_1 = 6/\ell^2; \quad C_2 = -6/\ell^3; \quad C_3 = 2/\ell; \\
C_4 = -3/\ell^2; \quad C_5 = 3/\ell^2; \quad C_6 = -2/\ell^3; \quad C_7 = -1/\ell; \\
C_8 = 1/\ell^2; \quad C_9 = -1/\ell; \quad C_{10} = 1/\ell; \quad C_{11} = \ell; \\
L_2 = \ell^2/2; \quad L_3 = \ell^3/3; \quad L_4 = \ell^4/4; \quad L_5 = \ell^5/5; \\
L_6 = \ell^6/6; \quad L_7 = \ell^7/7.
\]

\(\ell\) = length of the element; \(M\) = mass per unit length;
\(X_J\) = distance of the element from the axis of rotation;
\(E\) = distance between mass and elastic axes, \(e\);
\(SBETA = \sin\beta\); \(CBETA = \cos\beta\)

\[
KM_1S = \frac{k^2}{m_1}; \quad KM_2S = \frac{k^2}{m_2}; \quad \text{CTBETA} = \cos 2\beta
\]

\[
T = \int_{\ell}^{R} \Omega^2 mx \, dx; \quad \ell_e = \ell/2 + X_J
\]